Borel orbits of square 0 in $\mathfrak{sp}_{2n}(\mathbb{C})$ and orbital varieties

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June 19, 2015

Notation

- G – a simple Lie group over $\mathbb{C}$, $B \subset G$ – fixed Borel subgroup,
- R – the corresponding root system, R^+ – set of positive roots,
- $W = W(G, B)$ – Weyl group,
- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n} \oplus \mathfrak{h} = \mathrm{Lie}(B)$
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If G is of type A_n, B_n, C_n then $\mathcal{O}_u$ is defined completely by Jordan form of u (*Gerstenhaber, Kraft-Procesi*). The number of $G-$ orbits is finite.

Consider $\mathcal{O}_u \cap \mathfrak{n}$. This is a reducible, equidimensional Lagrangian subvariety of $\mathcal{O}_u$ so that in particular $\dim \mathcal{O}_u \cap \mathfrak{n} = 0.5 \dim \mathcal{O}_u$ (*Spaltenstein, Steinberg, Joseph*). Its irreducible components are called **orbital varieties**.

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The classification of such orbits is as follows (*Panyushev*):

- $u^2 = 0$ in cases $A_n,\ C_n$;
- $u^3 = 0$ with not more than one Jordan 3-block in cases $B_n,\ D_n$;

$B-$orbits of square 0 and sums of orthogonal roots in A_n

Different aspects of $B-$ orbits in a spherical orbit in the case of A_n were studied intensively (*Boos - Reineke, Di Francesco - Knutson - Zinn-Justin, Fresse - Melnikov, Ignatiev - Panov, Panyushev, Perrin - Smirnov, Rothbach, Stroppel, etc. etc*).

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Theorem

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Sums of orthogonal roots in A_n and link patterns

Recall that in case A_{n-1} we can choose B to be the group of upper-triangular invertible matrices (of det=1) and $\mathfrak{n}$ the algebra of strictly upper triangular matrices.
In this case root vectors for positive roots can be identified as $X_{e_i-e_j} = E_{i,j}$ where $1 \leq i < j \leq n$ and $E_{i,j}$ is an elementary matrix.
Recall also that $e_i - e_j$ and $e_k - e_l$ are strongly orthogonal iff $\{i,j\} \cap \{k,l\} = \emptyset$.
Thus,

$$\sum_{s=1}^{m} X_{\alpha_s} \longleftrightarrow (i_1, j_1) \ldots (i_m, j_m)$$

where $\alpha_s = e_{i_s} - e_{j_s}$.

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Thus,

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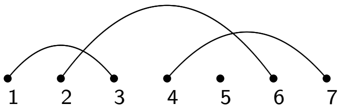
where $\alpha_s=e_{i_s}-e_{j_s}$.
We can visualise this set of disjoint 2-cycles as a graph on n points put on horizontal line with edges $(i_1,j_1)\dots(i_m,j_m)$ drawn as arcs.
Such an array is called a **link pattern**.

Example

For example, for $n = 7,\ k = 3$ and $\sigma = \{(1,3),\ (2,6),\ (4,7)\}$ one has

$P_\sigma =$ 

Combinatorics of link patterns and closures of B-orbits

So $B-$orbits of square 0 in $\mathfrak{n}$ are labelled by link patterns. We can read a lot of information on topology of $B-$orbits out of combinatorics of link pattern: dimensions, inclusions of the closures, smoothness of its closure in $G-$orbit, etc.

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We define a combinatorial order on link patterns $\preceq$ as follows: For $\sigma = (i_1, j_1)\ldots(i_s, j_s)$ and $\sigma' = (k_1, l_1)\ldots(k_t, l_t)$ put $\sigma' \preceq \sigma$ if for every $1 \leq a < b \leq n$ one has that the number of arcs of σ on the interval $[a, b]$ is greater or equal to the number of arcs of σ' on $[a, b]$.

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Theorem

(A.M.) For $B-$orbits $\mathcal{B}_\sigma,\ \mathcal{B}_{\sigma'}$ of square 0 in A_{n-1} one has $\mathcal{B}_{\sigma'} \subset \overline{\mathcal{B}}_\sigma$ iff $\sigma' \preceq \sigma$.

Applications to orbital varieties

Each orbital variety in a given $\mathcal{O}_u$ admits a unique dense $B-$orbit, so that considering the maximal $B-$orbits in $\mathcal{O}_u \cap \mathfrak{n}$ we get the data on orbital varieties.

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Remark: We have not used it but indeed link patterns are graphs of involutions in $W = S_n$.

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Symmetric link patterns

Recall that root vectors of positive roots in C_n are identified with
$X_{e_i-e_j} = E_{i,j} - E_{j+n,i+n} \longleftrightarrow (i,j)(2n+1-j, 2n+1-i)$
$X_{e_i+e_j} = E_{i,j+n} + E_{j,i+n} \longleftrightarrow (i, 2n+1-j)(j, 2n+1-i)$
$X_{2e_i} = E_{i,i+n} \longleftrightarrow (i, 2n+1-i)$
where $1 \leq i < j \leq n$.
Recall also that roots are strongly orthogonal in C_n iff
$\{i,j\} \cap \{k,l\} = \emptyset$ in the case of $e_i \pm e_j$ and $e_k \pm e_l$
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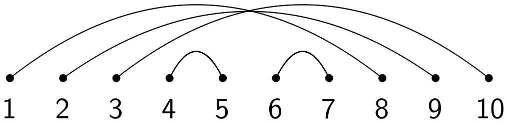
For example $X_{e_1+e_3} + X_{2e_2} + X_{e_4-e_5}$ in C_5 corresponds to



Ordering of symmetric link patterns and corollaries for orbital varieties

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(N.B - A.M.) The inclusions of $B-$*orbit closures of square 0 in* C_n *are defined by restriction of* $\preceq$ *on* $2n$ *points link patterns to the subset of symmetric link patterns.*

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Panyushev's conjecture for $B-$orbits in abelian nilradicals

D. Panyushev proposed a general approach to $B-$orbits in abelian nilradicals of all simple Lie algebras. They are always $B-$orbits in spherical orbits and can be labelled by special sums of strongly orthogonal roots.
In cases A_n and C_n these are orbits of square zero.
The sum of strongly orthogonal roots can be translated into an involution of W. In general different sums of strongly orthogonal roots can give the same involution, but this translation is 1:1 restricted to the subset of strongly orthogonal roots in abelian nilradicals. We label $B-$orbits by involutions σ in this subset.

Let $w_o \in W$ be its longest element. Let $\#(\sigma)$ be the number of 2-cycles in involution σ. Let $\leq$ denote Bruhat order on W and ℓ be length function on W.

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Conjecture

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- $\mathcal{B}_{\sigma'} \subset \overline{\mathcal{B}}_\sigma$ *if and only if* $w_o\sigma' w_o \leq w_o \sigma w_o$;
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The conjecture was known to be true for A_n. We have proven it for $B_n,\ C_n,\ D_n$. The only cases left are $E_6,\ E_7$.

Acknowlegment

Many Thanks!