Vertex operator algebras associated with $\mathbb{Z}_k$-codes

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$\mathbb{Z}_k$-code VOA or VOSA M_D

$k \geq 2$; integer

$M^0 = K(sl_2, k)$; parafermion VOA of type sl_2

simple, rational, C_2-cofinite,

of CFT-type VOA, c.c. $\frac{2(k-1)}{k+2}$

M^j, $0 \leq j \leq k-1$; simple current M^0-mod.

- $j \pmod{k}$
- top level 1 dim. wt $\frac{j(k-j)}{k}$
- $M^i \times M^j = M^{i+j}$; fusion rules

For $\xi = (i_1, \ldots, i_n), \quad \eta = (j_1, \ldots, j_n) \in (\mathbb{Z}_k)^n$,

$$(\xi|\eta) = \textstyle\sum_{p=1}^{n} i_p j_p \in \mathbb{Z}_k$$

D; $\mathbb{Z}_k$-code of length n (additive subgp. of $(\mathbb{Z}_k)^n$)

Case 1. $(\xi|\xi) = 0, \ {}^{\forall}\xi \in D$

Case 2. $k =$ even, $(\xi|\eta) \in \{0, \frac{k}{2}\}, \ {}^{\forall}\xi, \eta \in D$,

$(\xi|\xi) = \frac{k}{2}, \ {}^{\exists}\xi \in D$

$$M_\xi = M^{i_1} \otimes \cdots \otimes M^{i_n}, \quad \xi = (i_1, \ldots, i_n) \in D$$

$$M_D = \textstyle\bigoplus_{\xi \in D} M_\xi$$

Theorem.

Case 1. M_D; simple, rational, C_2-cofinite, of CFT-type VOA, c.c. $\frac{2n(k-1)}{k+2}$

Case 2. $M_D = M_D^{\bar{0}} \oplus M_D^{\bar{1}}$; simple VOSA with

$$M_D^{\bar{0}} = \bigoplus_{\substack{\xi \in D \\ (\xi|\xi)=0}} M_\xi \quad \text{as in Case 1}$$

$$M_D^{\bar{1}} = \bigoplus_{\substack{\xi \in D \\ (\xi|\xi)=k/2}} M_\xi$$

Remarks.

- M_D is D-graded simple current extension of
$$M_{\mathbf{0}} = M^0 \otimes \cdots \otimes M^0$$
- top level of $M_{(i_1,\ldots,i_n)}$; 1 dim. wt $\sum_{p=1}^{n} \frac{i_p(k-i_p)}{k}$
- $M_D \subset V_{\Gamma_D}$; lattice VOA
- M_D is generalized VA for general D
- M_D is known in cases $k = 2, 3$

Parafermion VOAs

$L_{\widehat{\mathfrak{g}}}(k,0)$; integrable h.w. module for $\widehat{\mathfrak{g}}$, level k

$$\begin{aligned} K(\mathfrak{g},k) &= \{v \in L_{\widehat{\mathfrak{g}}}(k,0) \,|\, h(m)v = 0, h \in \mathfrak{h}, m \geq 0\} \\ &= \text{commutant of Heisenberg alg. gen. by} \\ &\quad\ \text{Cartan subalg. } \mathfrak{h} \text{ in } L_{\widehat{\mathfrak{g}}}(k,0) \end{aligned}$$

In case $\mathfrak{g} = sl_2$

$M^{i,j}$ $(0 \le j < i \le k)$; irred. modules

$M^j = M^{k,j}$ $(0 \le j \le k-1)$; simple currents

- $k = 2,\ \ M^0 \cong L(\frac{1}{2}, 0)$
- $k = 3,\ \ M^0 \cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$
- $k = 4,\ \ M^0 \cong V_{\mathbb{Z}\beta}^{+},\ \ \langle \beta, \beta \rangle = 6$
- $k \ge 5,\ \ M^0 = \langle W^2, W^3, W^4, W^5 \rangle$; W_5-algebra

Examples of M_D

- $k=6,\ n=1,\ D=\{(0),(3)\}$

  $M_D = M^0 \oplus M^3 \cong L_{NS}(\frac{5}{4},0) \oplus L_{NS}(\frac{5}{4},3)$

  $L_{NS}(\frac{5}{4},0)$; $N=1$ superconformal alg. c.c. $\frac{5}{4}$

  (Neveu-Schwarz alg.)

  $L_{NS}(\frac{5}{4},3)$; irred. $L_{NS}(\frac{5}{4},0)$-mod. h.w. 3

- $k=9,\ n=1,\ D=\{(0),(3),(6)\}$

  $M_D = M^0 \oplus M^3 \oplus M^6 \cong U_{3C}$

- $k=5,\ n=2,\ D=\{(00),(12),(24),(31),(43)\}$

  $M_D \cong U_{5A}$

Realization of M_D in lattice VOA

$$\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$$

$$L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k \cong {A_1}^{\oplus k}$$

$$\gamma = \alpha_1 + \cdots + \alpha_k, \quad \langle \gamma, \gamma \rangle = 2k$$

$$N = \sum_{p=1}^{k-1} \mathbb{Z}(\alpha_p - \alpha_{p+1}) \cong \sqrt{2}A_{k-1}, \quad \langle N, \gamma \rangle = 0$$

Let H, E, $F \in V_L$ be

$$H = \gamma(-1)\mathbf{1}$$

$$E = e^{\alpha_1} + \cdots + e^{\alpha_k}$$

$$F = e^{-\alpha_1} + \cdots + e^{-\alpha_k}$$

$$V_L \supset V^{\mathrm{aff}} = \langle H, E, F \rangle \cong L_{\widehat{sl}_2}(k, 0)$$
$$V^{\mathrm{aff}} \supset \langle e^{\gamma}, e^{-\gamma} \rangle = V_{\mathbb{Z}\gamma}$$

$$V^{\mathrm{aff}} = \bigoplus_{j=0}^{k-1} M^j \otimes V_{\mathbb{Z}\gamma - j\gamma/k}$$

$$M^j = \{ v \in L_{\widehat{sl}_2}(k, 0) \,|\, \gamma(m)v = -2j\delta_{m,0}v,\ m \geq 0 \}$$
$$M^0 = \mathrm{Com}_{L_{\widehat{sl}_2}(k,0)}(V_{\mathbb{Z}\gamma}) = K(sl_2, k)$$

$$T = \mathrm{Com}_{V_L}(V^{\mathrm{aff}}) = \mathrm{Com}_{V_N}(M^0) \cong K(sl_k, 2)$$
$$V^{\mathrm{aff}} = \mathrm{Com}_{V_L}(T) = \{ v \in V_L \,|\, (\omega_T)_1 v = 0 \}$$

(ω_T; the conformal vector of T)

$$V_N = \mathrm{Com}_{V_L}(V_{\mathbb{Z}\gamma})$$

$$
\begin{tikzcd}[column sep=small]
 & & V_L = (V_{A_1})^{\otimes k} = L_{\widehat{sl}_2}(1,0)^{\otimes k} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & V_{\sqrt{2}A_{k-1}} = V_N \arrow[dl, dash] \arrow[dr, dash] & & V^{\mathrm{aff}} = L_{\widehat{sl}_2}(k,0) \arrow[dl, dash] \arrow[dr, dash] & \\
T & & M^0 & & V_{\mathbb{Z}\gamma}
\end{tikzcd}
$$

X^*; dual lattice of an integral lattice X

$R = N \oplus \mathbb{Z}\gamma$, $\quad R \subset L \subset L^* \subset R^*$

A linear isom. ψ_p on $V_{R^*} = M_{\mathbb{C}\otimes_{\mathbb{Z}} R}(1) \otimes \mathbb{C}\{R^*\}$

$$\psi_p : u \otimes e^{\alpha} \longmapsto u \otimes (e^{\alpha} e^{p\gamma/2k}) = u \otimes e^{\alpha + p\gamma/2k}$$

for $u \in M_{\mathbb{C}\otimes_{\mathbb{Z}} R}(1)$, $\alpha \in R^*$, $p \in \mathbb{Z}$

- $\psi_p : V_{R^*} \to V_{R^*}$ is V_N-module isom.
- $\psi_p(V_{R+\lambda}) = V_{R+\lambda+p\gamma/2k}$, $\lambda \in R^*$, $p \in \mathbb{Z}$

Remark. ψ_p is Δ-operator

$$N^j = N - j\alpha_1 + j\gamma/k \subset N^* \qquad (0 \le j \le k-1)$$

$$\begin{aligned} M^{(j)} &= \psi_{2j}(M^j) \\ &= \{v \in V_{N^j} \mid (\omega_T)_1 v = 0\} \\ &\cong M^j \text{ as } M^0\text{-modules} \end{aligned}$$

For $\xi = (i_1, \dots, i_n) \in (\mathbb{Z}_k)^n$,

$$N_\xi = N^{i_1} \oplus \cdots \oplus N^{i_n} \subset (N^*)^{\oplus n}$$

$$\langle \alpha, \beta \rangle \in -\frac{2}{k}(\xi|\eta) + 2\mathbb{Z} \quad \text{for} \quad \alpha \in N_\xi,\ \beta \in N_\eta$$

$\Gamma_D = \bigcup_{\xi \in D} N_\xi$

Case 1. Γ_D is positive def. even lattice

Case 2. Γ_D is positive def. odd integral lattice

$$V_{N_\xi} = V_{N^{i_1}} \otimes \cdots \otimes V_{N^{i_n}} \subset (V_{N^*})^{\otimes n}$$

$$V_{\Gamma_D} = \bigoplus_{\xi \in D} V_{N_\xi}$$

$$\begin{aligned} M_D &= \mathrm{Com}_{V_{\Gamma_D}}(T^{\otimes n}) \\ &= \{v \in V_{\Gamma_D} \mid (\omega_{T^{\otimes n}})_1 v = 0\} \\ &= \bigoplus_{\xi \in D} M_\xi \end{aligned}$$

with $M_\xi = M^{(i_1)} \otimes \cdots \otimes M^{(i_n)}$

Remarks.

- $N \cong \sqrt{2}A_{k-1}, \qquad N^*/N \cong (\mathbb{Z}_2)^{k-2} \times \mathbb{Z}_{2k}$

  $$\mathbb{Z}_k \longleftrightarrow M^{(j)} \ (0 \le j \le k-1)$$

- $M^{(i,j)} = \psi_{-(i-2j)}(M^{i,j}) \subset V_{N^*}$

  $\longrightarrow$ irred. M_D-modules $\subset V_{(\Gamma_D)^*}$

- $N \subset \Lambda$; Leech lattice if $(k-1) | 24$

  $\longrightarrow$ Relation with Monster

McKay's observation