The Arcetri model(s): exactly solvable spherical model(s) of interface growth

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Some themes **unrelated** to physical ageing :

Healthy Aging FOR DUMMIES
Stay fit and vigorous and feel great no matter your age!
Brent Agin, MD
Sharon Perkins, RN
A Reference for the Rest of Us!

Rita Levi-Montalcini
Die Vorzüge des Alters
Leistungsfähigkeit und geistige Aktivität ein Leben lang
PIPER

A Contemplative Guide to Growing Older and Wiser
Aging as a Spiritual Practice
Lewis Richmond
Foreword by Sylvia Boorstein, author of It's Easier Than You Think

The Aging Skeleton
Edited by
Clifford J. Rosen
Julie Glowacki
John P. Bilezikian
ACADEMIC PRESS

Sherwin B. Nuland
National Book Award-winning author of HOW WE DIE
the art of aging
A Doctor's Prescription for Well-Being

. . . **not** discussed here . . .

... but there also exist books on **physical ageing** ...
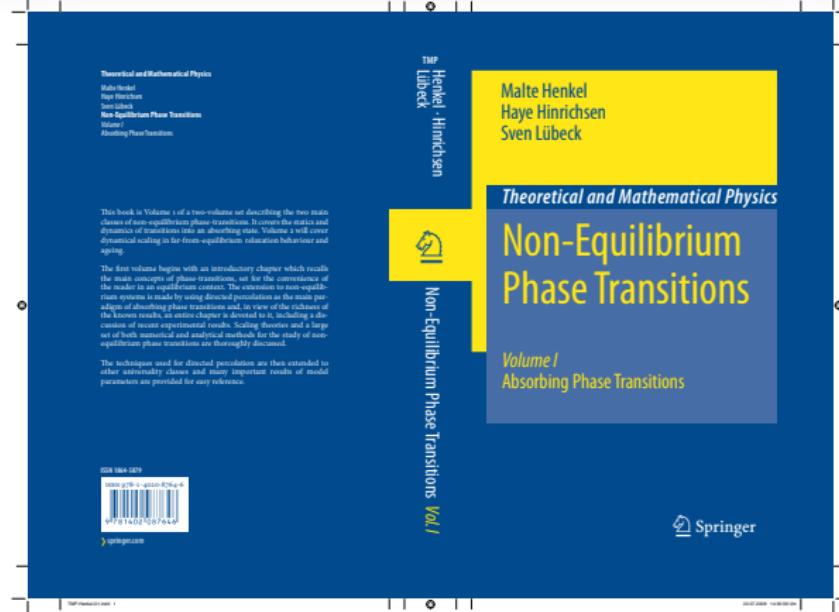
Vol. 1 : absorbing phase transitions – co-authors H. **Hinrichsen**, S. **Lübeck** 2009

Vol. 2 : ageing & local scaling — co-author M. **Pleimling** 2010

Overview :

0. Physical ageing : a reminder

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s   Struik '78
   discovery : ageing effects **reproducible** & **universal** !
occur in widely different systems
   (structural glasses, spin glasses, polymers, simple magnets, . . .)

Three defining properties of **ageing** :

1. slow relaxation (non-exponential !)
2. **no** time-translation-invariance (TTI)
3. dynamical scaling   without fine-tuning of parameters

Cooperative phenomenon, far from equilibrium

Consider a **simple ferromagnet** (no disorder, no 'frustration')

1. prepare system initially at high temperature $T \gg T_c > 0$
2. quench to temperature $T < T_c$ (or $T = T_c$)
   $\rightarrow$ non-equilibrium state
3. fix T and observe dynamics
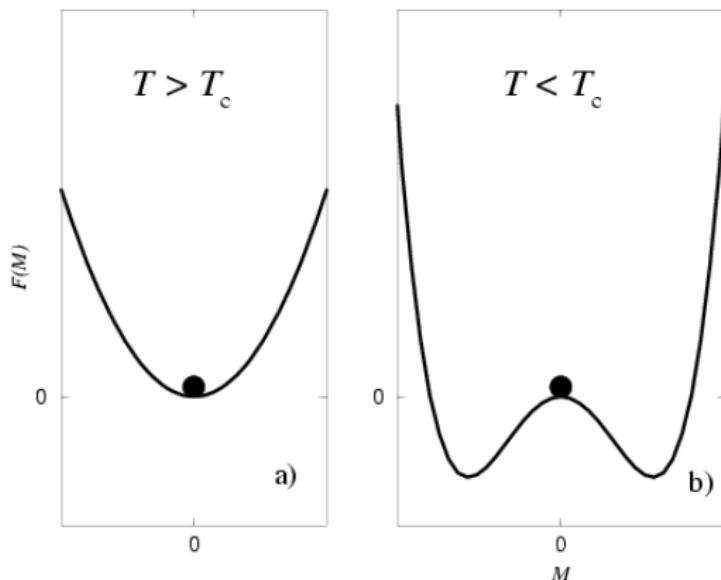
competition :
2 equivalent ground states
rapid local ordering
no global order
relaxation time ∞

$t = t_1$ $t = t_2 > t_1$
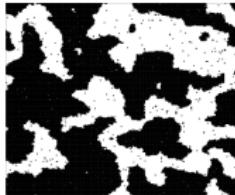
magnet $T < T_c$

$\longrightarrow$ ordered cluster
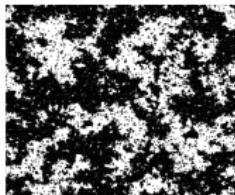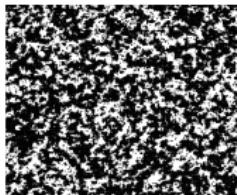
magnet $T = T_c$

$\longrightarrow$ correlated cluster

growth of ordered/correlated domains, of typical linear size

$$L(t) \sim t^{1/z}$$

dynamical exponent z : determined by equilibrium state
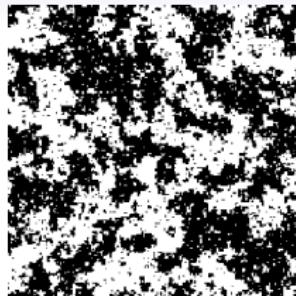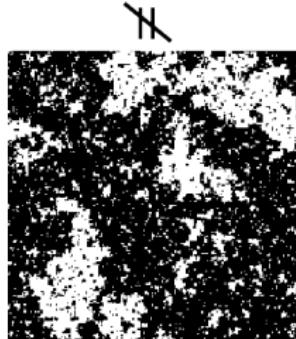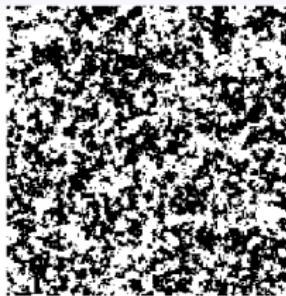
illustration of statistical self-similarity for different times $t_1 < t_2$

Two-time observables for simple magnets

time-dependent magnetisation = order-parameter = $\phi(t, \mathbf{r})$

two-time **correlator** $\quad C(t, s) := \langle \phi(t, \mathbf{r})\phi(s, \mathbf{r}) \rangle - \langle \phi(t, \mathbf{r}) \rangle \langle \phi(s, \mathbf{r}) \rangle$

two-time **response** $\quad R(t, s) := \left. \dfrac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{r})} \right|_{h=0} = \left\langle \phi(t, \mathbf{r})\widetilde{\phi}(s, \mathbf{r}) \right\rangle$

t : observation time, s : waiting time

a) system **at** equilibrium : **fluctuation-dissipation theorem** $\qquad$ KUBO

$$R(t - s) = \frac{1}{T}\frac{\partial C(t - s)}{\partial s} \quad , \quad T : \text{temperature}$$

b) far from equilibrium : C and R **independent** !

The **fluctuation-dissipation ratio** (FDR) $\qquad$ CUGLIANDOLO, KURCHAN, PARISI '94

$$X(t, s) := \frac{T R(t, s)}{\partial C(t, s)/\partial s}$$

measures the distance with respect to equilibrium : $\boxed{X_{\text{eq}} = X(t - s) = 1}$

Experiment : breaking of the FDT

spin glass $CdCr_{1.7}In_{0.3}S_4$, quenched to $T/T_c = 0.8$
susceptibility $\chi_{\mathrm{ZFC}}(t,s) = \int_s^t \mathrm{d}u\, R(t,u)$ over against correlator $C(t,s)$

D. HÉRISSON ET M. OCIO, Phys. Rev. Lett. **88**, 257202 (2002)
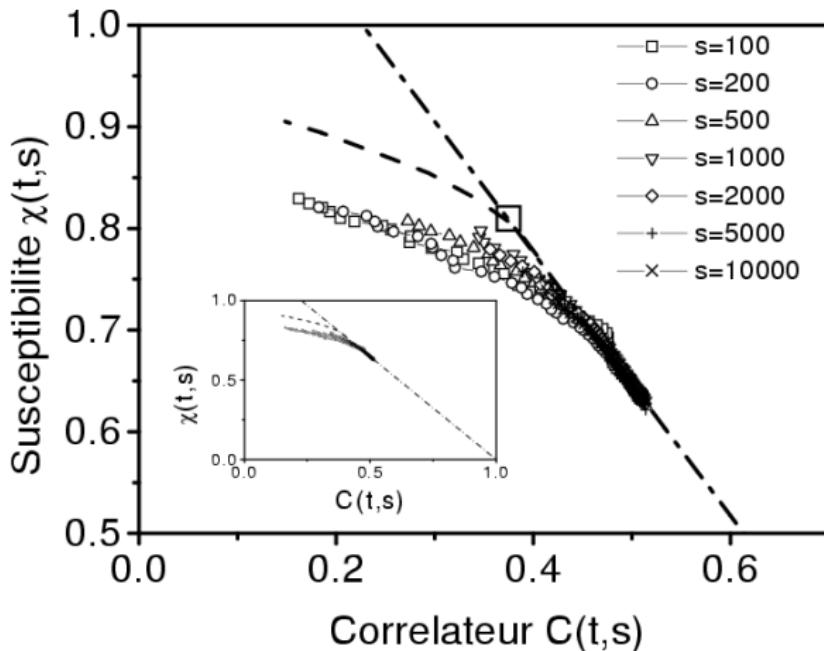
For quenches to $T \leq T_c$, system **never** reaches equilibrium

Scaling regime : $t, s \gg \tau_{\mathrm{micro}}$ **and** $t - s \gg \tau_{\mathrm{micro}}$

$$C(t,s) = s^{-b} f_C\left(\frac{t}{s}\right) \;,\; R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right)$$

asymptotics : $f_C(y) \sim y^{-\lambda_C/z}$, $f_R(y) \sim y^{-\lambda_R/z}$ for $y \gg 1$

λ_C : autocorrelation exponent, λ_R : autoresponse exponent,
z : dynamical exponent, a, b : ageing exponents

Constat : exponents & scaling functions are universal,
   i.e. independent of 'fine details'

may use simplified theoretical models to find their values

Dynamical scaling in the ageing 3D Ising model, $T < T_c$
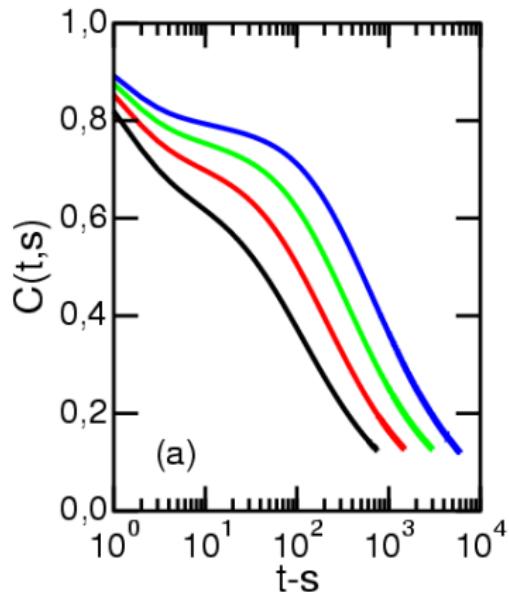
(a) no time-translation invariance
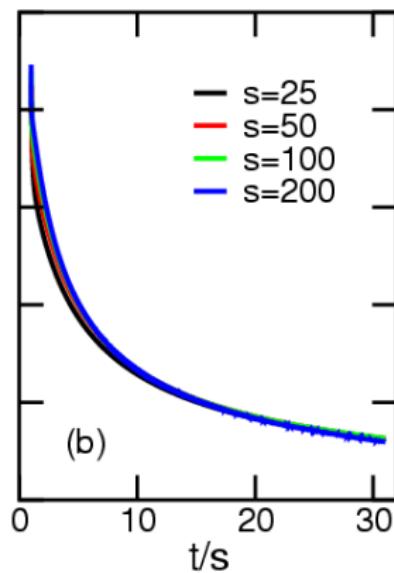
(b) dynamical scaling

$C(t,s)$: autocorrelation function, quenched to $T < T_c$
scaling regime : $t, s \gg \tau_{\mathrm{micro}}$ and $t - s \gg \tau_{\mathrm{micro}}$
data collapse evidence for **dynamical scale-invariance**   MH & PLEIMLING 10

shear creep compliance
J(t), 10⁻⁹m²/N

PS	PMMA
SAN	PPO
Imp. PS	Sugar
PVC-VA	shellac
Rigid PVC	lead
post chl. PVC	

± 2%

creep time t, sec
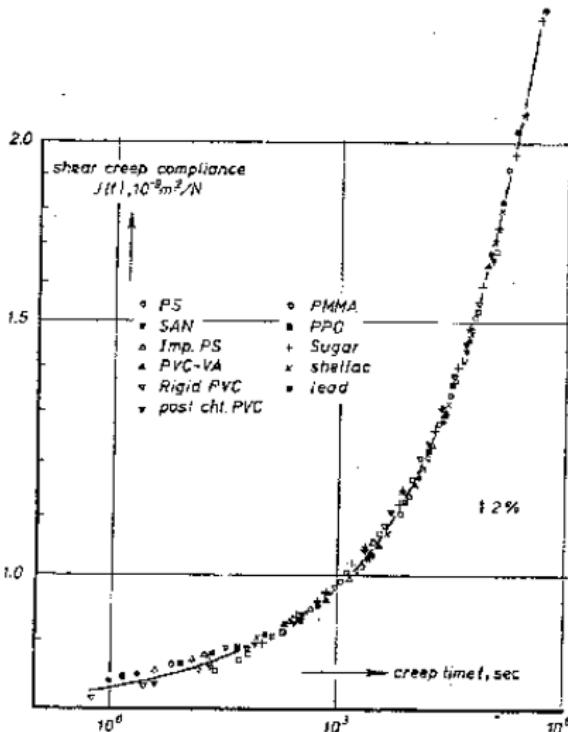
Master curves of mechanical response of distinct glassy materials are **identical**

independent of 'details'

$\longrightarrow$ Universality !

good for theorists ...

hint for hidden symmetry ?

STRUIK 78

since universal : use simplified models to find their form

Interface growth

deposition (evaporation) of particles on a substrate

$\rightarrow$ height profile $h(t, \mathbf{r})$ slope profile $\mathbf{u}(t, \mathbf{r}) = \boldsymbol{\nabla} h(t, \mathbf{r})$

p = deposition prob.

$1 - p$ = evap. prob.
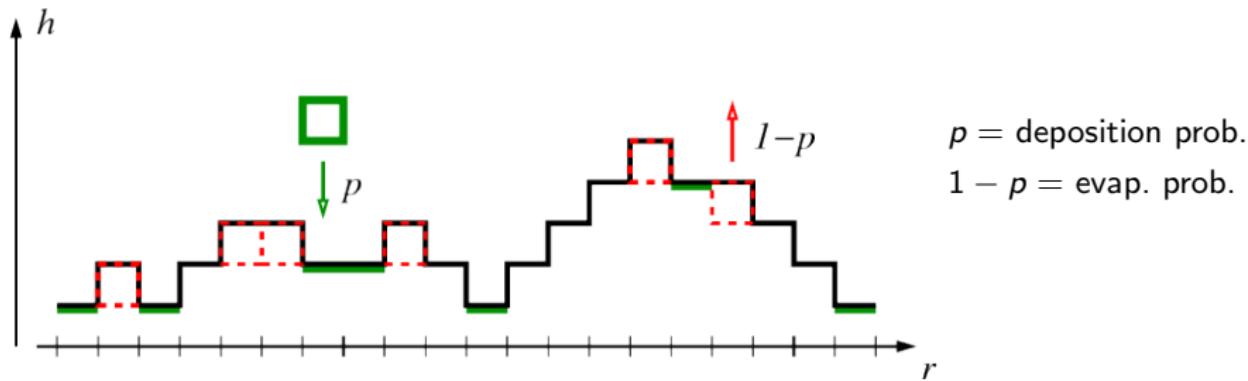
Questions :

* average properties of profiles & their fluctuations ?
* what about their relaxational properties ?
* are these also examples of physical ageing ?
? does dynamical scaling **always** exist ?

1. Magnets and growing interfaces : analogies

Common properties of critical and ageing phenomena :

* **collective** behaviour,
    very **large** number of interacting degrees of freedom
* **algebraic** large-distance and/or large-time behaviour
* described in terms of **universal** critical **exponents**
* very **few** relevant scaling operators
* justifies use of extremely **simplified mathematical models**
    with a remarkably rich and complex behaviour
* yet of **experimental significance**

Magnets

thermodynamic equilibrium state

order parameter $\phi(t, \mathbf{r})$

phase transition, at critical temperature T_c

variance :
$$\left\langle (\phi(t, \mathbf{r}) - \langle\phi(t)\rangle)^2 \right\rangle \sim t^{-2\beta/(\nu z)}$$

relaxation, after quench to $T \leq T_c$

autocorrelator
$$C(t, s) = \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}) \rangle_c$$

Interfaces

growth continues forever

height profile $h(t, \mathbf{r})$

same generic behaviour throughout

roughness :
$$w(t)^2 = \left\langle \left(h(t, \mathbf{r}) - \overline{h}(t)\right)^2 \right\rangle \sim t^{2\beta}$$

relaxation, from initial substrate :

autocorrelator $C(t, s) =$
$$\left\langle \left(h(t, \mathbf{r}) - \overline{h}(t)\right) \left(h(s, \mathbf{r}) - \overline{h}(s)\right) \right\rangle$$

ageing scaling behaviour :

when $t, s \to \infty$, and $y := t/s > 1$ fixed, expect

$$C(t, s) = s^{-b} f_C(t/s) \quad \text{and} \quad f_C(y) \overset{y \to \infty}{\sim} y^{-\lambda_C/z}$$

b, β, ν and dynamical exponent z : **universal** & related to stationary state

autocorrelation exponent λ_C : **universal** & independent of stationary exponents

Magnets

exponent value $b = \begin{cases} 0 & ; \quad T < T_c \\ 2\beta/\nu z & ; \quad T = T_c \end{cases}$

Interfaces

exponent value $b = -2\beta$

models :

(a) gaussian field
$\mathcal{H}[\phi] = -\frac{1}{2}\int d\mathbf{r}\,(\nabla\phi)^2$

(b) Ising model
$\mathcal{H}[\phi] = -\frac{1}{2}\int d\mathbf{r}\,\left[(\nabla\phi)^2 + \tau\phi^2 + \frac{g}{2}\phi^4\right]$

such that $\tau = 0 \leftrightarrow T = T_c$

dynamical Langevin equation (Ising) :

$$\begin{aligned} \partial_t \phi &= -D\frac{\delta\mathcal{H}[\phi]}{\delta\phi} + \eta \\ &= D\nabla^2\phi + \tau\phi + g\phi^3 + \eta \end{aligned}$$

$\eta(t,\mathbf{r})$ is the usual white noise, $\langle\eta(t,\mathbf{r})\eta(t',\mathbf{r}')\rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$

(a) Edwards-Wilkinson (EW) :
$\partial_t h = \nu\nabla^2 h + \eta$

(b) Kardar-Parisi-Zhang (KPZ) :

$$\partial_t h = \nu\nabla^2 h + \frac{\mu}{2}(\nabla h)^2 + \eta$$

phase transition exactly solved $\underline{d = 2}$
relaxation exactly solved $\underline{d = 1}$

growth exactly solved $\underline{d = 1}$

CALABRESE & LE DOUSSAL '11

ONSAGER '44, GLAUBER '63, ...

SASAMOTO & SPOHN '10

Question : obtain qualitative understanding by approximate treatment of the non-linearity ?

Ising model : **yes, certainly !** $\Rightarrow$ **spherical model** ! Berlin & Kac 52
Lewis & Wannier 52

(a) for a lattice model : replace Ising spins $\sigma_i = \pm 1 \mapsto S_i \in \mathbb{R}$, with (mean) spherical constraint $\sum_i S_i^2 = \mathcal{N}$
(b) for continuum field : replace $\phi^3 \mapsto \phi \langle \phi^2 \rangle$ and spherical constraint $\int d\mathbf{r} \langle \phi^2 \rangle \sim 1$.

Interest : analytically solvable for any d and in more general contexts than Ising model, all exponents **...** known exactly. Very useful to illustrate general principles in a specific setting. New universality class, distinct from the Ising model (O(N) model with $N \to \infty$).

Question : can one find a similar procedure, based on the KPZ equation ?
Are there new universality class(es) for interface growth ?
Behaviour different from the rather trivial EW-equation ?

2. Interface growth & KPZ class

deposition (evaporation) of particles on a substrate $\rightarrow$ height profile $h(t, \mathbf{r})$
generic situation : RSOS (restricted solid-on-solid) model   Kim & Kosterlitz 89

$p =$ deposition prob.
$1 - p =$ evap. prob.

here $p = 0.98$
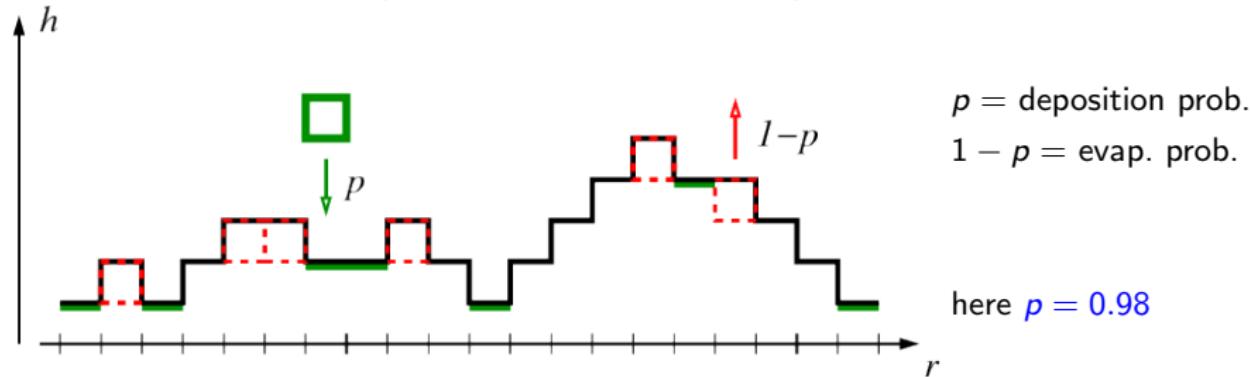
some universality classes :
(a) KPZ $\partial_t h = \nu \boldsymbol{\nabla}^2 h + \frac{\mu}{2} (\boldsymbol{\nabla} h)^2 + \eta$ Kardar, Parisi, Zhang 86
(b) EW $\partial_t h = \nu \boldsymbol{\nabla}^2 h + \eta$ Edwards, Wilkinson 82

η is a gaussian white noise with $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2 \nu T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

Family-Viscek scaling on a spatial lattice of extent L^d : $\overline{h}(t) = L^{-d} \sum_j h_j(t)$

FAMILY & VISCEK 85

$$w^2(t; L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \left\langle \left(h_j(t) - \overline{h}(t) \right)^2 \right\rangle = L^{2\alpha} f\left(tL^{-z} \right) \sim \begin{cases} L^{2\alpha} & \text{; if } tL^{-z} \gg 1 \\ t^{2\beta} & \text{; if } tL^{-z} \ll 1 \end{cases}$$

β : growth exponent, α : roughness exponent, $\boxed{\alpha = \beta z}$

two-time correlator :

$$C(t, s; \mathbf{r}) = \left\langle \left(h(t, \mathbf{r}) - \left\langle \overline{h}(t) \right\rangle \right) \left(h(s, \mathbf{0}) - \left\langle \overline{h}(s) \right\rangle \right) \right\rangle = s^{-b} F_C\left(\frac{t}{s}, \frac{\mathbf{r}}{s^{1/z}} \right)$$

with ageing exponent : $\boxed{b = -2\beta}$
KALLABIS & KRUG 96

expect for $\underline{y = t/s \gg 1}$: $F_C(y, \mathbf{0}) \sim y^{-\lambda_C/z}$ autocorrelation exponent

$1D$ relaxation dynamics, starting from an initially flat interface
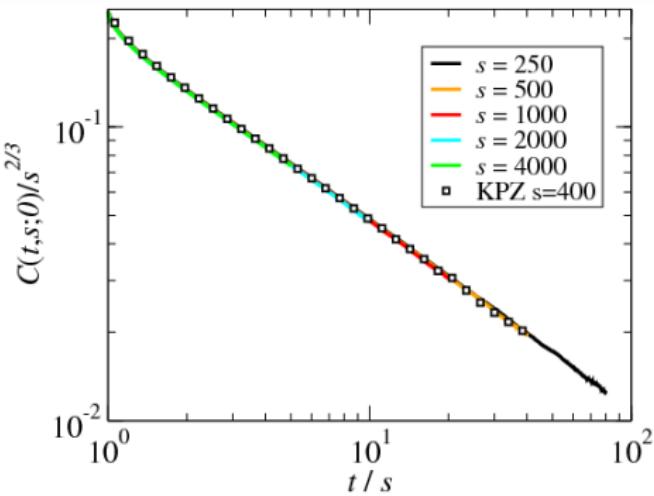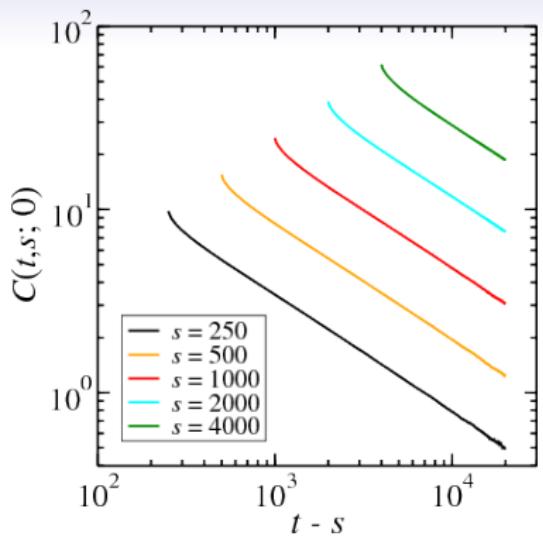
observe all **3** properties of **ageing** :
$$\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$$

confirm **simple ageing** for the $1D$ KPZ universality class

confirm expected exponents $b = -2/3$, $\lambda_C/z = 2/3$ *pars pro toto*

KALLABIS & KRUG 96 ; KRECH 97 ; BUSTINGORRY *et al.* 07-10 ; CHOU & PLEIMLING 10 ;
D'AQUILA & TÄUBER 11/12 ; MH, NOH, PLEIMLING 12 . . .

Values of some growth and ageing exponents in $1D$

model	z	a	b	$\lambda_R = \lambda_C$	β	α
KPZ	3/2	$-1/3$	$-2/3$	1	1/3	1/2
exp 1			$\approx -2/3^{\dagger}$	$\approx 1^{\dagger}$	0.336(11)	0.50(5)
exp 2	1.5(2)				0.32(4)	0.50(5)
EW	2	$-1/2$	$-1/2$	1	1/4	1/2

liquid crystals
(cancer) cell growth

Takeuchi, Sano, Sasamoto, Spohn 10/11/12

Huergo, Pasquale, Gonzalez, Bolzan, Arvia 12

$\dagger$ scaling holds only for flat interface

Two-time space-time responses and correlators consistent with
simple ageing for $1D$ KPZ

Similar results known for EW universality class

ROETHLEIN, BAUMANN, PLEIMLING 06

3. Interface growth & Arcetri models : heuristics

? KPZ $\longrightarrow$ intermediate model $\longrightarrow$ EW ?

preferentially exactly solvable, and this in $d \geq 1$ dimensions

inspiration : mean **spherical model** of a ferromagnet

BERLIN & KAC 52
LEWIS & WANNIER 52

Ising spins $\sigma_i = \pm 1$ obey $\sum_i \sigma_i^2 = \mathcal{N} = \#$ sites

spherical spins $S_i \in \mathbb{R}$ spherical constraint $\left\langle \sum_i S_i^2 \right\rangle = \mathcal{N}$

hamiltonian $\mathcal{H} = -J \sum_{(i,j)} S_i S_j - \lambda \sum_i S_i^2$ Lagrange multiplier λ

exponents **non**-mean-field for $2 < d < 4$ and $T_c > 0$ for $d > 2$

kinetics from Langevin equation

$$\partial_t \phi = -D \frac{\delta \mathcal{H}[\phi]}{\delta \phi} + \mathfrak{z}(t)\phi + \eta$$

time-dependent Lagrange multiplier $\mathfrak{z}(t)$ fixed from spherical constraint
all equilibrium and ageing exponents exactly known, for $T < T_c$ and $T = T_c$

RONCA 78, CONIGLIO & ZANNETTI 89, GODRÈCHE & LUCK '00, CORBERI, LIPPIELLO, FUSCO, GONNELLA & ZANNETTI 02-14 . . .

consider **RSOS**-adsorption process :

rigorous : continuum limit gives KPZ Bertini & Giacomin 97

$+ \quad + \quad - \quad + \quad + \quad - \quad - \quad + \quad - \quad + \quad -$
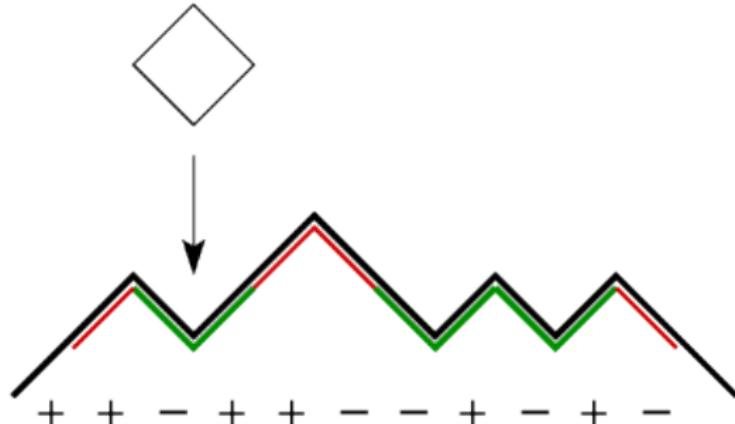
use **not** the heights $h_n(t) \in \mathbb{N}$ on a discrete lattice,
but rather the **slopes** $u_n(t) = \frac{1}{2}\left(h_{n+1}(t) - h_{n-1}(t)\right) = \pm 1$ **RSOS**

? let $u_n(t) \in \mathbb{R}$, & impose a spherical constraint $\boxed{\sum_n \langle u_n(t)^2 \rangle \overset{!}{=} \mathcal{N}}$ **?**

? consequences of the 'hardening' of a soft EW-interface by a 'spherical constraint' on the u_n **?**

KPZ equation for height $h(t,r)$:
$$\partial_t h = \nu \partial_r^2 h + \frac{\mu}{2}\left(\partial_x h\right)^2 + \eta$$

Burger's equation for slope $u(t,r) = \partial_r h(t,r)$:
$$\partial_t u = \nu \partial_r^2 u + \mu u \partial_r u + \partial_r \eta$$

model $\mathrm{A}I$:
$$\boxed{\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t) u + \partial_r \eta, \qquad \int \mathrm{d}r \left\langle u^2 \right\rangle \sim 1}$$
$$\mathfrak{z}(t) \sim \left\langle\!\left\langle \partial_r u \right\rangle\!\right\rangle \sim \text{curvature}$$

model $\mathrm{A}II$:
$$\boxed{\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t) \partial_r u + \partial_r \eta, \qquad \int \mathrm{d}r \left\langle u^2 \right\rangle \sim 1}$$
$$\mathfrak{z}(t) \sim \left\langle\!\left\langle u \right\rangle\!\right\rangle \sim \text{slope}$$

model $\mathrm{A}III$:
$$\boxed{\partial_t h = \nu \partial_r^2 h + \mathfrak{z}(t) \partial_r h + \eta, \qquad \int \mathrm{d}r \left\langle (\partial_r h)^2 \right\rangle \sim 1}$$
$$\mathfrak{z}(t) \sim \left\langle\!\left\langle \partial_r h \right\rangle\!\right\rangle \sim \text{slope}$$

? interface rough or smooth ?
? long-time properties and ageing behaviour ?
? does dynamical scaling resp. simple ageing always hold ?

4. First Arcetri model AI : simple ageing

slope $u(t,x) = \partial_x h(t,x)$ obeys Burgers' equation,
replace its non-linearity by a mean spherical condition $\Longrightarrow$

$$\partial_t u_n(t) = \nu\left(u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)\right) + \mathfrak{z}(t)u_n(t)$$
$$+ \frac{1}{2}\left(\eta_{n+1}(t) - \eta_{n-1}(t)\right)$$
$$\sum_n \left\langle u_n(t)^2 \right\rangle = N \qquad\qquad \langle\eta_n(t)\eta_m(s)\rangle = 2T\nu\delta(t-s)\delta_{n,m}$$

Extension to $d \geq 1$ dimensions : $\qquad\qquad\qquad \mathfrak{z}(t)$ Lagrange multiplier
define gradient fields $u_a(t,\mathbf{r}) := \nabla_a h(t,\mathbf{r})$, $a = 1,\ldots,d$:

$$\partial_t u_a(t,\mathbf{r}) = \nu\nabla_\mathbf{r}\cdot\nabla_\mathbf{r} u_a(t,\mathbf{r}) + \mathfrak{z}(t)u_a(t,\mathbf{r}) + \nabla_a\eta(t,\mathbf{r})$$
$$\sum_\mathbf{r}\sum_{a=1}^d \left\langle u_a(t,\mathbf{r})^2 \right\rangle = N^d$$

interface height : $\boxed{\widehat{u}_a(t,\mathbf{q}) = \mathrm{i}\sin q_a\, \widehat{h}(t,\mathbf{q})}$ $\qquad;\quad \mathbf{q}\neq\mathbf{0}$ in Fourier space

$$\omega(\mathbf{q}) = \sum_{a=1}^{d}(1 - \cos q_a), \qquad \mathbf{q} \neq \mathbf{0}$$

$$\widehat{h}(t, \mathbf{q}) = \widehat{h}(0, \mathbf{q})e^{-2t\omega(\mathbf{q})}\sqrt{\frac{1}{g(t)}} + \int_0^t \mathrm{d}\tau \, \widehat{\eta}(\tau, \mathbf{q})\sqrt{\frac{g(\tau)}{g(t)}} \, e^{-2(t-\tau)\omega(\mathbf{q})}$$

in terms of the auxiliary function $g(t) = \exp\left(-2\int_0^t \mathrm{d}\tau \, \mathfrak{z}(\tau)\right)$,

which satisfies Volterra equation

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau \, g(\tau)f(t-\tau) \;, \;\; f(t) := d\frac{e^{-4t}I_1(4t)}{4t}\left(e^{-4t}I_0(4t)\right)^{d-1}$$

* for $d=1$, identical to 'spherical spin glass', with $T = 2T_{\mathrm{SG}}$:
  hamiltonian $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}S_iS_j$; J_{ij} random matrix, its eigenvalues
  distributed according to Wigner's semi-circle law   CUGLIANDOLO & DEAN 95

a further auxiliary function : $F_{\mathbf{r}}(t) := \prod_{a=1}^{d} e^{-2t}I_{r_a}(2t)$ I_n : modified Bessel function
for initially uncorrelated heights and initially flat interface

height autocorrelator :

$C(t, s) = \langle h(t, \mathbf{r}) h(s, \mathbf{r}) \rangle_c = \frac{2 F_0(t+s)}{\sqrt{g(t) g(s)}} + \frac{2T}{\sqrt{g(t) g(s)}} \int_0^s \mathrm{d}\tau \, g(\tau) F_0(t + s - 2\tau)$

interface width : $w^2(t) = C(t, t) = \frac{2 F_0(2t)}{g(t)} + \frac{2T}{g(t)} \int_0^t \mathrm{d}\tau \, g(\tau) F_0(2t - 2\tau)$

slope autocorrelator :

$A(t, s) = \sum_{a=1}^d \langle u_a(t, \mathbf{r}) u_a(s, \mathbf{r}) \rangle_c = \frac{2 f((t+s)/2)}{\sqrt{g(t) g(s)}} + \int_0^s \mathrm{d}\tau \, \frac{2 T g(\tau)}{\sqrt{g(t) g(s)}} f((t+s)/2 - \tau)$

height response : $R(t, s; \mathbf{r}) = \left. \frac{\delta \langle h(t, \mathbf{r}) \rangle}{\delta j(s, \mathbf{0})} \right|_{j=0} = \Theta(t - s) \sqrt{\frac{g(s)}{g(t)}} \, F_{\mathbf{r}}(t - s)$

slope autoresponse : $Q(t, s; \mathbf{0}) = \Theta(t - s) \sqrt{\frac{g(s)}{g(t)}} \, f((t - s)/2)$

* correspondence of $1D$ $\mathrm{A}I$ **model with**
   spherical spin glass : $\boxed{\text{spins } S_i \leftrightarrow \text{slopes } u_n}$

spin glass autocorrelator $\boxed{C_{\mathrm{SG}}(t, s) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \overline{\langle S_i(t) S_i(s) \rangle} = A(t, s)}$

spin glass response $\boxed{R_{\mathrm{SG}}(t, s) = \sum_{i=1}^{\mathcal{N}} \left. \frac{\delta \overline{\langle S_i(t) \rangle}}{\delta h_i(s)} \right|_{h=0} = 2 Q(t, s)}$

* kinetics of heights $h_n(t)$ in model $\mathrm{A}I$ driven by phase-ordering of the
   spherical spin glass $\equiv 3D$ kinetic spherical model

phase transition : long-range correlated surface growth for $T \leq T_c$

$$\frac{1}{T_c(d)} = \frac{d}{2} \int_0^\infty \mathrm{d}t \; e^{-dt} t^{-1} I_1(t) I_0(t)^{d-1} \quad ; \quad T_c(1) = 2, \, T_c(2) = \frac{\pi}{\pi - 2}$$

Some results : $\boxed{\text{always simple ageing}}$ $\quad$ upper critical dimension $d^* = 2$

1. $T = T_c$, $d < 2$:

$\qquad$ **rough** interface, width $w(t) = t^{(2-d)/4} \Longrightarrow \beta = \frac{2-d}{4} > 0$

$\qquad$ ageing exponents $a = b = \frac{d}{2} - 1$, $\lambda_R = \lambda_C = \frac{3d}{2} - 1$; $z = 2$

$\boxed{\text{exponents } z, \beta, a, b \text{ same as } \text{EW}, \text{ but exponent } \lambda_C = \lambda_R \text{ different}}$

2. $T = T_c$, $d > 2$:

$\qquad$ **smooth** interface, width $w(t) = \text{cste.} \Longrightarrow \beta = 0$

$\qquad$ ageing exponents $a = b = \frac{d}{2} - 1$, $\lambda_R = \lambda_C = d$; $z = 2$

$\boxed{\text{same asymptotic exponents as } \text{EW}, \text{ but scaling functions are distinct}}$

3. $T < T_c$:

$\qquad$ **rough** interface, width $w^2(t) = (1 - T/T_c)t \Longrightarrow \beta = \frac{1}{2}$

$\qquad$ ageing exponents $a = \frac{d}{2} - 1$, $b = -1$, $\lambda_R = \lambda_C = \frac{d-2}{2}$; $z = 2$

Summary of results in the A/ model :

Captures at least some qualitative properites of growing interfaces.

* phenomenology of relaxation analogous to domain growth in simple magnets $\implies$ **dynamical scaling form of simple ageing**

* existence of a critical point $T_c(d) > 0$ for all $d > 0$ **as a magnet**

* at $T = T_c$, rough interface for $d < 2$, smooth interface for $d > 2$; upper critical dimension $d^* = 2$

* at $T = T_c$, $d < 2$, the stationary exponents (β, z) are those of EW, but the non-stationary ageing exponents are different

   explicit example for expectation from field-theory renormalisation group in domain growth of independent exponents $\lambda_{C,R}$

   different from EW and KPZ classes, where $\lambda_C = d$ for all $d < 2$ KRECH 97

* at $T = T_c$, $d > 2$, **distinct from** EW, although all exponents agree

* for $d = 1$, equivalent to $p = 2$ spherical spin glass

* at $T = T_c$ and $2 < d < 4$, same ageing behaviour as at the multicritical point of the bosonic pair-contact process with diffusion (BPCPD)

* distinct universality class for $T < T_c$

5. Second Arcetri model All : several length scales

$$\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t)\partial_r u + \partial_r \eta, \qquad \int \mathrm{d}r \, \langle u^2 \rangle \sim 1$$

requirement : stationary solution should remain roughly flat

but find $\nu u'' + \mathfrak{z} u' = 0 \implies u = u^{(0)} + u^{(1)} e^{-(\mathfrak{z}/\nu)r}$ exponential growth **?**

N.B. : equation of motion couples even and odd contributions to slope profile

decompose $\boxed{u(t,r) = a(t,r) + b(t,r)}$

with $\underline{a(t,r) = a(t,-r)}$ **even** and $\underline{b(t,r) = -b(t,-r)}$ **odd**

gives $\nu a'' + \mathfrak{z}b' = 0$, $\nu b'' + \mathfrak{z}a' = 0 \implies$ exponential growth as $r \to \pm\infty$ **?**

$u(t,r) = a(t,r) + b(t,r)$ with a even and b odd

construct pair of equations of motion, with an **important modification**

$$\partial_t a(t,r) = \nu \partial_r^2 a(t,r) + \mathfrak{z}(t)\partial_r b(t,r) + \partial_r \eta^-(t,r)$$
$$\partial_t b(t,r) = \nu \partial_r^2 b(t,r) - \mathfrak{z}(t)\partial_r a(t,r) - \partial_r \eta^+(t,r)$$
$$\langle \sum_r (a(t,r) + b(t,r))^2 \rangle = \mathcal{N}$$

with symmetrised noise $\eta^\pm(t,r) = \frac{1}{2}\left(\eta(t,r) \pm \eta(t,-r)\right)$

These are the defining equations of the model A*II*

gives $\nu a'' + \mathfrak{z}b' = 0$, $\nu b'' - \mathfrak{z}a' = 0 \implies \nu^2 a''' = -\mathfrak{z}^2 a'$, $\nu^2 b''' = -\mathfrak{z}^2 b'$
$\implies$ profiles remain bounded as $r \to \pm\infty$!

analogous procedure for third Arcetri model A*III*

initial condition :

interface flat on average, initial slopes uncorrelated,
spherical constraint respected

<u>work out spherical constraint</u> : let $Z(t) := \int_0^t \mathrm{d}\tau \, \mathfrak{z}(\tau)$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}k \, \cosh(2\sin k Z(t)) e^{-4\nu\omega(k)t}$$

$$+ \quad \frac{\nu T}{\pi} \int_{-\pi}^{\pi} \mathrm{d}k \, \sin^2 k \int_0^t \mathrm{d}\tau \, \cosh(2\sin k(Z(t) - Z(\tau))) e^{-4\nu\omega(k)(t-\tau)} =$$

concentrate on case $T = 0$: dynamics driven by initial fluctuations

much as in phase-ordering kinetics in simple magnets

spherical constraint : $\boxed{e^{4\nu t} = I_0(\sqrt{(4\nu t)^2 + (2Z(t))^2}\,)}$

asymptotic solution for $t \gg 1$: $Z(t) \simeq (\nu t \ln(\pi\nu t))^{1/2}$

slope response

$$R_{x,y}(t,s) = \left\langle \left. \frac{\partial a(t,x)}{\partial j^{+}(s,y)} \right|_{j=0} \right\rangle + \left\langle \left. \frac{\partial b(t,x)}{\partial j^{-}(s,y)} \right|_{j=0} \right\rangle$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}k \ \sin k e^{-2\omega(k)(t-s)} \sinh(\sin k(Z(t) - Z(s))) \cos k(x - y)$$

slope correlator

$$C_{x,y}(t,s) = \langle a(t,x)a(s,y) + b(t,x)b(s,y) \rangle$$

$$= \frac{1}{2\pi} \int_{\pi}^{\pi} \mathrm{d}k \ e^{-2\omega(k)(t+s)} \cosh(\sin k(Z(t) + Z(s))) \cos k(x - y)$$

both can be evaluated as sums of modified Bessel functions

analysis of the long-time scaling behaviour

it turns out that simple ageing is not obeyed !

rather, consider as a scaling variable $\boxed{\tau := t - s = ys \ln^{-\varsigma} \pi s}$

$\boxed{\text{scaling limit } t, s \to \infty \text{ with } y \text{ fixed and } \varsigma > 0}$ *'logarithmic sub-ageing'*

use $Z(t) \simeq \sqrt{t \ln \pi t}$ for $t \to \infty$:

slope autocorrelator $C(t,s) = C_{0,0}(t,s)$

$$
\begin{aligned}
C(t,s) &= \frac{I_0\left(2(t+s)\sqrt{(1 + (Z(t) + Z(s))^2/(2(t+s))^2}\,\right)}{I_0\left(2(t+s)\sqrt{1 + Z^2((t+s)/2)}\,\right)} \\
&\simeq \exp\left(-\frac{y^2}{32} \ln^{1-2\varsigma} \pi s\right)
\end{aligned}
$$

* try simple ageing $\varsigma = 0 : \implies$ no data collapse & multiscaling !
* only find dynamical scaling if $\varsigma = \frac{1}{2} > 0$
* same sub-ageing behaviour as in the $2D$ spherical magnet with conserved order parameter

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slope autoresponse $R(t,s) = R_{0,0}(t,s)$

$$R(t,s) \simeq \sqrt{\frac{2}{\pi}} \, s^{-1} y^{-3/2} \ln^{1+3\varsigma/2} \pi s$$

* looks very similar to simple ageing
* but additional logarithmic factor breaks dynamical scale-invariance

spatial equal-time correlator $C_n(t) = C_{n,0}(t,t)$

$$
\begin{aligned}
C_n(t) &= \frac{I_n\left(4t\sqrt{1+Z^2(t)/4t^2}\right)\cos\left(n\arctan Z(t)/2t\right)}{I_0\left(4t\sqrt{1+Z^2(t)/4t^2}\right)} \\
&\simeq \exp\left(-\left(\frac{n}{\sqrt{8t}}\right)^2\right)\cos\left(\frac{n}{\sqrt{2t/\ln \pi t}}\right)
\end{aligned}
$$

* find two marginally different length scales
* simple scaling ansatz leads to **multiscaling**
* analogue : spherical magnet at $T = 0$, conserved order-parameter CONIGLIO & ZANNETTI 89
  but the A/I model does not have a macroscopic conservation law !

6. Conclusions

* long-time dynamics of growing interfaces naturally evolves towards
    dynamical scaling & ageing
* phenomenology very similar to ageing phenomena in simple magnets
* subtleties in the precise scaling forms
* exactly solvable model with proven sub-ageing, although the $A||$ does
    not have a macroscopic conservation law !

> **proving dynamical symmetries can remain a delicate affair** !