News on SU(2|1) Supersymmetric Mechanics

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"Lie Theory and Its Applications in Physics"

XI International Workshop,
15 - 21 June 2015, Varna, Bulgaria

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Motivations and contents

Supersymmetric Quantum Mechanics (SQM) (E.Witten, 1983) is the simplest ($d = 1$) supersymmetric theory:

- Probes structure of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc;
- Extended SUSY in $d = 1$ ($\mathcal{N} \geq 2$) exhibits interesting specific features: dualities between various supermultiplets (J.S.Gates, Jr. & L.Rana, 1995, A.Pashnev & F.Toppan, 2001), nonlinear "cousins" of off-shell linear multiplets (E.I., S.Krivonos, O.Lechtenfeld, 2003, 2004), etc.

Symmetry group of the standard $\mathcal{N}$ extended SQM is

$$\{Q^A, Q^B\} = 2\delta^{AB} H\,, \quad [H, Q^A] = 0\,, \quad A, B = 1 \ldots \mathcal{N}\,. \tag{1}$$

Recently, there was a substantial interest in rigid supersymmetric theories based on curved analogs of the Poincaré supergroup in diverse dimensions (e.g.,T.Dumitrescu, G.Festuccia, N.Seiberg, 2011, 2012). There is the hope that their study will lead to a further progress in understanding, e.g., the generic gauge/gravity correspondence. It is interesting to consider "curved" analogs of SQM, based on some non-trival semi-simple supergroups.

The subject of my talk is the simplest version of such SQM based on the supergroup $SU(2|1)$.

- The universal way to construct supersymmetric theories is the Superspace Approach.
- Our aim is to construct the worldline superfield realizations of $SU(2|1)$ and to show that all off-shell multiplets of $\mathcal{N}=4, d=1$ supersymmetry have the well-defined $SU(2|1)$ analogs.
- The "week supersymmetry" models (A.Smilga, 2004) are based on the $SU(2|1)$ multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$.
- $SU(2|1)$ has also invariant chiral subspaces which are carriers of the chiral multiplets $(\mathbf{2},\mathbf{4},\mathbf{2})$. The relevant component actions naturally provide the bosonic $d=1$ Wess-Zumino-type terms.
- $SU(2|1)$ also admits a supercoset which is an analog of the harmonic analytic superspace of the $\mathcal{N}=4, d=1$ supersymmetry (E.I., O.Lechtenfeld, 2004). There, $SU(2|1)$ analogs of the analytic $\mathcal{N}=4$ superfields $(\mathbf{4},\mathbf{4},\mathbf{0})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$ live.
- The superconformal $D(2,1;\alpha)$ invariant models in the $SU(2|1)$ superspace naturally yield trigonometric realization of the $d=1$ conformal group $SO(2,1)$.

$SU(2|1)$ superspace

- The (central-extended) superalgebra $su(2|1)$:

$$\{Q^i, \bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$
$$\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[F, \bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k.$$

- The supercoset:

$$\frac{SU(2|1)}{SU(2)\times U(1)} \sim \frac{\{Q^i, \bar{Q}_j, H, I^i_j, F\}}{\{I^i_j, F\}}\,.$$

The superspace coordinates $\{t, \theta_i, \bar{\theta}^j\}$ are identified with the parameters associated with the coset generators. An element of this supercoset can be conveniently parametrized as

$$g = \exp\left(itH + i\tilde{\theta}_i Q^i - i\tilde{\bar{\theta}}^j \bar{Q}_j\right), \quad \tilde{\theta}_i = \left[1 - \frac{2m}{3}\left(\bar{\theta}\cdot\theta\right)\right]\theta_i\,.$$

- Transformation properties under $Q, \bar{Q}$

$$\delta\theta_i = \epsilon_i + 2m\,(\bar{\epsilon}\cdot\theta)\,\theta_i\,, \qquad \delta\bar{\theta}^j = \bar{\epsilon}^i - 2m\left(\epsilon\cdot\bar{\theta}\,\right)\bar{\theta}^i,$$
$$\delta t = i\left[\left(\epsilon\cdot\bar{\theta}\,\right) + (\bar{\epsilon}\cdot\theta)\right].$$

- Invariant integration measure

$$d\zeta = dtd^2\theta d^2\bar{\theta}(1 + 2m\,\bar{\theta}\cdot\theta)\,, \quad \delta d\zeta = 0\,.$$

- Generators

$$Q^i = -i\frac{\partial}{\partial\theta_i} + 2im\bar{\theta}^i\bar{\theta}^j\frac{\partial}{\partial\bar{\theta}^j} + \bar{\theta}^i\frac{\partial}{\partial t}\,,\ \bar{Q}_j = i\frac{\partial}{\partial\bar{\theta}^j} + 2im\theta_j\theta_k\frac{\partial}{\partial\theta_k} - \theta_j\frac{\partial}{\partial t}\,,$$
$$I^i_j = \left(\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{\delta^i_j}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),$$
$$F = \frac{1}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),\ H = i\partial_t.$$

- Covariant derivatives

$$\mathcal{D}^i = \left[1 + m\left(\bar{\theta}\cdot\theta\right) - \frac{3m^2}{4}\left(\bar{\theta}\cdot\theta\right)^2\right]\frac{\partial}{\partial\theta_i} - m\bar{\theta}^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar{\theta}^i\frac{\partial}{\partial t} + \ldots\,,$$

$$\bar{\mathcal{D}}_j = -\left[1 + m\left(\bar{\theta}\cdot\theta\right) - \frac{3m^2}{4}\left(\bar{\theta}\cdot\theta\right)^2\right]\frac{\partial}{\partial\bar{\theta}^j} + m\bar{\theta}^k\theta_j\frac{\partial}{\partial\bar{\theta}^k} + i\theta_j\frac{\partial}{\partial t} + \ldots\,.$$

Here “dots” stand for matrix $U(2)$ connection parts.

$(1,4,3)$ multiplet

- The $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet is described by the real neutral superfield $G(t,\theta,\bar\theta)$ satisfying

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\,\bar{\mathcal{D}}_j G = \varepsilon_{lj}\mathcal{D}^l\,\mathcal{D}^j G = 0 \quad \Rightarrow$$

$$G = x - mx\left(\bar\theta\cdot\theta\right)\left[1-2m\left(\bar\theta\cdot\theta\right)\right] + \frac{\ddot{x}}{2}\left(\bar\theta\cdot\theta\right)^2 - i\left(\bar\theta\cdot\theta\right)\left(\theta_i\,\dot\psi^i + \bar\theta^j\,\dot{\bar\psi}_j\right)$$
$$+\left[1-2m\left(\bar\theta\cdot\theta\right)\right]\left(\theta_i\,\psi^i - \bar\theta^j\,\bar\psi_j\right) + \bar\theta^j\theta_i\,B^i_j\,,\quad B^k_k = 0\,.$$

- The irreducible set of off-shell fields is $x(t), \psi^i(t), \bar\psi_i(t), B^i_j(t)(B^k_k = 0)$, In the limit $m = 0$ it is reduced to the ordinary $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfield.
- The ϵ transformation properties of the component fields:

$$\delta x = \left(\bar\epsilon\cdot\bar\psi\right) - \left(\epsilon\cdot\psi\right), \qquad \delta\psi^i = i\bar\epsilon^i\dot{x} - m\bar\epsilon^i x + \bar\epsilon^k B^i_k\,,$$
$$\delta B_{(ij)} = -2i\left[\epsilon_{(i}\dot\psi_{j)} + \bar\epsilon_{(i}\dot{\bar\psi}_{j)}\right] + 2m\left[\bar\epsilon_{(i}\bar\psi_{j)} - \epsilon_{(i}\psi_{j)}\right].$$

- Invariant off-shell Lagrangian

$$\mathcal{L} = -\int d^2\theta\,d^2\bar\theta\,\left(1+2m\,\bar\theta\cdot\theta\right)\,f(G)\,,\quad S = \int dt\mathcal{L}\,.$$

- After doing θ integral and eliminating the auxiliary field, the on-shell Lagrangian is

$$\mathcal{L} = \dot{x}^2 g(x) + i\left(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i\right)g(x) - \frac{1}{2}\left(\bar{\psi}_i\psi^i\right)^2\left[g''(x) - \frac{3\,(g'(x))^2}{2g(x)}\right]$$
$$-m^2x^2g(x) + 2m\bar{\psi}_i\psi^i g(x) + mx\bar{\psi}_i\psi^i g'(x).$$

- This Lagrangian can be simplified by passing to new variables $y(x), \zeta^i$,

$$\dot{x}^2 g(x) = \frac{1}{2}\dot{y}^2\,, \quad \Rightarrow y'(x) = \sqrt{2g(x)}\,,\ \zeta^i = \psi^i y'(x)\,,$$

We find

$$\mathcal{L} = \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) - \frac{m^2}{2}V^2(y) + m\bar{\zeta}_i\zeta^i V'(y)$$
$$-\frac{1}{2}\left(\bar{\zeta}_i\zeta^i\right)^2\partial_y\left(\frac{V'(y)-1}{V(y)}\right),\ V(y) := \frac{x(y)}{x'(y)}$$

- This Lagrangian is that defining the SQM model with “weak” $\mathcal{N} = 4$ supersymmetry (A.Smilga, 2004).

$(1,4,3)$ multiplet: quantization

- The simplest choice is $f(x) = \frac{x^2}{4}$:

$$\mathcal{L} = \frac{\dot{x}^2}{2} - \frac{m^2 x^2}{2} + \frac{i}{2}\left(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i\right) + m\bar{\psi}_i\psi^i,$$
$$\delta x = \left(\bar{\epsilon}\cdot\bar{\psi}\right) - \left(\epsilon\cdot\psi\right), \qquad \delta\psi^i = i\bar{\epsilon}^i\dot{x} - m\bar{\epsilon}^i x\,. \tag{2}$$

- The conserved Noether charges and Hamiltonian:

$$Q^i = \psi^i\left(p - imx\right), \qquad \bar{Q}_i = \bar{\psi}_i\left(p + imx\right),$$
$$F = \frac{1}{2}\psi^k\bar{\psi}_k, \qquad I^i_j = \psi^i\bar{\psi}_j - \frac{1}{2}\delta^i_j\psi^k\bar{\psi}_k.$$
$$H = \frac{p^2}{2} + \frac{m^2x^2}{2} + m\psi^i\bar{\psi}_i\,.$$

This H is $SU(2|1)$ extension of the harmonic oscillator Hamiltonian.

We quantize as

$$[\hat{x},\hat{p}] = i\,,\quad \{\hat{\psi}^i,\hat{\bar{\psi}}_j\} = \delta^i_j\,,\quad \hat{p} = -i\partial_x\,,\ \hat{\bar{\psi}}_j = \partial/\partial\hat{\psi}^j\,.$$

The quantum Hamiltonian and supercharges are

$$\begin{aligned}
\hat{H} \quad &= \quad \frac{1}{2}\,(\hat{p}+im\hat{x})\,(\hat{p}-im\hat{x}) + m\hat{\psi}^i\hat{\bar{\psi}}_i\,,\\
&\hat{Q}^i = \hat{\psi}^i\,(\hat{p}-im\hat{x})\,, \qquad \hat{\bar{Q}}_i = \hat{\bar{\psi}}_i\,(\hat{p}+im\hat{x})\,,\\
&\hat{F} = \frac{1}{2}\hat{\psi}^k\hat{\bar{\psi}}_k, \qquad \hat{I}^i_j = \hat{\psi}^i\hat{\bar{\psi}}_j - \frac{1}{2}\delta^i_j\hat{\psi}^k\hat{\bar{\psi}}_k\,.
\end{aligned}$$

They form the superalgebra $su(2|1)$.

Spectrum

- We construct the Hilbert space of wave functions in terms of the harmonic oscillator wave functions. The super wave-function $\Omega^{(\ell)}$ at the energy level $\ell, \ell \geq 2$, reveals the four-fold degeneracy

$$\Omega^{(\ell)} = a^{(\ell)}|\ell\rangle + b_i^{(\ell)}\,\psi^i|\ell-1\rangle + \frac{1}{2}\,c^{(\ell)}\,\varepsilon_{ij}\psi^i\psi^j|\ell-2\rangle, \qquad \ell \geq 2\,,$$

  where $|\ell\rangle, |\ell-1\rangle, |\ell-2\rangle$ are the harmonic oscillator functions.

- We treat the operators $\hat{p} \pm imx$ in $\hat{H}$ as the creation and annihilation operators and impose the standard conditions

$$\hat{\bar{\psi}}_k|\ell\rangle = 0, \qquad (\hat{p} - im\hat{x})\,|0\rangle = 0, \qquad (\hat{p} + im\hat{x})\,|\ell\rangle = |\ell+1\rangle.$$

  The spectrum of the Hamiltonian is then

$$\hat{H}\,\Omega^{(\ell)} = m\,\ell\,\Omega^{(\ell)}, \qquad m > 0.$$

- The ground state ($\ell = 0$) and the first excited states ($\ell = 1$) are special, they encompass non-equal numbers of bosonic and fermionic states:

$$\Omega^{(0)} = a^{(0)}\,|0\rangle\,, \qquad \Omega^{(1)} = a^{(1)}\,|1\rangle + b_i^{(1)}\,\psi^i|0\rangle\,.$$

$SU(2|1)$ representation content

- The ground state is annihilated by all $SU(2|1)$ generators including Q^i and $\bar{Q}_i$, so it is $SU(2|1)$ singlet.
- The states with $\ell = 1$ form the fundamental $(\mathbf{2}|\mathbf{1})$ representation of $SU(2|1)$. The action of the supercharges on them is

$$Q^i\,\psi^k|0\rangle = 0\,, \qquad \bar{Q}_i\,\psi^k|0\rangle = \delta_i^k\,|1\rangle\,,$$
$$Q^i\,|1\rangle = 2m\,\psi^i|0\rangle\,, \qquad \bar{Q}_i\,|1\rangle = 0\,.$$

- The states with $\ell > 1$ form the representations $(\mathbf{2}|\mathbf{2})$, with equal numbers of bosonic and fermionic states.
- $SU(2|1)$ Casimirs are nicely expressed as

$$m^2\,C_2 = \hat{H}\left(\hat{H} - m\right), \quad m^3\,C_3 = \hat{H}\left(\hat{H} - m\right)\left(\hat{H} - \frac{m}{2}\right)$$

  and so are fully specified by the energy spectrum of $\hat{H}$:

$$C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = (\ell - 1/2)\,(\ell - 1)\,\ell.$$

  The ground state with $\ell = 0$ is atypical, because Casimir operators are zero on it. On the states with $\ell = 1$ both Casimirs as well vanish, so these states also form an atypical (fundamental) $SU(2|1)$ representation. On the $\ell > 1$ states both Casimirs are non-zero, so these states form typical $SU(2|1)$ representations.

Chiral multiplet

- $SU(2|1)$ counterpart of the $\mathcal{N}=4, d=1$ chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ exists. This is due to the existence of the chiral coset

$$\frac{\{Q^i, \bar{Q}_j, H, I^i_k, F\}}{\{\bar{Q}_j, I^i_k, F\}} \sim (t_L, \theta_i)\,, \qquad t_L = t + \frac{i}{2m}\ln(1+2m\,\bar{\theta}\cdot\theta),$$

$$\delta\theta_i = \epsilon_i + 2m\,(\bar{\epsilon}\cdot\theta)\theta_i\,, \qquad \delta t_L = 2i\,(\bar{\epsilon}\cdot\theta)\,.$$

- The multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is described by the chiral superfield Φ

$$\bar{\mathcal{D}}_j\Phi = 0, \qquad \hat{I}^i_j\Phi = 0, \qquad \hat{F}\Phi = 2\kappa\Phi\,, \tag{3}$$

where in general $\kappa \neq 0$.

- The solution of this constraint is

$$\Phi = \left[1 + m\,(\bar{\theta}\cdot\theta)\right]^{-\kappa}\varphi_L(t_L,\theta), \quad \varphi_L(t_L,\theta) = z + \sqrt{2}\,\theta_i\xi^i + \varepsilon^{ij}\theta_i\theta_j F.$$

- The superfield φ_L and its components transform as

$$\delta^*\varphi_L = 4\kappa m\,(\bar{\epsilon}^j\theta_j)\,\varphi_L \;\Rightarrow$$

$$\delta z = -\sqrt{2}\,\epsilon_i\xi^i, \quad \delta\xi^i = \sqrt{2}\,i\bar{\epsilon}^i\nabla_t z - \sqrt{2}\,\varepsilon^{ik}\epsilon_k F,$$

$$\delta F = -\sqrt{2}\,\varepsilon_{ik}\bar{\epsilon}^k\left[m\xi^i + i\nabla_t\xi^i\right], \quad \nabla_t := \partial_t + 2i\kappa m\,.$$

Invariant Lagrangian

- General superfield Lagrangian reads

$$\mathcal{L}_k = \frac{1}{4}\int d^2\theta\, d^2\bar{\theta}(1 + 2m\,\bar{\theta}\cdot\theta)\, f(\Phi, \Phi^\dagger).$$

- Its component on-shell form is as follows:

$$\begin{aligned}\mathcal{L} \;=\;& g\dot{\bar{z}}\dot{z} + 2i\kappa m\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right)g - \frac{im}{2}\left(\dot{\bar{z}}f_{\bar{z}} - \dot{z}f_z\right) - \frac{i}{2}\left(\bar{\xi}\cdot\xi\right)\left(\dot{\bar{z}}g_{\bar{z}} - \dot{z}g_z\right)\\ &+\frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right)g - m^2 V - m\left(\bar{\xi}\cdot\xi\right)U + \frac{1}{2}\left(\bar{\xi}\cdot\xi\right)^2 R\,,\end{aligned}$$

with

$$\begin{aligned}V &= \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)f - \kappa^2\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)^2 f,\\ U &= \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)g - (1 - 2\kappa)\,g,\\ R &= g_{z\bar{z}} - \frac{g_z g_{\bar{z}}}{g}\,.\end{aligned}$$

It is invariant under

$$\delta z = -\sqrt{2}\,\epsilon_i\xi^i, \qquad \delta\xi^i = \sqrt{2}\,i\bar{\epsilon}^i\nabla_t z + \sqrt{2}\,\epsilon_k\xi^k\xi^i\,\frac{g_z}{g}.$$

- Bosonic Lagrangian

$$\mathcal{L} = g\dot{\bar{z}}\dot{z} + 2i\kappa m\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right)g - \frac{im}{2}\left(\dot{\bar{z}}f_{\bar{z}} - \dot{z}f_z\right) - m^2 V\,,$$
$$V = \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)f - \kappa^2\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)^2 f.$$

- The standard $\mathcal{N} = 4, d = 1$ kinetic term is deformed to a non-trivial Lagrangian with WZ-term, and potential term. The latter vanishes for $\kappa = 0$, however, the WZ term vanishes only in the limit $m = 0$.
- Thus the basic novel point compared to the standard $\mathcal{N} = 4$ Kähler sigma model for the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ is the necessary presence of the WZ term with the strength m, along with the Kähler kinetic term.

Model on a complex plane

- The Kähler potential

$$f\left(\Phi, \Phi^{\dagger}\right) = \Phi\Phi^{\dagger} \quad \Rightarrow$$

$$\mathcal{L} = \dot{\bar{z}}\dot{z} + im\left(2\kappa - \frac{1}{2}\right)\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right) + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right)$$
$$+ 2\kappa\left(2\kappa - 1\right)m^2\bar{z}z + \left(1 - 2\kappa\right)m\left(\bar{\xi}\cdot\xi\right).$$

The Lagrangian is invariant under

$$\delta z = -\sqrt{2}\,\epsilon_i\xi^i, \qquad \delta\xi^i = \sqrt{2}\,i\bar{\epsilon}^i\dot{z} - 2\sqrt{2}\,\kappa m\bar{\epsilon}^i z.$$

- The quantum Hamiltonian and supercharges:

$$\hat{H} = \bar{\nabla}_{\bar{z}}\nabla_z - 2\kappa m\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) + m\left(1 - 2\kappa\right)\hat{\eta}^k\hat{\bar{\eta}}_k$$

$$\hat{Q}^i = \sqrt{2}\,\hat{\eta}^i\nabla_z\,, \qquad \hat{\bar{Q}}_j = \sqrt{2}\,\hat{\bar{\eta}}_j\bar{\nabla}_{\bar{z}}\,,$$

$$\hat{F} = -2\kappa\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)\hat{\eta}^k\hat{\bar{\eta}}_k\,, \qquad \hat{I}^i_j = \hat{\eta}^i\hat{\bar{\eta}}_j - \frac{1}{2}\delta^i_j\hat{\eta}^k\hat{\bar{\eta}}_k\,.$$

They form $su(2|1)$ superalgebra, with

$$\nabla_z = -i\partial_z - \frac{i}{2}m\bar{z}\,, \bar{\nabla}_{\bar{z}} = -i\partial_{\bar{z}} + \frac{i}{2}mz\,, \left[\nabla_z, \bar{\nabla}_{\bar{z}}\right] = m.$$

Wave functions and spectrum

- We take advantage of the fact that there exists an extra $U(1)$ charge generator commuting with all $SU(2|1)$ generators

$$\hat{E} = -\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \hat{\eta}^k \hat{\bar{\eta}}_k \,.$$

- The relevant wave functions can be constructed in terms of bosonic eigenfunctions of this external generator

$$\Omega^{(\alpha)} = \bar{z}^{\alpha} A\,(z\bar{z})\,, \quad \hat{E}\,\Omega^{(\alpha)} = \alpha\,\Omega^{(\alpha)}\,, \tag{4}$$

α being some positive real number.

- Requiring this set to simultaneously form the full set of the eigenfunctions of the bosonic part of the Hamiltonian yields

$$\Omega^{(\alpha)} \to \Omega^{(\ell;\alpha)}\,, \quad \hat{H}\,\Omega^{(\ell;\alpha)} = m(\ell + 2\kappa\alpha)\,\Omega^{(\ell;\alpha)}$$
$$\Omega^{(\ell;\alpha)} = \bar{z}^{\alpha} e^{-\frac{mz\bar{z}}{2}} L_{\ell}^{(\alpha)}\,(m\,z\bar{z})\,,$$

where $L_{\ell}^{(\alpha)}$ are Laguerre polynomials and ℓ is Landau level.

- Acting by supercharges on $\Omega^{(\ell;\alpha)}$ and imposing the obvious vacuum condition,

$$\bar{\eta}_j\,\Omega^{(\ell;\alpha)} = 0 \;\Rightarrow\; \bar{Q}_j\,\Omega^{(\ell;\alpha)} = 0\,,$$

  we obtain other eigenstates of $\hat{H}$ and $\hat{E}$.
- The full set of eigenfunctions obtained in this way reads:

$$\Psi^{(\ell;\alpha)} = \left[a^{(\ell;\alpha)} + b_i^{(\ell;\alpha)}\eta^i\,\nabla_z + \frac{1}{2}\,c^{(\ell;\alpha)}\,\varepsilon_{ij}\eta^i\eta^j\,\nabla_z^2\right]\Omega^{(\ell;\alpha)}, \quad \ell \geq 2,$$

$$\Psi^{(1;\alpha)} = a^{(1;\alpha)}\,\Omega^{(1;\alpha)} + b_i^{(1;\alpha)}\eta^i\,\nabla_z\,\Omega^{(1;\alpha)}, \quad \Psi^{(0;\alpha)} = a^{(0;\alpha)}\,\Omega^{(0;\alpha)}.$$

- The ground state ($\ell = 0$) and the first excited states ($\ell = 1$) are special: they encompass non-equal numbers of bosonic and fermionic states.
- The wave functions with $\ell = 1$ form the fundamental representation of $SU(2|1)$ (one bosonic and two fermionic states), while those with $\ell \geq 2$ form typical $(\mathbf{2}|\mathbf{2})$ representations.

- Casimir operators:

$$m^2 C_2 = \left(\hat{H} - 2\kappa m\hat{E}\right)\left(\hat{H} - 2\kappa m\hat{E} - m\right),$$
$$m^3 C_3 = \left(\hat{H} - 2\kappa m\hat{E}\right)\left(\hat{H} - 2\kappa m\hat{E} - m\right)\left(\hat{H} - 2\kappa m\hat{E} - \frac{m}{2}\right)$$

$$C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = (\ell - 1/2)\,(\ell - 1)\,\ell\,.$$

- They are vanishing for the wave functions with $\ell = 0, 1$, confirming the interpretation of the corresponding representations as atypical, and are non-vanishing on the wave functions with $\ell \geq 2$, implying them to form typical representations of $SU(2|1)$, with equal numbers of bosonic and fermionic states.

Generalized $SU(2|1)$ chirality

- One can choose another $SU(2|1)$ coset as the basic superspace

$$\frac{SU(2|1) \rtimes U(1)_{\rm ext}}{SU(2) \times U(1)_{\rm ext}} = \frac{SU(2|1)}{SU(2)} \sim \frac{\{Q^i, \bar{Q}_j, \tilde{H}, I^i_j\}}{\{I^i_j\}}\,.$$

The Hamiltonian is now the full internal $U(1)$ generator $\tilde{H} = H - mF$.

- The covariant spinor derivatives $\mathcal{D}_i$, $\bar{D}^i$ are $U(1)$ inert and a generalized chirality condition can be imposed

$$(\cos\lambda\,\bar{\mathcal{D}}_i - \sin\lambda\,\mathcal{D}_i)\Phi = 0\,, \qquad (5)$$

λ being a new real parameter. No κ now, so (5) $\rightarrow \Phi = \varphi_L(\hat{t}_L, \hat{\theta}_i)$.

- The components of φ_L are now transformed with manifest t

$$\delta z = -\sqrt{2}\cos\lambda\,(\epsilon\cdot\xi)\,e^{\frac{i}{2}mt} + \sqrt{2}\sin\lambda\;(\bar{\epsilon}\cdot\xi)\,e^{-\frac{i}{2}mt},$$
$$\delta\xi^i = \sqrt{2}\,\bar{\epsilon}^i\,[i\cos\lambda\,\dot{z} - \sin\lambda\,B]\,e^{-\frac{i}{2}mt} - \sqrt{2}\,\epsilon^i\,[i\sin\lambda\,\dot{z} + \cos\lambda\,B]\,e^{\frac{i}{2}mt},$$
$$\delta B = \sqrt{2}\cos\lambda[i(\bar{\epsilon}\cdot\dot{\xi}) + \frac{m}{2}(\bar{\epsilon}\cdot\xi)]e^{-\frac{i}{2}mt} + \sqrt{2}\sin\lambda[i(\epsilon\cdot\dot{\xi}) - \frac{m}{2}(\epsilon\cdot\xi)]e^{\frac{i}{2}mt}$$

- The most general $SU(2|1)$ invariant action of $\varphi^a(t_L, \hat{\theta}),\ a = 1, \dots N,$:

$$S_{\rm kin} = \int dt\, \mathcal{L}_{\rm kin} = \frac{1}{4} \int d\zeta\, f(\varphi^a, \bar{\varphi}^{\bar{a}})\,.$$

- Its on-shell bosonic core

$$\mathcal{L}^{\rm on}_{\rm kin}| = g_{\bar{a}b}\dot{\bar{z}}^{\bar{a}}\dot{z}^b - \frac{i}{2} m \cos 2\lambda (\dot{\bar{z}}^{\bar{a}} f_{\bar{a}} - \dot{z}^a f_a) - \frac{m^2}{4} g^{\bar{a}b} \sin^2 2\lambda\, f_{\bar{a}} f_b\,.$$

recognized as the Lagrangian of the Kähler oscillator (S. Bellucci, A. Nersessian, 2003, 2004) extended by a coupling to an external magnetic field.

- The supercharges do not commute with the Hamiltonian $\tilde{H}$, but are still conserved due to their explicit t-dependence

$$\frac{d}{dt} Q^i = \partial_t Q^i + \{Q^i, \tilde{H}\} = 0, \qquad \frac{d}{dt} \bar{Q}_j = \partial_t \bar{Q}_j + \{\bar{Q}_j, \tilde{H}\} = 0\,.$$

Superconformal models

- The most general $\mathcal{N}=4, d=1$ superconformal group is $D(2,1;\alpha)$:

$$\{Q_{\alpha i i'}, Q_{\beta j j'}\} = 2\left(\epsilon_{ij}\epsilon_{i'j'}\,T_{\alpha\beta} + \alpha\,\epsilon_{\alpha\beta}\epsilon_{i'j'}\,J_{ij} - (1+\alpha)\,\epsilon_{\alpha\beta}\epsilon_{ij}L_{i'j'}\right),$$
$$[T_{\alpha\beta}, Q_{\gamma i i'}] = -i\,\epsilon_{\gamma(\alpha}Q_{\beta) i i'}\,, \qquad [T_{\alpha\beta}, T_{\gamma\delta}] = i\,(\epsilon_{\alpha\gamma}\,T_{\beta\delta} + \epsilon_{\beta\delta}\,T_{\alpha\gamma})\,,$$
$$[J_{ij}, Q_{\alpha k i'}] = -i\,\epsilon_{k(i}Q_{\alpha j) i'}\,, \qquad [J_{ij}, J_{kl}] = i\,(\epsilon_{ik}J_{jl} + \epsilon_{jl}J_{ik})\,,$$
$$[L_{i'j'}, Q_{\alpha i k'}] = -i\,\epsilon_{k'(i'}Q_{\alpha i j')}\,, \qquad [L_{i'j'}, L_{k'l'}] = i\,(\epsilon_{i'k'}L_{j'l'} + \epsilon_{j'l'}L_{i'k'})$$

- $Q_{\alpha i i'}$ are eight supercharges, the bosonic subalgebra is

$$su(2) \oplus su(2)' \oplus so(2,1) \equiv \{J_{ik}\} \oplus \{L_{i'k'}\} \oplus \{T_{\alpha\beta}\}$$

- At $\alpha = -1, 0$, the superalgebra $D(2,1;\alpha)$ is reduced to

$$D\,(2,1;\alpha) \cong PSU(1,1|2) \rtimes SU(2)_{\rm ext}$$

- How to implement $D(2,1;\alpha)$ in the $SU(2|1)$ superspaces? The crucial property is the existence of TWO different $su(2|1) \subset D(2,1;\alpha)$, so that the latter is a closure of these two.

- These are defined by the following relations

$$(\mathrm{I}).\quad \{Q^i, \bar{Q}_j\} = 2m(\mu)\, I^i_j + 2\delta^i_j[H(\mu) - m(\mu)\, F]\,,$$

$$m(\mu) := -\alpha\mu\,, \quad H(\mu) := \mathcal{H} + \mu F\,, \quad \mathcal{H} = \hat{H} + \frac{\mu^2}{4}\hat{K}\,,$$

$$(\hat{H}, \hat{K}) \in so(2,1)\,, \quad F \in su(2)'\,, \quad I^i_j \in su(2)\,,$$

$$(\mathrm{II}).\quad \{S^i, \bar{S}_j\} = 2m(-\mu)\, I^i_j + 2\delta^i_j[H(-\mu) - m(-\mu)\, F]$$

  The remaining $D(2,1;\alpha)$ generators appears in $\{Q, S\}$ and $\{Q, \bar{S}\}$.
- $SU(2|1)$ (I) is identified with the manifest superisometry of the $SU(2|1)$ superspace; then $SU(2|1)$ (II) is realized on the superspace coordinates and superfields as a hidden symmetry.
- Always the "trigonometric" realization of the $d = 1$ conformal generators:

$$\hat{H} = \frac{i}{2}\,[1 + \cos\mu t\,]\,\partial_t\,, \quad \hat{K} = \frac{2i}{\mu^2}\,[1 - \cos\mu t\,]\,\partial_t\,, \quad \hat{D} = \frac{i}{\mu}\sin\mu t\,\partial_t$$

- The basic constraints are $D(2,1;\alpha)$ covariant, at least for some special values of α. The multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is superconformal for any α, the chiral multiplet admits the superconformal symmetry only for $\alpha = 0, -1$.
- The superconformal subclasses of the general $SU(2|1)$ actions are singled out by requiring them to be even functions of μ, in accord with the above structure of $D(2,1;\alpha)$ as a closure of two $SU(2|1)$.

Some examples of superconformal actions

- The multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$:

$$S_{\rm conf} = -\int d\zeta\, f(G)\,,\;\; f(G) = \begin{cases} \frac{1}{8(\alpha+1)}\, G^{-\frac{1}{\alpha}} & \text{for } \alpha \neq -1, 0\,, \\ \frac{1}{8}\, G \ln G & \text{for } \alpha = -1 \end{cases}$$

The simplest choice is $\alpha = -1/2$, yields free Lagrangian

$$\mathcal{L}_{\rm conf} = \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i \dot{\zeta}^i - \dot{\bar{\zeta}}_i \zeta^i\right) + \tilde{B}^i_j \tilde{B}^j_i - \frac{\mu^2}{8}\, y^2$$

- The most general G superfield superconformally covariant constraints

$$\bar{\mathcal{D}}^2\, \tilde{G} = \mathcal{D}^2\, \tilde{G} = 0\,, \quad \left[\mathcal{D}, \bar{\mathcal{D}}\right] \tilde{G} = 4m\, \tilde{G} - 4c$$

At $c \neq 0$ covariant only under the $\alpha = -1$ supergroup. The bosonic sector of $\mathcal{L}_{\rm conf}$ contains the standard conformal potential:

$$\mathcal{L}^{\rm bos}_{(\alpha=-1)} = \frac{\dot{y}^2}{2} - \frac{\mu^2 y^2}{8} - \frac{c^2}{8y^2}$$

- The standard multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$:

$$\bar{\mathcal{D}}_i\Phi = 0\,, \qquad \mathcal{D}^i\bar{\Phi} = 0\,, \quad \tilde{F}\Phi = 2\kappa\Phi$$

The constraints are covariant only for $\alpha = -1$. At $\kappa \neq 0$:

$$S^{\rm conf}_{\rm kin} = \frac{1}{4}\int d\zeta\, f\left(\Phi,\bar{\Phi}\right)\,, \quad f = \left(\Phi\bar{\Phi}\right)^{\frac{1}{4\kappa}}$$

Neither WZ term, nor standard conformal potential appear. Oscillator term only. At $\kappa = 1/4$, the free Lagrangian:

$$\mathcal{L}_{(\kappa=1/4)} = \dot{\bar{z}}\dot{z} + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right) - \frac{\mu^2}{4}\,z\bar{z} \qquad (6)$$

- The generalized $(\mathbf{2},\mathbf{4},\mathbf{2})$:

$$(\cos\lambda\,\bar{\mathcal{D}}_i - \sin\lambda\,\mathcal{D}_i)\Phi = 0$$

Also covariant under $D(2,1;\alpha=-1)$ only.

- The only superconformally invariant superpotential in both cases is the holomorphic $S^{\rm conf}_{\rm pot} \sim \nu\int d\zeta_L \ln\varphi_L + {\rm c.c.}$ $\to$ standard conformal potential $\sim -\nu^2|z|^{-2}$. Extra potential $\sim \mu^2 z\bar{z}$ comes always from the sigma-model action.
- At any κ the conformal action is reduced, by a field redefinition, to the free one (6) plus the conformal potential $\sim |z|^{-2}$.

Harmonic $SU(2|1)$ superspace

- The harmonic extension of the standard $SU(2|1)$ superspace:

$$\{t,\theta_i,\bar\theta^j\} \;\Rightarrow\; \{t,\theta_i,\bar\theta^j,w_i^\pm\}\,,\quad w^{+i}w_i^- = 1\,,$$
$$\delta\theta_i = \epsilon_i + 2m\,\bar\epsilon^k\,\theta_k\,\theta_i\,,\;\delta\bar\theta^j = \bar\epsilon^j - 2m\,\epsilon_k\,\bar\theta^k\,\bar\theta^j,\;\delta t = i\left(\epsilon_k\,\bar\theta^k + \bar\epsilon^k\,\theta_k\right),$$
$$\delta w_i^+ = \lambda^{++}w_i^-,\;\lambda^{++} = -\,m\left(1 - m\bar\theta^l\theta_l\right)\left(\bar\theta^k\epsilon^j + \theta^k\bar\epsilon^j\right)w_k^+w_j^+,$$
$$\delta w_i^- = 0\,. \tag{7}$$

- Passing to the analytic basis

$$\{t,\theta_i,\bar\theta^j,w_i^\pm\} \;\Rightarrow\; \{t_{(A)},\theta^\pm,\bar\theta^\pm,w_i^\pm\}\,,$$
$$\delta\theta^+ = \epsilon^+ + m\,\bar\theta^+\theta^+\epsilon^-,\;\delta\bar\theta^+ = \bar\epsilon^+ - m\,\bar\theta^+\theta^+\bar\epsilon^-,\;\delta t_{(A)} = 2i\left(\epsilon^-\bar\theta^+ + \theta^+\bar\epsilon^-\right),$$
$$\delta\theta^- = \epsilon^- + 2m\,\bar\epsilon^-\theta^-\theta^+,\;\delta\bar\theta^- = \bar\epsilon^- + 2m\,\epsilon^-\bar\theta^-\bar\theta^+,$$
$$\delta w_i^+ = -\,m\left(\bar\theta^+\epsilon^+ + \theta^+\bar\epsilon^+\right)w_i^-,\;\delta w_i^- = 0\,,\;\epsilon^\pm = \epsilon^i w_i^\pm\,,\;\bar\epsilon^\pm = \bar\epsilon^i w_i^\pm$$

- The analytic $SU(2|1)$ superspace

$$\zeta_A := \left(t_{(A)},\bar\theta^+,\theta^+,w_i^\pm\right)\,,$$

is closed under the $SU(2|1)$ transformations!

- In the analytic basis, the algebra of covariant derivatives (both spinor and harmonic) contains the following closed subalgebra, the so called "CR-structure":

$$\{\mathcal{D}^+, \bar{\mathcal{D}}^+\} = -2m\,\mathcal{D}^{++}, \qquad \left[\mathcal{D}^{++}, \mathcal{D}^+\right] = \left[\mathcal{D}^{++}, \bar{\mathcal{D}}^+\right] = 0\,,$$

where

$$\mathcal{D}^+ = \frac{\partial}{\partial\theta^-} + m\,\bar\theta^-\mathcal{D}^{++}, \qquad \bar{\mathcal{D}}^+ = -\frac{\partial}{\partial\bar\theta^-} - m\,\theta^-\mathcal{D}^{++}$$

and $\mathcal{D}^{++} = w^{+i}\partial_{w^{-i}} + \ldots$ is one of three harmonic derivatives forming an $su(2)$ algebra (together with $\mathcal{D}^{--} = w^{-i}\partial_{w^{+i}} + \ldots$ and $\mathcal{D}^0 = w^{+i}\partial_{w^{+i}} - w^{-i}\partial_{w^{-i}} + \ldots$).

- Analytic superfields:

$$\mathcal{D}^+\varphi^{+q} = \bar{\mathcal{D}}^+\varphi^{+q} = 0\,, \;\Rightarrow\; \mathcal{D}^{++}\varphi^{+q} = 0\,.$$

- The simplest choice is $q = 1$, with q^{+A}, $\widetilde{q^{+A}} = \varepsilon_{AB}q^{+B}$, describing the off-shell multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$

$$q^{+a}\,(\zeta_A) = x^{ia}w_i^+ + \theta^+\psi^a + \bar\theta^+\bar\psi^a - 2i\theta^+\bar\theta^+\dot{x}^{ia}w_i^-\,.$$

It transforms as

$$\delta q^{+a} = -\,m\left(\bar\theta^+\epsilon^- + \theta^+\bar\epsilon^-\right)q^{+a} \quad \Rightarrow$$

$$\delta x^{ia} = -\,\epsilon^i\psi^a - \bar\epsilon^i\bar\psi^a, \quad \delta\bar\psi_a = 2i\epsilon_k\dot{x}^k_a - m\,\epsilon_k x^k_a\,, \quad \delta\psi^a = 2i\bar\epsilon^k\dot{x}^a_k + m\,\bar\epsilon^k x^a_k\,.$$

- The sigma-model type invariant action is

$$S\left(q^{\pm a}\right) = \int d\zeta_H \, L\left(q^{+a} q^{-}_a\right), \; q^{-a} = \mathcal{D}^{--} q^{+a}.$$

- In components:

$$\begin{aligned}\mathcal{L} \;\; = \;\; & G\left[\dot{x}^{ia}\dot{x}_{ia} + \frac{i}{2}\left(\bar{\psi}_a\dot{\psi}^a - \dot{\bar{\psi}}_a\psi^a\right) + \frac{m}{2}\,\psi^a\bar{\psi}_a\right] - \frac{i}{2}\,\dot{x}^{ia}\partial_{ic}G\left(\psi_a\bar{\psi}^c + \psi^c\bar{\psi}_a\right) \\ & - \frac{\Delta_x G}{16}\left(\bar{\psi}\,\right)^2(\psi)^2 + \frac{m}{2}\,x^2 G'\,\psi^a\bar{\psi}_a - \frac{m^2}{4}\,x^2\,G, \quad G = G(x^2)\,.\end{aligned}$$

- The simplest Lagrangian corresponds to $L = q^{+a} q^{+}_a$:

$$\mathcal{L}_{\rm free} = \dot{x}^{ia}\dot{x}_{ia} + \frac{i}{2}\left(\bar{\psi}_a\dot{\psi}^a - \dot{\bar{\psi}}_a\psi^a\right) + \frac{m}{2}\,\psi^a\bar{\psi}_a - \frac{m^2}{4}\,x^{ia}x_{ia}\,.$$

- As distinct from the flat $(\mathbf{4},\mathbf{4},\mathbf{0})$ case, no explicit $SU(2|1)$ invariant Wess-Zumino term can be constructed. An internal WZ term, coming from the kinetic sigma-model action, exists if $\tilde{F} q^{+a} \neq 0$.

Quantization in the free case

- The quantum Hamiltonian and supercharges:

$$H = -\frac{1}{4}\left(\partial^{ia} - mx^{ia}\right)\left(\partial_{ia} + mx_{ia}\right) + \frac{m}{2}\,\bar{\xi}_a\xi^a\,,$$
$$Q_i = -i\xi^a\left(\partial_{ia} - mx_{ia}\right), \qquad \bar{Q}_i = -i\bar{\xi}^a\left(\partial_{ia} + mx_{ia}\right),$$
$$F = -\frac{1}{2}\,\bar{\xi}_a\xi^a, \qquad I_{ik} = x^a_{(i}\,\partial_{k)a}\,,$$
$$E_{ab} = x^i_{(a}\,\partial_{ib)} - \bar{\xi}_{(a}\xi_{b)}\,, \quad [E_{ab}, E_{cd}] = \varepsilon_{cb}E_{ad} - \varepsilon_{ad}E_{cb}\,.$$

- The ground state conditions

$$\xi^a\,|0\rangle = 0\,, \quad \left(\partial_{ia} + mx_{ia}\right)|0\rangle = 0 \;\Rightarrow\; |0\rangle = e^{-\frac{m}{2}x^2}, \quad Q^i\,|0\rangle = \bar{Q}_i\,|0\rangle = 0\,.$$

- The general bosonic state $|\ell\,; s\rangle$ is defined as

$$|\ell\,; s\rangle = A_{(i_1 i_2 \ldots i_s)(a_1 a_2 \ldots a_s)}\nabla^{i_1 a_1}\nabla^{i_2 a_2}\ldots\nabla^{i_s a_s}\left(\nabla^{ia}\nabla_{ia}\right)^{\ell}|0\rangle, \qquad (8)$$

$s/2$ is the highest weight ("isospin") of the irreducible representation of the group $SU(2)_{\rm PG}$ acting on indices a. The spin content with respect to $SU(2)_{\rm int}$ is the same.

- The general wave function $\Omega^{(l;s)}$ is a collection of $|\ell\,;s\rangle$ and its fermionic descendants obtained as the result of action of $\bar{Q}_i$ on it. The spectrum is given by

$$H\,\Omega^{(\ell;s)} = \frac{m}{2}\,(2\ell + s)\,\Omega^{(\ell;s)}, \qquad m > 0\,.$$

- Casimirs are

$$m^2 C_2 = H\,(H + m) - \frac{m^2}{2}\,E^a_b E^b_a\,,\; m^3 C_3 = \left(H + \frac{m}{2}\right) C_2\,;$$
$$\frac{1}{2} E^a_b E^b_a \Omega^{(\ell;s)} = \frac{s}{2}(\frac{s}{2} + 1)\,\Omega^{(\ell;s)}.$$

- All cases with $\ell \geq 0$ correspond to the typical representations, with the degeneracy $4(s+1)^2$ and equal number of bosonic and fermionic states; The wave function $\Omega^{(0;s)}$ describes atypical representations, with $(s+1)^2$ bosonic and $s(s+1)$ fermionic states and the degeneracy $(2s+1)(s+1)$.

Mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet

- The standard $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet has the content (x^{ia},ξ^{a}), i being the doublet index of $SU(2)_{\rm int}$ and a - of $SU(2)_{\rm PG}$; the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet has the content (y^{A},ψ^{iA}), A being the index of some other $SU'(2)_{\rm PG}$.
- In the flat $\mathcal{N}=4, d=1$ supersymmetry these multiplets and the relevant SQM models are equivalent, up to switching two $SU(2)$ automorphism algebras. In the $SU(2|1)$ case the models based on these two multiplets cease to be equivalent.
- The striking feature of the mirror multiplet is the existence of the explicit $SU(2|1)$ invariant Wess-Zumino term:

$$\tilde{\mathcal{L}}_{\rm WZ} = 2\gamma\left\{i\left(\dot{y}^{A}\,\partial_{A}f - \dot{\bar{y}}^{A}\,\bar{\partial}_{A}f\right) - \frac{m}{2}\left(y^{A}\,\partial_{A}f + \bar{y}^{A}\bar{\partial}_{A}f\right) - \frac{1}{2}\,\psi^{iA}\psi^{B}_{i}\,\partial_{A}\bar{\partial}_{B}f\right\},$$

  where $f(y,\bar{y})$ obeys the constraints

$$\Delta_{y}f = 0\,,\quad m\left(y^{B}\partial_{B} - \bar{y}^{B}\bar{\partial}_{B}\right)f\,(y,\bar{y}) = 0\,.$$

  It is very interesting to study the corresponding SQM models.
- By analogy with the $\mathcal{N}=4, d=1$ case, one can expect that all set of self-consistent $SU(2|1)$ SQM models should follow from those associated with the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets through various versions of the Hamiltonian reduction.

Summary and outlook

- We reviewed a new type of $\mathcal{N} = 4$ supersymmetric mechanics which is based on the supergroup $SU(2|1)$. It is a deformation of the standard $\mathcal{N} = 4$ mechanics by a mass parameter m.
- We presented the superfield formalism on two different coset manifolds of $SU(2|1)$ treated as the real and chiral $SU(2|1)$, $d = 1$ superspaces. They are carriers of the off-shell multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. There are two non-equivalent types of the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplet.
- The $SU(2|1)$ SQM models reveal surprising features. For the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet, the oscillator-type potential terms with m as a mass are present in the sigma-model action. For the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ models, the kinetic term is accompanied by the $d = 1$ WZ term and potential terms.
- We constructed harmonic $SU(2|1)$ superspace and described two sorts of the “root” $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets in its framework. The mirror $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet admits an explicit $SU(2|1)$ invariant $d = 1$ super WZ term.
- In all cases the sets of the quantum states reveal deviations from the standard rule of equality of the bosonic and fermionic states, in accordance with the existence of atypical $SU(2|1)$ representations.
- Superconformally invariant subclasses of the $SU(2|1)$ actions were constructed, based on the property that the superconformal algebra $D(2,1;\alpha)$ is a closure of its two $su(2|1)$ subalgebras. For the $d = 1$ conformal generators - trigonometric realization automatically.

- *Some further lines of development*:

  (a) Multi-particle extensions: to take a few superfields of one or different types, to construct the relevant off- and on-shell actions, to quantize, to identify the relevant target bosonic geometries (m-deformed?), etc.

  (b) To inquire whether the remaining $\mathcal{N} = 4, d = 1$ multiplets (e.g. the multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$) have their $SU(2|1)$ counterparts and to construct the corresponding SQM models. By analogy with the flat $\mathcal{N} = 4, d = 1$ harmonic superspace approach (E.I., O.Lechtenfeld, 2003), one can expect that all such multiplets and the associate SQM models should follow from the "root" multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ through the well defined gauging procedure (F.Delduc, E.I., 2006, 2007).

  (c) To generalize all this to the next in complexity case of the supergroup $SU(2|2)$. It involves 8 supercharges and so can presumably be treated as a deformation of $\mathcal{N} = 8, d = 1$ supersymmetry (and of $\mathcal{N} = (4, 4), d = 2$ supersymmetry, in fact).

  (d) To establish possible links with the higher-dimensional theories with "curved" supersymmetries in the localization approach.

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THANK YOU FOR ATTENTION!