Petro Mohyla Black Sea State University, Mykolaiv, Ukraine
oksana.kuriksha@gmail.com

Lie symmetry analysis and solutions of generalized Kawahara equations

Oksana Kuriksha

Joint work with **Severin Pošta** and **Olena Vaneeva**

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Introduction

- It is widely known that Lie (point) symmetries of DEs give a powerful tool for finding exact solutions.
- At the same time it is much less known that Lie symmetries can be served also as a selection principle for equations which are important for applications among wide set of possible models. All fundamental equations of mathematical physics, for example, Maxwell or Schrödinger equation, have nontrivial symmetry properties, i.e., they admit multi-dimensional Lie invariance algebras.
- Moreover, many equations of mathematical physics can be derived just from requirement of invariance with respect to a transformation group.
- The important problem of classification of all possible Lie symmetry extensions for equations from a given class with respect to the equivalence group of the class is called the **group classification problem**.

Introduction

We study generalized Kawahara equations with time-dependent coefficients

$$u_t + \alpha(t)u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \qquad (1)$$

from the Lie symmetry point of view. It is a class of fifth-order nonlinear PDEs, where n is an arbitrary nonzero integer, α, β and σ are smooth nonvanishing functions of the variable t.

If α, β and σ are not functions but constants, the equation (1) becomes classical model (Kawahara, 1972) appearing in the solitary waves theory. It describes, e.g., the magneto-acoustic waves in plasmas and the waves with surface tension.

Introduction

We note that neither the classical Kawahara equation nor its generalization adduced above are integrable by the inverse scattering transform method.

We solve the group classification problem for variable-coefficient Kawahara equations in order:

1. to derive new variable coefficient nonlinear models admitting Lie symmetry extensions;
2. to construct solutions of these model equations.

Group classification problem

Statement of the group classification problem is the following.

Given a class of differential equations, to classify all possible cases of extension of Lie invariance algebras of such equations with respect to the equivalence group of the class.

A comprehensive investigation of this problem involves:

- Constructing the equivalence group of the entire class
- Finding the kernel of the maximal Lie invariance algebras
- Describing all possible inequivalent equations from the class that admit maximal Lie invariance algebras properly containing the kernel.

Equivalence transformations

So, we firstly look for equivalence transformations.
By Ovsiannikov, the equivalence group consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables are projectible on the space of these variables. After appearance of other kinds of equivalence group the one used by Ovsiannikov is called now **usual equivalence group**.

Equivalence transformations

If the transformations for independent and dependent variables involve arbitrary elements, then the corresponding equivalence group is called the **generalized equivalence group** (Meleshko, 1994). If new arbitrary elements appear to depend on old ones in a nonlocal way (e.g., new arbitrary elements are expressed via integrals of old ones), then the corresponding equivalence group is called **extended** (Ivanova, Popovych, Sophocleous, 2005). **Generalized extended equivalence group** possesses both the aforementioned properties.

Equivalence transformations

We search for equivalence transformations in the class of Kawahara equations using the direct method (Kingston, Sophocleous, 1998). The result is given in the following theorem.

Theorem

The usual equivalence group G of the class

$$u_t + \alpha(t)u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0 \qquad (2)$$

consists of the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u,$$

$$\tilde{\alpha}(\tilde{t}) = \frac{\delta_1}{{\delta_3}^n T_t}\alpha(t), \quad \tilde{\beta}(\tilde{t}) = \frac{\delta_1^3}{T_t}\beta(t), \quad \tilde{\sigma}(\tilde{t}) = \frac{\delta_1^5}{T_t}\sigma(t), \quad \tilde{n} = n,$$

where δ_j, $j = 1, 2, 3$, are arbitrary constants with $\delta_1\delta_3 \neq 0$, T is an arbitrary smooth function with $T_t \neq 0$.

Equivalence transformations

It appears that, if $n = 1$, then the class of variable-coefficient Kawahara equations admits equivalence transformations which do not belong to the usual equivalence group G and form a *generalized extended equivalence group*.

Equivalence transformations: Case $n = 1$

Theorem

The generalized extended equivalence group G_1 of the class

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0 \qquad (3)$$

is formed by the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = (x + \delta_1)X^1 + \delta_0, \quad \tilde{u} = \frac{\delta_2}{X^1}u - \delta_2\delta_3(x + \delta_1),$$

$$\tilde{\alpha}(\tilde{t}) = \frac{(X^1)^2}{\delta_2 T_t}\alpha(t), \quad \tilde{\beta}(\tilde{t}) = \frac{(X^1)^3}{T_t}\beta(t), \quad \tilde{\sigma}(\tilde{t}) = \frac{(X^1)^5}{T_t}\sigma(t),$$

where $X^1 = (\delta_3 \int \alpha(t)\mathrm{d}t + \delta_4)^{-1}$, δ_j, $j = 0, \ldots, 4$, are arbitrary constants with $\delta_2({\delta_3}^2 + {\delta_4}^2) \neq 0$; $T = T(t)$ is a smooth function with $T_t \neq 0$. The usual equivalence group of class (3) comprises the above transformations with $\delta_1 = \delta_3 = 0$.

Gauging of the arbitrary elements α, β and σ

Equivalence transformations allow us to simplify the group classification problem by gauging arbitrary elements, often reducing their number. For example, there is one arbitrary function $T(t)$ in the equivalence group G, therefore, we can gauge one arbitrary element, either α or β, or σ, to a simple constant value, e.g., to 1. The most convenient gauge $\alpha = 1$ can be done using the point transformation

$$\tilde{t} = \int \alpha(t)\,\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = u.$$

Then initial class is mapped to its subclass, where new arbitrary elements will be given by $\tilde{\alpha} = 1$, $\tilde{\beta} = \beta/\alpha$ and $\tilde{\sigma} = \sigma/\alpha$. Without loss of generality we can restrict ourselves to the study of the class with two variable coefficients

$$u_t + u^n u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0.$$

To derive the equivalence group for subclass of class

$$u_t + \alpha(t) u^n u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0$$

with $\alpha = 1$ we set $\tilde{\alpha} = \alpha = 1$ in the transformations presented above. So, we get the corollaries.

Corollary

The usual equivalence group $G_{\alpha=1}$ of the gauged class

$$u_t + u^n u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0$$

comprises the transformations

$$\tilde{t} = \delta_1 {\delta_3}^{-n} t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u,$$
$$\tilde{\beta}(\tilde{t}) = {\delta_1}^2 {\delta_3}^n \beta(t), \quad \tilde{\sigma}(\tilde{t}) = {\delta_1}^4 {\delta_3}^n \sigma(t),$$

where δ_j, $j = 0, 1, 2, 3$, are arbitrary constants with $\delta_1 \delta_3 \neq 0$.

Corollary

The usual equivalence group $G_{n=\alpha=1}$ of the class

$$u_t + uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0$$

consists of the transformations

$$\tilde{t} = \frac{at+b}{ct+d},\ \tilde{x} = \frac{e_2x + e_1t + e_0}{ct+d},\ \tilde{u} = \frac{e_2(ct+d)u - e_2cx - e_0c + e_1d}{\Delta},$$
$$\tilde{\beta} = \frac{{e_2}^3}{ct+d}\frac{\beta}{\Delta}, \quad \tilde{\sigma} = \frac{{e_2}^5}{(ct+d)^3}\frac{\sigma}{\Delta},$$

where a, b, c, d, e_0, e_1 and e_2 are arbitrary constants with $\Delta = ad - bc \neq 0$ and $e_2 \neq 0$, the tuple $(a, b, c, d, e_0, e_1, e_2)$ is defined up to nonzero multiplier and hence without loss of generality we can assume that $\Delta = \pm 1$.

Lie symmetries

Now we can start Lie symmetry classification. We search for operators of the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

which generate one-parameter groups of Lie symmetry transformations of equations from the class

$$u_t + u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0. \tag{4}$$

These operators satisfy the criterion of infinitesimal invariance, i.e., we require that the action of the fifth prolongation of the operator Q on Eq. (4) vanishes identically, on the manifold of solutions of this equation:

$$Q^{(5)}\{u_t + u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx}\}\big|_{u_t = u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx}} = 0.$$

Determining equations

The infinitesimal invariance criterion implies that the coefficients of Q have the form

$$\tau = \tau(t), \quad \xi = \xi(t,x), \quad \eta = \eta^1(t,x)u + \eta^0(t,x),$$

where τ, ξ, η^1 and η^0 are arbitrary smooth functions of their variables. The rest of determining equations have the form

$$\eta^1_x = 2\xi_{xx}, \quad 3(\eta^1_x - \xi_{xx})\beta + 5(2\eta^1_{xxx} - \xi_{xxxx})\sigma = 0,$$
$$\tau\sigma_t = (5\xi_x - \tau_t)\sigma, \quad \tau\beta_t = (3\xi_x - \tau_t)\beta + 10(\xi_{xxx} - \eta^1_{xx})\sigma,$$
$$\eta^1_x u^{n+1} + \eta^0_x u^n + (\eta^1_t + \eta^1_{xxx}\beta + \eta^1_{xxxxx}\sigma)u + \eta^0_t + \eta^0_{xxx}\beta + \eta^0_{xxxxx}\sigma = 0,$$
$$\xi_t = (\tau_t - \xi_x + n\eta^1)u^n + n\eta^0 u^{n-1} + (3\eta^1_{xx} - \xi_{xxx})\beta + (5\eta^1_{xxxx} - \xi_{xxxxx})\sigma.$$

The main difficulty of group classification is the need to solve classifying equations with respect to the coefficients of the operator Q and arbitrary elements simultaneously.

Some of the determining equations do not contain arbitrary elements and therefore can be integrated immediately.
The latter two equations can be split with respect to different powers of u. Special case of splitting arises if $n = 1$. Therefore, the cases $n \neq 1$ and $n = 1$ should be investigated separately.

$\mathbf{n \neq 1}$. The coefficients of the operator Q are of the form

$$\tau = c_1 t + c_2, \quad \xi = (c_1 + nc_0)x + c_3, \quad \eta = c_0 u,$$

where c_i, $i = 0, \ldots, 3$, are arbitrary constants, and the **classifying equations** which include arbitrary elements β and σ are

$$(c_1 t + c_2)\beta_t = (2c_1 + 3nc_0)\beta, \quad (c_1 t + c_2)\sigma_t = (4c_1 + 5nc_0)\sigma.$$

We solve them up to the corresponding equivalence and get the statement.

Result of group classification problem: $n \neq 1$

The kernel of the maximal Lie invariance algebras of equations from class

$$u_t + u^n u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0$$

coincides with the one-dimensional algebra $\langle \partial_x \rangle$. All possible $G_{\alpha=1}$-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by the cases 1–3 of the table

	$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
1	εt^ρ	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ 3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u$
2	εe^t	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ 3n\partial_t + nx\partial_x + u\partial_u$
3	ε	δ	$\partial_x,\ \partial_t$

Here $\delta = \pm 1 \bmod G_{\alpha=1}$, ρ is an arbitrary constant.

Determining equations: $n = 1$

n = 1. Solving determining equations we get that the coefficients of the operator Q are of the form

$$Q = (c_2 t^2 + 2c_1 t + c_0)\partial_t + ((c_2 t + c_1 + c_3)x + c_4 t + c_5)\,\partial_x, \\ + ((c_3 - c_1 - c_2 t)u + c_2 x + c_4)\,\partial_u,$$

where c_i, $i = 0, \dots, 5$, are arbitrary constants, and the **classifying equations** which includes arbitrary elements β and σ are

$$(c_2 t^2 + 2c_1 t + c_0)\beta_t = (c_2 t + c_1 + 3c_3)\beta, \\ (c_2 t^2 + 2c_1 t + c_0)\sigma_t = (3c_2 t + 3c_1 + 5c_3)\sigma.$$

Result of group classification problem: $n = 1$

The kernel of $A^{\max}$ of equations from the class

$$u_t + uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0$$

coincides with $\langle \partial_x, t\partial_x + \partial_u \rangle$. All $G_{n=\alpha=1}$-inequivalent cases of extension of $A^{\max}$ are exhausted by the cases 1–4 of the table

	$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	$\partial_x,\ t\partial_x + \partial_u$
1	εt^{ρ}	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ t\partial_x + \partial_u,\ 3t\partial_t + (\rho+1)x\partial_x + (\rho-2)u\partial_u$
2	εe^{t}	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ t\partial_x + \partial_u,\ 3\partial_t + x\partial_x + u\partial_u$
3	$\varepsilon(t^2+1)^{\frac{1}{2}} \times e^{3\nu \arctan t}$	$\delta(t^2+1)^{\frac{3}{2}} \times e^{5\nu \arctan t}$	$\partial_x,\ t\partial_x + \partial_u,$ $(t^2+1)\partial_t + (t+\nu)x\partial_x + ((\nu - t)u + x)\partial_u$
4	ε	δ	$\partial_x,\ t\partial_x + \partial_u,\ \partial_t$

Further application of the results obtained

The presented group classification gives all inequivalent generalized Kawahara equations with time-dependent coefficients

$$u_t + \alpha(t)u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0$$

for which the classical method of Lie reduction can be effectively used.

Classification of reductions

Case	Ansatz, $u=$	Reduced ODE
$1_{\rho\neq-1}$	$t^{\frac{\rho-2}{3n}}\varphi(xt^{-\frac{\rho+1}{3}})$	$\delta\varphi''''' + \varepsilon\varphi''' + \left(\varphi^n + \frac{\rho-1}{3}\omega\right)\varphi' + \frac{\rho-2}{3n}\varphi = 0$
$1_{\rho=-1}$	$t^{-\frac{1}{n}}\varphi(x - \frac{a}{n}\ln t)$	$\delta\varphi''''' + \varepsilon\varphi''' + \left(\varphi^n - \frac{a}{n}\right)\varphi' - \frac{1}{n}\varphi = 0$
2	$e^{\frac{1}{3n}t}\varphi(xe^{-\frac{1}{3}t})$	$\delta\varphi''''' + \varepsilon\varphi''' + \left(\varphi^n - \frac{1}{3}\omega\right)\varphi' + \frac{1}{3n}\varphi = 0$
Specific reductions for $n=1$		
0	$\varphi(t) + \dfrac{x}{t+a}$	$(\omega + a)\varphi' + \varphi = 0$
3	$\dfrac{e^{\rho\arctan t}}{\sqrt{t^2+1}}\varphi\left(\dfrac{xe^{-\rho\arctan t}}{\sqrt{t^2+1}}\right) + \dfrac{xt}{t^2+1}$	$\delta\varphi''''' + \varepsilon\varphi''' + (\varphi - \rho\omega)\varphi' + \rho\varphi + \omega = 0$

Invariant solutions

Exact solutions of the fifth-order ODEs presented in the Table are not known. At the same time behavior of solutions for variable coefficients models is what we are most interested in.

The Lie reductions obtained can be useful in seeking numerical solutions of the Kawahara equations accompanied with boundary conditions that are invariant with respect to the corresponding Lie symmetry algebras

A class of invariant BVP problems

Consider a class of generalized Kawahara equations with the coefficients being the power functions of t,

$$u_t + u^n u_x + \lambda t^\rho u_{xxx} + \delta t^{\frac{5\rho+2}{3}} u_{xxxxx} = 0,\ t > t_0, \quad x > 0,\ n \in \mathbb{N},$$

with the following boundary conditions

$$u(t,0) = \gamma_0 t^{\frac{\rho-2}{3n}}, \quad \left.\frac{\partial^i u(t,x)}{\partial x^i}\right|_{x=0} = \gamma_i t^{\frac{\rho-2-n(\rho+1)i}{3n}},\ t > t_0,\ i = 1,\dots,4,$$

where γ_i, $i = 0,\dots,4$, λ, δ are arbitrary constants with $\gamma_0\lambda\delta \neq 0$.

Both equation and boundary conditions are invariant with respect to the scaling symmetry operator

$$Q = 3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u.$$

Reduction of the VBP

Using the ansatz $u = t^{\frac{\rho-2}{3n}}\varphi(\omega)$ with invariant variable $\omega = xt^{-\frac{\rho+1}{3}}$ this problem reduces to the initial value problem (IVP) for a fifth-order ODE,

$$\delta\varphi''''' + \lambda\varphi''' + \left(\varphi^n - \tfrac{\rho+1}{3}\omega\right)\varphi' + \tfrac{\rho-2}{3n}\varphi = 0,$$
$$\varphi(0) = \gamma_0, \quad \left.\frac{\mathrm{d}^i\varphi(\omega)}{\mathrm{d}\omega^i}\right|_{\omega=0} = \gamma_i,\ i = 1,\dots,4.$$

After the problem for the latter IVP is solved numerically, then the corresponding solution of the initial BVP can be recovered using the above similarity transformation.

Example

Let's consider an example with physical application.

The following constant coefficient Kawahara equation

$$v_t + v_x + \frac{3}{2}\varepsilon v v_x + \frac{1}{2}\varkappa v_{xxx} + \frac{1}{2}\gamma v_{xxxxx} = 0$$

arises as a model describing the propagation of long nonlinear waves in the water covered by ice (A.V. Marchenko, 1988).

Here ε, $\varkappa$, and γ are constants connected to some physical properties such as the depth of the fluid, characteristic wavelength, densities of the fluid and ice, etc.

Example

The equation proposed by Marchenko was constant coefficient one (because ice thickness was assumed to be constant).
We consider the situation when the ice thickness growth in time possessing the law $h \sim \sqrt{t}$, which for certain weather conditions is in well agreement with the experimental data obtained for the real sea, and solve the problem for some typical values such as $\lambda \approx 100\mathrm{m}$ wavelength, $H \approx 10\mathrm{m}$ depth of water, etc.
Then the model equation takes the form

$$u_t + uu_x + \lambda t^{\frac{1}{2}} u_{xxx} + \delta t^{\frac{3}{2}} u_{xxxxx} = 0,$$

and the boundary conditions, being considered by us, are

$$u(t,0) = \gamma_0 t^{-\frac{1}{2}}, \ u_x(t,0) = 0,$$
$$u_{xx}(t,0) = 0, \ u_{xxx}(t,0) = 0, \ u_{xxxx}(t,0) = 0.$$

Example

We reduce this BVP to IVP for an ODE, then solve it numerically and recover the corresponding solution for BVP using similarity transformation.

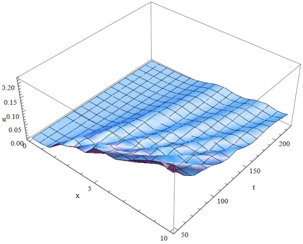

Figure: Solution of BVP.

Thank you for your attention!