Extension fullness and derived category $\mathcal{O}$

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Joint work with V. Mazorchuk

K. Coulembier, V. Mazorchuk.
Some homological properties of category $\mathcal{O}$. III,
Accepted in Advances in Mathematics.

K. Coulembier, V. Mazorchuk.
Extension fullness of the categories of Gelfand-Zeitlin and Whittaker modules,
SIGMA 11 (2015), 016, 17 pages.

K. Coulembier, V. Mazorchuk.
Dualities and derived equivalences for category $\mathcal{O}$,
In preparation.

Content:

(1) Intro: Category $\mathcal{O}$ as example of a highest weight category

(2) General concepts

- Ringel duality
- Derived categories
- Extension fullness

(3) Results

- Ringel duality for parabolic category $\mathcal{O}$
- Derived equivalences for category $\mathcal{O}$
- Extension fullness

Outline

Category $\mathcal{O}$

Consider a reductive Lie algebra $\mathfrak{g}$ (finite dimensional, complex), with triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The BGG category $\mathcal{O}$ is the full subcategory of the category of all modules, of modules which are

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- finitely generated
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Similarly $\mathcal{O}^{\mathfrak{p}}$ for a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$

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Simple, standard and projective modules

Verma module for every $\lambda \in \mathfrak{h}^*$

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Category $\mathcal{O}$ decomposes into blocks which each contain a finite amount of simple modules (described by Weyl group orbits), we denote such a block by $\mathcal{O}_\lambda$ with $\lambda \in \mathfrak{h}^*$ the highest of the highest weights of the simple modules.
$\mathcal{O}_\lambda \cong A_\lambda$-mod, for some finite dimensional algebra A_λ

Highest weight categories / quasi-hereditary algebras A finite dimensional algebra is quasi-hereditary iff there is some poset P such that there are modules $\Delta(\mu) \in A$-mod (called standard) for all $\mu \in P$ such that

- $\dim \mathrm{End}_A(\Delta(\mu)) = 1$
- $\mathrm{Hom}_A(\Delta(\mu_1), \Delta(\mu_2)) \neq 0$, then $\mu_1 \leq \mu_2$
- $\mathrm{Ext}^1_A(\Delta(\mu_1), \Delta(\mu_2)) \neq 0$, then $\mu_1 < \mu_2$
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Then there is also a dual notion of costandard modules $\nabla(\mu)$ and injective modules have costandard filtrations.
For category $\mathcal{O}$ this is governed by the duality functor.

Outline

For a quasi-hereditary algebra A with poset P, there exist precisely $|P|$ non-isomorphic indecomposable modules which admit both standard and costandard filtrations. They are known (unfortunately) as tilting modules $T(\mu)$.

The Ringel dual is defined as

$$R(A) := \mathrm{End}_A(T)^{\mathrm{opp}}$$

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Properties:

- $R(R(A)) \cong A$
- $R(A)$ inherits a quasi-hereditary structure from A
- A and $R(A)$ are 'derived equivalent'

Outline

Derived categories

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The previous result then reads

$$\mathcal{D}^b(A\text{-mod}) \;\cong\; \mathcal{D}^b(R(A)\text{-mod})$$

for a quasi-hereditary algebra A.

Yoneda extensions

Procedure introduced by Yoneda to define functors

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By a result of Verdier

$$\mathrm{Ext}^k_{\mathcal{C}}(M,N)\cong\mathrm{Hom}_{\mathcal{D}^b(\mathcal{C})}(M,N[k]).$$

Thus, also without injectives/projectives, the derived category allows to define/calculate extensions.

Outline

Definition

An abelian category $\mathcal{C}$ with exact full subcategory $\imath : \mathcal{B} \hookrightarrow \mathcal{C}$. We say that $\mathcal{B}$ is an extension full subcategory if $\imath$ induces an isomorphism

$$\imath : \operatorname{Ext}^k_{\mathcal{B}}(M, N) \to \operatorname{Ext}^k_{\mathcal{A}}(M, N)$$

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Example 1 and application

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Category $\mathcal{O}$ is extension full in the category of weight modules (Delorme). Consequently

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Example 2 (Cline-Parshall-Scott)

For A quasi-hereditary, the Serre subcategory of A-mod generated by the simples corresponding to a (co)-ideal in the poset is extension full in A-mod.

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The principal block in category $\mathcal{O}^{\mathfrak{p}}$ for any parabolic subalgebra $\mathfrak{p}$ of any reductive Lie algebra is Ringel self-dual.

Category $\mathcal{O}^{\mathfrak{p}}$, as a whole, is Ringel self dual for $\mathfrak{sl}(n)$, for any $\mathfrak{p}$.

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Theorem (K.C. and V. Mazorchuk)

Every block in category $\mathcal{O}^{\mathfrak{p}}$ for any parabolic subalgebra $\mathfrak{p}$ of any reductive Lie algebra is Ringel dual to **another** *block in the same category.*

$\longrightarrow$ These blocks are *not* always equivalent.

Outline

Here we consider only Lie algebras of type A. First set $\mathfrak{g} = \mathfrak{sl}(n)$. Any parabolic subalgebra corresponds to a composition of n. The Borel subalgebra $\mathfrak{b}$ corresponds to $(1, 1, \cdots, 1)$. Similarly, any dominant integral weight defines a partition. Moreover $\mathcal{O}_\lambda \cong \mathcal{O}_{\lambda'}$ if λ, λ' correspond to the same composition by W. Soergel.

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Theorem (M. Khovanov)

For $\mathfrak{g} = \mathfrak{sl}(n)$ and two compositions q_1 and q_2 of n which are the same up to ordering,

$$\mathcal{D}^b(\mathcal{O}_0^{q_1}) \cong \mathcal{D}^b(\mathcal{O}_0^{q_2}).$$

$\longrightarrow$ Note that in general here $\mathcal{O}_0^{q_1} \ncong \mathcal{O}_0^{q_2}$.

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$\longrightarrow$ The proof is very geometric, relying on description in terms of perverse sheaves on partial flag varieties.

Theorem (K.C. and V. Mazorchuk)

For $\mathfrak{g} = \mathfrak{sl}(n)$ and four compositions p_1, p_2, q_1 and q_2 of n such that p_1 and p_2 resp. q_1 and q_2 are the same up to ordering,

$$\mathcal{D}^b(\mathcal{O}^{q_1}_{p_1}) \cong \mathcal{D}^b(\mathcal{O}^{q_2}_{p_2}).$$

$\longrightarrow$ The proof is entirely algebraic, leading to an explicit description of the functor giving the equivalence.

Theorem (K.C. and V. Mazorchuk)

Consider two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ of type A, with respective Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$. Then there is a gradable derived equivalence

$$\mathcal{D}^b(\mathcal{O}_\lambda(\mathfrak{g},\mathfrak{b})) \;\cong\; \mathcal{D}^b(\mathcal{O}_{\lambda'}(\mathfrak{g}',\mathfrak{b}'))$$

for dominant $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$ if and only if there exists an isomorphism $\varphi : W_\Lambda \to W_{\Lambda'}$ such that $G \cap W_{\Lambda,\lambda} \cong \varphi(G) \cap W_{\Lambda',\lambda'}$ for any maximal simple subgroup G of W_Λ.

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Outline

Proposition (K.C. and V. Mazorchuk)

$\imath : \mathcal{B} \hookrightarrow \mathcal{C}$ is extension full if and only if $\imath$ induces a full and faithful functor

$$\imath : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{C})$$

Theorem (K.C. and V. Mazorchuk)

- *$\mathcal{O}$ is extension full in the category of weight modules*
- *thick $\mathcal{O}$ is extension full*
- *the category of generalised weight modules is extension full*
- *the category of Gelfand-Zeitlin modules is extension full*
- *the category of Whittaker modules is extension full*