# Perturbative quantum field theory meets number theory

Ivan Todorov

Bulgarian Academy of Sciences,

Lie Theory and its application in Physics

A family of functions and numbers is gradually uncovered
- first independently and recently in collaboration -
by mathematicians and by particle theorists:

(Chen's) iterated integrals $\leftrightarrow$ Feynman amplitudes

(Kontsevich-Zagier) periods $\leftrightarrow$ residues of poles of regularized primitively divergent amplitudes; values of scattering amplitudes at rational ratios of momenta and masses.

This family includes hyperlogarithms and multiple zeta values and has a rich algebraic structure and interesting (partly conjectural) properties.

Aim: introduction and a broad overview of the subject.

Residues of primitively divergent amplitudes

- Periods in position space renormalization
- Primitive conformal amplitudes

Double shuffle algebra of hyperlogarithms

Formal multizeta values

Single-valued hyperlogarithms. Applications

Outlook

# Residues of primitively divergent amplitudes

A position space Feynman integrand $G(\vec{x})$ in a massless QFT is a rational homogeneous function of $\vec{x} \in \mathbb{R}^{\mathbb{N}}$. If $G$ corresponds to a connected graph with $V$ vertices in $D = 4$ then $N = 4(V - 1)$. The integrand is *superficially divergent* if $G$ defines a homogeneous density in $\mathbb{R}^N$ of non-positive degree:

$$G(\lambda\vec{x})\, d^N\lambda x = \lambda^{-\kappa} G(\vec{x})\, d^N x\ , \quad \kappa \geq 0\ , \quad x \in \mathbb{R}^N \setminus \mathcal{D} \quad (\lambda > 0)\ ; \tag{1.1}$$

$\kappa$ is called (superficial) *degree of divergence*. In a scalar QFT with massless propagators a connected graph with a set $\mathcal{L}$ of internal lines gives rise to a Feynman amplitude that is a multiple of the product

$$G(\vec{x}) = \prod_{(i,j)\in\mathcal{L}} \frac{1}{x_{ij}^2}\ . \tag{1.2}$$

For a *primitively divergent* amplitude
the following proposition serves as a definition of both
the *residue ResG* and of a *renormalized amplitude* $G^{\rho}(\vec{x})$.
(N. Nikolov, R. Stora, I.T., 2011-2014)

**Proposition 1.** *If $G(\vec{x})$ (1.2) is primitively divergent then for any smooth norm $\rho(\vec{x})$ on $\mathbb{R}^N$ one has*

$$[\rho(\vec{x})]^{\epsilon} G(\vec{x}) - \frac{1}{\epsilon}(Res\, G)(\vec{x}) = G^{\rho}(\vec{x}) + O(\epsilon). \qquad (1.3)$$

Here $Res\, G$ is a distribution with support at the origin. Its calculation is reduced to the case $\kappa = 0$ of a logarithmically divergent graph by using the identity

$$(Res\, G)(\vec{x}) = \frac{(-1)^{\kappa}}{\kappa!}\, \partial_{i_1} ... \partial_{i_\kappa} Res\, (x^{i_1} ... x^{i_\kappa} G)(\vec{x}) \qquad (1.4)$$

where summation is assumed (from 1 to N) over the repeated indices $i_1, ..., i_\kappa$. If $G$ is homogeneous of degree $-N$ then

$$(Res\, G)(\vec{x}) = res\,(G)\, \delta(\vec{x}) \quad (\text{for } \partial_i(x^i G) = 0)\,. \qquad (1.5)$$

Here the numerical residue *res G* is given by an integral over the hypersurface $\Sigma_\rho = \{\vec{x} |\, \rho(\vec{x}) = 1\}$:

$$res\, G = \frac{1}{\pi^{N/2}} \int_{\Sigma_\rho} G(\vec{x}) \sum_{i=1}^{N} (-1)^{i-1} x^i dx^1 \wedge ... \hat{dx^i} ... \wedge dx^N, \quad (1.6)$$

(a hat over an argument meaning, as usual, that this argument is omitted). The residue *res G* is independent of the (transverse to the dilation) surface $\Sigma_\rho$ since the form in the integrant is closed in the projective space $\mathbb{P}^{N-1}$.

- Note that N is even, in fact, divisible by 4, so that $\mathbb{P}^{N-1}$ is orientable.
- The functional *res G* is a *period* according to the definition of Maxim Kontsevich and Don Zagier. Such residues are often called ”Feynman” or ”quantum” periods in the present context.
- The same numbers appear in the expansion of the renormalization group beta function (Broadhurst-Kreimer, Schnetz 1990’s).

The convention of accompanying the 4D volume $d^4x$ by a $\pi^{-2}$ yields **rational residues for one- and two-loop graphs**.

For graphs with three or higher *number of loops* $\ell$ one encounters **multiple zeta values of overall weight** not exceeding $2\ell - 3$.

The only residues at three, four and five loops (in the $\varphi^4$ theory) are integer multiples of $\zeta(3), \zeta(5)$ and $\zeta(7)$, respectively.

The first double zeta value, $\zeta(5,3)$, appears at six loops (with a rational coefficient). All *known* residues were (up to 2013) rational linear combinations of multiple zeta values.

A seven loop graph was demonstrated by Panzer in 2014 to involve what Broadhurst calls *multiple Deligne values* - hyperlogarithms at sixth roots of unity.

The definition of a period is deceptively simple: a complex number is a period if its real and imaginary parts can be written as absolutely convergent integrals of rational functions with rational coefficients in domains given by polynomial inequalities with rational coefficients. The set $\mathbb{P}$ of all periods would not change if we replace everywhere in the definition "rational" by "algebraic". If we denote by $\bar{\mathbb{Q}}$ the *field of algebraic numbers* (the inverse of an algebraic number being also algebraic) then we would have the inclusions

$$\mathbb{Q} \subset \bar{\mathbb{Q}} \subset \mathbb{P} \subset \mathbb{C}. \tag{1.7}$$

The periods form a ring: the inverse of a period needs not be a period. Feynman amplitudes in an arbitrary QFT can be normalized in such a way that their values at rational ratios of dimensional parameters (momenta and masses) become periods (Bogner, Weinzierl, 2009). The set of all periods is countable. Examples of periods include the transcendentals

$$\pi = \iint_{x^2+y^2\leq 1} dxdy, \ \ln n = \int_1^n \frac{dx}{x}, \ n = 2, 3, ..., \tag{1.8}$$

as well as the values of iterated integrals at algebraic arguments. They include both the classical MZVs as well as multiple Deligne values. The basis $e$ of natural logarithms, the Euler constant $\gamma = -\Gamma'(1)$, as well as $\ln(\ln n), \ln(\ln(\ln n))$, ..., and $1/\pi$ are believed (but not proven) not to be periods.

There are infinitely many primitively divergent 4-point graphs (while there is a single primitive 2-point graph - corresponding to the self-energy amplitudes $(x_{12}^2)^{-3}$). A remarkable sequence of $\ell$-loop graphs ($\ell \geq 3$) with four external lines, the *zig-zag graphs*, can be characterized by their *n*-point vacuum completions $\bar{\Gamma}_n,\ n = \ell + 2$ as follows. $\bar{\Gamma}_n$ admits a closed *Hamiltonian cycle* that passes through all vertices in consecutive order such that each vertex *i* is also connected with $i \pm 2$ (*mod n*). These graphs were conjectured by Broadhurst and Kreimer in 1995 and proven by Brown and Schnetz in 2012 to have residues

$$\begin{aligned} Per(\bar{\Gamma}_{\ell+2}) &= \frac{4 - 4^{3-\ell}}{\ell} \binom{2\ell - 2}{\ell - 1} \zeta(2\ell - 3) \quad \text{for } \ell = 3, 5, \dots ; \\ &= \frac{4}{\ell} \binom{2\ell - 2}{\ell - 1} \zeta(2\ell - 3) \quad \text{for } \ell = 4, 6, \dots . \end{aligned} \tag{1.9}$$

# Primitive conformal amplitudes

Each primitively divergent Feynman amplitude $G(x_1, ..., x_4)$ defines a conformally covariant (locally integrable) function away from the small diagonal $x_1 = ... = x_4$. On the other hand, every four points, $x_1, ..., x_4$, can be confined by a conformal transformation to a 2-plane. Then we can represent each euclidean point $x_i$ by a complex number $z_i$ so that

$$x_{ij}^2 = |z_{ij}|^2 = (z_i - z_j)(\bar{z}_i - \bar{z}_j). \tag{1.10}$$

The correspondence between 4-vectors $x$ and complex numbers $z$ can be written (in spherical coordinates) in the form:

$$x = r(cos\rho\, e + sin\rho\, n)\,,\ e^2 = 1 = n^2\,,\ en = 0\,,\ r \geq 0. \tag{1.11}$$

$$z = re^{i\rho} \to x^2 (= r^2) = z\bar{z}\,, \quad (x-e)^2 = |1-z|^2 = (1-z)(1-\bar{z}) \tag{1.12}$$

$$\int_{n \in \mathbb{S}^2} \frac{d^4x}{\pi^2} = |z - \bar{z}|^2 \frac{d^2z}{\pi}\,, \int_{\mathbb{S}^2} \delta(x) d^4x = \delta(z) d^2z\,. \tag{1.13}$$

For a primitively divergent graph with four distinct external vertices in the $\varphi^4$ theory the amplitude (integrated over the internal vertices) has scale dimension 12 and can be written in the form:

$$G(x_1, ..., x_4) = \frac{g(u, v)}{\prod_{i<j} x_{ij}^2} = \frac{F(z)}{\prod_{i<j} |z_{ij}|^2} \tag{1.14}$$

where the indices run in the range $1 \leq i < j \leq 4$, the variables $u, v$, and (the complex) $z$ are conformally invariant crossratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z} \,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = |1 - z|^2 \,, \quad z = \frac{z_{12} z_{34}}{z_{13} z_{24}} \,. \tag{1.15}$$

The crossratios $z$ and $\bar{z}$ are the simplest realizations of the argument $z$ of the hyperlogarithmic functions. They also appear (as a consequence of the so called *dual conformal invariance*) in the expressions of momentum space integrals like

$$T(p_1^2, p_2^2, p_3^2) = \int \frac{d^4 k}{\pi^2 k^2 (p_1 + k)^2 (k - p_3)^2} = \frac{F(z)}{p_3^2} \qquad (1.16)$$

where $p_1 + p_2 + p_3 = 0$, $\frac{p_1^2}{p_3^2} = z\bar{z}$, $\frac{p_2^2}{p_3^2} = |1 - z|^2$.

# Double shuffle algebra of hyperlogarithms

Let $\sigma_0 = 0, \sigma_1, ..., \sigma_N$ be distinct complex numbers corresponding to an alphabet $X = \{e_0, ..., e_N\}$. Let $X^*$ be the set of words $w$ in this alphabet including the empty word $\emptyset$. The hyperlogarithm $L_w(z)$ is a Chen's iterated integral defined recursively in the punctured complex plane
$D = \mathbb{P}^1(\mathbb{C}) \setminus \Sigma$, $\Sigma = \{\sigma_0, ..., \sigma_N, \infty\}$
by the differential equations:

$$\frac{d}{dz} L_{w\sigma}(z) = \frac{L_w(z)}{z-\sigma}\,, \quad \sigma \in \Sigma\,, \quad L_\emptyset = 1\,, \tag{2.1}$$

and the initial condition

$$L_w(0) = 0 \quad \text{for} \quad w \neq 0^n(= 0\ldots 0), \quad L_{0^n}(z) = \frac{(\ln z)^n}{n!}, \; L_\emptyset = 1. \tag{2.2}$$

Denoting by $\sigma^n$ a word of $n$ consecutive $\sigma$'s we find, for $\sigma \neq 0$,

$$L_{\sigma^n}(z) = \frac{(\ln(1-\frac{z}{\sigma}))^n}{n!}. \tag{2.3}$$

There is a correspondence between the finite alphabet of complex numbers $\sigma_i$ and the set of positive integers:

$$(-1)^d L_{\sigma_1 0^{n_1-1} \dots \sigma_d 0^{n_d-1}}(z) = Li_{n_1,\dots,n_d}(\frac{\sigma_2}{\sigma_1}, \dots, \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d}) \qquad (2.4)$$

where $Li_{n_1,\dots,n_d}$ is given by the multiple power series

$$Li_{n_1,\dots,n_d}(z_1, \dots, z_d) = \sum_{1 \le k_1 < \dots < k_d} \frac{z_1^{k_1} \dots z_d^{k_d}}{k_1^{n_1} \dots k_d^{n_d}}. \qquad (2.5)$$

More generally, we have

$$(-1)^d L_{0^{n_0} \sigma_1 0^{n_1-1} \dots \sigma_d 0^{n_d-1}}(z) =$$

$$\sum_{\substack{k_0 \ge 0 \; k_i \ge n_i, 1 \le i \le d \\ k_0 + \dots + k_d = n_0 + \dots + n_d}} (-1)^{k_0+n_0} \prod_{i=1}^{d} \binom{k_i - 1}{n_i - 1} L_{0^{k_0}}(z) \, L_{k_1 - k_r}(\frac{\sigma_2}{\sigma_1}, \dots, \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d}).$$

In particular,
$L_{01}(z) = Li_2(z) - \ln z Li_1(z) = Li_2(z) + \ln z \ln(1-z)$.
The number of letters $|w| = n_0 + ... + n_d$ of a word w defines its *weight*, while the number d of non zero letters is its *depth*. The set $X^*$ of words can be equipped with a commutative *shuffle product* $w ⧢ w'$ defined recursively:

$$\emptyset ⧢ w = w (= w ⧢ \emptyset)\,, \quad au ⧢ bv = a(u ⧢ bv) + b(au ⧢ v) \quad (2.7)$$

where $u, v, w$ are (arbitrary) words while $a, b$ are letters (note that the empty word $\emptyset$ is *not* a letter).

We denote by

$$\mathcal{O}_\Sigma = \mathbb{C}\big[z, \Big(\frac{1}{z-\sigma_i}\Big)_{i=1,\dots,N}\big] \tag{2.8}$$

the ring of regular functions on $D$. Extending by $\mathcal{O}_\Sigma$ linearity the correspondence $w \to L_w$ one proves that it defines a homomorphism of shuffle algebras $\mathcal{O}_\Sigma \otimes \mathbb{C}(X) \to \mathcal{L}_\Sigma$ where $\mathcal{L}_\Sigma$ is the $\mathcal{O}_\Sigma$ span of $L_w, w \in X^*$.
The shuffle product is commutative as displayed by

$$L_{u ⧢ v} = L_u L_v. \tag{2.9}$$

In particular, the dilogarithm $Li_2(z)$ given by (2.5) for $d = 1, n_1 = 2$ disappears from the shuffle product:

$$L_{0 ⧢ 1}(z) = L_{01}(z) + L_{10}(z) = L_0(z) L_1(z). \tag{2.10}$$

If the shuffle relations are suggested by the expansion of products of iterated integrals, the product of series expansions suggests the (also commutative) *stuffle product*. Here is the rule for the product of depth one and depth two factors:

$$Li_{n_1,n_2}(z_1,z_2)Li_{n_3}(z_3) = Li_{n_1,n_2,n_3}(z_1,z_2,z_3) + Li_{n_1,n_3,n_2}(z_1,z_3,z_2) + Li_{n_3,n_1,n_2}(z_3,z_1,z_2) + Li_{n_1,n_2+n_3}(z_1,z_2z_3) + Li_{n_1+n_3,n_2}(z_1z_3,z_2). \quad (2.11)$$

The multiple polylogarithms of one variable (with $z_1 = ... = z_{d-1} = 1$) span a shuffle but not a stuffle algebra. The stuffle product also respects the weight but (in contrast to the shuffle) only filters the depth (the depth of each term in the right hand side does not exceed the sum of depths of the factors in the left hand side).

It is convenient to rewrite the definition of hyperlogarithms in terms of a formal series $L(z)$ with values in the (free) tensor algebra $\mathbb{C}\langle X\rangle$ (the complex vector space generated by all words in $X^*$) which satisfies the *Knizhnik-Zamoldchikov (K-Z) equation*:

$$L(z) := \sum_{w} L_w(z)w, \ \frac{d}{dz}L(z) = L(z)\sum_{i=0}^{N} \frac{e_i}{z-\sigma_i}. \qquad (2.12)$$

For $X$ consisting of two letters $e_0, e_1 \leftrightarrow \sigma_0 = 0, \sigma_1 = 1$, $L(z)$ generates the *classical multipolylogarithms*; its value at $z = 1, Z := L(1)$ is the generating series of MZVs. The *monodromy* of L around the points 0 and 1 is:

$$\mathcal{M}_0\, L(z) = e^{2\pi i e_0}\, L(z)\,, \quad \mathcal{M}_1\, L(z) = Z\, e^{2\pi i e_1}\, Z^{-1}\, L(z),\ Z = \sum_w \zeta_w w, \tag{2.13}$$

so that $\mathcal{M}_0 L_{0^n}(z) = L_{0^n}(z) + 2\pi i L_{0(n-1)}(z)$, $\mathcal{M}_1 Li_n(z) = Li_n(z) - 2\pi i L_{0(n-1)}(z)$. Indeed, $L(z)$ is the unique solution of the K-Z equation obeying the "initial" condition

$$L(z) = e^{e_0 \ln z} h_0(z),\ h_0(0) = 1, \tag{2.14}$$

$h_0(z)$ being a formal power series in the words in $X^*$ that is holomorphic in $z$ in the neighborhood of $z = 0$. There exists a counterpart $h_1(z)$ of $h_0$, holomorphic around $z = 1$ and satisfying $h_1(1) = 1$ such that

$$L(z) = Z\, e^{e_1 \ln(1-z)}\, h_1(z). \tag{2.15}$$

The weight of consecutive terms in the expansion of $L(z)$ (2.15) is the sum of the weights of hyperlogarithms and the zeta factors. It is thus convenient to first review of the double shuffle algebra of MZVs.

# Formal multizeta values

The MZV $\zeta(n_1, ..., n_d)$ is defined as $Li_{n_1,...,n_d}(1)$ whenever the corresponding series converges:

$$(-1)^d \zeta_{10^{n_1-1}...10^{n_d-1}} = \zeta(n_1, ..., n_d) = \sum_{1 \leq k_1 < ... < k_d} \frac{1}{k_1^{n_1} ... k_d^{n_d}} \text{ for } n_d > 1. \tag{3.1}$$

The convergent MZVs of a given weight satisfy a number of shuffle and stuffle identities. Looking for instance at the shuffle (sh) and the stuffle (st) products of two $-\zeta_{10} = \zeta(2)$ we find:

$$sh : \zeta_{10}^2 = 4\zeta_{1100} + 2\zeta_{1010} (= 4\zeta(1,3) + 2\zeta(2,2));$$
$$st : \zeta(2)^2 = 2\zeta(2,2) + \zeta(4);$$
$$\text{hence } \zeta(4) = 4\zeta(1,3) = \zeta(2)^2 - 2\zeta(2,2). \tag{3.2}$$

There are no non-zero convergent words of weight 1 and hence no shuffle or stuffle relations of weight 3. On the other hand, already Euler has discovered the relation: $\zeta(1,2) = \zeta(3)$. Thus shuffle and stuffle relations among convergent words do not exhaust all known relations among MZVs of a given weight. The divergent zeta values for $n_d = 1$ cancel in the difference between the shuffle and stuffle products $u \sqcup\!\sqcup v - u * v$. For instance, at weight 3 we have

$$\zeta((1)\sqcup\!\sqcup(2)) = 2\zeta(1,2)+\zeta(2,1);\ \zeta((1)*(2)) = \zeta(1,2)+\zeta(3)+\zeta(2,1). \tag{3.3}$$

Extending the homomorphism $w \to \zeta(w)$ as a homomorphism of both the shuffle and the stuffle algebras to divergent words, assuming, in particular, that $\zeta((1) \sqcup\!\sqcup (2)) = \zeta((1) * (2)) = \zeta(1)\zeta(2)$ and taking the difference of the two equations (3.3) we observe that all divergent zeta's cancel and we recover Euler's identity.

Translation between the two-letter alphabet $\{0, 1\}$ (used as lower indices) and the infinite alphabet of all positive integers appearing (in parentheses) as arguments of zeta:

$$\vec{n} = (n_1, ..., n_d) \leftrightarrow (-1)^n \rho(\vec{n}) \text{ for } \rho(\vec{n}) = 10^{n_1 - 1} ... 10^{n_d - 1}. \quad (3.4)$$

Using this correspondence one obtains, in particular, the first relation (3.3).

One introduces *shuffle regularized MZVs* using the following.
*Proposition 3*. *There is a unique way to define a set of real numbers* $I(a_0; a_1, ..., a_n; a_{n+1})$ *for any* $a_i \in \{0, 1\}$*, such that*

$$(i)\ \ I(0; 1, a_2, ....a_{n-1}, 0; 1) = \zeta_{1a_2...a_{n-1}0}$$
$$(ii)\ \ I(a_0; a_1; a_2) = 0, I(a_0, a_1) = 1 \ \textit{for all}\ a_0, a_1, a_2 \in \{0, 1\};$$
$$(iii)\ I(a_0; a_1, ..., a_r; a_{n+1}) I(a_0; a_{r+1}, ..., a_{r+s}) = \sum_{\sigma \in \Sigma(r,s)} I(a_0; a_{\sigma(1)}, ..., a_{\sigma(r+s)}; a_{n+1}) \, (r + s = n);$$
$$(iv)\ \ I(a; a_1, ..., a_n; a) = 0 \ \textit{for}\ n > 0;$$
$$(v)\ \ I(a_0; a_1, ..., a_n; a_{n+1}) = (-1)^n I(a_{n+1}; a_n, ..., a_1; a_0);$$
$$(vi)\ \ I(a_0; a_1, ..., a_n; a_{n+1}) = I(1 - a_{n+1}; 1 - a_n, ..., 1 - a_1; 1 - a_0) \quad (3.5$$

$\Sigma(r, s)$ is the set of permutations that preserve the order of the first $r$ and the last $s$ indices among them. Eq. $\zeta(n_1, ..., n_d) = (-1)^d I(0; \rho(\vec{n}); 1)$ then defines the shuffle regularized zeta values for all $n_d \geq 1$. Condition (ii) implies, in particular, $\zeta(1) = 0$.

In fact, it suffices to add a condition involving multiplication by the divergent word (1),

$$\zeta((1) ⧢ w - (1) * w) = 0 \text{ for all convergent words } w, \qquad (3.6)$$

to the shuffle and stuffle relations among convergent words in order to obtain all known relations among MZVs of a given weight. For $w = (n), n \geq 2$ (a word of depth 1), Eq. (3.6) gives

$$\zeta((1) ⧢ (n) - (1) * (n)) = \sum_{i=1}^{n-1} \zeta(i, n+1-i) - \zeta(n+1) = 0 \quad (3.7)$$

(a relation known to Euler). The discovery (and the proof) that

$$\zeta(2n) = -\frac{B_{2n}}{2(2n)!}(2\pi i)^{2n},\ B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},\ (-1)^{n-1}B_{2n} \in \mathbb{Q}_{>0}, \qquad (3.8)$$

where $B_n$ are the (Jacob) Bernoulli numbers, was among the first that made Euler famous. Nothing is known about the transcendentality of $\zeta(n)$ (or of $\frac{\zeta(n)}{\pi^n}$) for odd n. We introduce following Leila Schneps the notion of the $\mathbb{Q}$-algebra $\mathcal{FZ}$ of *formal MZVs* $F\zeta$ which satisfy the above relations.

The algebra $\mathcal{FZ} = \bigoplus_n \mathcal{FZ}_n$ is *weight graded* and

$$\mathcal{FZ}_0 = \mathbb{Q}, \mathcal{FZ}_1 = \{0\}, \mathcal{FZ}_2 = \langle\zeta(2)\rangle, \mathcal{FZ}_3 = \langle\zeta(3)\rangle, \mathcal{FZ}_4 = \langle\zeta(4)\rangle, \\ \mathcal{FZ}_5 = \langle\zeta(5), \zeta(2)\zeta(3)\rangle, \mathcal{FZ}_6 = \langle\zeta(2)^3, \zeta(3)^2\rangle, \\ \mathcal{FZ}_7 = \langle\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)\rangle, \quad (3.9$$

where $< x, y, ... >$ is the $\mathbb{Q}$ vector space spanned by $x, y, ...$ (and we have replaced $F\zeta$ by $\zeta$ in the right hand side for short). **The main conjecture in the theory of MZVs** is that *the graded algebra $\mathcal{FZ}$ is isomorphic to the algebra $\mathcal{Z}$ of MZVs*. It is a strong conjecture. If true it would imply that there is no linear relation among MZVs of different weights over the rationals (in particular, that all $\zeta(n)$ are irrational). We follow the physicists' practice to treat this conjecture as established and to omit the $F$'s in the notation for (formal) MZVs.

*Examples*: E1. In order to see that the space $\mathcal{Z}_4$ of weight four zeta values is 1-dimensional we should add to Eqs. (3.2) the relation (3.7) for $n = 3$ and its depth three counterpart:

$$\zeta((1) ⧢ (1,2) - (1) * (1,2)) = \zeta(1,1,2) - \zeta(1,3) - \zeta(2,2) = 0. \tag{3.10}$$

This allows to express all zeta values of weight four as (positive) integer multiples of $\zeta(1,3)$.
E2. The shuffle and the stuffle products corresponding to $\zeta(2)\zeta(3)$ give two relations which combined with (3.7) for $n = 4$ allow to express the three double zeta values of weight five in terms of simple ones:

$$\zeta(1,4) = 2\zeta(5) - \zeta(2)\zeta(3),\ \zeta(2,3) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5),$$
$$\zeta(3,2) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3). \tag{3.11}$$

In general, the number of convergent words of weight $n$ in the alphabet $\{0, 1\}$ is $2^{n-2}$. We see from Eq. (3.9) the number of relations also grows fast: there are six relations among the eight MZVs at weight five; 14 such relations at weight six, 29, at weight seven. One first needs a double zeta value, say $\zeta(3, 5)$, in order to write a basis (of four elements) at weight eight (there being 60 relations among the elements of $\mathcal{F}\mathcal{Z}_8$).

Question: dimension of weight spaces $d_n = dim Z_n =$?

To answer it we will introduce (a pedestrian version of) *motivic zeta values*.

Consider the *concatenation algebra*

$$\mathcal{C} = \mathbb{Q}\langle f_3, f_5, ...\rangle, \tag{3.12}$$

the free algebra over $\mathbb{Q}$ on the countable alphabet $\{f_3, f_5, ....\}$. If we could identify the formal zeta values with the algebra

$$\mathcal{C}[f_2] = \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \tag{3.13}$$

we would be able to compute the dimension $d_n$ of $\mathcal{Z}_n$ for any n. Indeed, the generating (*Hibert-Poincaré*) series for the dimensions $d_n^{\mathcal{C}}$ of the weight n subspace of $\mathcal{C}$ is given by

$$\sum_{n\geq 0} d_n^{\mathcal{C}} t^n = \frac{1}{1 - t^3 - t^5 - ...} = \frac{1 - t^2}{1 - t^2 - t^3} \tag{3.14}$$

while the Hilbert-Poincaré series of the second factor $\mathbb{Q}[f_2]$ in (3.13) is $(1 - t^2)^{-1}$. Multiplying the two we obtain - for the "motivic zeta values" - the $d_n$ conjectured by Don Zagier:

$$\sum_{n\geq 0} d_n t^n = \frac{1}{1 - t^2 - t^3}, \; d_0 = 1, d_1 = 0, d_2 = 1, d_{n+2} = d_n + d_{n-1}. \tag{3.15}$$

Taking the identities among (formal) zeta values into account we can write the generating series $Z$ of MZV (also called *Drinfeld's associator*) in terms of multiple commutators of $e_0, e_1$:

$$Z = 1 + \zeta(2)[e_0, e_1] + \zeta(3)[[e_0, e_1], e_0 + e_1] + .... \qquad (3.16)$$

The concatenation algebra $\mathcal{C}$, identified with the quotient

$$\mathcal{C} = \mathcal{C}[f_2]/\mathbb{Q}[f_2], \tag{3.17}$$

can be turned into a Hopf algebra with the *deconcatenation coproduct*:

$$\Delta(f_{i_1}...f_{i_r}) = 1 \otimes f_{i_1}...f_{i_r} + f_{i_1}...f_{i_r} \otimes 1 + \sum_{k=1}^{r-1} f_{i_1}...f_{i_k} \otimes f_{i_{k+1}}...f_{i_r}. \tag{3.18}$$

This coproduct can be extended to the trivial comodule $\mathcal{C}[f_2]$ (3.13) by setting

$$\Delta : \mathcal{C}[f_2] \to \mathcal{C} \otimes \mathcal{C}[f_2],\ \Delta(f_2) = 1 \otimes f_2 \tag{3.19}$$

(and assuming that $f_2$ commutes with $f_{odd}$). Remarkably, there appear to be a one-to-one (non-canonical) correspondence between the bases of the weight spaces $\mathcal{Z}_n$ and $\mathcal{C}[f_2]_n$:

$$\begin{aligned}&\langle\zeta(2)\rangle \leftrightarrow \langle f_2\rangle; \langle\zeta(3)\rangle \leftrightarrow \langle f_3\rangle; \langle\zeta(2)^2\rangle \leftrightarrow \langle f_2^2\rangle;\\ &\langle\zeta(5), \zeta(2)\zeta(3)\rangle \leftrightarrow \langle f_5, f_2 f_3 (= f_3 f_2)\rangle; \langle\zeta(2)^3, \zeta(3)^2\rangle \leftrightarrow \langle f_2^3, f_3 \sqcup\!\sqcup f_3\rangle;\\ &\langle\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)\rangle \leftrightarrow \langle f_7, f_2 f_5, f_2^2 f_3\rangle, .... \end{aligned} \tag{3.20}$$

A sharpening of the above main conjecture for MZVs reads: *The algebra $\mathcal{Z}$ of MZVs is weight-graded and (non-canonically) isomorphic as graded Hopf algebra with* $\mathcal{C}[f_2]$. If true it would imply that the (infinite sequence of) numbers $\pi, \zeta(3), \zeta(5), ...$ are transcendental algebraically independent over the rationals. It would also fix the dimension of the weight spaces $\mathcal{Z}_n$ to be equal to $d_n$ (3.15). Presently, we only know that this is true for $n = 0, 1, 2, 3, 4$; in general, it is proven that

$$dim \mathcal{Z}_n \leq d_n \; \forall n; \quad dim \mathcal{Z}_n = d_n \quad \text{for} \quad n \leq 4 \; . \tag{3.21}$$

*Remark* The validity of the above sharpened conjecture would imply, in particular, that $\zeta(2n+1)$ are primitive elements of the Hopf algebra of MZVs:

$$\Delta(\zeta(2n+1)) = \zeta(2n+1) \otimes 1 + 1 \otimes \zeta(2n+1). \qquad (3.22)$$

Eq. (3.8) precludes the possibility of extending this property to even zeta values. Indeed, it implies the relation

$\zeta(2n) = b_n \zeta(2)^n,\ b_n = \frac{(24)^n |B_{2n}|}{2(2n)!}$

which is only compatible with

$\Delta\zeta(2) = 1 \otimes \zeta(2)$.

If for weights $n \leq 7$ one can express all MZVs in terms of (products of) simple zeta values (of depth one) for $n \geq 8$ this is no longer possible. Brown (2012) has established that the *Hoffman elements* $\zeta(n_1, ..., n_d)$ with $n_i \in \{2, 3\}$ form a basis of motivic zeta values for all $n$.

The coproduct for MZV, described in the Remark, extends to hyperlogarithms and can be formulated in terms of the regularized iterated integrals of Proposition 3. Here is the coproduct of a classical polylogarithm:

$$\Delta Li_n(z) = Li_n(z) \otimes 1 + \sum_{k=0|}^{n-1} \frac{(\ln z)^k}{k!} \otimes Li_{n-k}(z). \qquad (3.23)$$

Specializing to $z = 1$ in (3.23) for even n leads to a contradiction unless we factor $\mathcal{L}_\Sigma$ by $\ln(-1) = i\pi (= \sqrt{-6\zeta(2)})$:

$$\mathcal{H} := \mathcal{L}_\Sigma / i\pi \mathcal{L}_\Sigma \quad \text{so that} \quad \mathcal{L}_\Sigma = \mathcal{H}[i\pi] \, . \qquad (3.24)$$

The coaction $\Delta$ is defined on the comodule $\mathcal{L}_\Sigma$ as follows:

$$\Delta : \mathcal{L}_\Sigma \to \mathcal{H} \otimes \mathcal{L}_\Sigma \, , \quad \Delta(i\pi) = 1 \otimes i\pi \, . \qquad (3.25)$$

The asymmetry of the coproduct is also reflected in its relation to differentiation and to the discontinuity $disc_\sigma = \mathcal{M}_\sigma - 1$:

$$\Delta(\frac{\partial}{\partial z} F) = (\frac{\partial}{\partial z} \otimes id)\Delta F \, , \quad \Delta(disc_\sigma F) = (id \otimes disc_\sigma)\Delta F \, . \qquad (3.26)$$

# Single-valued hyperlogarithms. Applications

Knowing the action of the monodromy $M_{\sigma_i}$ around each singular point Brown constructs *single valued hyperlogarithms* in the tensor product of $\mathcal{L}_\Sigma$ with its complex conjugate. We shall spell out this construction for classical multiple polylogarithms $\mathcal{L}_\Sigma = \mathcal{L}_c$, defined as $\mathcal{O}$-linear combination of $L_w(z)$ for $w$, words in the "Morse alphabet" $X = \{e_0, e_1\} \leftrightarrow \{0, 1\}$, where $\mathcal{O} = \mathbb{C}[z, \frac{1}{z}, \frac{1}{z-1}]$. The tensor product $\bar{\mathcal{L}}_c \otimes \mathcal{L}_c$ contains functions of $(\bar{z}, z)$ transforming under arbitrary representations of the monodromy group including the trivial one, - i.e. the *single-valued multiple polylogarithms* (SVMPs).

We introduce an $\bar{\mathcal{O}}\mathcal{O}$ basis of homogeneous SVMPs $P_w(z)$ and will denote by

$$P_X(z) = \sum_{w \in X*} P_w(z) w \tag{4.1}$$

its generating series. Their significance stems from the fact that a large class of euclidean Feynman amplitudes are given by single valued hyperlogarithms. The following theorem is established by Francis Brown in 2004.

*Theorem 4. There exists a unique family of single-valued functions $\mathcal{P}_w(z), w \in X*, z \in \mathbb{C} \setminus \{0,1\}$ such that their generating function (4.1) satisfies the following Knizhnik-Zamolodchikov equations and initial condition:*

$$\partial P_X(z) = P_X(z)(\frac{e_0}{z} + \frac{e_1}{z-1}), \partial := \frac{\partial}{\partial z}\ ,\ \bar{\partial} := \frac{\partial}{\partial \bar{z}},$$
$$\bar{\partial} P_X(z) = (\frac{e_0}{\bar{z}} + \frac{e'_1}{1-\bar{z}})P_X(z),\ Z_{-e_0,-e'_1} e'_1 Z^{-1}_{-e_0,-e'_1} =$$
$$Z_{e_0,e_1} e_1 Z^{-1}_{e_0,e_1},\ \ P_X(z) \sim e^{e_0 \ln(z\bar{z})} \quad \textit{for} \quad z \sim 0\ . \quad (4.2)$$

*The functions $P_w(z)$ are linearly independent over $\bar{\mathcal{O}}\mathcal{O}$ and satisfy the shuffle relations. Every element of their linear span has a primitive with respect to $\frac{\partial}{\partial z}$, and every single valued function $F(z) \in \bar{\mathcal{L}}_c \mathcal{L}_c$ can be written as a unique $\bar{\mathcal{O}}\mathcal{O}$-linear combination of $P_w(z)$.*

The equation for $e_1'$ is dictated by the expression for the monodromy of $L_w(z)$ (2.13) around $z = 1$ and can be solved recursively in terms of elements of the Lie algebra over the *ring of zeta integers* $\mathbb{Z}[\mathcal{Z}]$, generated by $e_0, e_1$ and their multiple commutators. The result is:

$$e_1' = e_1 + 2\zeta(3)[[[e_0, e_1], e_1]\,,\; e_0 + e_1] + \zeta(5)(...) + ...\,, \quad (4.3)$$

where the parenthesis multiplying $\zeta(5)$ consists of eight bracket words of weight six. It follows that $e_1' = e_1$ for words of weight not exceeding three or depth not exceeding one.

For words involving (repeatedly) a single letter we have

$$P_{0^n}(z) = \frac{(\ln \bar{z}z)^n}{n!},\ \ P_{1^n}(z) = \frac{(\ln |1-z|^2)^n}{n!}. \tag{4.4}$$

The depth-one-weight-two SVMPs, which satisfy the differential equations

$$\begin{aligned} \partial P_{01} &= \frac{P_0}{z-1}\ , \quad \bar{\partial} P_{01} = \frac{P_1}{\bar{z}}\ (P_{01}(0) = 0 = P_{10}(0))\ , \\ \partial P_{10} &= \frac{P_1}{z}\ , \quad \bar{\partial} P_{10} = \frac{P_0}{\bar{z}-1}\ , \end{aligned} \tag{4.5}$$

are given by

$$\begin{aligned} P_{01} &= L_{10}(\bar{z}) + L_{01}(z) + L_0(\bar{z})L_1(z) = \\ &\quad Lie_2(z) - Li_2(\bar{z}) + \ln \bar{z}z \ln(1-z), \\ P_{10} &= L_{01}(\bar{z}) + L_{10}(z) + L_1(\bar{z})L_0(z) = \\ &\quad Li_2(\bar{z}) - Li_2(z) + \ln \bar{z}z \ln(1-\bar{z})\ . \end{aligned} \tag{4.6}$$

They obey the shuffle relation $P_{01} + P_{10} = P_0 P_1$ so that the only new weight two function is their difference,

$$P_{01} - P_{10} = 2(Li_2(z) - Li_2(\bar{z}) + \ln \bar{z}z \ln \frac{1-z}{1-\bar{z}} = 4iD(z) \ , \quad (4.7)$$

proportional to the *Bloch-Wigner dilogarithm*,
$D(z) = Im(Li_2(z) + \ln(1-z) \ln |z|)$.

One can also write down depth-one SVMPs of arbitrary weight encountered in the expression $F_n(z)$ for the graphical function associated with the *wheel diagram with* $(n+1)$ *spokes*, first computed by Broadhurst in 1985:

$$F_n(z) = (-1)^n \frac{P_{0^{n-1}10^n}(z) - P_{0^n10^{(n-1)}}(z)}{z-\bar{z}} =$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n} P_{0^{n-k}}(z) \frac{Li_{n+k}(z) - Li_{n+k}(\bar{z})}{z-\bar{z}} \, . \qquad (4.8)$$

The period of the wheel amplitude is given by the limit of this expression for $z \to 1$

$$F_n(1) = \binom{2n}{n} Li_{2n-1}(1) = \binom{2n}{n} \zeta(2n-1). \qquad (4.9)$$

Just like MZVs appear as values at $z = 1$ of multiple polylogarithms the values at one of SVMPs define *single-valued periods* (Brown, 2013) with applications in QFT (and in superstrings: Stieberger, 2013). Their generating function is

$$Z^{sv} = P_{e_0,e_1}(1) = 1 + 2\zeta(3)[e_0,[e_1,e_0]] + 2\zeta(5)(...) + ... \Rightarrow \zeta^{sv}(2) = 0. \qquad (4.10)$$

The structure of a graded Hopf algebra of the family of hyperlogarithms allows to read off there symmetry properties from the simpler properties of ordinary logarithms, as illustrated by Duhr's derivation of the inversion formula for the dilog:
$Li_2\left(\frac{1}{x}\right) = i\pi \ln x - Li_2(x) - \frac{1}{2}\ln^2 x + 2\zeta(2)$.
SVMPs are symmetric under the group $\mathcal{S}_3$ of Möbius transformations, that permute the singular points $\{0, 1, \infty\}$. $\mathcal{S}_3$ is generated by two involutions, $s_1 : z \to 1 - z$, $s_2 : z \to \frac{1}{z}$ such that $s_1 s_2 : z \to \frac{z-1}{z}$, $(s_1 s_2)^3 = 1$. The formal power series $P_{e_0,e_1}(z)$ obeys simple symmetry relations under $s_1$ and $s_2$:

$$P_{e_0,e_1}(1-z) = P_{e_0,e_1}(1)P_{e_1,e_0}(z),$$
$$P_{e_0,e_1}(\frac{1}{z}) = P_{e_0,-e_0-e_1}(1)P_{-e_0-e_1,e_1}(z). \tag{4.11}$$

According to (4.10) the first factor in the right hand side of (4.11) does not contribute to the transformation law of SVMPs of weight one and two; $s_1$ just permutes the indices 0 and 1 while $P_0(\frac{1}{z}) = -P_0(z), P_1(\frac{1}{z}) = P_1(z) - P_0(z)$ and

$$P_{01}(\frac{1}{z}) = P_{00}(z) - P_{01}(z),\ P_{10}(\frac{1}{z}) = P_{00}(z) - P_{10}(z)$$
$$\Rightarrow D(\frac{1}{z}) = -D(z) \quad (4.12)$$

where $D(z)$ is the Bloch-Wigner dilogarithm (4.7).

As a simple example, one can calculate - without really integrating - the integral

$$I(x_1, x_2, x_3, x_4) = \int \frac{d^4x}{\pi^2} \prod_{i=1}^{4} \frac{1}{(x - x_i)^2} = \frac{f(u, v)}{x_{13}^2 x_{24}^2}, \qquad (4.13)$$

where $u, v$ are the crossratios (1.15). Using the conformal invariance of $f(u, v)$ we can set
$x_1 \to \infty, x_2 = e\,(e^2 = 1), x_4 = 0; x_3^2 = \bar{z}z, (X_3 - e)^2 = |1 - z|^2$.
Applying to the result the 4-dimensional Laplacian with respect to $x_3$ which acts on $F(z) = f(u, v)$ as
$\frac{1}{4}\Delta_3 F(z) = \frac{1}{z-\bar{z}}\bar{\partial}\partial[(z - \bar{z})F(z)]$, we obtain:

$$\bar{\partial}\partial[(z - \bar{z})F(z)] = \frac{\bar{z} - z}{\bar{z}z|1 - z|^2} = \frac{1}{\bar{z}(z - 1)} - \frac{1}{z(\bar{z} - 1)}$$

$$\Rightarrow F(z) = \frac{P_{01}(z) - P_{10}(z)}{z - \bar{z}}. \qquad (4.14)$$

Thus $F(z)$ is given by (4.8) for $n = 1$, $(z - \bar{z})F(z)$ being the only odd with respect to complex conjugation SVMP of weight two. We note that the odd denominator $z - \bar{z}$ also comes from the Jacobian $J$ of the change of integration variables $\{x^\alpha\} \to \{D_i = (x - x_i)^2\}$, $\alpha, i = 1, ..., 4$ in (4.13):

$$I(x_1, ..., x_4) = \frac{1}{\pi^2} \int \frac{1}{J} \prod_{i=1}^{4} \frac{dD_i}{D_i}, \; J = det\left(\frac{\partial D_i}{\partial x^\alpha}\right). \qquad (4.15)$$

Indeed, at the singularity $D_i = 0$ we have

$$J|_{D_i=0} = 4x_{13}^2 x_{24}^2 \sqrt{2(u + v + uv) - 1 - u^2 - v^2} = 4x_{13}^2 x_{24}^2 \sqrt{-(z - \bar{z})^2} \qquad (4.16)$$

Integrals of the type of (4.13) have been calculated long ago by more conventional methods by Ussyukina. The present techniques have been applied to calculate a (previously unknown) 3-loop correlator.

# Outlook

Multidimensional Feynman integrals give rise to a family of functions and numbers with the structure of a differential graded double shuffle Hopf algebra. It is displayed most readily for conformally invariant position space amplitudes in a massless QFT.

The dimensions of weight spaces of MZVs (which exhaust the Feynman periods up to six loops in the massless $\varphi^4$ theory) do not exceed - and are conjectured to coincide with - their motivic counterparts studied by Francis Brown. At seven loops first appear values of hyperlogarithms at sixth root of unity. In the two-loop sunrise integral with massive propagators) multiple elliptic polylogarithms are involved (Bloch, Vanhove, Adams, Bogner, Weinzierl); they are also expected for higher loop massless integrals.

We have not touched upon the application of cluster algebras to multileg on shell Feynman amplitudes. A remarkable step in this direction was made by Goncharov and a group of four physicists, 2013.

As hyperlogarithms and associated numbers do not suffice for expressing massive or higher order Feynman amplitudes, mathematicians and physicists are exploring their (elliptic polylog and modular form) generalizations (Brown, 2014-2015).

Connections of MZVs with other part of mathematics (including the Grothendieck-Teichmüller Lie algebra, mixed Tate motives and modular forms) are surveyed, for instance, by Schneps.

Recent developments and perspectives are surveyed in Francis Brown's lecture at the 2014 Intentional Congress of Mathematicians (1407.5165).

# References


- D.J. Broadhurst, D. Kreimer, Knots and numbers in $\phi^4$ to 7 loops and beyond, *Int. J. Mod. Phys.* **6C** (1995) 519-524; Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, *Phys. Lett.* **B393** (1997) 403-412; hep-th/9609128.
- D.J. Broadhurst, O.Schnetz, Algebraic geometry informs perturbative quantum field theory, *Proc. Sci.* **211** (2014) 078; arXiv:1409.5570 [hep-th].
- F. Brown, Single-valued hyperlogarithms and unipotent differential equations, IHES notes, 2004.
- F. Brown, On the decomposition of motivic multiple zeta values, *Galois-Teichmüller Theory Arithmetic Geometry*, H. Nakamura, F. Pop, L. Schneps, A. Tamogawa (eds.) *Advanced Studies in Pure Mathematics* **63** (2012) 31-58; arXiv:1102.1310v2.
- F. Brown, Single-valued periods and multiple zeta values, arXiv:1309.5309 [math.NT].

- K.T. Chen, Iterated path integrals, *Bull. Amer. Math. Soc.* **83** (1977) 831-879.
- J. Drummond, C. Duhr, B. Eden, P. Heslop, J. Pennington, V.A. Smirnov, Leading singularities and off shell conformal amplitudes, *JHEP* **1308** (2013) 133; arXiv:1303.6909v2.
- C. Duhr, Mathematical aspects of scattering amplitudes, arXiv:1411.7538 [hep-ph].
- J.K. Golden, A.B. Goncharov, M. Spradlin, C. Vergu, A. Volovich, Motivic amplitudes and cluster coordinates, arXiv:1305.1617 [hep-th].
- M. Kontsevich, D. Zagier, Periods, in:*Mathematics - 20101 and beyond*, Springer, Berlin et al. 2001, pp. 771-808.
- N.M. Nikolov, R. Stora, I. Todorov, Euclidean configuration space renormalization, residues and dilation anomaly, *LT9*, Ed. V.K. Dobrev, Springer Japan, Tokyo 2013, pp. 127-147.

- N.M. Nikolov, R. Stora, I. Todorov, Renormalization of massless Feynman amplitudes as an extension problem for associate homogeneous distributions, *Rev. Math. Phys.* **26**:4 (2014) 1430002; arXiv:1307.6854 [hep-th].
- E. Panzer, Feynman integrals via hyperlogarithms, *Proc. Sci.* bf 211 (2014) 049; arXiv:1407.0074 [hep-ph].
- L. Schneps, Survey of the theory of multiple zeta values, 2011.
- O. Schnetz, Quantum periods: A census of $\phi^4$ transcendentals, arXiv:0801.2856v2.
- O. Schnetz, Graphical functions and single-valued multiple polylogarithms, arXiv:1302.6445 [math.NT].
- I. Todorov, Polylogarithms and multizeta values in massless Feynman amplitudes, *LT 10*, ed. V. Dobrev, Springer, Tokyo 2014; pp. 155-176; Bures-sur-Yvette, IHES/P/14/10.
- I. Todorov, Renormalized position space amplitudes in a massless QFT, CERN-PH-TH-2015-016.