On Finite W-algebras and Yangians

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1. Introduction

- A *finite W-algebra* is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semi-simple Lie algebra or a classical Lie superalgebra and $e \in \mathfrak{g}$ is an even nilpotent element.

- A finite W-algebra is a generalization of the universal enveloping algebra $U(\mathfrak{g})$.

- It is a quantization of the Poisson algebra of functions on the Slodowy slice at e to the orbit $Ad(G)e$, where $\mathfrak{g} = Lie(G)$.

- Due to recent results of I. Losev, A. Premet and others, finite W-algebras play a very important role in description of primitive ideals.

- Finite W-algebras for semi-simple Lie algebras were introduced by A. Premet.

- Finite W-algebras for Lie algebras and superalgebras have been extensively studied by mathematicians and physicists: L. Fehér, C. Briot, E. Ragoucy, A. Premet, I. Losev, V. Ginzburg, W. L. Gan, J. Brundan, J. Brown, S. Goodwin, W. Wang, L. Zhao, Y. Zeng, B. Shu.

2. Finite W-algebras for Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with reductive even part $\mathfrak{g}_{\bar{0}}$,

$\chi \in \mathfrak{g}_{\bar{0}}^* \subset \mathfrak{g}^*$ be an even element in the coadjoint representation,

$G_{\bar{0}}$ be the algebraic reductive group of $\mathfrak{g}_{\bar{0}}$.

Definition. χ is **nilpotent** if the closure of the $G_{\bar{0}}$-orbit in $\mathfrak{g}_{\bar{0}}^*$ contains zero.

Let $\mathfrak{g}^\chi$ be the **annihilator** of χ in $\mathfrak{g}$:

$$\mathfrak{g}^\chi = \{x \in \mathfrak{g} \mid \chi([x, \mathfrak{g}]) = 0\}$$

Definition. A good $\mathbb{Z}$-grading for χ is a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ satisfying the following two conditions:

(1) $\chi(\mathfrak{g}_j) = 0$ if $j \neq -2$,

(2) $\mathfrak{g}^\chi$ belongs to $\bigoplus_{j\geq 0} \mathfrak{g}_j$.

- $\chi([\cdot,\cdot]) : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C}$ is a non-degenerate (super)skew-symmetric even bilinear form on $\mathfrak{g}_{-1}$.

- Let $\mathfrak{l}$ be a maximal isotropic (Lagrangian) subspace with respect to this form.

Let $\mathfrak{m} = (\bigoplus_{j \leq -2} \mathfrak{g}_j) \oplus \mathfrak{l}$. The restriction of χ to $\mathfrak{m}$

$$\chi : \mathfrak{m} \longrightarrow \mathbb{C}$$

defines a one-dimensional representation $C_\chi =< v >$ of $\mathfrak{m}$.

Definition. The generalized Whittaker module

is the induced $\mathfrak{g}$-module

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} C_\chi \cong U(\mathfrak{g})/I_\chi,$$

where I_χ is the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$.

Definition. The finite W-algebra associated to the nilpotent element χ is

$$W_\chi := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}$$

• Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ be the natural projection. Then

$$W_\chi = (Q_\chi)^{\mathfrak{m}} = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid \mathrm{ad}(a)y \in I_\chi \text{ for all } a \in \mathfrak{m}\}$$

The algebra structure on W_χ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $\mathrm{ad}(a)y_i \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1, 2$.

• As in the Lie algebra case, the superalgebras W_χ are all isomorphic for different choices of good $\mathbb{Z}$-gradings and maximal isotropic subspaces $\mathfrak{l}$.

• If $\mathfrak{g}$ admits an even non-degenerate invariant supersymmetric bilinear form, then $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\chi(x) = (e|x)$ for some nilpotent $e \in \mathfrak{g}_{\bar{0}}$.

• By *the Jacobson–Morozov theorem* e can be included in $\mathfrak{sl}(2) =< e, h, f >$.

The linear operator ad$h$ defines a **Dynkin** $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \mathrm{ad}h(x) = jx\}.$$

Remark.

(1) The Dynkin $\mathbb{Z}$-grading is good for χ,

(2) $e \in \mathfrak{g}_2$.

Example. Let $\chi = 0$. Then $\mathfrak{m} = 0$,

$$Q_\chi = U(\mathfrak{g}), \quad W_\chi = U(\mathfrak{g}).$$

Definition. A nilpotent $\chi \in \mathfrak{g}_{\bar{0}}^*$ is called **regular** if the $G_{\bar{0}}$-orbit of χ has maximal dimension, i.e. the dimension of $\mathfrak{g}_{\bar{0}}^\chi$ is minimal.

Theorem. (*B. Kostant, 1978*)

For a regular nilpotent χ and a reductive Lie algebra $\mathfrak{g}$ the algebra W_χ is isomorphic to the center of $U(\mathfrak{g})$.

- Theorem of Kostant does not hold for Lie superalgebras,

  since W_χ must have a non-trivial odd part, and the center of $U(\mathfrak{g})$ is even.

• J. Brown, J. Brundan and S. Goodwin proved that the finite W-algebra for the *general linear Lie superalgebra* $\mathfrak{g} = \mathfrak{gl}(l|k)$ associated to **regular (principal)** nilpotent element is a certain truncation of a shifted version of the super-Yangian $Y(\mathfrak{gl}(1|1))$.
(Algebra Numb. Theory, 2013).

• We described the finite W-algebra for the *queer* Lie superalgebra $Q(n)$ associated to **regular** nilpotent element as a quotient of the super-Yangian $Y(Q(1))$.
(To arrear in *Adv. Math.*)

Recall: For a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, the **Yangian** of $\mathfrak{g}$ is an infinite-dimensional *Hopf algebra* $Y(\mathfrak{g})$. It is a deformation of the universal enveloping algebra of the Lie algebra of polynomial currents of $\mathfrak{g}$.

3. The super-Yangian of $\mathfrak{gl}(1|1)$

$$\mathfrak{gl}(1|1) = \{A = \left(\begin{array}{c|c} c & b \\ \hline c & d \end{array}\right) \mid a, b, c, d \in \mathbb{C}\} \qquad [A, B] = AB - (-1)^{p(A)p(B)}BA$$

Definition. The **super-Yangian** $Y = Y(\mathfrak{gl}(1|1))$ is an associative unital superalgebra over $\mathbb{C}$ with a countable set of generators

$$t_{i,j}^{(r)} \text{ where } i, j = 1, 2, \text{ and } r \geq 0.$$

The $\mathbb{Z}_2$-grading of Y is defined by

$$p(t_{i,j}^{(r)}) = p(i) + p(j).$$

We employ the formal series:

$$t_{i,j}(u) = \sum_{r \geq 0} t_{i,j}^{(r)} u^{-r} \in Y[[u^{-1}]].$$

- Relations in Y:

$$(u - v)[t_{i,j}(u), t_{k,l}(v)] =$$
$$(-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}((t_{k,j}(u)t_{i,l}(v) - t_{k,j}(v)t_{i,l}(u)).$$

4. Principal W-algebra W_π

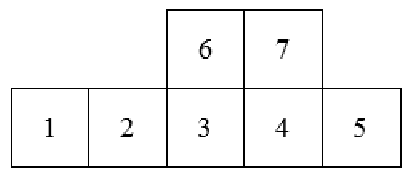
$\mathfrak{g} = \mathfrak{gl}(l|k)$ is a general linear Lie superalgebra (assume that $l \geq k$)

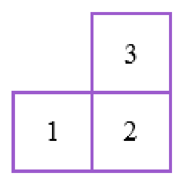
Definition. π is a two-rowed **pyramid:**

k is the number of boxes in the 1-st row,

l is the number of boxes in the 2-nd row.

Example.

$\mathfrak{g} = \mathfrak{gl}(5|2)$ $(l = 5,\ k = 2)$ $\qquad \pi =$ 

$\mathfrak{g} = \mathfrak{gl}(2|1)$ $(l = 2,\ k = 1)$ $\qquad \pi =$ 

- Pyramid π defines $\mathbb{Z}$**-grading** on $\mathfrak{g}$:

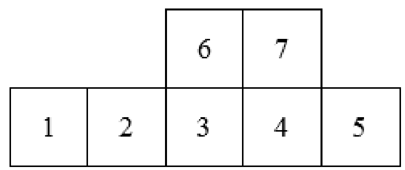
$$\mathfrak{g} = \oplus_{r \in \mathbb{Z}} \mathfrak{g}(r) \qquad \deg(e_{i,j}) := col(j) - col(i), \qquad \mathfrak{h} := \mathfrak{g}(0)$$

• The explicit **principal** nilpotent element e is

$$e := \sum_{i,j} e_{i,j} \in \mathfrak{g}_{\bar{0}}$$

summing over all adjacent pairs of boxes in π.

Example.

$\mathfrak{g} = \mathfrak{gl}(5|2)$ $\pi =$ 

$e = e_{1,2} + e_{2,3} + e_{3,4} + e_{4,5} + e_{6,7}$

• It is a **good** $\mathbb{Z}$-grading for e:

$$e \in \mathfrak{g}(1) \text{ and } \mathfrak{g}^e := \mathrm{Ker}(ade) \subseteq \oplus_{r\geq 0}\mathfrak{g}(r)$$

• Set $\chi(x) := (e|x) \quad$ for $x \in \mathfrak{g}$

Remark. We double the degree to agree with the previous definition of a good $\mathbb{Z}$-grading.

Definition. The principal W-algebra W_π associated to the pyramid π

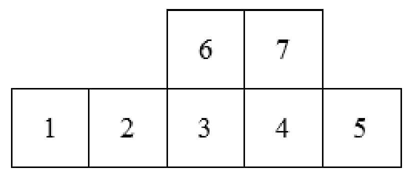
is defined as usual.

Definition. Matrix $\sigma = \begin{pmatrix} 0 & s_{1,2} \\ s_{2,1} & 0 \end{pmatrix}$ is **compatible** with pyramid π if π has
$s_{2,1}$ columns of hight one on its *left* side and
$s_{1,2}$ columns of hight one on its *right* side,
or if $k = 0$ and $l = s_{2,1} + s_{1,2}$.

- $l = k + s_{1,2} + s_{2,1}$

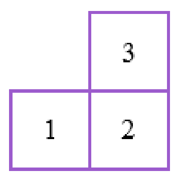
Example.

$\mathfrak{g} = \mathfrak{gl}(5|2)$, $l = 5, k = 2$ $\qquad \pi =$



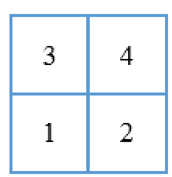
$\qquad \sigma = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

$\mathfrak{g} = \mathfrak{gl}(2|1)$, $l = 2, k = 1$ $\qquad \pi =$



$\qquad \sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathfrak{g} = \mathfrak{gl}(2|2)$, $l = 2, k = 2$ $\qquad \pi =$



$\qquad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Theorem. *(Brown-Brundan-Goodwin, 2012)*

The principal W-algebra W_π for $\mathfrak{gl}(l|k)$ is isomorphic to the **shifted super-Yangian** Y^l_σ **of level** l.

Idea of Proof. Define **the Miura transform:**

$$\mu : W_\pi \longrightarrow U(\mathfrak{h}) \cong U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes U(\mathfrak{gl}(1|1))^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}},$$

where $U(\mathfrak{gl}_1) := \mathbb{C}[e_{1,1}]$.

μ is injective and

$$\mu(W_\pi) = Y^l_\sigma \cong Y_\sigma / I^l_\sigma,$$

where Y_σ is the **shifted super-Yangian** (a subalgebra of Y),

and I^l_σ is a two-sided ideal.

• $Y^l_\sigma \subset U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes U(\mathfrak{gl}(1|1))^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}}$ is the image of Y_σ under homomorphism defined using the comultiplication on Y and the evaluation map $Y \to U(\mathfrak{gl}(1|1))$.

5. The queer Lie superalgebra $\mathfrak{g} = \mathbf{Q}(\mathbf{n})$

$$Q(n) = \{\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \mid A, B \text{ are } n \times n \text{ matrices}\}$$

$e_{i,j}$ and $f_{i,j}$ are standard bases in A and B respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right)$$

$z = \sum_{i=1}^{n} e_{i,i}$ is a central element

$Q(n)$ admits an **odd** nondegenerate $\mathfrak{g}$-invariant super symmetric bilinear form

$$(x|y) := otr(xy) \text{ for } x, y \in \mathfrak{g},$$

where $otr \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = trB.$

Let $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = \sum_{i=1}^{n-1} e_{i,i+1}, \quad h = \mathrm{diag}(n-1, n-3, \ldots, 3-n, 1-n), \quad f = \sum_{i=1}^{n-1} i(n-i)e_{i+1,i}.$$

e is a regular nilpotent element.

h defines an **even** Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices are

$$\left(\begin{array}{cccc|cccc}
0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\
-2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \\
\hline
0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\
-2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0
\end{array}\right)$$

Replace $e = \sum_{i=1}^{n-1} e_{i,i+1}$ by $E = \sum_{i=1}^{n-1} f_{i,i+1} \Rightarrow E$ is odd.

Define $\chi \in \mathfrak{g}_{\bar{0}}^*$ by $\chi(x) = (x|E)$.

$$\mathfrak{m} := \bigoplus_{j=2}^{n} \mathfrak{g}_{2-2j}.$$

It is generated by $e_{i+1,i}$ and $f_{i+1,i}$ where $i = 1, \ldots, n-1$

$$\chi(e_{i+1,i}) = 1, \quad \chi(f_{i+1,i}) = 0$$

- The left ideal I_χ and W_χ are defined as usual.

$$\mathfrak{p} := \bigoplus_{j=0}^{n-1} \mathfrak{g}_{2j}$$

is a **parabolic subalgebra** of $\mathfrak{g}$,

$\mathfrak{g}_0$ is a **Cartan subalgebra**:

$$\mathfrak{g}_0 = \mathfrak{h} =< x_i, \xi_i \mid i = 1, \ldots, n >, \text{ where } x_i = e_{i,i}, \ \xi_i = f_{i,i}$$

• Since the good $\mathbb{Z}$-grading is *even*, then W_χ can be regarded as a *subalgebra* of $U(\mathfrak{p})$.

6. The Harish-Chandra homomorphism

$$U(\mathfrak{p})^+ := \bigoplus_{i>0} U(\mathfrak{p})_i$$

is a two sided ideal in $U(\mathfrak{p})$ and $U(\mathfrak{p})/U(\mathfrak{p})^+ \cong U(\mathfrak{g}_0) = U(\mathfrak{h})$.

- Let $\vartheta : U(\mathfrak{p}) \longrightarrow U(\mathfrak{h})$ be the natural projection.

Theorem. *(P–S)* The restriction of ϑ on W_χ is injective:

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{h}).$$

• A. Sergeev defined by induction the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ belonging to $U(Q(n))$:

$$e_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} e_{k,j}^{(m-1)} + (-1)^{m+1} \sum f_{i,k} f_{k,j}^{(m-1)},$$

$$f_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} f_{k,j}^{(m-1)} + (-1)^{m+1} \sum f_{i,k} e_{k,j}^{(m-1)}.$$

Theorem. *(P–S)*
W_χ has n even generators: $\pi(e_{n,1}^{(n+k-1)})$ and n odd generators: $\pi(f_{n,1}^{(n+k-1)})$, $k = 1, \ldots, n$.

The images of these generators under the Harish-Chandra homomorphismare are the following elements in $U(\mathfrak{h})$:

$$\vartheta(\pi(e_{n,1}^{(n+k-1)})) = [\sum_{i_1 \geq i_2 \geq \ldots \geq i_k} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \ldots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{even},$$
$$\vartheta(\pi(f_{n,1}^{(n+k-1)})) = [\sum_{i_1 \geq i_2 \geq \ldots \geq i_k} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \ldots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{odd}.$$

7. The super-Yangian of $\mathbf{Q(n)}$

- Super-Yangian $Y(Q(n))$ was studied by M. Nazarov and A. Sergeev.

- $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$t_{i,j}^{(r)} \text{ where } r = 1, 2, \dots \text{ and } i, j = \pm 1, \pm 2, \dots, \pm n$$

The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$:

$$p(t_{i,j}^{(r)}) = p(i) + p(j), \text{ where } p(i) = 0 \text{ if } i > 0 \text{ and } p(i) = 1 \text{ if } i < 0$$

- We employ the formal series:

$$t_{i,j}(u) = \sum_{r\geq 0} t_{i,j}^{(r)} u^{-r} \in Y(Q(n))[[u^{-1}]].$$

- The relations in $Y(Q(n))[[u^{-1}, v^{-1}]]$:

$$\begin{aligned}(u^2 - v^2)[t_{i,j}(u), t_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\ = (u+v)(t_{k,j}(u)T_{i,l}(v) - t_{k,j}(v)T_{i,l}(u)) \\ -(u-v)(t_{-k,j}(u)T_{-i,l}(v) - t_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}\end{aligned}$$

- We also have the relations

$$t_{i,j}(-u) = t_{-i,-j}(u)$$

- From now on we will use only $Y(Q(1))$.

$$Q(1) = \{ \left(\begin{array}{c|c} a & b \\ \hline b & a \end{array} \right) \mid a, b \in \mathbb{C} \}$$

$Y(Q(1))$ is generated by $t_{1,1}^{(k)}$ (even) and $t_{-1,1}^{(k)}$ (odd) for $k = 1, 2, \ldots$

The Main Theorem. *(P–S)* There exists a surjective homomorphism:

$$\varphi : Y(Q(1)) \longrightarrow W_\chi$$

defined as follows:

$$\varphi(t_{1,1}^{(k)}) = (-1)^k \pi(e_{n,1}^{(n+k-1)}),$$

$$\varphi(t_{-1,1}^{(k)}) = (-1)^k \pi(f_{n,1}^{(n+k-1)}), \quad \text{for } k = 1, 2, \ldots$$

- $Y(Q(1))$ is a Hopf superalgebra with comultiplication given by the formula

$$\Delta(t_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} t_{i,k}^{(s)} \otimes t_{k,j}^{(r-s)}.$$

- **The opposite comultiplication:**

$$\Delta^{op}(t_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} t_{k,j}^{(r-s)} \otimes t_{i,k}^{(s)}.$$

- Homomorphism of associative algebras:

$\Delta_n^{op} : Y(Q(1)) \to Y(Q(1))^{\otimes n}$

$$\Delta_n^{op} := \Delta_{n-1,n}^{op} \cdots \circ \Delta_{2,3}^{op} \circ \Delta^{op}$$

- **The evaluation homomorphism** $U : Y(Q(1)) \to U(Q(1))$:

$$t_{1,1}^{(r)} \mapsto (-1)^r e_{1,1}^{(r)}, \quad t_{-1,1}^{(r)} \mapsto (-1)^r f_{1,1}^{(r)}.$$

- Identify $U(\mathfrak{h}) \subset U(Q(n))$ with $U(Q(1))^{\otimes n}$ by setting

$$x_i \mapsto 1^{\otimes n-i} \otimes e_{1,1} \otimes 1^{\otimes i-1}, \quad \xi_i \mapsto 1^{\otimes n-i} \otimes f_{1,1} \otimes 1^{\otimes i-1}.$$

- Obtain the homomorphism

$$\left(U^{\otimes n} \circ \Delta_n^{op}\right) : Y(Q(1)) \longrightarrow U(\mathfrak{h}),$$

where

$$\left(U^{\otimes n} \circ \Delta_n^{op}\right)(t_{1,1}^{(k)} + t_{-1,1}^{(k)}) = (-1)^k \sum_{i_1 \geq i_2 \geq \ldots \geq i_k} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \ldots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})$$

Proof of the Main Theorem.

We use that the Harish-Chandra homomorphism $\vartheta : W_\chi \to U(\mathfrak{h})$ is injective.

We also have that

$$\Big(U^{\otimes n} \circ \Delta_n^{op}\Big)(t_{1,1}^{(k)}) = (-1)^k \vartheta(\pi(e_{n,1}^{(n+k-1)})) \qquad (even)$$

$$\Big(U^{\otimes n} \circ \Delta_n^{op}\Big)(t_{-1,1}^{(k)}) = (-1)^k \vartheta(\pi(f_{n,1}^{(n+k-1)})) \qquad (odd)$$

Hence

$$\varphi := \vartheta^{-1} \circ U^{\otimes n} \circ \Delta_n^{op}$$

is a surjective homomorphism $\varphi : Y(Q(1)) \longrightarrow W_\chi$.

Remark. The proof of above theorem uses the same method as in *J. Brundan, A. Kleshchev, Adv. Math., 2006*, but we start with

$$U : Y(Q(1)) \to U(Q(1))$$

instead of the **usual evaluation map**

$$ev : Y(Q(1)) \to U(Q(1))$$

given by

$$t_{1,1}^{(1)} \mapsto -e_{1,1}, \quad t_{-1,1}^{(1)} \mapsto -f_{1,1}, \quad t_{i,j}^{(r)} \mapsto 0 \text{ for } r \geq 2$$

It is desirable to construct a homomorphism

$$Y(Q(1)) \longrightarrow W_\chi$$

using the evaluation map, but at the moment we do not have suitable generators in W_χ.

Conjecture. Let $\mathfrak{g}$ be a basic classical Lie superalgebra and e be regular nilpotent. Then it is possible to find a set of generators of W_χ such that even generators commute, and the commutators of odd generators are in the center of $U(\mathfrak{g})$.

• We proved this conjecture for $\mathfrak{g} = Q(n)$. The proof is based on the surjective homomorphism:

$$\varphi : Y(Q(1)) \longrightarrow W_\chi,$$

and the following relation in $Y(Q(1))$:

• If $r + s$ is even, then

$$[t_{1,1}^{(r)}, t_{1,1}^{(s)}] = 0.$$

8. The case of a regular nilpotent χ

- If χ is **regular**, then $\mathfrak{g}_0 = \mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$.

$$\mathfrak{h} =< x_i, \xi_i \mid i = 1, \dots, n >, \text{ where } x_i = e_{i,i}, \ \xi_i = (-1)^{i+1} f_{i,i}$$

x_i lie in the center of $U(\mathfrak{h})$ and $[\xi_i, \xi_i] = 2x_i$.

- $U(\mathfrak{h}_{\bar{0}}) = \mathbb{C}[x_1, \dots, x_n]$ coincides with the center of $U(\mathfrak{h})$.

Theorem *(P–S)* Let M be a simple W_χ-module. Then

$$\dim M \leq 2^{k+1}, \text{ where } k = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The proof is based on the *Amitsur–Levitzki theorem.*

Theorem. *(A–L)* If $A_1, \dots, A_{2n}$ are $n \times n$ matrices, then

$$\sum_{\sigma \in S_{2n}} sgn(\sigma) A_{\sigma(1)} \dots A_{\sigma(2n)} = 0.$$

Idea of Proof.

(1) $U(\mathfrak{h})$ satisfies Amitsur–Levitzki identity, i.e. for any $u_1, \dots, u_{2^{k+1}} \in U(\mathfrak{h})$

$$\sum_{\sigma \in S_{2^{k+1}}} sgn(\sigma) u_{\sigma(1)} \dots u_{\sigma(2^{k+1})} = 0. \qquad (*)$$

(2) W_χ satisfies Amitsur–Levitzki identity, since $W_\chi \cong \vartheta(W_\chi) \subset U(\mathfrak{h})$.

(3) Consider M as a module over the associative algebra W_χ, forgetting the $\mathbb{Z}_2$-grading. Then either M is simple or M is a direct sum of two non-homogeneous simple submodules $M_1 \oplus M_2$.

(a) In the former case $\dim M \leq 2^k$.

Assume $\dim M > 2^k$. Let V be a subspace of dimension $2^k + 1$. By density theorem for any $X_1, \ldots, X_{2^{k+1}} \in \mathrm{End}_{\mathbb{C}}(V)$ one can find $u_1, \ldots, u_{2^{k+1}}$ in W_χ such that $(u_i)_{|V} = X_i$ for all $i = 1, \ldots, 2^{k+1}$. Since $\mathrm{End}_{\mathbb{C}}(V)$ does not satisfy (*) we obtain a contradiction.

(b) In the latter case, we can prove in the same way that $\dim M_1 \leq 2^k$ and $\dim M_2 \leq 2^k$. Therefore $\dim M \leq 2^{k+1}$.

Theorem. *(P–S)*
For a basic classical Lie superalgebra or $Q(n)$ and a regular nilpotent χ, all irreducible representations of W_χ are finite-dimensional:

$$\dim M \leq 2^{k+1}$$

Let d be the **defect** of $\mathfrak{g}$.

In other cases we set $k = d$ or $k = d + 1$.

- $k = d$, if $\mathfrak{g}$ is of type I: $\mathfrak{g} = \mathfrak{sl}(m|n), \mathfrak{osp}(2|2n)$,

  or $\mathfrak{g}$ is of type II and $\dim(\mathfrak{g}_{\bar{1}}^{\chi})$ is even: $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m \geq n$,
  $\mathfrak{osp}(2m|2n)$ for $m \leq n$, G_3.

- $k = d + 1$, if $\mathfrak{g}$ is of type II and $\dim(\mathfrak{g}_{\bar{1}}^{\chi})$ is odd:

  $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m < n$, $\mathfrak{osp}(2m|2n)$ for $m > n$, $D(2,1;\alpha)$, F_4.

Idea of Proof.

1) If the good $\mathbb{Z}$-grading of $\mathfrak{g}$ with respect to χ is **even**,
then there is an injective homomorphism

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{g}_0).$$

2) If the good $\mathbb{Z}$-grading is **odd**,
then there is an injective homomorphism

$$\vartheta : W_\chi \longrightarrow \bar{W}^{\mathfrak{s}}_\chi,$$

where $\bar{W}^{\mathfrak{s}}_\chi$ is *"the finite W-algebra"* of $\mathfrak{s}$:

$\mathfrak{s}$ is the Levi subalgebra of a parabolic subalgebra $\mathfrak{p}$, such that $\mathfrak{n}^- \subset \mathfrak{m} \subset \mathfrak{p}^-$, where $\mathfrak{n}^-$ is the nilradical of the opposite parabolic $\mathfrak{p}^-$.

$\bar{W}^{\mathfrak{s}}_\chi = (U(\mathfrak{s}) \otimes_{U(\mathfrak{m}^{\mathfrak{s}})} C_\chi)^{\mathfrak{m}^{\mathfrak{s}}}$, where $\mathfrak{m}^{\mathfrak{s}} = \mathfrak{m} \cap \mathfrak{s}$, χ is the restriction of χ on $\mathfrak{s}$.

3) One can show that if χ is regular, then $U(\mathfrak{g}_0)$ (correspondingly, $\bar{W}^{\mathfrak{s}}_\chi$) satisfies Amitsur–Levitzki identity. Hence W_χ satisfies Amitsur–Levitzki identity.

9. References

[1] C. Briot, E. Ragoucy, *W-superalgebras as truncations of super-Yangians*, J. Phys. A**36** (2003), no. 4, 1057-1081.

[2] J. Brown, J. Brundan, S. Goodwin, Principal W-algebras for $GL(m|n)$, Algebra Numb. Theory **7** (2013), 1849–1882.

[3] J. Brundan, A. Kleshchev, Shifted Yangians and finite W-algebras, *Adv. Math.* **200** (2006), 136–195.

[4] B. Kostant, *On Wittaker vectors and representation theory*, Invent. Math. **48** (1978) 101-184.

[5] M. Nazarov, Yangian of the "strange" Lie superalgebras, Quantum groups, 90–97, *Lecture Notes in Math.* **1510**, Springer, 1992.

[6] M. Nazarov, A. Sergeev, *Centralizer construction of the Yangian of the queer Lie superalgebra*, Studies in Lie Theory, 417-441, Progr. Math. **243** (2006).

[7] E. Poletaeva, V. Serganova, *On Kostant's theorem for the Lie superalgebra $Q(n)$*. To appear in Advances of Mathematics. arXiv:1403.3866v1.

[8] A. Premet, *Special transverse slices and their enveloping algebras*, Adv. Math. **170** (2002) 1–55.

[9] A. Sergeev *The centre of envelloping algebra for Lie superalgebra $Q(n, \mathbb{C})$*, Letters in Math. Phys. **7** (1983) 177-179.

[10] Y. Zeng and B. Shu, Finite W-superalgebras for basic classical Lie superalgebras; arXiv:1404.1150v2.