Classifying $A_{\mathfrak{q}}(\lambda)$ modules by their Dirac cohomology

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The corresponding K are $SO(n) \subset SL(n, \mathbb{R})$; $U(p) \times U(q) \subset U(p, q)$; $U(n) \subset Sp(2n, \mathbb{R})$; $(O(p) \times O(q))_0 \subset O(p, q)_0$.

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which are compatible
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$\mathfrak{k}_0 =$ the Lie algebra of K.)

Such M can be decomposed under K as

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M is a Harish-Chandra module if it is finitely generated and all $m_\delta < \infty$.

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$\mathfrak{g}$ has a basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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The possible irreducible modules are

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \cdots \\ k & k+2 & k+4 & \cdots \end{array} \tag{1}$$

$$\begin{array}{cccc} \cdots & \bullet & \bullet & \bullet \\ \cdots & -k-4 & -k-2 & -k \end{array} \tag{2}$$

$$\begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ -n & -n+2 & \cdots & n \end{array} \tag{3}$$

$$\begin{array}{ccccc} \cdots & \bullet & \bullet & \bullet & \cdots \\ \cdots & i-2 & i & i+2 & \cdots \end{array} \tag{4}$$

where $k > 0$, $n \geq 0$ and i are integers.

A dot = a K-type = a 1-dim h-eigenspace. The numbers are the h-eigenvalues.

e raises the eigenvalue by 2, f lowers by 2.

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The simplest nontrivial element of $Z(\mathfrak{g})$ is the Casimir element

$$\mathrm{Cas}_{\mathfrak{g}} = \sum b_i d_i,$$

where b_i and d_i are dual bases of $\mathfrak{g}$ with respect to the Killing form.

The Clifford algebra for G

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

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Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B: the associative algebra with 1, generated by $\mathfrak{p}$, with relations

$$xy + yx = -2B(x, y).$$

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D is independent of b_i and K-invariant.

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$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$,
defined by $\mathfrak{k} \hookrightarrow U(\mathfrak{g})$ and $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$.

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If M is unitary, then D is self adjoint wrt an inner product. So

$$H_D(M) = \operatorname{Ker} D = \operatorname{Ker} D^2,$$

and $D^2 \geq 0$ (Parthasarathy's Dirac inequality).

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The modules corresponding to picture (4) have $H_D = 0$.

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Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. View $\mathfrak{t}^* \subset \mathfrak{h}^*$ via extension by 0 over $\mathfrak{a}$.

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Then the inf. character of M is $\gamma + \rho_{\mathfrak{k}}$ up to Weyl group $W_{\mathfrak{g}}$.

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- Relations to characters and branching (Huang-P-Zhu)

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- Can construct reps with $H_D \neq 0$ via "algebraic Dirac induction" (P.-Renard; Prlić)
- There is a translation principle for the Euler characteristic of H_D, i.e., the Dirac index (Mehdi-P.-Vogan).

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If X is unitary, then

$$H(\mathfrak{g}, K; X) = \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)).$$

(Or twice this if $\dim \mathfrak{p}$ is odd.)

$A_{\mathfrak{q}}(\lambda)$ modules

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They can also be characterized by their infinitesimal characters and the structure of K-types, which lie in a certain cone.

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Positive root systems for $(\mathfrak{g}, \mathfrak{h})$, $(\mathfrak{g}, \mathfrak{t})$, $(\mathfrak{k}, \mathfrak{t})$ and $(\mathfrak{l}, \mathfrak{t})$ are chosen in a compatible way. ρ and $\rho_{\mathfrak{k}}$ are the half sums of positive roots for $(\mathfrak{g}, \mathfrak{h})$ respectively $(\mathfrak{k}, \mathfrak{t})$.

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$W(\mathfrak{l},\mathfrak{t})^1$ consists of the elements of the Weyl group $W(\mathfrak{l},\mathfrak{t})$ which take the dominant $\mathfrak{l}$-chamber into the dominant $\mathfrak{l} \cap \mathfrak{k}$-chamber.

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Furthermore, it is related to Dirac induction and the issue of reconstructing modules from their Dirac cohomology.

Answer

Yes, unless the real form $\mathfrak{l}_0$ of $\mathfrak{l}$ has factors $\mathfrak{so}(2n, 1)$, $\mathfrak{sp}(p, q)$ or the nonsplit $\mathfrak{f}_4$. (Huang-P., with help from Vogan.)

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In particular, the answer is always yes if $\mathfrak{g}$ is of type A, D, E or G. It is also always yes if $(\mathfrak{g}, \mathfrak{k})$ is Hermitian.

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The question boils down to the issue whether $W(\mathfrak{l},\mathfrak{t})^1$ generates $W(\mathfrak{l},\mathfrak{t})$. The answer involves the study of modifications of Vogan diagrams by simple noncompact reflections.

Example: $\mathfrak{g}_0 = \mathfrak{so}(2n, 1)$

For each $k = 1, \ldots, n$, there is a θ-stable parabolic subalgebra $\mathfrak{q}_k$ with Levi factor modulo center equal to $\mathfrak{so}(2k, 1)$.

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The modules $A_{\mathfrak{q}_k}(0)$ are different, but they all have the same Dirac cohomology, consisting of two K-types:

$$(n - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2}) \quad \text{and} \quad (n - \frac{1}{2}, \dots, \frac{3}{2}, -\frac{1}{2}).$$

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(We also have two discrete series representations, each with a single K-type in the Dirac cohomology.)

THANK YOU!