Symmetries of Boltzmann equation

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Boltzmann Transport Equation: BOLTZMANN 1872

$\Rightarrow$ statistical behaviour of a thermodynamic system out of equilibrium. Classical example - a fluid with temperature gradient in space causing heat to flow from hotter to colder region, by random transport of particles. BTE arises by considering the probability a number of particle all occupy small region centered at the tip of position vector **r** and have very nearly equal small changes in momentum from a momentum vector **p**(velocity **v**), at an instant of time $dN = f(t, \mathbf{r}, \mathbf{p}) d^3\mathbf{r} d^3\mathbf{p}$.

For identical particles the BTE reads

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{coll} .$$

$\mathbf{F}(\mathbf{r}, t)$ is the force field acting on the particles in the fluid and m is the mass of the particles

On the RHS is a collision term, which describes the effect of collisions:knowledge of the statistics particles obey.

Collision term

Boltzmann solution of the problem:

- only binary collisions need to be considered - dilute gas
- on the rate of collisions, influence of F is negligible
- velocity and position are uncorrelated - molecular chaos

Collision term can be written as a momentum-space integral over the product of one-particle distribution functions:

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \int\int g I(g,\Omega)[f(\mathbf{p}'_A,t)f(\mathbf{p}'_B,t) - f(\mathbf{p}_A,t)f(\mathbf{p}_B,t)]d\Omega d\mathbf{p}_A,$$

$g = |\mathbf{p}_B - \mathbf{p}_A| = |\mathbf{p}'_B - \mathbf{p}'_A|$, $I(g,\Omega)$ is the differential cross section of the collision, in which the relative momenta of the colliding particles turns through an angle θ into the element of the solid angle $d\Omega$, due to the collision.

stationary solution $=$ equilibrium distribution represent famaus **Boltzmann distribution:** $f_0(\mathbf{v}) = \left[\frac{m}{2\pi kT}\right]^{3/2} e^{-mv^2/(2kt)}$.

Relation with conformal invariance

Starting point $\Rightarrow$ an extension of local scaling, with a dynamical exponent $z = 1$ towards an infinite dimensional algebra(in $d = 1$ space dimensions) which is inequivalent to standard conformal invariance HENKEL02,10.

It is constructed by pair of non-commuting Virasoro algebras (without central charge) with following non-vanishing commutation relations

$$[X_n, X_{n'}] = (n - n')X_{n+n'}, \quad [X_n, Y_m] = (n - m)Y_{n+m},$$
$$[Y_m, Y_{m'}] = A_{10}(m - m')Y_{m+m'}, \quad n, n', m, m' \in \mathbb{Z}$$

and realization HENKEL02,10:

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - A_{10}^{-1}[(t + A_{10}r)^{n+1} - t^{n+1}]\partial_r \\ &- (n+1)xt^n - \frac{n+1}{2}\frac{B_{10}}{A_{10}}[(t + A_{10}r)^n - t^n] \\ Y_{k-1} &= -(t + A_{10}r)^k\partial_r - \frac{k}{2}B_{10}(t + A_{10}r)^{k-1}, \quad k = m + 1. \end{aligned}$$

Consider a finite-dimensional subalgebra $\langle X_{\pm1,0}, Y_{\pm1,0}\rangle$ with a representation($A_{10} =: \beta + \gamma =: \mu$,$B_{10} =: 2\gamma$)

$$
\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - r\partial_r - x \\
X_1 &= -t^2\partial_t - 2tr\partial_r - \mu r^2\partial_r - 2xt - 2\gamma r \\
Y_{-1} &= -\partial_r, \quad Y_0 = -t\partial_r - \mu r\partial_r - \gamma \\
Y_1 &= -t^2\partial_r - 2\mu tr\partial_r - \mu^2 r^2\partial_r - 2\gamma t - 2\gamma\mu r
\end{aligned}
$$

It follows that $\phi = \phi(x, \mu, \gamma)$ and also that the above representation acts as a dynamical symmetry on the equation

$$\hat{S}\phi(t, r) = (\mu\partial_t - \partial_r)\phi(t, r) = 0.$$

Really, it is easy to calculate

$$
\begin{aligned}
&[\hat{S}, Y_{\pm1,0}] = [\hat{S}, X_{-1}] = 0 \\
&[\hat{S}, X_0] = -\hat{S}, \quad [\hat{S}, X_1] = -2t\hat{S} + 2(\mu x - \gamma).
\end{aligned}
$$

Now consider the following equation

$$\hat{L}f = (\mu\partial_t + v\partial_r)f(t, r, v) = 0$$

where v is an additional variable. This equation represents first two terms of Boltzmann equation in one space dimensions, and for $v = constant$ coincides with previous equation. This suggests that dynamical symmetry algebra of BTE is a representation of discussed conformal algebra, but realized in terms of an additional variable v, related to the velocity(momentum) of the particles.

How to find such a representation?

Velocity representation of conformal algebra

Dynamical symmetry algebra of equation

$$\hat{L}f = (\mu\partial_t + v\partial_r)f(t,r,v) = 0$$

we shall describe using two basic ones: **time-translation and dynamical scaling**

$$\begin{aligned} X_{-1} &= -\partial_t, \quad [\hat{L}, X_{-1}] = 0 \\ X_0 &= -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - x, \quad [\hat{L}, X_0] = -\hat{L}. \end{aligned}$$

Next we write the general form of X_n hierarchy

$$X_n = -a_n(t,r,v)\partial_t - b_n(t,r,v)\partial_r - c_n(t,r,v)\partial_v - d_n(t,r,v).$$

We find X_n from the following three condition:

- X_n is a symmetry of equation $\hat{L}f = 0 : [\hat{L}, X_n] = \lambda_n \hat{L}$.
- Action of X_0 must recover the original operator: $[X_0, X_n] = \alpha_{0n} X_n$
- X_{-1} acts as annihilation operator $[X_{-1}, X_n] = \alpha_{-1,n} X_{n-1}$

Finite-dimensional representations

For $n = 1$ we anticipate $\alpha_{1,-1} = \alpha_0 = 2$, obtain $\alpha_{10} = 1$ and the most general form of $X_1 = -a_1\partial_t - b_1\partial_r - c_1\partial_v - d_1$

$$
\begin{aligned}
a_1(t,r,v) &= t^2 + A_{12}r^2v^{-2} + A_{110}rv^{\frac{2z-1}{1-z}} + A_{100}v^{\frac{2z}{1-z}} \\
b_1(t,r,v) &= \frac{2}{z}tr + \left(\frac{A_{12}}{\mu} + \frac{z-2}{z}\mu\right)r^2v^{-1} + B_{110}rv^{\frac{z}{1-z}} + B_{100}v^{\frac{z+1}{1-z}} \\
c_1(t,r,v) &= \frac{2}{z}(1-z)(vt-\mu r) + (B_{110} - \frac{A_{110}}{\mu})v^{\frac{z}{1-z}} \\
d_1(t,r,v) &= \frac{2}{z}xt - \frac{2}{z}\mu xrv^{-1} + D_0v^{\frac{z}{1-z}}
\end{aligned}
$$

Construction of Y_m hierarchies from

- X_1 acts as creation operator in both hierarchies
  $[X_1, Y_{-1}] = Y_0, \quad [X_1, Y_0] = Y_1$
- Y_{-1} must act as annihilation operator also in both hierarchies
  $[Y_{-1}, Y_0] = [Y_{-1}, [Y_{-1}, X_1]] = Y_{-1}$. It follows

$$Y_{-1} = -v\partial_r$$

$$
\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -t^2\partial_t - \left(\frac{2}{z}tr + \frac{z-2}{z}\mu r^2 v^{-1}\right)\partial_r - \frac{2(1-z)}{z}(vt-\mu r)\partial_v \\
&- \frac{2}{z}xt + \frac{2}{z}\mu x r v^{-1} \\
Y_{-1} &= -v\partial_r, \quad Y_0 = -(tv - \frac{\mu}{z}r)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
Y_1 &= -\left(t^2 v - \frac{2}{z}\mu tr - \frac{z-2}{z}\mu^2 r^2 v^{-1}\right)\partial_r - \frac{2}{z}(z-1)\mu(vt-\mu r)\partial_v \\
&+ \frac{2}{z}\mu x t - \frac{2}{z}\mu^2 x r v^{-1}
\end{aligned}
$$

The above generators give a finite-dimensional(6) representation of conformal algebra for an arbitrary dynamical exponent z

In addition they act as dynamical symmetry algebra of the equation

$$\hat{L}f = (\mu\partial_t + v\partial_r)f(t, r, v) = 0,$$

for any $f = f(t, r, v)$, as it can be verified through the relations

$$[\hat{L}, X_{-1}] = [\hat{L}, Y_{-1}] = [\hat{L}, Y_0] = [\hat{L}, Y_1] = 0$$
$$[\hat{L}, X_0] = -\hat{L}, \quad [\hat{L}, X_1] = -2t\hat{L}.$$

In particular, for $z = 1$ one recovers the usual realization of conformal algebra acting on function on t, r only.

Case B: $A_{12} \neq 0, A_{110} \neq 0, A_{100} \neq 0, B_{110} \neq 0, B_{100} \neq 0, D_0 \neq 0$

Modification of generators:

$$\begin{aligned}
\bar{X}_1 &= X_1 + \widetilde{X}_1, \quad \bar{Y}_0 = Y_0 + \widetilde{Y}_0 \\
\widetilde{X}_1 &= -\left(A_{12} r^2 v^{-2} + A_{110} r v^{\frac{2z-1}{1-z}} + A_{100} v^{\frac{2z}{1-z}}\right)\partial_t \\
&- \left(\frac{A_{12}}{\mu} r^2 v^{-1} + B_{110} r v^{\frac{z}{1-z}} + B_{100} v^{\frac{z+1}{1-z}}\right)\partial_r \\
&- (B_{110} - \frac{A_{110}}{\mu}) v^{\frac{z}{1-z}} \partial_v - D_0 v^{\frac{z}{1-z}}, \\
\widetilde{Y}_0 &= \frac{1}{2}[\widetilde{X}_1, Y_{-1}] = -(A_{12} r v^{-1} + \frac{1}{2} A_{110} v^{-1+1/(1-z)})\partial_t \\
&- \frac{1}{2\mu}(2A_{12} r + A_{110} v^{1/(1-z)})\partial_r.
\end{aligned}$$

Now, calculating $[\bar{Y}_0, Y_{-1}] = -\mu Y_{-1} + A_{12} X_{-1} + \frac{A_{12}}{\mu} Y_{-1}$, we conclude that $A_{12} = 0$ or we shall obtain an extension of usual conformal algebra!!!

case B1: $A_{12} = 0$

$$
\begin{aligned}
\bar{X}_{-1} &= X_{-1}, \quad \bar{X}_0 = X_0, \quad \bar{Y}_{-1} = Y_{-1} \\
\bar{X}_1 &= -(t^2 + A_{110} r v^{\frac{2z-1}{1-z}} + \frac{A_{110}^2}{4\mu^2} v^{\frac{2z}{1-z}})\partial_t \\
&- \left(\frac{2}{z} tr + \frac{z-2}{z} \mu r^2 v^{-1} + \frac{A_{110}}{\mu} r v^{\frac{z}{1-z}} + \frac{A_{110}^2}{4\mu^3} v^{\frac{z+1}{1-z}} \right) \partial_r \\
&- \frac{2(1-z)}{z} (vt - \mu r)\partial_v - \frac{2}{z} xt + \frac{2}{z} \mu x r v^{-1} \\
\bar{Y}_0 &= -\frac{A_{110}}{2} v^{\frac{z}{1-z}} \partial_t - (tv - \frac{\mu}{z} r + \frac{A_{110}}{2\mu} v^{\frac{1}{1-z}})\partial_r - \frac{z-1}{z} \mu v \partial_v + \mu \frac{x}{z} \\
\bar{Y}_1 &= -A_{110}(t v^{\frac{z}{1-z}} - \mu r v^{\frac{2z-1}{1-z}})\partial_t \\
&- \left(t^2 v - \frac{2}{z} \mu tr - \frac{z-2}{z} \mu^2 r^2 v^{-1} + \frac{A_{110}}{\mu} (t v^{\frac{1}{1-z}} - \mu r v^{\frac{z}{1-z}}) \right) \partial_r \\
&- \frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \frac{2}{z} \mu x t - \frac{2}{z} \mu^2 x r v^{-1}
\end{aligned}
$$

This is again a finite-dimensional representation of conformal algebra for arbitrary $z \neq 1$ and A_{110} and acts as symmetry algebra of the same equation

$$\hat{L}f = (\mu\partial_t + v\partial_r)f = 0,$$

for any $f = f(t, r, v)$, justified by the relations

$$[\hat{L}, X_{-1}] = [\hat{L}, Y_{-1}] = [\hat{L}, Y_0] = [\hat{L}, Y_1] = 0$$
$$[\hat{L}, X_0] = -\hat{L}, \quad [\hat{L}, X_1] = -(2t + \frac{A_{110}}{\mu} v^{z/(1-z)})\hat{L}.$$

This representation do not exist for $z = 1$, because all terms with A_{110} give infinity in this case.

Using **case B1**and after performing A LOT of COMMUTATORS we conclude that discussed generators in this case close into Lie algebra only and only if $A_{12} = \mu$ and $A_{110} = 0$(correspondingly all other constant are zero). The form of generators is the following

$$
\begin{aligned}
\mathcal{X}_{-1} &= X_{-1}, \quad \mathcal{X}_0 = X_0, \quad \mathcal{Y}_{-1} = Y_{-1} \\
\mathcal{X}_1 &= -\left(t^2 + \mu r^2 v^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{z + \mu(z-2)}{z} r^2 v^{-1}\right)\partial_r \\
&- \frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z}xt + \frac{2}{z}\mu x r v^{-1} \\
\mathcal{Y}_0 &= -\mu r v^{-1}\partial_t - \left(tv - (\frac{\mu}{z} - 1)r\right)\partial_r - \frac{z-1}{z}\mu v \partial_v + \mu\frac{x}{z} \\
\mathcal{Y}_1 &= -\mu\left(2trv^{-1} + (1-\mu)r^2 v^{-2}\right)\partial_t \\
&- \left(t^2 v - \frac{2}{z}(z-\mu)tr + \frac{z(1-\mu) - (z-2)\mu^2}{z} r^2 v^{-1}\right)\partial_r \\
&- \frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \frac{2}{z}\mu x t - \frac{2}{z}\mu^2 x r v^{-1}
\end{aligned}
$$

This Lie algebra is defined for an arbitrary z, through the following non-zero commutation relations

$$
\begin{aligned}
&[\mathcal{X}_n, \mathcal{X}_{n'}] = (n - n')\mathcal{X}_{n+n'}, \quad [\mathcal{X}_n, \mathcal{Y}_m] = (n - m)\mathcal{Y}_{n+m} \\
&[\mathcal{Y}_m, \mathcal{Y}_{m'}] = (m - m')\left(\mu\mathcal{X}_{m+m'} + (1 - \mu)\mathcal{Y}_{m+m'}\right), \\
&n, n', m, m' \in [-1, 0, 1].
\end{aligned}
$$

It gives more general symmetry algebra of the equation

$$\hat{L}f = (\mu\partial_t + v\partial_r)f = 0,$$

for any $f = f(t, r, v)$, verified by the commutators

$$
\begin{aligned}
&[\hat{L}, X_{-1}] = [\hat{L}, Y_{-1}] = [\hat{L}, Y_0] = [\hat{L}, Y_1] = 0 \\
&[\hat{L}, X_0] = -\hat{L}, \quad [\hat{L}, X_1] = -2(t + rv^{-1})\hat{L}.
\end{aligned}
$$

Symmetries of collisionless Boltzmann equation

Consider the collisionless Boltzmann:

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + F(t,r,v)\partial_v)\, f(t,r,v) = 0$$

We shall try to determine $F(t,r,v)$, denoting the "force" term, such that it is compatible with the symmetries of the compleat BTE.

Invariance under basic symmetries:

- time translations $X_{-1} = -\partial_t$, it follows $[X_{-1}, \hat{B}] = -\dot{F} = 0 \rightarrow F = F(r,v)$
- dynamical scaling $X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{v}{\alpha}\partial_v - \frac{x}{z}$

$$[X_0, \hat{B}] = -\hat{B}, \quad \alpha = \frac{z}{1-z}$$
$$(r\partial_r + (1-z)v\partial_v - (1-2z))\, F(r,v) = 0$$

$\Rightarrow F(r,v) = r^{1-2z}\varphi(r^{z-1}v)$, where $\varphi(u = r^{z-1}v)$ is an arbitrary function.

Convenient change: $(t, r, v) \to (t, r, u)$. In the new coordinates:

$$X_{-1} = -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{x}{z}$$
$$Y_{-1} = -r^{1-z}u\partial_r - r^{-z}\Phi(u)\partial_u, \quad \Phi(u) = (z-1)u^2 + \varphi(u)$$

$$\hat{B}f(t, r, u) = \left(\mu\partial_t + r^{1-z}u\partial_r + r^{-z}\Phi(u)\partial_u\right) f(t, r, u) = 0$$

We suggest that a finite dimensional representation of the symmetry algebra $X_{0,\pm1}$, $Y_{0,\pm1}$ of above equation must satisfy

$$[X_n, X_{n'}] = (n - n')X_{n+n'}$$
$$[X_n, Y_m] = (n - m)Y_{n+m}, \quad n, n', m = 0, \pm1$$
$$[Y_0, Y_{-1}] = pX_{-1} + sY_{-1},$$
$$[Y_1, Y_{-1}] = nX_0 + qY_0,$$
$$[Y_1, Y_0] = NX_1 + QY_1,$$

In fact, the commutator with X_{-1} and X_0 shall give the dependence on t and r, respectively in unknown generators. We obtain

$$
\begin{aligned}
Y_0 &= -r^z a_0(u)\partial_t - (r^{1-z}u + r b_0(u))\partial_r \\
&\quad -(r^{-z}\Phi(u)t + c_0(u))\partial_u - d_0(u) \\
X_1 &= -(t^2 + r^{2z}a_{12}(u))\partial_t - ((2/z)tr + r^{z+1}b_{12}(u))\partial_r \\
&\quad -r^z c_{12}(u)\partial_u - (2/z)xt - r^z d_{12}(u) \\
Y_1 &= -\left(2tr^z a_0(u) + r^{2z}A(u)\right)\partial_t \\
&\quad -\left(t^2 r^{1-z}u + 2tr b_0(u) + r^{z+1}B(u)\right)\partial_r \\
&\quad -\left(t^2 r^{-z}\Phi(u) + 2tc_0(u) + r^z C(u)\right)\partial_u + (2/z)\mu xt - r^z D(u), \\
A(u) &= 2zb_0 a_{12} + c_0 a_{12}' - za_0 b_{12} - a_0' c_{12} \\
C(u) &= zb_0 c_{12} + c_0 c_{12}' - c_0' c_{12} - a_{12}\Phi \\
B(u) &= \frac{2}{z}a_0 + zb_0 b_{12} + c_0 b_{12}' - ua_{12}' - b_0' c_{12} \\
D(u) &= \frac{2}{z}xa_0 + zb_0 d_{12} + c_0 d_{12}'.
\end{aligned}
$$

The analogue of extended Galilei algebra

Consider the algebra$\{X_{-1}, X_0, Y_{-1}, Y_0\}$. It is closed if

1. $zua_0(u) + \Phi(u)a_0'(u) - p = 0$
2. $zub_0(u) + \Phi(u)b_0'(u) - c_0(u) - su = 0$
3. $\Phi'(u)c_0 - \Phi(u)c_0'(u) + (s - zb_0)\Phi = 0$
4. $\Phi(u)d_0'(u) = 0$

$$\Phi^2(u)b_0''(u)+zu\Phi(u)b_0'(u)+(2z\Phi(u)-zu\Phi'(u))b_0(u)-2s\Phi(u) = 0,$$

$$\begin{aligned} Y_{01} &= -r^z a_0\partial_t - (r^{1-z}u + rb_{01})\partial_r - (r^{-z}\Phi(u)t + c_{01})\partial_u - d_0 \\ Y_{02} &= -r^z a_0\partial_t - (r^{1-z}u + rb_{02})\partial_r - (r^{-z}\Phi(u)t + c_{02})\partial_u - d_0 \end{aligned}$$

$$[Y_0, X_{-1}] = Y_{-1}, \quad [X_0, X_{-1}] = X_{-1}$$

$$[Y_0, Y_{-1}] = \frac{p-\mu^2}{\mu}Y_{-1} + pX_{-1}$$

Representations of conformal algebra - symmetry algebra of collisionless Boltzman equation

Adding the generators X_1, Y_1 to the analogue of extended Galilei algebra, one must also satisfy

5. $\lambda_{X_1}(t, r, u) = -2(t + r^z a_0(u))$
6. $c_{12}(u) = (2/z)\mu - (u/\mu)(2za_{12}(u) + \Phi(u)a'_{12}(u))$
   $+(2zub_{12} + \Phi(u)b'_{12}(u))$
7. $zuc_{12} + \Phi c'_{12}(u) - c_{12}(u)\Phi'(u) + zb_{12}(u)\Phi(u) - 2c_0(u) = 0$
8. $zud_{12}(u) + \Phi(u)d'_{12}(u) + (2/z)\mu x = 0$
9. $\Phi^2(u)b''_{12}(u) + 3zu\Phi(u)b'_{12}(u)$
   $+z[2zu^2 + 3\Phi(u) - 2u\Phi'(u)]b_{12}(u) + (2\mu/z)(zu - \Phi'(u))$
   $-(u/\mu)\Phi^2(u)a''_{12}(u) - [3zu^2 + 2\Phi(u)](\Phi/\mu)a'_{12}(u)$
   $-[zu^2 + 3\Phi(u) - u\Phi'(u)](2zu/\mu)a_{12}(u) = 0$

10. $2zua_{12}(u) + \Phi(u)a_{12}'(u) - 2a_0(u) = 0$

11. $2zub_{12}(u) + \Phi(u)b_{12}'(u) - c_{12}(u) - 2b_0(u) = 0$

12. $b_0(u) = (u/\mu)a_0(u) - \mu/z$

13. $c_0(u) = (\Phi/\mu)a_0(u)$

14. $d_0(u) = const. = -\mu x/z.$

15. $n = 2p, \quad q = 2s$

16. $2zuA(u) + \Phi(u)A'(u) - 2sa_0(u) = 0$

17. $2zuB(u) + \Phi(u)B'(u) - C(u) - 2(p/z + sb_0(u)) = 0$

18. $zuC(u) + \Phi(u)C'(u) - \Phi'(u)C(u) + z\Phi(u)B(u) - 2sc_0(u) = 0$

19. $zuD(u) + \Phi(u)D'(u) - (2x/z)(p - \mu s) = 0$

20. $N = p, \quad Q = s$

21. $(s - zb_0)A(u) - c_0 A'(u) + za_0(u)B(u) + a_0'(u)C(u)$
$+pa_{12}(u) - 2a_0^2 = 0$

22. $(s - zb_0)B(u) - c_0 B'(u) + uA(u) + b_0' C(u) + pb_{12}(u)$
$-2a_0(u)b_0(u) = 0$

23. $(s - zb_0 + c_0'(u))C(u) - c_0 C'(u) + \Phi(u)A(u)$
$+pc_{12}(u) - 2a_0(u)c_0(u) = 0$

24. $(s - zb_0)D(u) - c_0 D'(u) + pd_{12}(u) + \dfrac{2a_0(u)\mu x}{z} = 0$

25. $2z(b_{12}(u)A(u) - a_{12}(u)B(u)) + c_{12}(u)A'(u) - a_{12}'(u)C(u)$
$+2a_0(u)a_{12}(u) = 0$

26. $(2/z)A(u) - c_{12}(u)B'(u) + b_{12}' C(u) - 2b_0(u)a_{12}(u) = 0$

27. $(zb_{12} - c_{12}'(u))C + c_{12} C'(u) - zc_{12} B + 2c_0 a_{12} = 0$

28. $(2x/z)(\mu a_{12}(u) + A(u)) + zd_{12}(u)B(u) + d_{12}'(u)C(u)$
$-zb_{12} D(u) - c_{12}(u)D'(u) = 0$

Case C: $\Phi(u) = (z-1)u^2 + \varphi(u) = 0, \quad \varphi(u) = (1-z)u^2$

$$
\begin{aligned}
a_0(u) &= \frac{p}{z}u^{-1}, \quad b_0 = const. = \frac{p}{z\mu} - \frac{\mu}{z} \\
c_0(u) &= 0, \quad d_0 = const. = -\frac{\mu}{z}x \\
a_{12}(u) &= \frac{p}{z^2}u^{-2}, \quad b_{12}(u) = \frac{1}{\mu z^2}(p-\mu^2)u^{-1} \\
c_{12}(u) &= 0, \quad d_{12}(u) = -\frac{2\mu x}{z^2}u^{-1} \\
A(u) &= \frac{p}{\mu z^2}(p-\mu^2)u^{-2}, \quad B(u) = \frac{1}{\mu^2 z^2}\left(p(p-\mu^2)+\mu^4\right)u^{-1} \\
C(u) &= 0, \quad D(u) = \frac{2\mu^2 x}{z^2}u^{-1}.
\end{aligned}
$$

One can verify that the above results satisfy the system.

The representation of the algebra, valid for an arbitrary z is

$$
\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{x}{z} \\
X_1 &= -\left(t^2 + \frac{p}{z^2} r^{2z} u^{-2}\right)\partial_t - \left(\frac{2}{z} tr + \frac{p-\mu^2}{z^2\mu} r^{z+1} u^{-1}\right)\partial_r \\
&\quad -\frac{2}{z} xt + \frac{2\mu x}{z^2} r^z u^{-1}, \\
Y_{-1} &= -r^{1-z} u \partial_r, \\
Y_0 &= -\frac{p}{z} r^z u^{-1} \partial_t - \left(t r^{1-z} u + \frac{p-\mu^2}{z\mu} r\right)\partial_r + \frac{\mu x}{z}, \\
Y_1 &= -\left(\frac{2p}{z} t r^z u^{-1} + \frac{p(p-\mu^2)}{z^2\mu} r^{2z} u^{-2}\right)\partial_t \\
&\quad -\left(t^2 r^{1-z} u + 2\frac{p-\mu^2}{z\mu} tr + \frac{p(p-\mu^2)+\mu^4}{z^2\mu^2} r^{z+1} u^{-1}\right)\partial_r \\
&\quad +\frac{2}{z}\mu x t - \frac{2\mu^2 x}{z^2} r^z u^{-1}.
\end{aligned}
$$

Finite-dimensional representation of conformal algebra

Putting $z = 1 \to \Phi(u) = 0, u = r^0 v = cnst = 1$

$$
\begin{aligned}
X_1 &= -\left(t^2 + pr^2\right)\partial_t - \left(2tr + \frac{p-\mu^2}{\mu} r^2\right)\partial_r - 2xt + 2\mu xr, \\
Y_0 &= -pr\partial_t - \left(t + \frac{p-\mu^2}{\mu} r\right)\partial_r + \mu x \\
Y_1 &= -\left(2ptr + \frac{p(p-\mu^2)}{\mu} r^2\right)\partial_t \\
&\quad - \left(t^2 + 2\frac{p-\mu^2}{\mu} tr + \frac{p(p-\mu^2)+\mu^4}{\mu^2} r^2\right)\partial_r \\
&\quad + 2\mu xt - 2\mu^2 xr.
\end{aligned}
$$

$$[X_n, X_{n'}] = (n-n')X_{n+n'}, \quad [X_n, Y_m] = (n-m)Y_{n+m}$$

$$[Y_m, Y_{m'}] = (m-m')\left(pX_{m+m'} + \frac{p-\mu^2}{\mu} Y_{m+m'}\right)$$

$$
\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -\left(t^2 + \frac{p}{z^2}r^2v^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{p-\mu^2}{z^2\mu}r^2v^{-1}\right)\partial_r \\
&- (1-z)\left(\frac{2}{z}tv + \frac{p-\mu^2}{z^2\mu}r\right)\partial_v - \frac{2}{z}xt + \frac{2\mu x}{z^2}rv^{-1}, \\
Y_{-1} &= -v\partial_r - (1-z)r^{-1}v^2\partial_v, \\
Y_0 &= -\frac{p}{z}rv^{-1}\partial_t - \left(tv + \frac{p-\mu^2}{z\mu}r\right)\partial_r \\
&\quad -(1-z)\left(tr^{-1}v^2 + \frac{p-\mu^2}{z\mu}v\right)\partial_v + \frac{\mu x}{z},
\end{aligned}
$$

$$
\begin{aligned}
Y_1 \;=\; & -\left(\frac{2p}{z}trv^{-1} + \frac{p(p-\mu^2)}{z^2\mu}r^2v^{-2}\right)\partial_t \\
& -\left(t^2v + 2\frac{p-\mu^2}{z\mu}tr + \frac{p(p-\mu^2)+\mu^4}{z^2\mu^2}r^2v^{-1}\right)\partial_r \\
& -(1-z)\left(t^2r^{-1}v^2 + 2\frac{p-\mu^2}{z\mu}tv + \frac{p(p-\mu^2)+\mu^4}{z^2\mu^2}r\right)\partial_v \\
& +\frac{2}{z}\mu xt - \frac{2\mu^2 x}{z^2}rv^{-1}.
\end{aligned}
$$

The above generators close a finite-dimensional representation of conformal algebra for arbitrary z. They act as dynamical symmetry algebra of the BTE:

$$
\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + (1-z)r^{-1}v^2\partial_v)f(t,r,v) = 0
$$

$$
[\hat{B}, X_0] = -\hat{B}, \quad [\hat{B}, X_1] = -2\left(t + \frac{p}{z\mu}rv^{-1}\right)\hat{B}
$$

$$
[\hat{B}, Y_0] = -(p/\mu)\hat{B}, \quad [\hat{B}, Y_1] = -2\left(\frac{p}{\mu}t + \frac{p}{z\mu^2}rv^{-1}\right)\hat{B}.
$$

Case D: $p=0, \Phi(u)\neq 0$. Leads to $a_0=0$ and
$b_0 = const. = -\mu/z, \quad c_0=0, \quad d_0=-\mu x/z; A(u)=a_{12}(u)=0$
$B(u)=-\mu b_{12}(u), \quad C(u)=-\mu c_{12}(u), \quad D(u)=-\mu d_{12}$.
$c_{12}(u)=2zub_{12}(u)+((z-1)u^2+\varphi(u))b'_{12}(u)+2\mu/z$. Then for
$\Phi(u)=(z-1)u^2+\varphi(u)$, $b_{12}(u), d_{12}(u)$ must satisfy

$$[(z-1)u^2+\varphi(u)]^2 b''_{12}(u)+3zu[(z-1)u^2+\varphi(u)]b'_{12}(u)$$
$$+z[(z+1)u^2-2u\varphi'(u)+3\varphi(u)]b_{12}(u)+[(2-z)u-\varphi'(u)]2\mu/z =$$
$$=0$$

$$zud_{12}(u)+[(z-1)u^2+\varphi(u)]d'_{12}(u)+2\mu x/z=0.$$

It follows that for any triplet $(\varphi(u), b_{12}(u), d_{12}(u))$, solution of the system, the generators close the conformal algebra and act as dynamical symmetry algebra of the Boltzmann equation, with a quite general "force" term:

$$\hat{B}f(t,r,v)=(\mu\partial_t+v\partial_r+r^{1-2z}\varphi(u)\partial_v)f(t,r,v)=0.$$

BTE with a physical force term

Physical requirement: $F(r,v) = F(r) \Rightarrow \varphi(u) = \varphi_0 = const.$

$$[(z-1)u^2+\varphi_0]^2 b''_{12}(u) + 3zu[(z-1)u^2+\varphi_0]b'_{12}(u)$$
$$+z[(z+1)u^2+3\varphi_0]b_{12}(u) + 2\mu\frac{2-z}{z}u = 0$$
$$zud_{12}(u) + [(z-1)u^2+\varphi_0]d'_{12}(u) + 2\mu x/z = 0$$

For $z=2$ we have relatively simple solution of the system:

$$\begin{aligned}
b_{12}(u) &= b_{120}\frac{u}{(u^2+\varphi_0)^2} + b_{121}\frac{u^2-\varphi_0}{(u^2+\varphi_0)^2} \\
d_{12}(u) &= = -\mu x\frac{u}{u^2+\varphi_0} \\
\hat{B}f(t,r,v) &= (\mu\partial_t + v\partial_r + r^{-3}\varphi_0\partial_v)f(t,r,v) = 0.
\end{aligned}$$