L. Schwartz 1954

"Impossibility of the multiplication of distributions"

"Multiplication of distributions is impossible in any mathematical context, possibly different of distribution theory"

in other words: the impossibility has its origin in the foundations of mathematics: it is not a defect of distribution theory

To prove this claim L. S. stated a list of "axioms" to be satisfied by any multiplication of distributions: he proved a contradiction in his axioms.

We are going to see that the unique cause of the impossibility is the statement of one axiom in a non natural way. Everything lies in the two formulas: let H= Heaviside function

$$\int_{\mathbb{R}} (H^2 - H)(x)\,\varphi(x)\,dx = 0 \quad \forall \varphi \in \mathcal{C}_c^\infty \quad , \quad \int_{\mathbb{R}} (H^2 - H)(x)\,H'(x)\,dx = -\frac{1}{6} \neq 0$$

$$\int (H^2 - H)(x)\, \varphi(x)\, dx = 0 \quad \forall \varphi \in \mathcal{C}_c^\infty$$ follows from classical integration

(1) $$\int (H^2 - H)(x)\, H'(x)\, dx = \left[\frac{H^3}{3} - \frac{H^2}{2} \right]_{-\infty}^{+\infty} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} \neq 0$$

is a formal calculation justified when H is an idealization.

They show that in a context in which one could multiply distributions

$$\int F(x)\varphi(x)\, dx = 0 \ \forall \varphi \in \mathcal{C}_c^\infty \not\Longrightarrow F = 0$$

L. S. has chosen axioms that imply $H^2 - H = 0$

This gives the impossibility with (1)

L. S. chose his axioms to ensure coherence with the classical operations of derivation and multiplication

Is $\int (H^2 - H) H' \, dx \neq 0$ coherent with classical mathematics?

Let a sequence of calculations that make sense within $\mathcal{D}'$ let $f \in \mathcal{D}'$ be the result. Do the same calculations in $\mathcal{D}'$ in a slightly different way. Let $f_1 \in \mathcal{D}'$ be the result.

One has $\forall \varphi \in \mathcal{C}_c^\infty$ $\boxed{\int f_1(x) \varphi(x) dx = \int f(x) \varphi(x) dx}$

Now let A be an algebra containing $\mathcal{D}'$ and let $F \in A$ be the result of the same calculations in A.

$H^2 \neq H$ shows that one can have $F \neq f$. But coherence is (rather than $F = f$):

$\forall \varphi \in \mathcal{C}_c^\infty$ $\boxed{\int F(x) \varphi(x) dx = \int f(x) \varphi(x) dx}$

in A (We know this is weaker than $F = f$ in A).

Adopting the axiom

$$\int F(x)\,\varphi(x)\,dx = \int f(x)\,\varphi(x)\,dx \quad \forall \varphi \in \mathcal{C}_c^\infty$$

in place of the axiom

$$F = f \quad \text{in } A$$

there is no more impossibility and coherence is perfectly satisfied.

Indeed $\int (H^2 - H)\,\delta\,dx = 0$ is irrealistic

$H^2 - H$ is not defined at $x = 0$

" $\delta(0) = +\infty$ "

" $\int (H^2 - H)\,\delta\,dx = 0 \times \infty$ "

$\int (H^2 - H)\,H'\,dx = -\frac{1}{6}$ is obtained as a limit of smoothed H.

I constructed such a context in 1981

$$\Omega \subset \mathbb{R}^n \text{ open}, \quad \underbrace{\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^0(\Omega) \subset \underbrace{\mathcal{D}'(\Omega) \subset}_{\text{vector subspace}} \mathcal{G}(\Omega)}_{\text{subalgebra}}$$

induced partial derivatives

$$\mathcal{C}^0(\Omega) \xrightarrow{i} \mathcal{G}(\Omega) \text{ and } \mathcal{C}^0(\Omega) \text{ is not a subalgebra}$$

i.e. $i(fg) \neq i(f)\cdot i(g)$

but $\forall \varphi \in \mathcal{C}_c^\infty(\Omega) \quad \int i(fg)(x)\varphi(x) = \int f(x)g(x)\varphi(x)dx$

One observes in this way, through integration on test functions (not algebraically) coherence with all classical calculations.

The construction is very natural:
recall on Mikusinski (or sequential) definition of distributions

$$E = \left\{ (f_\varepsilon)_{\varepsilon \to 0},\ f_\varepsilon \in \mathcal{C}^\infty(\Omega), \,/\ \forall \varphi \in \mathcal{C}_c^\infty(\Omega)\ \int f_\varepsilon(x)\,\varphi(x)\,dx \xrightarrow[\varepsilon \to 0]{} \text{a limit in } \mathbb{R} \text{ or } \mathbb{C} \right\}$$

vector space
$\cup$

$$F = \left\{ (f_\varepsilon)_{\varepsilon \to 0} \ /\ \int f_\varepsilon(x)\,\varphi(x)\,dx \xrightarrow[\varepsilon \to 0]{} 0 \right\}$$

vector subspace of E

Theorem: $\mathcal{D}'(\Omega) = E/F$.

A natural modification gives $\mathcal{G}(\Omega)$

$\mathcal{A}$ = differential algebra far larger than E

$\cup$

$E = \{ (f_\varepsilon) \ / \ \int f_\varepsilon(x)\varphi(x)\,dx \to$ a limit

$\cup$

$F = \{ (f_\varepsilon) \ / \ \int f_\varepsilon(x)\varphi(x) \to 0$

$\cup$

$\mathcal{I}$ = ideal of $\mathcal{A}$ far smaller than F

$H^2 - H \in F$ and $H^2 - H \notin \mathcal{I}$

$\mathcal{G}(\Omega) := \mathcal{A}/\mathcal{I}$

Example of new predictions in physics

recall: in $\mathcal{G}(\Omega)$ two concepts play the role of equality

the equality $F = G$: all operations $+, \times, \frac{\partial}{\partial x}$ are allowed

the "association" $F \approx G$: $\times$ not allowed

$\Leftrightarrow \int (F-G)\varphi \, dx = 0 \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega)$

example: $\sqrt{\delta} \approx 0 \;\not\Rightarrow\; \underbrace{\sqrt{\delta}\,\sqrt{\delta}}_{= \delta} \approx \underbrace{0\sqrt{\delta}}_{= 0}$

$F = G \;\overset{\not\Leftarrow}{\Rightarrow}\; F \approx G$

example $H^2 \approx H$ and $H^2 \neq H$

system modelling strong collisions (armour of battle tanks)

in 1-D

$\rho_t + (\rho u)_x = 0$ mass conservation

$(\rho u)_t + (\rho u^2)_x + (p - s)_x = 0$ momentum "

$(\rho e)_t + (\rho e u)_x + [(p - s) u]_x = 0$ total energy "

$s_t + u s_x - \frac{4}{3} \mu(|s|) u_x = 0$ a differential form of Hooke's law

$p = \phi(\rho, e - \frac{u^2}{2})$ Mie-Grüneisen state law

Products $H \times \delta$ in case of shock waves

in $\underset{\downarrow}{u}\ \underset{\downarrow}{s_x}$ and $\underbrace{\mu(|s|)}_{\downarrow}\ \underset{\downarrow}{u_x}$

$\quad\ H \quad \delta \qquad\qquad\ \ H \quad \delta$

Problem: give jump conditions so as to construct a Godunov scheme ?

use of $\mathcal{G}$ because products of distributions.

the solution that was found :

| state the basic conservation laws with = in $\mathcal{G}$ (valid $\approx 10^{-9}m$)

| state the "secondary" equations (Hooke's law, Mie Grüneisen) with $\approx$ in $\mathcal{G}$ (valid at a far larger scale only)

has given well defined jump conditions (J F Colombeau lecture Notes in Math 1532)

Perfect agreement (qualitative and quantitative) with observations

<u>Basic point</u>. using the fact that $\mathcal{G}(\Omega)$ is a <u>more refined</u> mathematical context (2 concepts = and $\approx$) one states the laws of physics in a more refined way
$\Longrightarrow$ resolution of ambiguities in multiplication of distributions.

QFT : the canonical Hamiltonian formalism (Heisenberg – Pauli 1927)

The canonical Hamiltonian formalism is a formal solution of the interacting field equation

$$\left(-\partial_t^2 + \sum_{i=1}^{3} \partial_{x_i}^2 - m^2\right) \phi(x,t) = g\, \phi^N(x,t)$$

$$\phi(x,0) = \phi_0(x) = \text{Free Field operator}$$

The solution $\phi(x,t)$ = interacting field operator is

$$\phi(x,t) = e^{-i(t-\tau) H_0(\tau)}\, \phi_0(x,\tau)\, e^{i(t-\tau) H_0(\tau)}$$

where $H_0(\tau)$ is the Hamiltonian operator in which the free field has been inserted

$$\Longrightarrow \quad \phi(x,t) = S_\tau(t)^{-1}\, \phi_0(x,t)\, S_\tau(t)$$

$S := S_{-\infty}(+\infty)$ is called the scattering operator

If ϕ_1, ϕ_2 are normalized states

$|\langle \phi_1, S\phi_2 \rangle| =$ proba that ϕ_2 becomes ϕ_1 after interaction

S depends on $g \in \mathbb{R}$, $g =$ coupling constant

Phys. did a formal development $S = id + gS_1 + g^2 S_2 + \cdots$

and S_2 gave "infinite values" 1927

Renormalization theory (1950 $\rightarrow$) extracts finite answers

perfect agreement with experiments in QED (1950)

What can be done with $\mathcal{G}(\Omega)$?

in these formal calculations two different difficulties are mixed

- multiplications of distributions treated as $\mathcal{C}^\infty$ functions
- unbounded operators treated as bounded operators

the free field $\phi_0(x,t)$ is a distribution and

$$\phi_0(\varphi) = \int \phi_0(x,t)\,\varphi(x,t)\,dx\,dt$$ is an unbounded operator

The $\mathcal{G}$ context is adapted to multiplication of distributions but does not improve the problems with unbounded operators.

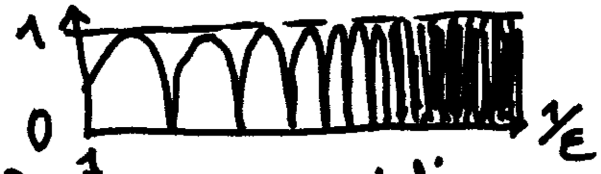
The exponentials in $\phi(x,t) = e^{-i(t-\tau)H_0(\tau)} \phi_0(x,\tau)\, e^{i(t-\tau)H_0(\tau)}$

make sense (from a proof of self-adjointness of $H_0(\tau)$)

One obtains a scattering operator S:

$|\langle \phi_1, S\phi_2 \rangle|$ is an oscillating value between 0 and 1, with a parameter $\varepsilon \to 0$

example $\left|\cos \frac{q}{\varepsilon}\right|$, $q \neq 0$



To such objects $f\left(\frac{1}{\varepsilon}\right)$ with f a quasi periodic function one can associate a real number which is their limit in probability

$$\left|\cos \frac{q}{\varepsilon}\right| \sim \frac{2}{\pi}$$

easy verification on a computer

In QFT I was asked to compute the value $|\langle \phi_1, S\phi_2 \rangle|$ in the well known cases : anomalous magnetic moment of the electron...

This can be done only numerically
? Reproduce the whole construction ?
I was unable a priori : computation in a Hilbert space etc. I do not know if this is realistic or not.

Colombeau - Gsponer 108 pages in arXiv 2008

Gsponer died in 2009 and the job was stopped.