# Operator Product Expansion

S. Hollands

UNIVERSITÄT LEIPZIG

based on joint work with J. Holland and Ch. Kopper

Varna
18 June 2015

European Research Council
Established by the European Commission

Commun.Math.Phys.313 (2012) 257-290, J.Math.Phys.54 (2013) 072302,

arXiv:1401.3144, arXiv:1411.1785

# Introduction

## Operator Product Expansion [Wilson '69]

Products of composite fields can be expanded as

$$\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\underbrace{\ldots}_{\text{Spectators}}\rangle \sim \sum_B \underbrace{\mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)}_{\text{OPE coefficients}}\langle \mathcal{O}_B(x_N)\ldots\rangle$$

# Introduction

## Operator Product Expansion [Wilson '69]

Products of composite fields can be expanded as

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \underbrace{\ldots}_{\text{Spectators}} \rangle \sim \sum_B \underbrace{\mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N)}_{\text{OPE coefficients}} \langle \mathcal{O}_B(x_N) \ldots \rangle$$

- Asymptotic short distance expansion:
  Difference vanishes in the limit $x_i \to x_N$ for all $i \leq N$

# Introduction

## Operator Product Expansion [Wilson '69]

Products of composite fields can be expanded as

$$\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) \underbrace{\ldots}_{\text{Spectators}} \rangle \sim \sum_B \underbrace{\mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)}_{\text{OPE coefficients}} \langle \mathcal{O}_B(x_N)\ldots\rangle$$

- Asymptotic short distance expansion:
  Difference vanishes in the limit $x_i \to x_N$ for all $i \leq N$
- Practical application e.g. in deep-inelastic scattering
- Plays fundamental role in conformal field theory
  (Conformal bootstrap, "Vertex operator algebras", ...)
- Plays fundamental role in QFTCST
  (State-independent definition of QFT!)

# Topics of today's talk:

1. In what sense does the OPE converge? $N$-point functions $\leftrightarrow$ $1$-point functions & OPE coefficients
2. What are algebraic relations between OPE coefficients? Vertex algebras in $d$-dims.
3. A novel recursion scheme for OPE coefficients New self-consistent construction method

# Topics of today's talk:

1. In what sense does the OPE converge? $N$-point functions $\leftrightarrow$ $1$-point functions & OPE coefficients

2. What are algebraic relations between OPE coefficients?
   Vertex algebras in $d$-dims.

3. A novel recursion scheme for OPE coefficients
   New self-consistent construction method

Model: Perturbative, Euclidean $\varphi_4^4$-theory

# The model: Euclidean $\varphi^4$-theory

- Correlation functions are defined via the path integral

$$\langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \exp\left[-S\right] \, \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \, ,$$

where the action is given by

$$S(\varphi) := \int \mathrm{d}^4 x \left( \frac{1}{2} (\partial_\mu \varphi)^2(x) + \frac{m^2}{2} \varphi^2(x) + g\varphi(x)^4 - \text{counterterms} \right)$$

# The model: Euclidean $\varphi^4$-theory

- Correlation functions are defined via the path integral

$$\langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \exp\left[-S\right] \, \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \,,$$

  where the action is given by

$$S(\varphi) := \int \mathrm{d}^4 x \left( \frac{1}{2} (\partial_\mu \varphi)^2(x) + \frac{m^2}{2} \varphi^2(x) + g\varphi(x)^4 - \text{counterterms} \right)$$

- Composite operator insertions $\mathcal{O}_A(x) = (\partial^{v_1}\varphi \dots \partial^{v_n}\varphi)(x)$ (renormalized)

# The model: Euclidean $\varphi^4$-theory

- Correlation functions are defined via the path integral

$$\langle \mathcal{O}_{A_1}(x_1)\dots\mathcal{O}_{A_N}(x_N)\rangle \; := \mathcal{N} \int \mathcal{D}\varphi \, \exp\left[-S\right] \, \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\,,$$

  where the action is given by

$$S(\varphi) := \int \mathrm{d}^4x \left( \frac{1}{2}(\partial_\mu\varphi)^2(x) + \frac{m^2}{2}\varphi^2(x) + g\varphi(x)^4 - \text{counterterms} \right)$$

- Composite operator insertions $\mathcal{O}_A(x) = (\partial^{v_1}\varphi\dots\partial^{v_n}\varphi)(x)$ (renormalized)

- OPE coefficients can be defined a la Zimmermann or a la Keller-Kopper

# The model: Euclidean $\varphi^4$-theory

- Correlation functions are defined via the path integral

$$\langle \mathcal{O}_{A_1}(x_1)\dots\mathcal{O}_{A_N}(x_N)\rangle := \mathcal{N}\int \mathcal{D}\varphi\, \exp\left[-S\right]\, \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\,,$$

  where the action is given by

$$S(\varphi) := \int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu\varphi)^2(x) + \frac{m^2}{2}\varphi^2(x) + g\varphi(x)^4 - \text{counterterms}\right)$$

- Composite operator insertions $\mathcal{O}_A(x) = (\partial^{v_1}\varphi\dots\partial^{v_n}\varphi)(x)$ (renormalized)
- OPE coefficients can be defined a la Zimmermann or a la Keller-Kopper
- We use a "renormalization group flow equation" approach [Wilson, Polchinski, Kopper-Keller-Salmhofer]

# Outline

# Outline

# The OPE factorises

## Theorem (Holland-SH)

*At any arbitrary but fixed loop order:*

$$\mathcal{C}_{A_1 \dots A_N}^{B}(x_1, \dots, x_N) = \sum_C \mathcal{C}_{A_1 \dots A_M}^{C}(x_1, \dots, x_M) \mathcal{C}_{C A_{M+1} \dots A_N}^{B}(x_M, \dots, x_N)$$

*holds on the domain* $\frac{\max_{1 \le i \le M} |x_i - x_M|}{\min_{M < j \le N} |x_j - x_M|} < 1$. *(Sum over* $C$ *absolutely convergent !)*

# The OPE factorises

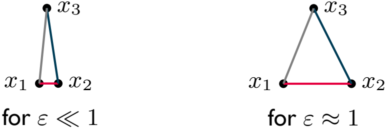
## Theorem (Holland-SH)

*At any arbitrary but fixed loop order:*

$$\mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N) = \sum_C \mathcal{C}^{C}_{A_1\ldots A_M}(x_1,\ldots,x_M)\mathcal{C}^{B}_{C A_{M+1}\ldots A_N}(x_M,\ldots,x_N)$$

*holds on the domain* $\frac{\max_{1\le i\le M}|x_i-x_M|}{\min_{M<j\le N}|x_j-x_M|} < 1$. *(Sum over* $C$ *absolutely convergent !)*

For $N=3$: $\varepsilon = \frac{|x_1-x_2|}{|x_2-x_3|} < 1$



# The OPE factorises

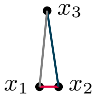
## Theorem (Holland-SH)

*At any arbitrary but fixed loop order:*

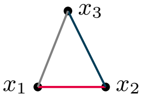
$$\mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N)=\sum_C \mathcal{C}^{C}_{A_1\ldots A_M}(x_1,\ldots,x_M)\mathcal{C}^{B}_{C A_{M+1}\ldots A_N}(x_M,\ldots,x_N)$$

*holds on the domain* $\frac{\max_{1\leq i\leq M}|x_i-x_M|}{\min_{M<j\leq N}|x_j-x_M|}<1$. *(Sum over* $C$ *absolutely convergent !)*

For $N=3$: $\varepsilon=\frac{|x_1-x_2|}{|x_2-x_3|}<1$




for $\varepsilon \ll 1$




for $\varepsilon \approx 1$

This shows associativity really holds!

- Vertex Algebras (Borcherds property) also in $4d$.
- $\mathcal{C}^{B}_{A_1\ldots A_N}$ uniquely determined in terms of $\mathcal{C}^{B}_{A_1 A_2}$
- "Bootstrap construction" of OPE coefficients possible

# Outline

# Bound on OPE remainder I

## Theorem (Holland-Kopper-SH)

*At any perturbation order $r$ and for any $D \in \mathbb{N}$,*

$$\Big|\overbrace{\Big\langle\Big(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\leq D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\Big)\underbrace{\hat{\varphi}(p_1)\cdots\hat{\varphi}(p_n)}_{\text{Spectator fields}}\Big\rangle}^{\text{OPE-Remainder}}\Big|$$

# Bound on OPE remainder I

## Theorem (Holland-Kopper-SH)

*At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that*

$$\overbrace{\left| \left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\dim[B] \le D} \mathcal{C}^{B}_{A_1 \dots A_N}(x_1, \dots, x_N)\, \mathcal{O}_B(x_N) \right) \underbrace{\hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n)}_{\text{Spectator fields}} \right\rangle \right|}^{\text{OPE-Remainder}}$$

$$\le \frac{M^{n-1}}{\sqrt{D!}} \frac{\left( K M \max\limits_{1 \le i \le N} |x_i - x_N| \right)^{D+1}}{\min\limits_{1 \le i < j \le N} |x_i - x_j|^{\sum_i \dim[A_i] + 1}} \cdot \sup\left(1, \frac{|P|}{\sup(m, \kappa)}\right)^{(D+2)(r+5)}$$

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale

# Bound on OPE remainder I

## Theorem (Holland-Kopper-SH)

*At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that*

$$\overbrace{\left|\left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\dim[B] \le D} \mathcal{C}^B_{A_1 \dots A_N}(x_1, \dots, x_N)\, \mathcal{O}_B(x_N) \right) \underbrace{\hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n)}_{\text{Spectator fields}} \right\rangle\right|}^{\text{OPE-Remainder}}$$

$$\le \frac{M^{n-1}}{\sqrt{D!}} \frac{\left( K M \max_{1 \le i \le N} |x_i - x_N| \right)^{D+1}}{\min_{1 \le i < j \le N} |x_i - x_j|^{\sum_i \dim[A_i] + 1}} \cdot \sup\left(1, \frac{|P|}{\sup(m, \kappa)}\right)^{(D+2)(r+5)}$$

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale
- $|P| = \sup_i |p_i|$: maximal momentum of spectators

# Bound on OPE remainder I

## Theorem (Holland-Kopper-SH)

*At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that*

$$\Big|\overbrace{\Big\langle \Big( \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\le D} \mathcal{C}^{B}_{A_1\dots A_N}(x_1,\dots,x_N)\,\mathcal{O}_B(x_N)\Big)\underbrace{\hat{\varphi}(p_1)\cdots\hat{\varphi}(p_n)}_{\text{Spectator fields}}\Big\rangle}^{\text{OPE-Remainder}}\Big|$$

$$\le \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\left(KM \max\limits_{1\le i\le N}|x_i - x_N|\right)^{D+1}}{\min\limits_{1\le i<j\le N}|x_i - x_j|^{\sum_i \dim[A_i]+1}}\cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale
- $|P| = \sup_i |p_i|$: maximal momentum of spectators
- $\kappa := \inf(\mu, \varepsilon)$, where $\varepsilon = \min_{I\subset\{1,\dots,n\}} |\sum_I p_i|$
  $\varepsilon$: distance of $(p_1, \dots, p_n)$ to "exceptional" configurations

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \le \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max\limits_{1\le i\le N} |x_i - x_N|\right)^{D+1}}{\min\limits_{1\le i<j\le N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(KM \max_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

1. Massive fields ($m > 0$): Bound is finite for arbitrary $p_1, \ldots, p_n$

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(KM \max_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

1. Massive fields ($m > 0$): Bound is finite for arbitrary $p_1, \ldots, p_n$
2. Massless fields: Bound is finite only for non-exceptional $p_1, \ldots, p_n$

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max\limits_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min\limits_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

1. Massive fields ($m > 0$): Bound is finite for arbitrary $p_1, \ldots, p_n$
2. Massless fields: Bound is finite only for non-exceptional $p_1, \ldots, p_n$
3. Bound vanishes as $D \to \infty$

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

1. Massive fields ($m > 0$): Bound is finite for arbitrary $p_1, \ldots, p_n$
2. Massless fields: Bound is finite only for non-exceptional $p_1, \ldots, p_n$
3. Bound vanishes as $D \to \infty \Rightarrow$ OPE converges at any finite distances!

# Conclusions from bound on OPE remainder

$$\text{"OPE remainder"} \leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(KM \max_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

1. Massive fields ($m > 0$): Bound is finite for arbitrary $p_1, \ldots, p_n$
2. Massless fields: Bound is finite only for non-exceptional $p_1, \ldots, p_n$
3. Bound vanishes as $D \to \infty \Rightarrow$ OPE converges at any finite distances!
4. Convergence is slow if...
   - $|P|$ is large ("energy scale" of spectators)

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max\limits_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min\limits_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

1. Massive fields ($m > 0$): Bound is finite for arbitrary $p_1, \ldots, p_n$
2. Massless fields: Bound is finite only for non-exceptional $p_1, \ldots, p_n$
3. Bound vanishes as $D \to \infty \Rightarrow$ OPE converges at any finite distances!
4. Convergence is slow if...
   - $|P|$ is large (“energy scale” of spectators)
   - maximal distance of points $x_i$ from reference point $x_N$ is large

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \, \frac{\left(KM \max_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

1. Massive fields ($m > 0$): Bound is finite for arbitrary $p_1, \ldots, p_n$
2. Massless fields: Bound is finite only for non-exceptional $p_1, \ldots, p_n$
3. Bound vanishes as $D \to \infty \Rightarrow$ OPE converges at any finite distances!
4. Convergence is slow if...
   - $|P|$ is large (“energy scale” of spectators)
   - maximal distance of points $x_i$ from reference point $x_N$ is large
   - ratio of max. and min. distances is large, e.g. for $N = 3$

# Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

## Theorem (Holland-Kopper-SH)

*At any perturbation order* $r$ *and for any* $D \in \mathbb{N}$,

$$\Big|\Big\langle \Big(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\leq D} \mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\Big)\varphi(f_1)\cdots\varphi(f_n)\Big\rangle\Big|$$

# Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

### Theorem (Holland-Kopper-SH)

*At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that*

$$\left|\left\langle\Big(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\le D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\Big)\varphi(f_1)\cdots\varphi(f_n)\right\rangle\right|$$

$$\le \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\Big(KM\max_{1\le i\le N}|x_i - x_N|\Big)^{D+1}}{\min_{1\le i<j\le N}|x_i - x_j|^{\sum_i \dim[A_i]+1}}\sum_{s_1+\ldots+s_N=0}^{(D+2)(r+5)}\prod_{i=1}^{n}\frac{\|\hat{f}_i\|_{\frac{s_i}{2}}}{M^{s_i}}$$

$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields

$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}|(p^2+M^2)^s\hat{f}(p)|$ (Schwartz norm)

# Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

## Theorem (Holland-Kopper-SH)

*At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that*

$$\left|\left\langle\Big(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\leq D} \mathcal{C}^B_{A_1\dots A_N}(x_1,\dots,x_N)\,\mathcal{O}_B(x_N)\Big)\varphi(f_1)\cdots\varphi(f_n)\right\rangle\right|$$

$$\leq \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\Big(KM\max\limits_{1\leq i\leq N}|x_i - x_N|\Big)^{D+1}}{\min\limits_{1\leq i<j\leq N}|x_i - x_j|^{\sum_i \dim[A_i]+1}} \sum_{s_1+\ldots+s_N=0}^{(D+2)(r+5)} \prod_{i=1}^{n} \frac{\|\hat{f}_i\|_{\frac{s_i}{2}}}{M^{s_i}}$$

$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields
$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4} |(p^2 + M^2)^s \hat{f}(p)|$ (Schwartz norm)

1. Bound is finite for any $f_i \in \mathcal{S}(\mathbb{R}^4)$ (*Schwartz space*)
   OPE remainder is a tempered distribution

# Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

## Theorem (Holland-Kopper-SH)

*At any perturbation order* $r$ *and for any* $D \in \mathbb{N}$, *there exists a* $K > 0$ *such that*

$$\left|\left\langle\left(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\leq D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\right)\varphi(f_1)\cdots\varphi(f_n)\right\rangle\right|$$

$$\leq \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\left(KM\max\limits_{1\leq i\leq N}|x_i - x_N|\right)^{D+1}}{\min\limits_{1\leq i<j\leq N}|x_i - x_j|^{\sum_i \dim[A_i]+1}}\sum_{s_1+\ldots+s_N=0}^{(D+2)(r+5)}\prod_{i=1}^{n}\frac{\|\hat{f}_i\|_{\frac{s_i}{2}}}{M^{s_i}}$$

$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields
$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}|(p^2+M^2)^s\hat{f}(p)|$ (Schwartz norm)

1. Bound is finite for any $f_i \in \mathcal{S}(\mathbb{R}^4)$ (*Schwartz space*)
   OPE remainder is a tempered distribution
2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$:

# Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

## Theorem (Holland-Kopper-SH)

*At any perturbation order* $r$ *and for any* $D \in \mathbb{N}$*, there exists a* $K > 0$ *such that*

$$\left|\left\langle\left(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\le D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\right)\varphi(f_1)\cdots\varphi(f_n)\right\rangle\right|$$

$$\le \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\left(KM\max_{1\le i\le N}|x_i - x_N|\right)^{D+1}}{\min_{1\le i<j\le N}|x_i - x_j|^{\sum_i \dim[A_i]+1}}\,\sup\left(1,\frac{|P|}{M}\right)^{(D+2)(r+5)}$$

$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields
$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}|(p^2 + M^2)^s\hat{f}(p)|$ (Schwartz norm)

1. Bound is finite for any $f_i \in \mathcal{S}(\mathbb{R}^4)$ (*Schwartz space*)
   OPE remainder is a tempered distribution
2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$:

# Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

## Theorem (Holland-Kopper-SH)

*At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that*

$$\left|\left\langle\left(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\le D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\right)\varphi(f_1)\cdots\varphi(f_n)\right\rangle\right|$$

$$\le \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\left(KM\max\limits_{1\le i\le N}|x_i-x_N|\right)^{D+1}}{\min\limits_{1\le i<j\le N}|x_i-x_j|^{\sum_i \dim[A_i]+1}}\ \sup\left(1,\frac{|P|}{M}\right)^{(D+2)(r+5)}$$

$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields
$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}|(p^2+M^2)^s\hat{f}(p)|$ (Schwartz norm)

1. Bound is finite for any $f_i \in \mathcal{S}(\mathbb{R}^4)$ (*Schwartz space*)
   OPE remainder is a tempered distribution
2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$: Bound vanishes as $D \to \infty$
   $\Rightarrow$ OPE converges at any finite distances!

# Outline

# Motivation for a new construction method

Textbook method (roughly):

- Write down correlation function with operator insertions
- Perform short distance/large momentum expansion (in some clever way)
- Argue that the coefficients obtained this way are state independent

# Motivation for a new construction method

Textbook method (roughly):

- Write down correlation function with operator insertions
- Perform short distance/large momentum expansion (in some clever way)
- Argue that the coefficients obtained this way are state independent

Not entirely satisfying:

- Relies on correlation functions $\Rightarrow$ OPE not 'fundamental'
- State independence not obvious
- Hard to study general properties of OPE

# Recursion formula (for mass $m > 0$)

## Theorem (Hollands-JH)

*Coupling constant derivatives of OPE coefficients in $g\varphi^4$-theory can be expressed as*

$$\partial_g \mathcal{C}^{B}_{A_1\dots A_N}(x_1,\dots,x_N) = -\int \mathrm{d}^4 y \Big[ \mathcal{C}^{B}_{\varphi^4 A_1\dots A_N}(y,x_1,\dots,x_N) - \sum_{i=1}^{N} \sum_{[C]\leq[A_i]} \mathcal{C}^{C}_{\varphi^4 A_i}(y,x_i)\, \mathcal{C}^{B}_{A_1\dots\widehat{A_i}\, C\dots A_N}(x_1,\dots,x_N) - \sum_{[C]<[B]} \mathcal{C}^{C}_{A_1\dots A_N}(x_1,\dots,x_N)\, \mathcal{C}^{B}_{\varphi^4 C}(y,x_N) \Big].$$

# Recursion formula (for mass $m > 0$)

## Theorem (Hollands-JH)

*Coupling constant derivatives of OPE coefficients in $g\varphi^4$-theory can be expressed as*

$$\partial_g \, \mathcal{C}_{A_1 \ldots A_N}^{B}(x_1, \ldots, x_N) = -\int \mathrm{d}^4 y \Big[ \mathcal{C}_{\varphi^4 A_1 \ldots A_N}^{B}(y, x_1, \ldots, x_N) - \sum_{i=1}^{N} \sum_{[C] \leq [A_i]} \mathcal{C}_{\varphi^4 A_i}^{C}(y, x_i) \, \mathcal{C}_{A_1 \ldots \widehat{A_i} \, C \ldots A_N}^{B}(x_1, \ldots, x_N) - \sum_{[C] < [B]} \mathcal{C}_{A_1 \ldots A_N}^{C}(x_1, \ldots, x_N) \, \mathcal{C}_{\varphi^4 C}^{B}(y, x_N) \Big] .$$

- Compute OPE coefficients to any perturbation order by iteration.
  Initial data: Coefficients of free theory.

# Recursion formula (for mass $m > 0$)

## Theorem (Hollands-JH)

*OPE coefficients at perturbation order* $(r+1)$ *can be expressed as*

$$(\mathcal{C}_{r+1})^{B}_{A_1\dots A_N}(x_1,\dots,x_N) = -\int \mathrm{d}^4y \Big[ (\mathcal{C}_r)^{B}_{\varphi^4 A_1\dots A_N}(y,x_1,\dots,x_N)$$

$$- \sum_{s=0}^{r}\sum_{i=1}^{N}\sum_{[C]\leq[A_i]} (\mathcal{C}_s)^{C}_{\varphi^4 A_i}(y,x_i)\,(\mathcal{C}_{r-s})^{B}_{A_1\dots\widehat{A_i}\,C\dots A_N}(x_1,\dots,x_N)$$

$$- \sum_{s=0}^{r}\sum_{[C]<[B]} (\mathcal{C}_s)^{C}_{A_1\dots A_N}(x_1,\dots,x_N)\,(\mathcal{C}_{r-s})^{B}_{\varphi^4 C}(y,x_N)\Big].$$

- Compute OPE coefficients to any perturbation order by iteration.
  Initial data: Coefficients of free theory.

# Recursion formula (for mass $m > 0$)

## Theorem (Hollands-JH)

*Coupling constant derivatives of OPE coefficients in $g\varphi^4$-theory can be expressed as*

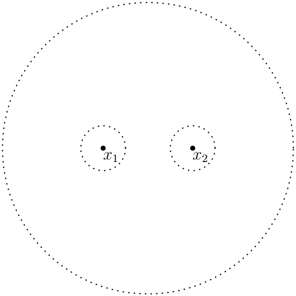
$$\partial_g \mathcal{C}^{B}_{A_1\dots A_N}(x_1,\dots,x_N) = -\int \mathrm{d}^4 y \Big[ \mathcal{C}^{B}_{\varphi^4 A_1\dots A_N}(y,x_1,\dots,x_N) - \sum_{i=1}^{N} \sum_{[C]\le[A_i]} \mathcal{C}^{C}_{\varphi^4 A_i}(y,x_i)\, \mathcal{C}^{B}_{A_1\dots\widehat{A_i}\, C\dots A_N}(x_1,\dots,x_N) - \sum_{[C]<[B]} \mathcal{C}^{C}_{A_1\dots A_N}(x_1,\dots,x_N)\, \mathcal{C}^{B}_{\varphi^4 C}(y,x_N) \Big] .$$

- Compute OPE coefficients to any perturbation order by iteration.
  Initial data: Coefficients of free theory.
- State independence obvious.
  No other objects enter the construction.
- The formula depends on the renormalisation conditions.
  (Here BPHZ)

# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \mathcal{C}_{\varphi^4 A_1 A_2}^{B}(y, x_1, x_2) - \sum_{[C] \leq [A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C] \leq [A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C] < [B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

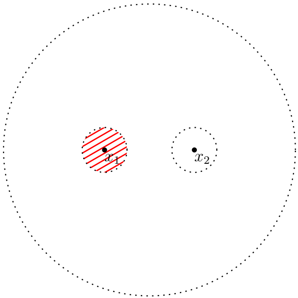


# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4y \Big[ \mathcal{C}_{\varphi^4 A_1 A_2}^{B}(y, x_1, x_2) - \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

UV-region I ($y \approx x_1$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises

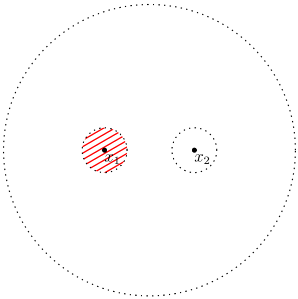


# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \sum_{[C]=0}^{\infty} \mathcal{C}_{\varphi^4 A_1}^{C}(y,x_1)\, \mathcal{C}_{C A_2}^{B}(x_1,x_2) - \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y,x_1)\, \mathcal{C}_{C A_2}^{B}(x_1,x_2)$$

$$- \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y,x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1,x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1,x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y,x_2) \Big]$$

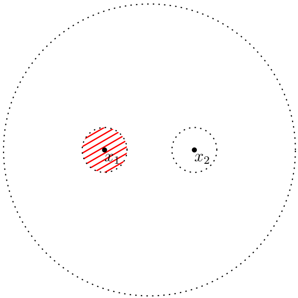
UV-region I ($y \approx x_1$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises



# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \sum_{[C]>[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

UV-region I ($y \approx x_1$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises $\Rightarrow$ divergences cancel

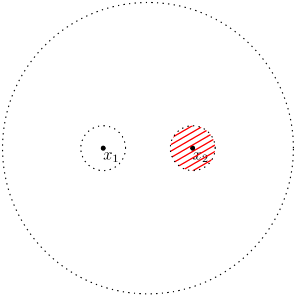


# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \mathcal{C}_{\varphi^4 A_1 A_2}^{B}(y, x_1, x_2) - \sum_{[C] \leq [A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2)$$

$$- \sum_{[C] \leq [A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C] < [B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

UV-region II ($y \approx x_2$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises

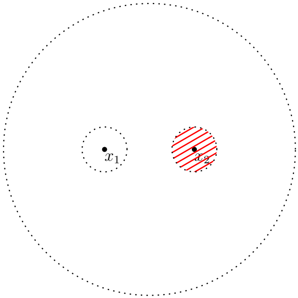


# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \sum_{[C]>[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

UV-region II ($y \approx x_2$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises $\Rightarrow$ divergences cancel

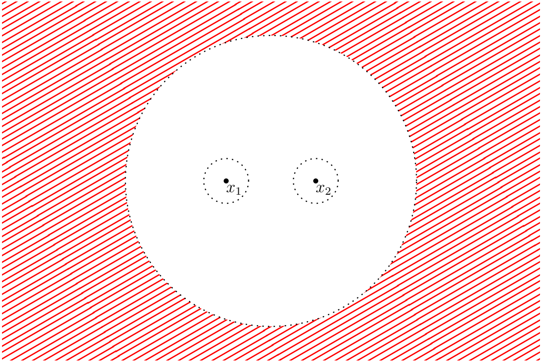


# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \sum_{[C]\geq[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) \Big]$$

IR-region ($|y - x_2| \gg |x_1 - x_2|$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises

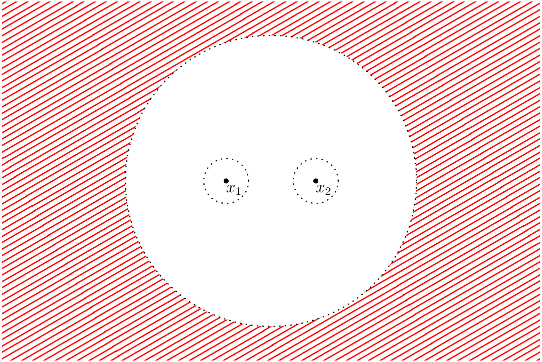


# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \sum_{[C]\geq[B]} \mathcal{C}^{C}_{A_1 A_2}(x_1, x_2)\, \mathcal{C}^{B}_{\varphi^4 C}(y, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}^{C}_{\varphi^4 A_1}(y, x_1)\, \mathcal{C}^{B}_{C A_2}(x_1, x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}^{C}_{\varphi^4 A_2}(y, x_2)\, \mathcal{C}^{B}_{A_1 C}(x_1, x_2) \Big]$$

IR-region ($|y - x_2| \gg |x_1 - x_2|$): $\mathcal{C}^{B}_{\varphi^4 A_1 A_2}$ factorises $\Rightarrow$ divergences cancel

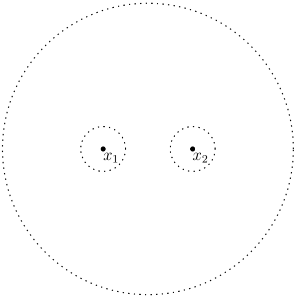


# Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[ \mathcal{C}_{\varphi^4 A_1 A_2}^{B}(y, x_1, x_2) - \sum_{[C] \leq [A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C] \leq [A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C] < [B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

The integral is absolutely convergent due to the factorisation property.



# Conclusions & Outlook

In Euclidean perturbation theory, we found that:

1. The OPE converges at finite distances.
2. The OPE factorises (associativity).
3. The OPE satisfies a recursion formula.

# Conclusions & Outlook

In Euclidean perturbation theory, we found that:

1. The OPE converges at finite distances.
2. The OPE factorises (associativity).
3. The OPE satisfies a recursion formula.

## Possible Generalisations

- Gauge theories (in progress)
- Curved manifolds
- Minkowski space
- ...

## Applications of the Recursion Formula

- Does the algorithm facilitate computations?
- Does the perturbation series for OPE coefficients converge?