Algebraic structures related to sets of Hahn polynomials and the corresponding oscillator models

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Outline

1. Introduction
   - Definitions and notations
   - Introductory example
   - Classification
2. Applications
   - Finite quantum oscillators
   - Eigenvalue testmatrices
   - “New” polynomials

Definitions and notations

Hahn polynomial of degree $n \in \{0, 1, \ldots, N\}$

$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -n, \; n+\alpha+\beta+1, \; -x \\ \alpha+1, \; -N \end{matrix} ; 1 \right)$$

Generalized hypergeometric series:

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}$$

Pochhammer symbol $(a)_0 = 1$

$$(a)_k = a(a+1)\cdots(a+k-1)$$

Three-term recurrence relation and difference equation

Definitions and notations

Discrete orthogonality relation

$$\sum_{x=0}^{N} w(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N)Q_{n'}(x;\alpha,\beta,N) = h_n(\alpha,\beta,N)\,\delta_{n,n'}$$

with $\alpha, \beta > -1$ (or $< -N$)

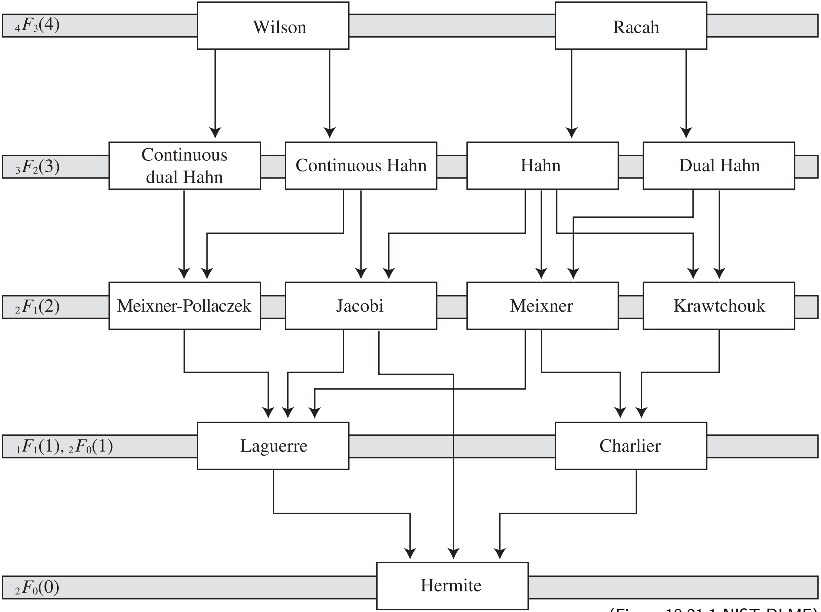
$$w(x;\alpha,\beta,N) = \binom{\alpha+x}{x}\binom{N+\beta-x}{N-x} \qquad (x = 0,1,\ldots,N)$$

$$h_n(\alpha,\beta,N) = \frac{(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(N-n+1)_n N!}$$

orthonormal Hahn functions

$$\tilde{Q}_n(x;\alpha,\beta,N) \equiv \frac{\sqrt{w(x;\alpha,\beta,N)}\,Q_n(x;\alpha,\beta,N)}{\sqrt{h_n(\alpha,\beta,N)}}$$

Askey-scheme of hypergeometric orthogonal polynomials



(Figure 18.21.1 NIST DLMF)

Introductory example

Lie algebra $\mathfrak{su}(2)$ with basis J_0, J_+, J_-

$$[J_0, J_\pm] = \pm J_\pm \qquad {J_0}^\dagger = J_0$$
$$[J_+, J_-] = 2J_0 \qquad {J_\pm}^\dagger = J_\mp$$

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deformation of $\mathfrak{su}(2)$

$$[J_0, J_\pm] = \pm J_\pm \qquad {J_0}^\dagger = J_0$$
$$[J_+, J_-] = 2J_0 + c\,P \qquad {J_\pm}^\dagger = J_\mp$$

parameter c and P parity operator

$$P^2 = 1 \qquad [P, J_0] = 0 \qquad \{P, J_\pm\} = 0 \qquad P^\dagger = P$$

$\cong$ Bannai-Ito algebra 1 parameter

Unitary finite-dimensional irreps

On basis $|j,-j\rangle$, $|j,-j+1\rangle$, …, $|j,j\rangle$ with $\epsilon = \pm 1$

odd dim, j integer

$$P|j,m\rangle = \epsilon(-1)^{j+m}\,|j,m\rangle \qquad \hat{c} = c\,\epsilon/(2j+1)$$

$$J_0|j,m\rangle = (m - \hat{c}/2)\,|j,m\rangle$$

$$J_\pm|j,m\rangle = \begin{cases} \sqrt{(j \mp m \pm \hat{c})(j \pm m + 1)}\,|j,m\pm 1\rangle & \text{if } j+m \text{ odd} \\ \sqrt{(j \mp m)(j \pm m + 1 \mp \hat{c})}\,|j,m\pm 1\rangle & \text{if } j+m \text{ even} \end{cases}$$

even dim, j half-integer

$$P\,|j,m\rangle = \epsilon(-1)^{j+m+1}\,|j,m\rangle,$$

$$J_0\,|j,m\rangle = m\,|j,m\rangle$$

$$J_\pm|j,m\rangle = \begin{cases} \sqrt{(j \mp m)(j \pm m + 1)}\,|j,m\pm 1\rangle & \text{if } j \pm m \text{ odd} \\ \sqrt{(j \mp m)(j \pm m + 1) + c\,\epsilon}\,|j,m\pm 1\rangle & \text{if } j \pm m \text{ even} \end{cases}$$

Finite oscillator model

Finite-dim represenation of $\mathfrak{su}(2)$ deformation

$$[J_0, J_\pm] = \pm J_\pm \qquad {J_0}^\dagger = J_0$$
$$[J_+, J_-] = 2J_0 + c\,P \qquad {J_\pm}^\dagger = J_\mp$$

Position, momentum and Hamiltonian operator

$$\hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{i}{2}(J_+ - J_-), \qquad \hat{H} = J_0 + j + \frac{\hat{c}+1}{2}$$

are self-adjoint and satisfy

$$[\hat{H}, \hat{q}] = -i\hat{p} \qquad [\hat{H}, \hat{p}] = i\hat{q} \qquad (1 = \hbar = m = \omega)$$

On $|j, -j\rangle$, $|j, -j+1\rangle$, ..., $|j, j\rangle$ the spectrum of $\hat{H}$ is linear

$$n + \frac{1}{2} \qquad (n = 0, 1, \ldots, 2j)$$

Matrix form position operator

On $|j, m\rangle$

$$2\hat{q} = \begin{pmatrix} 0 & M_0 & 0 & \cdots & 0 \\ M_0 & 0 & M_1 & \cdots & 0 \\ 0 & M_1 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & M_{2j-1} \\ 0 & 0 & & M_{2j-1} & 0 \end{pmatrix}$$

with

$$M_k = \begin{cases} \sqrt{(k+1-\hat{c})(2j-k)} & \text{if } k \text{ odd} \\ \sqrt{(k+1)(2j-k+\hat{c})} & \text{if } k \text{ even} \end{cases}$$

Eigenvalues of the position operator $\hat{q}$

$$-j, -j+1, \ldots, -1, 0, 1, 2, \ldots, j-1, j$$

How?

Introductory example

Matrix

$$M = \begin{pmatrix} 0 & M_0 & & & \\ M_0 & 0 & M_1 & & \\ & M_1 & 0 & M_2 & \\ & & M_2 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

with

$$M_k = \begin{cases} \sqrt{(k+1+b)(2j-k)} & \text{if } k \text{ odd} \\ \sqrt{(k+1)(2j-k+a)} & \text{if } k \text{ even} \end{cases}$$

Eigenvalues

$$\pm\sqrt{k(k+a+b)} \quad \text{for } k \in \{0, \dots, j\}$$

Special case: $b = -a$

Introductory example

Eigenvectors

$$\boxed{MU = UD}$$

Orthogonal matrix $U = (U_{rs})_{0 \leq r,s \leq 2j}$

$$a = 2\alpha + 1$$
$$b = 2\beta + 1$$

$$U_{2r,j-s} = U_{2r,j+1+s} = \frac{(-1)^r}{\sqrt{2}} \tilde{Q}_s(r; \alpha, \beta+1, j)$$

$$U_{2r+1,j-s} = -U_{2r+1,j+1+s} = -\frac{(-1)^r}{\sqrt{2}} \tilde{Q}_s(r; \alpha+1, \beta, j))$$

Rows alternate Hahn polynomials with different parameters

$$\tilde{Q}_n(x; \alpha, \beta, N) \quad \text{and} \quad \tilde{Q}_n(x; \hat{\alpha}, \hat{\beta}, N)$$

eigenvectors of two-diagonal matrix with simple eigenvalues

Classification

Hahn polynomials with different parameters

$$\tilde{Q}_n(x) \equiv \tilde{Q}_n(x; \alpha, \beta, N) \quad \text{and} \quad \hat{Q}_n(x) \equiv \tilde{Q}_n(x; \hat{\alpha}, \hat{\beta}, N)$$

combined into new set with
- two-diagonal Jacobi matrix
- simple eigenvalues

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Underlying mechanism: pair of relations

$$\hat{a}(x)\, \hat{Q}_n(x-1) + \hat{b}(x)\, \hat{Q}_n(x) = d(n)\, \tilde{Q}_n(x)$$
$$a(x)\, \tilde{Q}_n(x) + b(x)\, \tilde{Q}_n(x+1) = \hat{d}(n)\, \hat{Q}_n(x)$$

Key property: coeff LHS no n, RHS no x

Classification

Hahn I $\hat{Q}_n(x) \equiv Q_n(x;\alpha+1,\beta+1,N-1)$

$$Q_n(x) - Q_n(x+1) = \frac{n(n+\alpha+\beta+1)}{N(\alpha+1)}\hat{Q}_{n-1}(x)$$

$$\begin{array}{l}(N-x-1)(x+\alpha+2)\hat{Q}_{n-1}(x+1)\\ -(x+1)(N-x+\beta)\hat{Q}_{n-1}(x)\end{array} = N(\alpha+1)Q_n(x+1)$$

Hahn II $\hat{Q}_n(x) \equiv Q_n(x;\alpha,\beta,N-1)$

$$\begin{array}{c}(x-\beta-N)Q_n(x)\\ -(x+\alpha+1)Q_n(x+1)\end{array} = -\frac{(N-n)(n+\alpha+\beta+N+1)}{N}\hat{Q}_n(x)$$

$$(x+1)\hat{Q}_n(x) - (x-N+1)\hat{Q}_n(x+1) = NQ_n(x+1)$$

Hahn III $\hat{Q}_n(x) \equiv Q_n(x;\alpha+1,\beta-1,N)$

$$-(x-\beta-N)Q_n(x) + (x-N)Q_n(x+1) = \frac{(n+\alpha+1)(n+\beta)}{(\alpha+1)}\hat{Q}_n(x)$$

$$-(x+1)\hat{Q}_n(x) + (x+\alpha+2)\hat{Q}_n(x+1) = (\alpha+1)Q_n(x+1)$$

Algebraic structures

$$J_+ = 2\begin{pmatrix} 0 & 0 & & \\ M_0 & 0 & 0 & \\ & M_1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

$$J_- = \text{upper diagonal}$$

$$J_0 = \mathrm{diag}(-N, \dots, N)$$

$$P = \mathrm{diag}(1, -1, 1, -1, \dots)$$

These matrices satisfy

$$[J_0, J_\pm] = \pm J_\pm \qquad P^2 = 1 \qquad PJ_0 = J_0 P \qquad PJ_\pm = -J_\pm P$$

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Hahn I

$$[J_+, J_-] = 2J_0 + 2(\alpha+\beta+1)J_0P - (2N+1)(\alpha-\beta)P + (\alpha-\beta)I$$

Hahn II

$$[J_+, J_-] = -2J_0 + 2(\alpha+\beta+2N+1)J_0P + (2N+1)(\alpha-\beta)P - (\alpha-\beta)I$$

Hahn III

$$[J_+, J_-] = 2J_0 + 2(\alpha-\beta)J_0P - c(\alpha,\beta,2N+2)P + (\alpha-\beta)I$$

Special cases

Representations of deformations $\mathfrak{su}(2)$

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$\alpha = -\frac{1}{2} = \beta \rightarrow$ reduce to $\mathfrak{su}(2)$

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$\alpha = \beta \rightarrow$ deformation of $\mathfrak{su}(2)$

$$[J_0, J_\pm] = \pm J_\pm \qquad {J_0}^\dagger = J_0$$
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parameter c and P parity operator

$$P^2 = 1 \qquad [P, J_0] = 0 \qquad \{P, J_\pm\} = 0 \qquad P^\dagger = P$$

Finite oscillator model

Position, momentum and Hamiltonian operator

$$\hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{i}{2}(J_+ - J_-), \qquad \hat{H} = J_0 + X$$

are self-adjoint and satisfy

$$[\hat{H}, \hat{q}] = -i\hat{p} \qquad [\hat{H}, \hat{p}] = i\hat{q} \qquad (1 = \hbar = m = \omega)$$

$\hat{H}$ linear spectrum

$$n + \frac{1}{2} \qquad (n = 0, 1, \ldots, 2j)$$

Matrix form position operator

$$\hat{q} = \begin{pmatrix} 0 & M_0 & 0 & \cdots & 0 \\ M_0 & 0 & M_1 & \cdots & 0 \\ 0 & M_1 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & M_{2j-1} \\ 0 & 0 & & M_{2j-1} & 0 \end{pmatrix}$$

Eigenvalue testmatrices

Sylvester-Kac matrix, the Kac matrix, the Clement matrix, ...

$$\begin{pmatrix} 0 & 1 & & & & \\ N & 0 & 2 & & & \\ & N-1 & 0 & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & 0 & N \\ & & & & 1 & 0 \end{pmatrix}$$

Tridiagonal

zero diagonal

simple entries

Eigenvalues

$$-N, -N+2, -N+4, \ldots, N-4, N-2, N$$

Eigenvalue testmatrices

N even: Extension with parameter a

$$\begin{pmatrix} 0 & 1+a & & & & \\ N & 0 & 2 & & & \\ & N-1-a & 0 & 3+a & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & 0 & N \\ & & & & 1-a & 0 \end{pmatrix}$$

Tridiagonal

zero diagonal

simple entries

Eigenvalues

$$-N, -N+2, -N+4, \ldots, N-4, N-2, N$$

Eigenvalue testmatrices

N even: Extension two parameters

$$\begin{pmatrix} 0 & 1+a & & & & & \\ N & 0 & 2 & & & & \\ & N-1+b & 0 & 3+a & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3+b & 0 & N-1+a & \\ & & & & 2 & 0 & N \\ & & & & & 1+b & 0 \end{pmatrix}$$

Eigenvalues

$$0, \pm\sqrt{(2k)(2k+a+b)} \qquad \text{for } k \in \{1, \ldots, N/2\}$$

Special case: $b = -a$

Eigenvalue testmatrices

N odd: Extension two parameters

$$\begin{pmatrix} 0 & 1+a & & & & & \\ N+b & 0 & 2 & & & & \\ & N-1 & 0 & 3+a & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3+b & 0 & N-1 & \\ & & & & 2 & 0 & N+a \\ & & & & & 1+b & 0 \end{pmatrix}$$

Eigenvalues

$$\pm\sqrt{(2k+1+a)(2k+1+b)} \quad \text{for } k \in \{0,\ldots,(N-1)/2\}$$

Special case: $b = a$

New polynomials

Let $\gamma > -1$, $\delta > -1$, and consider the $2N+2$ polynomials

$$P_{2n}(q) = \frac{(-1)^n}{\sqrt{2}} Q_n(q^2 - \gamma - 1; \gamma, \delta, N)$$

$$P_{2n+1}(q) = -\frac{(-1)^n}{\sqrt{2}} \sqrt{\frac{(n+\gamma+1)(n+\gamma+\delta+1)(2n+2+\gamma+\delta)}{(n+N+\gamma+\delta+2)(2n+\gamma+\delta+1)}} \times \frac{1}{(\gamma+1)} q\, Q_n(q^2 - \gamma - 1; \gamma+1, \delta, N)$$

satisfy discrete orthogonality relation

$$\sum_{q \in S} \binom{q^2 - 1}{q^2 - \gamma - 1} \binom{N - q^2 + \gamma + \delta + 1}{N - q^2 + \gamma + 1} P_n(q) P_{n'}(q) = h_{\lfloor n/2 \rfloor}(\gamma, \delta, N) \delta_{n,n'}$$

with

$$S = \{-\sqrt{N+\gamma+1}, \ldots, -\sqrt{\gamma+1}, \sqrt{\gamma+1}, \ldots, \sqrt{N+\gamma+1}\}$$

New polynomials

Let $\gamma > -1$, $\delta > -1$, and consider the $2N+1$ polynomials:

$$P_{2n}(q) = \frac{(-1)^n}{\sqrt{2}} Q_n(q^2; \gamma, \delta, N)$$

$$P_{2n+1}(q) = -\frac{(-1)^n}{\sqrt{2}} \sqrt{\frac{(n+\gamma+\delta+1)(2n+\gamma+\delta+2)}{(2n+\gamma+\delta+1)}} \times \frac{\sqrt{(N-n)(n+\gamma+1)}}{(\gamma+1)N} q \, Q_n(q^2-1; \gamma+1, \delta, N-1)$$

satisfy discrete orthogonality relation

$$\sum_{q\in S} \binom{q^2+\gamma}{q^2} \binom{N-q^2+\delta}{N-q^2} (1+\delta_{q,0}) P_n(q) P_{n'}(q) = h_{\lfloor n/2 \rfloor}(\gamma, \delta, N) \delta_{n,n'}$$

with

$$S = \{-\sqrt{N}, -\sqrt{N-1}, \ldots, -1, 0, 1, \ldots, \sqrt{N-1}, \sqrt{N}\}$$

Conclusion

Classification

- Hahn
- Dual Hahn
- Racah

Applications

- Algebraic structure for oscillator models
- Eigenvalue testmatrices
- New polynomials

Thank you