About Filiform Lie Algebras of order 3

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Outline

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1. Introduction

Filiform Lie superalgebras

The definition of *filiform Lie superalgebras* is well known, a class of nilpotent Lie superalgebras with important properties.

In particular every filiform Lie superalgebra can be obtained by a deformation of the model filiform Lie superalgebra $L^{n,m}$.

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Lie algebras of order F

- A classical use of the generalizations of Lie theory is in the study of symmetries in physics. Nowadays, symmetries are not limited to the geometrical ones of space-time, because there are other new ones associated with internal degrees of freedom of particles and fields.
- Thus the generalizations of Lie theory that have been proven to be physically relevant are, among others, color Lie (super)algebras and **Lie algebras of order F**.

Outline

V is $\mathbb{Z}_2-$graded $\quad V = V_{\bar{0}} \oplus V_{\bar{1}}$.
$X \in V$ *homogeneous of degree* $\alpha \;\; X \in V_\alpha$
$X \in V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are also called even (resp. odd).

Definition

A **Lie superalgebra** ($\mathfrak{g}$,[.,.]) $\mathfrak{g}$ $\mathbb{Z}_2$-graded vector space,
$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, bracket product [,] verifying:

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta(mod2)} \qquad \alpha, \beta \in \mathbb{Z}_2$.
- $[X, Y] = -(-1)^{\alpha\cdot\beta}[Y, X] \qquad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
- $(-1)^{\gamma\alpha}[X,[Y,Z]]+(-1)^{\alpha\beta}[Y,[Z,X]]+(-1)^{\beta\gamma}[Z,[X,Y]] = 0$
  $X \in \mathfrak{g}_\alpha, \;\; Y \in \mathfrak{g}_\beta, \;\; Z \in \mathfrak{g}_\gamma \;\; \alpha, \beta, \gamma \in \mathbb{Z}_2$

graded Jacobi identity $(J_g(X, Y, Z))$

Descending Central Sequence is a sequence defined by

$$(\mathcal{C}^k(\mathfrak{g})),\ k \in \mathbb{N} \cup \{0\}$$
$$\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$$
$$\mathcal{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})], \quad i \in \mathbb{N}$$

$\mathfrak{g}$ is **Nilpotent** $\Leftrightarrow \exists m \in \mathbb{N} : \ \mathcal{C}^m(\mathfrak{g}) = \{0\}, \mathcal{C}^{m-1}(\mathfrak{g}) \neq \{0\}$

$\mathfrak{g}$ is Nilpotent if the descending central sequence is stabilized in zero.

The smallest integer verifying this condition is called the **Nilindex**. (Nilindex m)

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Definition

Let $F \in \mathbb{N}^*$. A $\mathbb{Z}_F$-graded $\mathbb{C}$-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_{F-1}$ is called a complex **Lie algebra of order** F if the following hold:

(1) $\mathfrak{g}_0$ is a complex Lie algebra.

(2) For all $i = 1, \ldots, F-1$, $\mathfrak{g}_i$ is a representation of $\mathfrak{g}_0$. If $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_i$, then $[X, Y]$ denotes the action of $X \in \mathfrak{g}_0$ on $Y \in \mathfrak{g}_i$ for all $i = 1, \ldots, F-1$.

(3) For all $i = 1, \ldots, F-1$, there exists an F-Linear, $\mathfrak{g}_0$-equivariant map, $\{\cdots\} : \mathcal{S}^{\mathcal{F}}(\mathfrak{g}_i) \longrightarrow \mathfrak{g}_0$, where $\mathcal{S}^{\mathcal{F}}(\mathfrak{g}_i)$ denotes the F-fold symmetric product of $\mathfrak{g}_i$.

(4) For all $X_i \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_k$, the following “Jacobi identities" hold:

Definition

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0. \quad (1)$$

$$[[X_1, X_2], Y_3] + [[X_2, Y_3], X_1] + [[Y_3, X_1], X_2] = 0. \quad (2)$$

$$[X, \{Y_1, \dots, Y_F\}] = \{[X, Y_1], \dots, Y_F\} + \dots + \{Y_1, \dots, [X, Y_F]\}. \quad (3)$$

$$\sum_{j=1}^{F+1} [Y_j, \{Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_{F+1}\}] = 0. \quad (4)$$

Corollary

We observe that a Lie algebra of order 1 *it is just a Lie algebra and a Lie algebra of order* 2 *it is a Lie superalgebra. Thus, Lie algebras of order* F *can be seen as a generalization of Lie algebras and superalgebras.*

Theorem

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F, with $F > 1$. For any $i = 1, \ldots, F-1$, the subspaces $\mathfrak{g}_0 \oplus \mathfrak{g}_i$ inherits the structure of a Lie algebra of order F. We call these type of algebras **elementary Lie algebras of order** *F.*

We will restrict our study to elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Definition

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an elementary Lie algebra of order 3 and let $A = (\mathfrak{g}_0 \wedge \mathfrak{g}_0) \oplus (\mathfrak{g}_0 \wedge \mathfrak{g}_1) \oplus S^3(\mathfrak{g}_1)$. The linear map $\psi : A \longrightarrow \mathfrak{g}$ is called an **infinitesimal deformation** (Gerstenhaber deformations) of $\mathfrak{g}$ if it satisfies

$$\mu \circ \psi + \psi \circ \mu = 0$$

and

$$\psi \circ \psi = 0$$

with μ representing the law of $\mathfrak{g}$.

Outline

Definition

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F. $\mathfrak{g}_i$ is called a **$\mathfrak{g}_0$-filiform module** if there exists a decreasing subsequence of vector subspaces in its underlying vectorial space V, $V = V_m \supset \cdots \supset V_1 \supset V_0$, with dimensions $m, m-1, \ldots 0$, respectively, $m > 0$, and such that $[\mathfrak{g}_0, V_{i+1}] = V_i$.

Definition

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F. Then $\mathfrak{g}$ is a **filiform Lie algebra of order** F if the following conditions hold:

(1) $\mathfrak{g}_0$ is a filiform Lie algebra.

(2) $\mathfrak{g}_i$ has structure of $\mathfrak{g}_0$-filiform module, for all i, $1 \leq i \leq F-1$

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ADAPTED BASIS

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie algebra of order 3. If $\mathfrak{g}$ is a **filiform** *Lie algebra of order 3, then there exists an adapted basis of $\mathfrak{g}$, namely $\{X_0, \dots, X_n, Y_1, \dots, Y_m, Z_1, \dots, Z_p\}$ with $\{X_0, X_1, \dots, X_n\}$ a basis of $\mathfrak{g}_0$, $\{Y_1, \dots, Y_m\}$ a basis of $\mathfrak{g}_1$ and $\{Z_1, \dots, Z_p\}$ a basis of $\mathfrak{g}_2$, such that:*

$$\begin{cases} [X_0, X_i] = X_{i+1}, \ [X_0, Y_j] = Y_{j+1}, \ [X_0, Z_k] = Z_{k+1}, \\ [X_i, X_j] = \sum_{k=0}^{n} C_{ij}^k X_k, \ [X_i, Y_j] = \sum_{k=1}^{m} D_{ij}^k Y_k, \ [X_i, Z_j] = \sum_{k=1}^{p} E_{ij}^k Z_k, \\ \{Y_i, Y_j, Y_l\} = \sum_{k=0}^{n} F_{ijl}^k X_k, \ \{Z_i, Z_j, Z_l\} = \sum_{k=0}^{n} G_{ijl}^k X_k, \end{cases}$$

X_0 is called the characteristic vector.

The model filiform Lie algebra of order 3

$$\mu_0 : \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1 \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1 \\ [X_0, Z_k] = Z_{k+1} & 1 \leq k \leq p-1 \end{cases}$$

with $\{X_0, X_1, \ldots, X_n, \; Y_1, \ldots, Y_m, \; Z_1, \ldots, Z_p\}$ a basis of μ_0.

$\Downarrow$

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Definition

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an elementary Lie algebra of order 3 and let $A = (\mathfrak{g}_0 \wedge \mathfrak{g}_0) \oplus (\mathfrak{g}_0 \wedge \mathfrak{g}_1) \oplus S^3(\mathfrak{g}_1)$. The linear map $\psi : A \longrightarrow \mathfrak{g}$ is called a **pre-infinitesimal deformation** of $\mathfrak{g}$ if it satisfies

$$\mu \circ \psi + \psi \circ \mu = 0$$

with μ representing the law of $\mathfrak{g}$.

$Z(\mu_0)$ = all the pre-infinitesimal deformations of μ_0 that vanish on the characteristic vector X_0

$$\begin{aligned} Z(\mu_0) &= Z(\mu_0) \cap Hom(\mathfrak{g}_0 \wedge \mathfrak{g}_0, \mathfrak{g}_0) \oplus Z(\mu_0) \cap Hom(\mathfrak{g}_0 \wedge \mathfrak{g}_1, \mathfrak{g}_1) \\ &\quad \oplus Z(\mu_0) \cap Hom(S^3(\mathfrak{g}_1), \mathfrak{g}_0) \\ &:= A \oplus B \oplus C \end{aligned}$$

Then, as the vector space of pre-infinitesimal deformations called $Z(\mu_0)$ is equal to $A \oplus B \oplus C$ we will restrict our study to each vector subspace. Of all of them, **the most important vector subspace will be C because any pre-infinitesimal deformation ψ belonging to C verifies that $\psi \circ \psi = 0$, i.e. ψ is an infinitesimal deformation.** Thus, $\mu_0 + \psi$ will be a filiform elementary Lie algebra of order 3 with $\psi \in C$.

We have obtained the dimension and a basis of C:

- for n arbitrary and $m = 3$
- for $m = 4$ and n even.

Outline

$\mathfrak{sl}(2,\mathbb{C})$-module Method

$\mathfrak{sl}(2,\mathbb{C}) =< X_-, H, X_+ >$ with the following commutation relations:

$$[X_+, X_-] = H, \quad [H, X_+] = 2X_+, \quad [H, X_-] = -2X_-$$

Let V be a n-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module, $V =< e_1, \ldots, e_n >$. Then, up to isomorphism there exists a unique structure of an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module in V given in a basis $e_1, \ldots, e_n$ as follows:

$$\begin{cases} X_+ \cdot e_i = e_{i+1}, & 1 \leq i \leq n-1, \\ X_+ \cdot e_n = 0, & \\ H \cdot e_i = (-n+2i-1)e_i, & 1 \leq i \leq n. \end{cases}$$

e_n is the maximal vector of V and its weight, called the highest weight of V, is equal to $n-1$.

$\mathfrak{sl}(2,\mathbb{C})$-module Method

Let $V_0, V_1, \ldots, V_k$ be $\mathfrak{sl}(2,\mathbb{C})$-modules, then the space $\mathrm{Hom}(\otimes_{i=1}^k V_i, V_0)$ is a $\mathfrak{sl}(2,\mathbb{C})$-module in the following natural manner:

$$(\xi\cdot\varphi)(x_1,\ldots,x_k) = \xi\cdot\varphi(x_1,\ldots,x_k) - \sum_{i=1}^{i=k} \varphi(x_1,\ldots,\xi\cdot x_i, x_{i+1},\ldots,x_n)$$

with $\xi \in \mathfrak{sl}(2,\mathbb{C})$ and $\varphi \in \mathrm{Hom}(\otimes_{i=1}^k V_i, V_0)$. An element $\varphi \in \mathrm{Hom}(V_1 \otimes V_1 \otimes V_1, V_0)$ is said to be invariant if $X_+ \cdot \varphi = 0$, i.e.

$$X_+\cdot\varphi(x_1,x_2,x_3) - \varphi(X_+\cdot x_1,x_2,x_3) - \varphi(x_1,X_+\cdot x_2,x_3) - \varphi(x_1,x_2,X_+\cdot x_3) = 0 \quad (5)$$

$\forall x_1, x_2, x_3 \in V_1$.

Note that $\varphi \in \mathrm{Hom}(V_1 \otimes V_1 \otimes V_1, V_0)$ is invariant if and only if φ is a maximal vector.

$\mathfrak{sl}(2,\mathbb{C})$-module Method

Let $V_0, V_1, \ldots, V_k$ be $\mathfrak{sl}(2,\mathbb{C})$-modules, then the space $\mathrm{Hom}(\otimes_{i=1}^{k} V_i, V_0)$ is a $\mathfrak{sl}(2,\mathbb{C})$-module in the following natural manner:

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$\mathfrak{sl}(2,\mathbb{C})$-module Method

The model filiform elementary Lie algebra of order 3 $\mu_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with basis $\{X_0, X_1, \dots, X_n, Y_1, \dots, Y_m\}$. It can be seen that a pre-infinitesimal deformation φ belonging to C will be a symmetric multi-linear map:

$$\varphi : S^3(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0/\mathbb{C}X_0$$

such that

$$[X_0, \varphi(Y_i, Y_j, Y_k)] - \varphi([X_0, Y_i], Y_j, Y_k) - \varphi(Y_i, [X_0, Y_j], Y_k) - \\ -\varphi(Y_i, Y_j, [X_0, Y_k]) = 0 \qquad (6)$$

with $1 \leq i \leq j \leq k \leq m$

$\mathfrak{sl}(2,\mathbb{C})$-module Method

We are going to consider the structure of irreducible $\mathfrak{sl}(2,\mathbb{C})$-module in $V_0 =< X_1, \dots, X_n >= \mathfrak{g}_0/\mathbb{C}X_0$ and in $V_1 =< Y_1, \dots, Y_n >= \mathfrak{g}_1$, thus in particular:

$$\begin{cases} X_+ \cdot X_i = X_{i+1}, \ 1 \le i \le n-1, & X_+ \cdot X_n = 0, \\ X_+ \cdot Y_j = Y_{j+1}, \ 1 \le j \le m-1, & X_+ \cdot Y_m = 0. \end{cases}$$

We identify the multiplication of X_+ and X_i in the $\mathfrak{sl}(2,\mathbb{C})$-module $V_0 =< X_1, \dots, X_n >$, with the bracket $[X_0, X_i]$ in $\mathfrak{g}_0$. Analogously, we identify $X_+ \cdot Y_j$ and $[X_0, Y_j]$.

$$\Downarrow$$

The expressions (5) and (6) are equivalent.

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The expressions (5) and (6) are equivalent.

$\mathfrak{sl}(2,\mathbb{C})$-module Method

Lemma

Any symmetric multi-linear map φ, $\varphi : S^3V_1 \longrightarrow V_0$ will be an element of C if and only if φ is a maximal vector of the $\mathfrak{sl}(2,\mathbb{C})$-module $\mathrm{Hom}(S^3V_1, V_0)$, with $V_0 = \langle X_1, \dots, X_n\rangle$ and $V_1 = \langle Y_1, \dots, Y_m\rangle$.

Corollary

As each irreducible $\mathfrak{sl}(2,\mathbb{C})$-module has (up to nonzero scalar multiples) an unique maximal vector, then the dimension of C is equal to the number of summands of any decomposition of $\mathrm{Hom}(S^3V_1, V_0)$ into the direct sum of irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules.

For Further Reading

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