# A class of infinite-dimensional representations of the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ and the parastatistics Fock space

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Work in collaboration with N.I. Stoilova.

# Overview

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   - Old history
   - Recent history
2. Parastatistics algebra $\mathfrak{osp}(2m+1|2n)$
   - The LSA $\mathfrak{osp}(2m+1|2n)$
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   - Action of para-operators on this basis
   - Structure of $V(p)$
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3. Outlook

# Bosons and Fermions

Commutator: $[A, B] = AB - BA$
Anti-commutator: $\{A, B\} = AB + BA$
Graded: $[\![A, B]\!] = AB - (-1)^{\langle A \rangle \langle B \rangle} BA \quad (\langle A \rangle = \deg(A) \in \{0, 1\})$

- $n$ pairs of Bose operators $B_i^{\pm}$:

$$[B_i^-, B_j^+] = \delta_{ij} \qquad \text{all other } [\cdot, \cdot] \text{ zero}$$

- Fock space characterized by $(B_i^{\pm})^{\dagger} = B_i^{\mp}$ and $B_i^- |0\rangle = 0$
- Basis vectors (orthogonal, normalized):

$$|k_1, \ldots, k_n\rangle = \frac{(B_1^+)^{k_1} \cdots (B_n^+)^{k_n}}{\sqrt{k_1! \cdots k_n!}} |0\rangle$$

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- $m$ pairs of Fermi operators $F_i^{\pm}$:

$$\{F_i^-, F_j^+\} = \delta_{ij} \qquad \text{all other } \{\cdot,\cdot\} \text{ zero}$$

- Fock space characterized by $(F_i^{\pm})^{\dagger} = F_i^{\mp}$ and $F_i^-|0\rangle = 0$

# Parabosons and parafermions

- $n$ pairs of parabosons $b_j^{\pm}$ $(j = 1, \ldots, n)$ [Green 1953]:

$$[\{b_j^{\xi}, b_k^{\eta}\}, b_l^{\epsilon}] = (\epsilon - \eta)\delta_{kl} b_j^{\xi} + (\epsilon - \xi)\delta_{jl} b_k^{\eta}$$

- Fock space $V(p)$ characterized by $(b_j^{\pm})^{\dagger} = b_j^{\mp}$ and $b_j^{-}|0\rangle = 0$ and [Greenberg & Messiah 1965]

$$\{b_j^{-}, b_k^{+}\}|0\rangle = p\,\delta_{jk}\,|0\rangle$$

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- $m$ pairs of parafermion $f_j^{\pm}$ $(j = 1, \ldots, m)$ [Green 1953]:

$$[[f_j^{\xi}, f_k^{\eta}], f_l^{\epsilon}] = |\epsilon - \eta|\delta_{kl} f_j^{\xi} - |\epsilon - \xi|\delta_{jl} f_k^{\eta}$$

- Fock space $W(p)$ characterized by $(f_j^{\pm})^{\dagger} = f_j^{\mp}$ and $f_j^{-}|0\rangle = 0$ and [Greenberg & Messiah 1965]

$$[f_j^{-}, f_k^{+}]|0\rangle = p\,\delta_{jk}\,|0\rangle$$

# Paraboson and parafermion algebra

**Theorem (LA by generators and relations) [Kamefuchi & Takahishi 1962; Ryan & Sudarshan 1963]**

The Lie algebra (LA) generated by $2m$ elements $f_j^{\pm}$ subject to the parafermion triple relations is $\mathfrak{so}(2m+1)$. The Fock space $W(p)$ is the unitary irreducible representation of $\mathfrak{so}(2m+1)$ with lowest weight $(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$.

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**Theorem (LSA by generators and relations) [Ganchev & Palev 1980]**

The Lie superalgebra (LSA) generated by $2n$ odd elements $b_j^{\pm}$ subject to the paraboson triple relations is $\mathfrak{osp}(1|2n)$. The Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$.

# Parastatistics, parastatistics algebra

Simultaneous system: can be combined in 4 ways [Greenberg, Messiah]. The most interesting case: *relative para-fermi*.
With $c_j = f_j$ $(1 \le j \le m)$ and $c_{m+j} = b_j$ $(1 \le j \le n)$: parastatistics relations.

$$[\![[\![c_j^+, c_k^-]\!], c_l^+]\!] = 2\delta_{kl} c_j^+, \qquad [\![[\![c_j^+, c_k^+]\!], c_l^+]\!] = 0$$
$$[\![c_j^-, [\![c_k^+, c_l^-]\!]]\!] = 2\delta_{jk} c_l^-, \qquad [\![[\![c_j^-, c_k^-]\!], c_l^-]\!] = 0$$

or

$$[\![[\![c_j^\xi, c_k^\eta]\!], c_l^\epsilon]\!] = -2\delta_{jl}\delta_{\epsilon,-\xi}\epsilon^{\langle l\rangle}(-1)^{\langle k\rangle\langle l\rangle} c_k^\eta + 2\epsilon^{\langle l\rangle}\delta_{kl}\delta_{\epsilon,-\eta} c_j^\xi$$

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$$[\![ c_j^-, [\![ c_k^+, c_l^- ]\!] ]\!] = 2\delta_{jk} c_l^-, \qquad [\![ [\![ c_j^-, c_k^- ]\!], c_l^- ]\!] = 0$$

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**Theorem [Palev 1982]**

The Lie superalgebra (LSA) generated by $2m$ even elements $f_j^\pm$ and $2n$ odd elements $b_j^\pm$ subject to the above relations is $\mathfrak{osp}(2m+1|2n)$. The Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(2m+1|2n)$ with lowest weight $[-\frac{p}{2}, \ldots, -\frac{p}{2} | \frac{p}{2}, \ldots, \frac{p}{2}]$.

# Structure of Fock space

- Green's ansatz: $p$-fold tensor product of ordinary boson/fermion Fock space & extract an irreducible component. Ongoing work: Daskaloyannis 2007, Salom 2013.
- Characters of paraboson or parastatistics space [Loday & Popov 2009]
- Basis and paraboson operator actions in $V(p)$ for $\mathfrak{osp}(1|2n)$: Lievens, Stoilova, Van der Jeugt 2008.
- Basis and parafermion operator actions in $W(p)$ for $\mathfrak{so}(2m+1)$: Stoilova, Van der Jeugt 2008.
- Extension to infinite number of parabosons or parafermions [Stoilova, Van der Jeugt 2009]

# The LSA $\mathfrak{osp}(2m+1|2n)$

Lie superalgebra $\mathfrak{osp}(2m+1|2n)$: matrices of size $(m+m+1+n+n)\times(m+m+1+n+n)$:

$$\begin{pmatrix} a & b & u & x & x_1 \\ c & -a^t & v & y & y_1 \\ -v^t & -u^t & 0 & z & z_1 \\ y_1^t & x_1^t & z_1^t & d & g \\ -y^t & -x^t & -z^t & f & -d^t \end{pmatrix}$$

($b$ and $c$ skew-symmetric, $g$ and $f$ are symmetric)
blue even
red odd

Cartan subalgebra $\mathfrak{h}$: $h_i = e_{ii} - e_{i+m,i+m}$ $(i=1,\ldots,m)$, $h_{m+j} = e_{2m+1+j,2m+1+j} - e_{2m+1+n+j,2m+1+n+j}$ $(j=1,\ldots,n)$.
Even/odd generators:

$$c_j^+ = f_j^+ = \sqrt{2}(e_{j,2m+1} - e_{2m+1,j+m}),$$
$$c_j^- = f_j^- = \sqrt{2}(e_{2m+1,j} - e_{j+m,2m+1}),$$
$$c_{m+j}^+ = b_j^+ = \sqrt{2}(e_{2m+1,2m+1+n+j} + e_{2m+1+j,2m+1}),$$
$$c_{m+j}^- = b_j^- = \sqrt{2}(e_{2m+1,2m+1+j} - e_{2m+1+n+j,2m+1}).$$

# Subalgebras of $\mathfrak{osp}(2m+1|2n)$

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Even/odd generators:

$$c_j^+ = f_j^+ = \sqrt{2}(e_{j,2m+1} - e_{2m+1,j+m}),$$
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$$c_{m+j}^+ = b_j^+ = \sqrt{2}(e_{2m+1,2m+1+n+j} + e_{2m+1+j,2m+1}),$$
$$c_{m+j}^- = b_j^- = \sqrt{2}(e_{2m+1,2m+1+j} - e_{2m+1+n+j,2m+1}).$$

# Subalgebras of $\mathfrak{osp}(2m+1|2n)$

Obvious even subalgebra: $\mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$

$$[c_i^\xi, c_k^\eta], \quad c_l^\epsilon, \qquad (i,k,l = 1,\ldots,m;\ \xi,\eta,\epsilon = \pm)$$
$$\{c_{m+j}^\xi, c_{m+s}^\eta\}, \qquad (j,s = 1,\ldots,n;\ \xi,\eta = \pm).$$

Subalgebra $\mathfrak{u}(m|n) \equiv \mathfrak{gl}(m|n)$: $(m+n)^2$ elements

$$[\![c_j^-, c_k^+]\!] \qquad (j,k = 1,\ldots,m+n)$$

Cartan subalgebra $\mathfrak{h}$: (recall: $[\![c_j^-, c_k^+]\!]|0\rangle = p\,\delta_{jk}$)

$$[c_i^-, c_i^+] = -2h_i \quad (i = 1,\ldots,m),$$
$$\{c_{m+j}^-, c_{m+j}^+\} = 2h_{m+j} \quad (j = 1,\ldots,n).$$

So $|0\rangle$ is vector of weight $[-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2}]$.

# Induced module construction

- $[\![c_j^-, c_k^+]\!]|0\rangle = p\,\delta_{jk}\,|0\rangle$ ⇔ trivial one-dim. $\mathfrak{u}(m|n)$ module of weight $[-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2}]$
- $c_j^-|0\rangle = 0$ ⇔ extend $\mathfrak{u}(m|n)$ to a parabolic subalgebra

$$\mathfrak{P} = \text{span}\{c_j^-, [\![c_j^+, c_k^-]\!], [\![c_j^-, c_k^-]\!] \mid j,k = 1,\ldots,m+n\},$$

  and extend $\mathbb{C}|0\rangle$ to trivial $\mathfrak{P}$ module by requiring $c_j^-|0\rangle = 0$ $(j = 1,\ldots,m+n)$

- Induced module: $\overline{V}(p) = \text{Ind}_{\mathfrak{P}}^{\mathfrak{osp}(2m+1|2n)}\mathbb{C}|0\rangle$
- basis vectors: ($k_{i,m+j} \in \{0,1\}$, others in $\mathbb{Z}_+$) [PBW-theo]

$$(c_1^+)^{k_1}\cdots(c_{m+n}^+)^{k_{m+n}}([\![c_1^+, c_2^+]\!])^{k_{12}}([\![c_1^+, c_3^+]\!])^{k_{13}}\cdots$$
$$\cdots([\![c_{m+n-1}^+, c_{m+n}^+]\!])^{k_{m+n-1,m+n}}|0\rangle$$

**Irreducible module**

$V(p) = \overline{V}(p)/M(p)$ (max. nontrivial submodule)

# Basis of $\overline{V}(p)$

Character of representations: formal series of terms

$$\nu x_1^{j_1} x_2^{j_2} \dots x_m^{j_m} y_1^{j_{m+1}} y_2^{j_{m+2}} \dots y_n^{j_{m+n}},$$

with $[j_1, \dots, j_m | j_{m+1}, \dots, j_{m+n}]$ the weight and $\nu$ the dimension of the corresp. weight space.

Character of $\mathbb{C}|0\rangle$ is just

$$(x_1)^{-p/2} \cdots (x_m)^{-p/2} (y_1)^{p/2} \cdots (y_n)^{p/2},$$

By earlier basis (or PBW):

$$\mathrm{char}\, \overline{V}(p) = \frac{(x_1)^{-p/2} \cdots (x_m)^{-p/2} (y_1)^{p/2} \cdots (y_n)^{p/2} \prod_{i,j}(1 + x_i y_j)}{\prod_i (1 - x_i) \prod_{i<k} (1 - x_i x_k) \prod_j (1 - y_j) \prod_{j<l} (1 - y_j y_l)}$$

# Expansion in supersymmetric Schur functions

## [Cummins, King 1987]

$$\frac{\prod_{i,j}(1+x_i y_j)}{\prod_i(1-x_i)\prod_{i<k}(1-x_i x_k)\prod_j(1-y_j)\prod_{j<l}(1-y_j y_l)} = \sum_{\lambda\in\mathcal{H}} s_\lambda(x_1,\ldots,x_m|y_1,\ldots,y_n) = \sum_{\lambda\in\mathcal{H}} s_\lambda(\mathbf{x}|\mathbf{y})$$

Sum is over the set $\mathcal{H}$ of all partitions $\lambda$ satisfying the $(m|n)$-hook condition $\lambda_{m+1} \leq n$.

$s_\lambda(\mathbf{x}|\mathbf{y})$ is the supersymmetric Schur function.

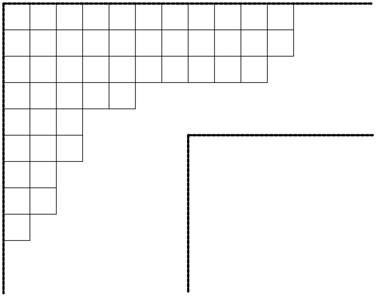
$s_\lambda(\mathbf{x}|\mathbf{y})$ are characters of (covariant) $\mathfrak{u}(m|n)$ repres. $V([\Lambda^\lambda])$

Relation: partition $\lambda=(\lambda_1,\lambda_2,\ldots)$ / highest weight $\Lambda^\lambda$
$\equiv [\mu]^r \equiv [\mu_{1r},\ldots,\mu_{mr}|\mu_{m+1,r}\ldots,\mu_{rr}]$ $(r=m+n)$:

$$\mu_{ir}=\lambda_i, \qquad \mu_{m+j,r}=\max\{0,\lambda'_j-m\}$$

Gives branching of the $\mathfrak{osp}(2m+1|2n)$ to $\mathfrak{u}(m|n)$ for $\overline{V}(p)$ (and thus also basis!).

# Covariant represenations of $\mathfrak{u}(m|n)$



$m = 5,\ n = 7,\ r = m + n = 12$
$u(5|7)$
$\lambda = (11, 11, 10, 5, 3, 3, 2, 2, 1)$
weight $[\mu] = [\mu]^r$:
$[11, 11, 10, 5, 3|4, 3, 1, 0, 0, 0, 0]$

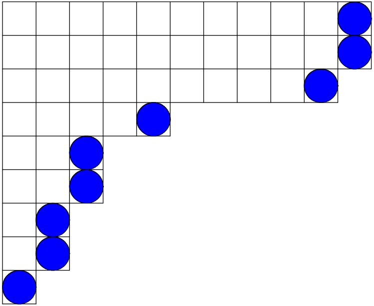
# Basis for $\overline{V}(p)$

Use Gelfand-Zetlin basis (GZ) for $\mathfrak{u}(m|n)$, based on $\mathfrak{u}(m|n) \supset \mathfrak{u}(m|n-1) \supset \cdots \supset \mathfrak{u}(m|1) \supset \mathfrak{u}(m) \supset \mathfrak{u}(m-1) \supset \cdots \supset \mathfrak{u}(1)$ [Stoilova, Van der Jeugt 2010]

$$|p;\mu) \equiv |\mu) \equiv |\mu)^r = \left| \begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array} \right) =$$

$$\left| \begin{array}{lllllll} \mu_{1r} & \cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots & \mu_{r-1,r} & \mu_{rr} \\ \mu_{1,r-1} & \cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots & \mu_{r-1,r-1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & & \\ \mu_{1,m+1} & \cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & & \\ \mu_{1m} & \cdots & \mu_{m-1,m} & \mu_{mm} & & & & \\ \mu_{1,m-1} & \cdots & \mu_{m-1,m-1} & & & & & \\ \vdots & \iddots & & & & & & \\ \mu_{11} & & & & & & & \end{array} \right)$$

# Decomposition $u(m|n)$ to $\mathfrak{u}(m|n-1)$



$$s_\lambda(\mathbf{x}|\mathbf{y}) = \sum_\sigma s_\sigma(\mathbf{x}|y_1, \ldots, y_{n-1})\; y_n^{|\lambda|-|\sigma|}$$

sum over all $\sigma$ such that $\lambda - \sigma$ is a vertical strip
Transfer to weights:

$$[\mu]^r \to \oplus[\mu']^{r-1}$$

# Basis for $\overline{V}(p)$

Use Gelfand-Zetlin basis (GZ) for $\mathfrak{u}(m|n)$, based on $\mathfrak{u}(m|n) \supset \mathfrak{u}(m|n-1) \supset \cdots \supset \mathfrak{u}(m|1) \supset \mathfrak{u}(m) \supset \mathfrak{u}(m-1) \supset \cdots \supset \mathfrak{u}(1)$ [Stoilova, Van der Jeugt 2010]

$$|p;\mu) \equiv |\mu) \equiv |\mu)^r = \left| \begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array} \right) =$$

$$\left| \begin{array}{llllllll} \mu_{1r} & \cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots & \mu_{r-1,r} & \mu_{rr} \\ \mu_{1,r-1} & \cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots & \mu_{r-1,r-1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & & \\ \mu_{1,m+1} & \cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & & \\ \mu_{1m} & \cdots & \mu_{m-1,m} & \mu_{mm} & & & & \\ \mu_{1,m-1} & \cdots & \mu_{m-1,m-1} & & & & & \\ \vdots & \iddots & & & & & & \\ \mu_{11} & & & & & & & \end{array} \right)$$

# Basis for $\overline{V}(p)$

In such a $\mu$-triangle, all $\mu_{ij} \in \mathbb{Z}_+$, satisfying conditions such as

- betweenness conditions ($1 \leq i \leq j \leq m-1$ or $m+1 \leq i \leq j \leq r-1$)

$$\mu_{i,j+1} \geq \mu_{ij} \geq \mu_{i+1,j+1}$$

- $\theta$-conditions ($1 \leq i \leq m$, $m+1 \leq s \leq r$)

$$\mu_{is} - \mu_{i,s-1} \equiv \theta_{i,s-1} \in \{0, 1\}$$

Vacuum vector $|0\rangle$ is such a triangle of all zeros, has weight $[-\frac{p}{2}, \dots, -\frac{p}{2} | \frac{p}{2}, \dots, \frac{p}{2}]$
Weight of a general vector $|\mu)$ follows from:

$$h_k|\mu) = \left( \mp\frac{p}{2} + \sum_{j=1}^{k} \mu_{jk} - \sum_{j=1}^{k-1} \mu_{j,k-1} \right) |\mu),$$

(minus sign when $1 \leq k \leq m$; plus sign when $m+1 \leq k \leq m+n$)

# Action of para-operators on this basis

$(c_1^+, c_2^+, \ldots, c_r^+)$ is $\mathfrak{u}(m|n)$ tensor of rank $(1, 0, \ldots, 0)$

$$c_j^+ \sim \begin{array}{l} 10\cdots000 \\ 10\cdots00 \\ \cdots \\ 0\cdots0 \\ \cdots \\ 0 \end{array}$$

- row 1, row 2, …, row $j-1$ are zero (start from the bottom)
- other rows are $10\cdots0$

Tensor product rule in $\mathfrak{u}(m|n)$:

$$([\mu]^r) \otimes (10\cdots0) = ([\mu]^r_{+1}) \oplus ([\mu]^r_{+2}) \oplus \cdots \oplus ([\mu]^r_{+r})$$

where

- $([\mu]^r) = (\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr})$
- $([\mu]^r_{\pm k}) = (\mu_{1r}, \ldots, \mu_{kr} \pm 1, \ldots, \mu_{rr})$

# Action via reduced matrix elements

$$(\mu'|c_j^+|\mu) = \left( \begin{array}{c} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{array} \middle| c_j^+ \middle| \begin{array}{c} [\mu]^r \\ |\mu)^{r-1} \end{array} \right)$$

$$= \left( \begin{array}{c} [\mu]^r \\ |\mu)^{r-1} \end{array} ; \begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \middle| \begin{array}{c} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{array} \right) \times ([\mu]^r_{+k}||c^+||[\mu]^r)$$

$$= (\mathfrak{u}(m|n) \text{ CGC}) \times (\text{reduced matrix element})$$

- These special $\mathfrak{u}(m|n)$ CGC's are explicitly known & have been computed recently [Stoilova, Van der Jeugt 2010]
- Unknown expressions:

$$G_k([\mu]^r) = G_k(\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}) = ([\mu]^r_{+k}||c^+||[\mu]^r)$$

# Action via reduced matrix elements (ctd)

$$c_j^+|\mu) = \sum_{k,\mu'} \left( \begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array} ; \begin{array}{l} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \middle| \begin{array}{l} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{array} \right) G_k([\mu]^r) \left| \begin{array}{l} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{array} \right)$$

- add $+1$ in one position $k$ of top row $r$
- from $|\mu)^{r-1}$ to $|\mu')^{r-1}$: add $+1$ in one position at rows $r-1$, $r-2$, …, $j$
- rows 1 upto $j-1$ remain unchanged
- pattern in rhs satisfies betweenness and $\theta$-conditions

$$c_j^-|\mu) = \sum_{k,\mu'} \left( \begin{array}{l} [\mu]^r_{-k} \\ |\mu')^{r-1} \end{array} ; \begin{array}{l} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \middle| \begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array} \right) G_k([\mu]^r_{-k}) \left| \begin{array}{l} [\mu]^r_{-k} \\ |\mu')^{r-1} \end{array} \right).$$

So all that is left is the determination of the unknown functions $G_k$.

# Reduced matrix elements: main result

Very technical computation.
Started from the action:

$$\{c_r^-, c_r^+\}|\mu) = 2h_r|\mu) = \Big(p + 2(\sum_{j=1}^{r} \mu_{jr} - \sum_{j=1}^{r-1} \mu_{j,r-1})\Big)|\mu)$$

Result for $k \leq m$ and $k$ even:

$$G_k(\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}) =$$

$$\left(-\frac{(\mathcal{E}_m(\mu_{kr}+m-n-k)+1)\prod_{j\neq k=1}^{m}(\mu_{kr}-\mu_{jr}-k+j)}{\prod_{j\neq \frac{k}{2}=1}^{\lfloor m/2\rfloor}(\mu_{kr}-\mu_{2j,r}-k+2j)(\mu_{kr}-\mu_{2j,r}-k+2j+1)}\right)^{1/2}$$

$$\times \prod_{j=1}^{n}\left(\frac{\mu_{kr}+\mu_{m+j,r}+m-j-k+2}{\mu_{kr}+\mu_{m+j,r}+m-j-k+2-\mathcal{E}_{m+\mu_{m+j,r}}}\right)^{1/2}$$

Similar expressions for $k \leq m$ and $k$ odd, and for $k > m$.

# Structure of $V(p)$

- All factors under square roots are positive. The only factor that may become zero is

$$p - \mu_{kr} + k - 1 \qquad (k \leq m \text{ and } k \text{ odd})$$

- For $k = 1$ this is $p - \mu_{1r}$; here $\mu_{1r}$ is the largest integer in the first row of the GZ-pattern.
- Starting from the vacuum vector (GZ-pattern with zeros), one can raise the entries in the GZ-pattern by applying the $c_j^+$.
- When $\mu_{1r}$ has reached the value $p$ it can no longer be increased.
- As a consequence, all vectors $|\mu)$ with $\mu_{1r} > p$ belong to the submodule $M(p)$.

**Theorem: $V(p)$ and action**

An orthonormal basis for the space $V(p)$ is given by the vectors $|\mu)$ with $\mu_{1r} \leq p$. The action of the para-operators is as given earlier.

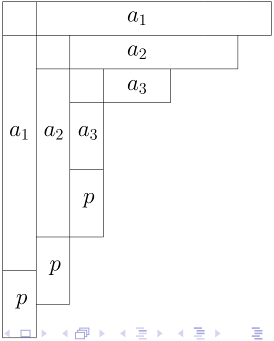
# Character of $V(p)$

$\mu_{1r} \leq p$ is equivalent to $\lambda_1 \leq p$, therefore:

**Theorem:** $\mathrm{char}\, V(p)$

$$\mathrm{char}\, V(p) = \mathbf{x}^{-p/2}\mathbf{y}^{p/2} \sum_{\lambda \in \mathcal{H},\ \lambda_1 \leq p} s_\lambda(\mathbf{x}|\mathbf{y})$$

Character formula? As in $\mathfrak{osp}(1|2n)$ case.




$\mathcal{P}_p$ is set of all partitions of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 + p & a_2 + p & \cdots & a_r + p \end{pmatrix}$$

in Frobenius notation.

# Character formula for $V(p)$

## Theorem: character formula

$$\mathrm{char}\, V(p) = \mathbf{x}^{-p/2}\mathbf{y}^{p/2} \times \frac{\prod_{i,j}(1+x_i y_j) \sum_{\sigma \in \mathcal{H}_p} (-1)^{(|\sigma| - \mathrm{r}(\sigma)(p-1))/2} s_\sigma(\mathbf{x}|\mathbf{y})}{\prod_i (1-x_i) \prod_{i<k}(1-x_i x_k) \prod_j (1-y_j) \prod_{j<l}(1-y_j y_l)}$$

Sum is over all partitions in $\mathcal{H}_p = \mathcal{H} \bigcap \mathcal{P}_p$.

Proof: product rule for supersymmetric Schur functions $s_\lambda(\mathbf{x}|\mathbf{y})$ is the same as product rule for ordinary Schur functions $s_\lambda(\mathbf{x})$ (Littlewood-Richardson rule).

(For ordinary case: identities proved by King in 2007 [King 2013]).

(See another proof of character formula in [Loday & Popov 2009]).

# Outlook

What about combining $m$ parafermions with $n$ parabosons with relative paraboson relation?

- Tolstoy (2014): a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded superalgebra $\mathfrak{osp}(1, 2m|2n, 0)$.
- Representations? Structure?

# Outlook

System with an infinite number of parafermions and parabosons: $f_i$, $b_i$ $(i = 1, 2, \ldots)$?

- Clearly, in the basis $|p;\mu)$ based upon $\mathfrak{u}(m|n)$ one cannot let $m$ and $n$ go to infinity (and use some stability).
- Idea: use a GZ-basis based on the branching

  $$\mathfrak{osp}(2n+1|2n) \supset \mathfrak{u}(n|n) \supset \mathfrak{u}(n|n-1) \supset \mathfrak{u}(n-1|n-1) \supset \mathfrak{u}(n-1|n-2) \supset \mathfrak{u}(n-2|n-2) \supset \cdots \supset \mathfrak{u}(1|1) \supset \mathfrak{u}(1)$$

  Only Lie superalgebras. Branching is known: GZ-basis can be described.
- Action (even for $\mathfrak{u}(n|n)$ generators in this basis) is not yet known.
- Limit for $n$ to infinity should be possible in this basis.

# Overview

1 Introduction: parabosons and parafermions
- Ancient history
- Old history
- Recent history

2 Parastatistics algebra $\mathfrak{osp}(2m+1|2n)$
- The LSA $\mathfrak{osp}(2m+1|2n)$
- Subalgebras of $\mathfrak{osp}(2m+1|2n)$
- Induced module construction
- Basis of $\overline{V}(p)$
- GZ-basis for $\mathfrak{u}(m|n)$
- Action of para-operators on this basis
- Structure of $V(p)$
- Character of $V(p)$

3 Outlook