Deformation of the $\mathfrak{su}(2)$ Lie algebra through the parity operator and its realization as finite-discrete quantum oscillator models

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Joint work with...

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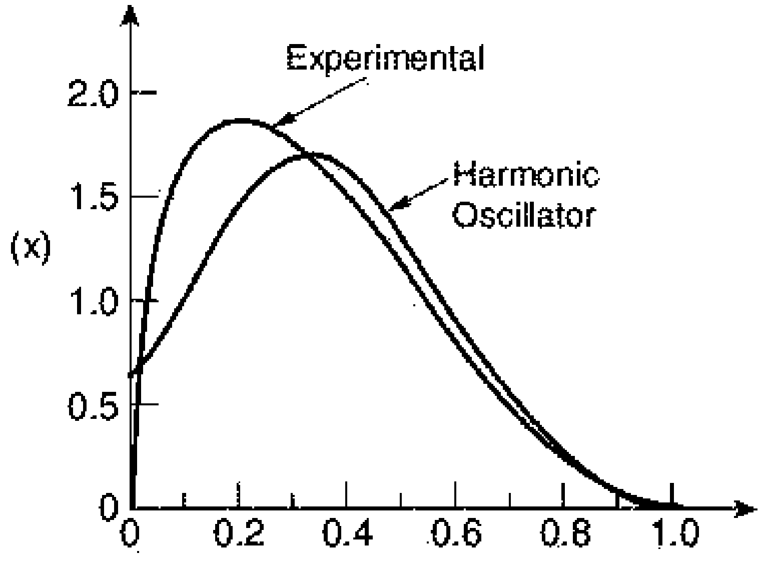
Importance of the Harmonic Oscillator



- Caption from **Y.S. Kim, M.E. Noz, Workshop on Branes and Generalized Dynamics (Argonne, Illinois, USA, October 2003)**

Importance of the Harmonic Oscillator - 2

- *Some years ago Professor P.A.M. Dirac and his wife Margit came to my home for dinner. Robert and Edie Dressler stopped by for a while. In the course of our conversation I remarked that I had read that Einstein said it would be enough if we could understand the electron. Dirac replied that it would be enough if students could understand the harmonic oscillator...*

Sylvan Charles Bloch, 'Introduction to Classical and Quantum Oscillators', New York: Wiley-Interscience (1997)

The algebra $\mathfrak{su}(2)$ under the $\mathcal{CP}$ deformed symmetry

The algebra $\mathfrak{su}(2)_{\mathcal{CP}}$ is a unital algebra with basis elements J_0, J_+, J_-, $\mathcal{C}$ and $\mathcal{P}$ subject to the following relations:

- The operator $\mathcal{C}$ commutes with all basis elements.
- $\mathcal{P}$ is a parity operator satisfying $\mathcal{P}^2 = 1$ and

$$[\mathcal{P}, J_0] = \mathcal{P}J_0 - J_0\mathcal{P} = 0, \qquad \{\mathcal{P}, J_\pm\} = \mathcal{P}J_\pm + J_\pm\mathcal{P} = 0.$$

- The $\mathfrak{su}(2)$ commutation relations are $\mathcal{CP}$ deformed:

$$[J_0, J_\pm] = \pm J_\pm,$$
$$[J_+, J_-] = 2J_0\left(\mathcal{CP} - 1\right).$$

Actions of the basis elements J_0, J_+ and J_-

$$J_0|j,m\rangle = m\,|j,m\rangle,$$

$$J_+|j,m\rangle = \begin{cases} \sqrt{(j-m)(j-m-1)}\,|j,m+1\rangle, & j+m \text{ even;} \\ \sqrt{(j+m)(j+m+1)}\,|j,m+1\rangle, & j+m \text{ odd,} \end{cases}$$

$$J_-|j,m\rangle = \begin{cases} \sqrt{(j+m)(j+m-1)}\,|j,m-1\rangle, & j+m \text{ even;} \\ \sqrt{(j-m)(j-m+1)}\,|j,m-1\rangle, & j+m \text{ odd.} \end{cases}$$

Actions of the operators $\mathcal{C}$ and $\mathcal{P}$

$$\mathcal{C}|j,m\rangle = 2j\;|j,m\rangle,$$

$$\mathcal{P}|j,m\rangle = (-1)^{j+m}\;|j,m\rangle.$$

- Note that j must be integer in order to be a representation;
- Then, $J_+|j,j\rangle = 0$ and $J_-|j,-j\rangle = 0$

A one-dimensional oscillator model based on the algebra $\mathfrak{su}(2)_{\mathcal{CP}}$

$$\hat{H} = J_0 + j + \frac{1}{2}, \qquad \hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{\mathrm{i}}{2}(J_+ - J_-).$$

They satisfy the Heisenberg equations:

$$\left[\hat{H}, \hat{q}\right] = -\mathrm{i}\hat{p}, \quad \left[\hat{H}, \hat{p}\right] = \mathrm{i}\hat{q}.$$

Then $\hat{H}|j,m\rangle = (m + j + \frac{1}{2})|j,m\rangle$.
Energy spectrum $E_n = n + \frac{1}{2} \qquad (n = 0, 1, \ldots, 2j)$.

Krawtchouk polynomials

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n, -x \\ -N\end{matrix}; \frac{1}{p}\right).$$

Orthogonality holds for discrete values of x:

$$\sum_{x=0}^{N} w(x,N)\, K_n(x;p,N)\, K_{n'}(x;p,N) = h(n,N)\, \delta_{n,n'},$$

$$w(x,N) = \binom{N}{x} p^x (1-p)^x \quad (x = 0, 1, \ldots, N),$$

$$h(n,N) = \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p}\right)^n.$$

Here, $(a)_k = a(a+1)\cdots(a+k-1)$ are Pochhammer symbols.

The symmetric Krawtchouk polynomials - difference equations

$$j\left(2j-1\right)K_{j+n}\left(2\left(x+1\right);\tfrac{1}{2},2j\right)=-\left(x+1\right)\left(2x+1\right)K_{j+n-1}\left(2x;\tfrac{1}{2},2\left(j-1\right)\right)$$

$$+\left(j-x-1\right)\left(2j-2x-3\right)K_{j+n-1}\left(2\left(x+1\right);\tfrac{1}{2},2\left(j-1\right)\right);$$

$$2\left(j+n\right)\left(j-n\right)K_{j+n-1}\left(2x;\tfrac{1}{2},2\left(j-1\right)\right)=j\left(2j-1\right)K_{j+n}\left(2x;\tfrac{1}{2},2j\right)$$

$$-j\left(2j-1\right)K_{j+n}\left(2\left(x+1\right);\tfrac{1}{2},2j\right).$$

Position wavefunctions

Even case $j + m = 2n$ and positive values of position:

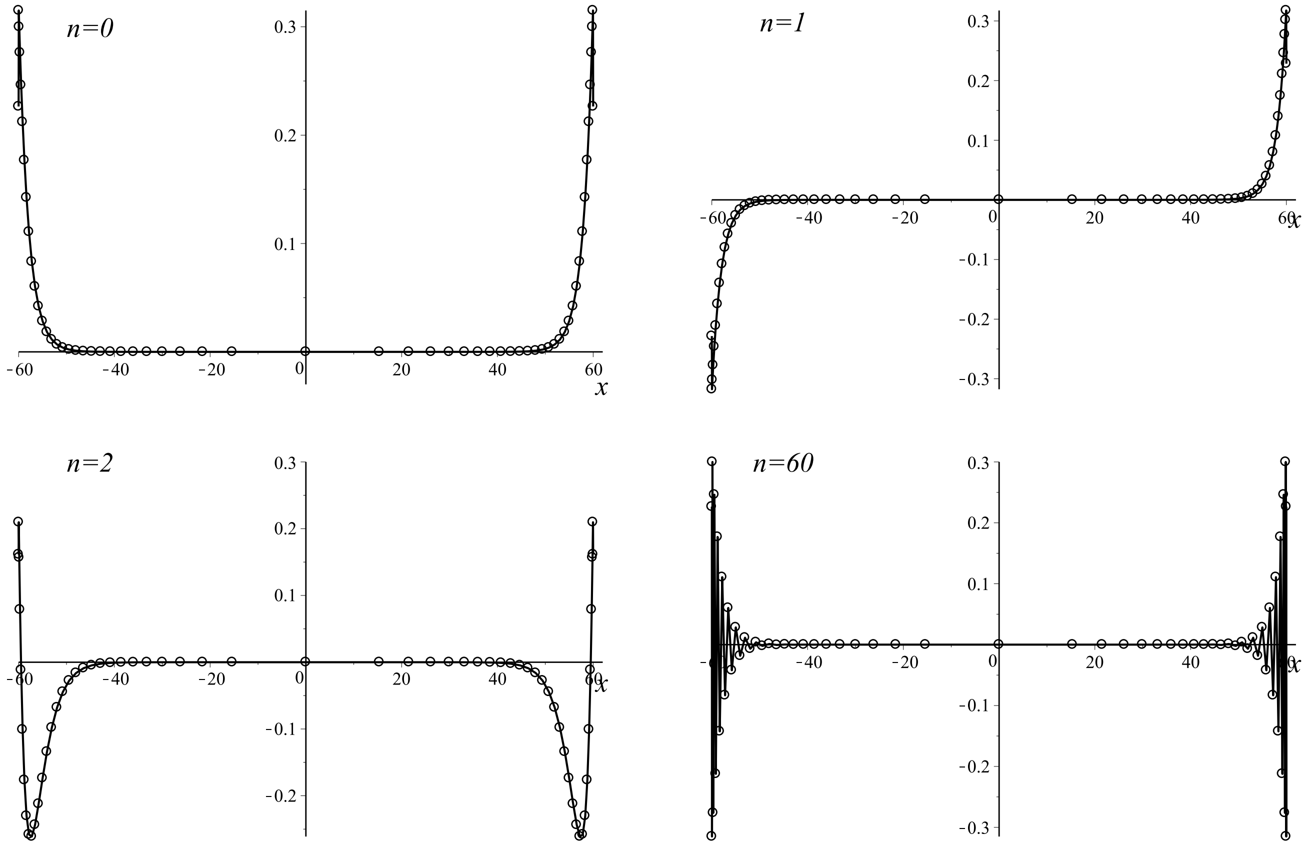
$$\Psi_{2n}(q_k) = (-1)^n 2^{k-j} \tilde{K}_{2n}(k; \tfrac{1}{2}, 2j), \quad n = 0, 1, \ldots, j, \quad k = 1, \ldots, j-1,$$

Odd case $j + m = 2n + 1$ and positive values of position:

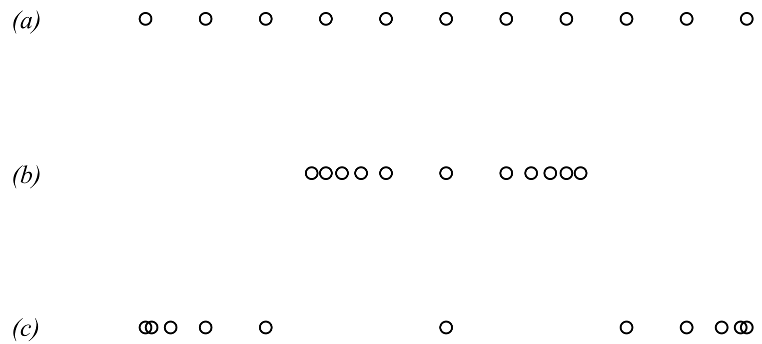
$$\Psi_{2n+1}(q_k) = (-1)^n 2^{k-j} \tilde{K}_{2n}(k-1; \tfrac{1}{2}, 2j-2).$$

One can extend these computations and obtain similar expressions for zero and negative values of the positions.

Position wavefunctions - 2



Eigenvalues of the position operator $\hat{q}$



Spectrum of the position operator for $j = 5$, (a) in the case of the $\mathfrak{su}(2)$ model, (b) in the case of the $\mathfrak{sl}(2|1)$ model and (c) in the case of the $\mathfrak{su}(2)_{\mathcal{CP}}$ model.

$$q_{\pm(j-k)} = \pm\sqrt{(j-k)\,(j+k)}, \quad k = 0, 1, 2, \ldots, j.$$

Generalization of the position eigenvalues

We suppose the following generalization:

$$q_{\pm(j-k)} = \pm\sqrt{(j-k)\,(j+k+\alpha+\beta+1)},$$

$$k = 0, 1, 2, \ldots, j.$$

$$\alpha > -1, \quad \beta > -1$$

There exists certain general algebra.

Generalization of the algebra $\mathfrak{su}(2)_{\mathcal{CP}}$

Basis elements J_0, J_+, J_-, $\mathcal{C}$ and $\mathcal{P}$ of the new algebra are subject to the following relations:

- The operator $\mathcal{C}$ commutes with all basis elements.
- $\mathcal{P}$ is a parity operator satisfying $\mathcal{P}^2 = 1$ and

$$[\mathcal{P}, J_0] = \mathcal{P}J_0 - J_0\mathcal{P} = 0, \qquad \{\mathcal{P}, J_\pm\} = \mathcal{P}J_\pm + J_\pm\mathcal{P} = 0.$$

- The $\mathfrak{su}(2)$ commutation relations are deformed by the following manner:

$$[J_0, J_\pm] = \pm J_\pm,$$

$$[J_+, J_-] = 2J_0\left[CP + (\alpha + \beta)\,P - 1\right] + (CP - 1)\,(\alpha - \beta)\,.$$

Generalized actions of J_0, J_+, J_- and $\mathcal{C}$

$$J_0|j,m\rangle = m\,|j,m\rangle,$$

$$J_+|j,m\rangle = \begin{cases} \sqrt{(j-m)(j-m+2\beta)}\,|j,m+1\rangle, & j+m \text{ even;} \\ \sqrt{(j+m+2\alpha+1)(j+m+1)}\,|j,m+1\rangle, & j+m \text{ odd,} \end{cases}$$

$$J_-|j,m\rangle = \begin{cases} \sqrt{(j+m)(j+m+2\alpha)}\,|j,m-1\rangle, & j+m \text{ even;} \\ \sqrt{(j-m+2\beta+1)(j-m+1)}\,|j,m-1\rangle, & j+m \text{ odd.} \end{cases}$$

$$\mathcal{C}|j,m\rangle = (2j+1)\,|j,m\rangle$$

Special case

Special case holds, when $\alpha = \beta$ and commutation relations become simpler:

$$[J_+, J_-] = 2J_0\,(CP + 2\alpha P - 1)$$

Realization is possible in terms of the Hahn polynomials by using the following difference euqtions for them!

Hahn polynomials - difference equations

$$j \cdot Q_k(n;\alpha,\beta,j) = n \cdot Q_k(n-1;\alpha,\beta,j-1)$$

$$+ (j-n)\,Q_k(n;\alpha,\beta,j-1)\,,$$

$$\frac{(j-k)(j+k+\alpha+\beta+1)}{j} Q_k(n;\alpha,\beta,j-1)$$

$$= (j+\beta-n)\,Q_k(n;\alpha,\beta,j) + (n+\alpha+1)\,Q_k(n+1;\alpha,\beta,j)\,.$$

Properties

- Finite-discrete analogue of the Fourier transform;
- Possible limits of the model

Thank you for attention!