Localization and the Canonical Commutation Relations

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ABSTRACT

Let $\mathbb{R}[x_i; \partial_j]$ denote the Weyl algebra with generators x_i and ∂_j satisfying the canonical commutation relations $[\partial_i, x_j] = \delta_{ij}$. It is well-known $so(3)$ or $so(1,2)$ can be realized as quadratic polynomial algebras in the x_i and ∂_j $(i, j = 1, 2, 3)$. What is not well-known is that by using localization the converse statements are also true. Here we construct a homomorphism of the Weyl algebra into a certain localization of the universal enveloping algebra, $U(so(1,2))$, of $so(1,2)$. Using this homomorphism we construct representations of the canonical commutation relations from certain representations of $so(1,2)$ with the appropriate symmetry and skew-symmetry properties required by physical considerations. We have also established analogous results in higher dimensions.

Localization

Powerful tool in mathematics to relate different algebraic strucures with underlying similarities.

- Gelfand-Kirillov conjecture (*Dokl. Akad. Nauk SSR* **167** 503 (1966))

  The field of quotients of the universal enveloping algebra of a Lie algebra over an algebraically closed field is isomorphic to some skew field extension of a Weyl algebra. (The Weyl algebra of index n over a field k is $k[x_1, x_2, x_3, \ldots, x_n; \partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \ldots, \partial/\partial x_n]$ where the x_i are coordinate functions of an n dimensional vector space.)

- Semisimple Lie Groups and Associated Semidirect Products

  P. Bozek, M. Havlíček, O. Navrátil, Charles Univ. Preprint (1985): $*$-isomorphism between certain commutative algebraic extensions of the Lie fields of $SO(1,4)$ and the Poincaré group. Similar results for all $SO(p,q)$ groups with $p+q \leq 5$ (p or $q \geq 1$) and for certain q groups and supergroups e.g. $U_q(so(2,1))$, $U_q(\mathfrak{osp}(1|2))$.

$SO(p+2,q)$, $\mathfrak{so}(p+2,q)$ and the Weyl Algebras

Let M denote the real vector space $\mathbb{R}^{p+q+2}$ of dimension $n = p+q+2$ with quadratic form

$Q(x) = x_{-1}^2 + x_0^2 + x_1^2 + x_2^2 + \ .\ . + x_p^2 - x_{p+1}^2 - \ . \ . \ . - x_{p+q}^2$

$\beta_0 = \mathrm{diag}(1,\ 1,\ .\ .\ 1,\ -1,\ -1,\ .\ .\ ,\ -1)$ ($p+2$ entries being $+1$'s and q entries being -1's)

- $SO(p+2,q) = \{g \in SL(n,\mathbb{R}) | g^\dagger \beta_0 g = \beta_0\}$
- Connected component: $SO_0(p+2,q)$
- $\mathfrak{so}(p+2,q)$: $X \in \mathfrak{sl}(n,\mathbb{R})$ such that $X^\dagger \beta_0 + \beta_0 X = 0$
- Weyl Algebra of index m: $\mathbb{R}[x_1, x_2, ...x_m; \partial_1, \partial_2, ...\partial_m]$

Some Facts about $SO_0(p+2,q)$ and $\mathfrak{so}(p+2,q)$

- Basis of $\mathfrak{g} = \mathfrak{so}(p+2,q)$: $\mathbf{L}_{ij}$ $(i,\ j \ = \ -1, 0, \ .\ .\ .\ ,\ p+q,\ i \ < \ j)$. Let $\mathbf{L}_{ij} \ = \ -\mathbf{L}_{ji}$ for $i > j$, and the $\mathbf{L}_{ij}$ satisfy the following commutation relations:

$$[\mathbf{L}_{ab},\ \mathbf{L}_{bc}] \ = \ -e_b\ \mathbf{L}_{ac}$$

with

$$e_{-1} = e_0 = e_1 = e_2 = \ \ldots = e_p = 1,\ e_{p+q} = e_{p+2} = \ \ldots = e_{p+q} = -1.$$

All other commutators vanish.

- Quadratic Casimir Operator: $\mathbf{C}_2 = \frac{1}{2}\sum_{i,j=-1}^{p+q} \mathbf{L}_{ij}\ \mathbf{L}^{ji} \ =$

$$-\frac{1}{2}\sum_{i,j=-1}^{p} \mathbf{L}_{ij}\,\mathbf{L}_{ij} \ - \frac{1}{2}\sum_{i,j=p+1}^{p+q} \mathbf{L}_{ij}\,\mathbf{L}_{ij} \ + \sum_{i=-1}^{p}\ \sum_{j=1}^{q} \mathbf{L}_{i,p+j}\,\mathbf{L}_{i,p+j} \ .$$

- Cartan Involution (β_0 presentation)

  $\theta(g) = g^{-1\dagger}$ for $g \in G = SO_0(p+2,q)$ and the corresponding Cartan involution on $\mathfrak{g}$ is $\theta(X) = -X^\dagger$ $(X \in \mathfrak{g})$.

- Let $\mathcal{K} = \{X \in \mathcal{G} | \theta(X) = X\}$, and $\mathcal{P} = \{X \in \mathcal{G} | \theta(X) = -X\}$.

- Let K and P be the subgroups of G whose Lie algebras are $\mathcal{K}$ and $\mathcal{P}$.

- In β_0 presentation $K = \left\{ g \in G \,\middle|\, g = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \right\}$ where $k_1 \in SO(p+2)$ and $k_2 \in SO(q)$.

- Let A be the one parameter subgroup of G generated by $\mathbf{L}_{-1,p+1}$

- Fact: G is generated by K and A i.e. all possible products of elements of K and A are dense in G.

Embedding of $\mathfrak{so}(p+2,q)$ into $\mathbb{R}\left[x_1, x_2, ...x_{p+q+2}; \partial_1, \partial_2, ...\partial_{p+q+2}\right]$

- G acts naturally on M as pseudo-rotations of pseudo-Euclidean space:

  (i) $SO(p+2)$ rotations with generators $\mathbf{L}_{ij}$ $(i,j=-1,0,...p,\ i<j)$

  (ii) $SO(q)$ rotations with generators $\mathbf{L}_{k\ell}$ $(k,\ell=p+1,...p+q,\ k<\ell)$

  (iii) a hyperbolic rotation in the -1, $p+q$ plane of M with generator $\mathbf{L}_{-1,p+q}$ (scaling transformation).

- This action of G on M gives rise to the left regular action of G on $C^\infty(M)$ i.e.

$$G \ni g \to \pi(g) : \pi(g)f(x) = f(g^{-1}x)$$

  where $x \in M$ and $f(x) \in C^\infty(M)$. Denote by $d\pi$ the corresponding action of $\mathfrak{g}$ on $C^\infty(M)$.

- Proposition: $d\pi$ defines an isomorphism of $\mathfrak{so}(p+2,q)$ onto its image in $\mathbb{R}[x_1, x_2, ...x_n; \partial_1, \partial_2, ...\partial_n]$ for $n = p+q+2$.

SO(2,1) and Some of its Subgroups

- $$SO(2) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \middle| \theta \in [-\pi, \pi) \right\}$$

- $$SO_0(1,1) = \left\{ \begin{bmatrix} \mathrm{ch}\beta & \mathrm{sh}\beta \\ \mathrm{sh}\beta & \mathrm{ch}\beta \end{bmatrix} \middle| \beta \in \mathbb{R} \right\}$$

- Some Subgroups of $SO_0(2,1)$:

$$K = \left\{ k(u) = \begin{bmatrix} u_1 & u_2 & 0 \\ -u_2 & u_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \; u = \begin{pmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{pmatrix} \in SO(2) \right\}$$

$$A = \left\{ a(\tau) = \begin{bmatrix} \mathrm{ch}(\tau) & 0 & \mathrm{sh}(\tau) \\ 0 & 1 & 0 \\ \mathrm{sh}(\tau) & 0 & \mathrm{ch}(\tau) \end{bmatrix} \middle| \tau \in \mathbb{R} \right\}$$

- Some Additional Subgroups

$$H = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{p}_0 & \hat{p} \\ 0 & \hat{p} & \hat{p}_0 \end{bmatrix} \middle| \begin{pmatrix} \hat{p}_0 & \hat{p} \\ \hat{p} & \hat{p}_0 \end{pmatrix} \in SO_0(1,1) \right\}$$

$$N = \left\{ n(a) = \begin{bmatrix} 1 - \frac{a^2}{2} & a & \frac{a^2}{2} \\ -a & 1 & a \\ -\frac{a^2}{2} & a & 1 + \frac{a^2}{2} \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

$$\tilde{N} = \left\{ \tilde{n}(x) = \begin{bmatrix} 1 - \frac{x^2}{2} & -x & -\frac{x^2}{2} \\ x & 1 & x \\ \frac{x^2}{2} & x & 1 + \frac{x^2}{2} \end{bmatrix} \middle| x \in \mathbb{R} \right\}$$

$$M = C_A(K) = \left\{ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Decompositions of SO(2,1)

- Iwasawa Decomposition

$$SO_0(2,1) \cong KAN$$

- Bruhat Decomposition

$$SO_0(2,1) \cong \tilde{N}MAN$$

- Parabolic Subgroup: $P = MAN$ (Langlands Decomposition)
- Hannabuss/Sekiguchi Decomposition

(Hannabuss: $SO(1,n)$, 1971; Sekiguchi: $SO(p,q)$, $Sp(n,\mathbb{R})$ etc., 1980)

$$SO_0(2,1) \cong H'AN \quad (H' = N_H(G))$$

- Let $Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $H' = H \cup HQ$ where $H = SO_0(1,1)$.

Relationship between Various Decompositions

- For $h \in SO_0(1,1)$, Iwasawa decomposition gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{p}_0 & \hat{p} \\ 0 & \hat{p} & \hat{p}_0 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & 0 \\ -u_2 & u_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{ch}(\tau) & 0 & \mathrm{sh}(\tau) \\ 0 & 1 & 0 \\ \mathrm{sh}(\tau) & 0 & \mathrm{ch}(\tau) \end{bmatrix} \begin{bmatrix} 1-\frac{a^2}{2} & a & \frac{a^2}{2} \\ -a & 1 & a \\ -\frac{a^2}{2} & a & 1+\frac{a^2}{2} \end{bmatrix}$$

- Solve for $\hat{p}_0$, $\hat{p}$, τ and a in terms of u_1, $u_2 \in K/M$:

$$\begin{aligned} \hat{p}_0 &= \frac{1}{u_1} \qquad \Rightarrow u_1 > 0 \\ \hat{p} &= \frac{u_2}{u_1} \\ e^{\tau} &= \frac{1}{u_1} \\ a &= u_2 \end{aligned}$$

- Similarly Iwasawa decomposition for $h' \in H\ Q$ gives:

$$\begin{aligned} \hat{p}_0 &= -\frac{1}{u_1} \qquad \Rightarrow u_1 < 0 \\ \hat{p} &= -\frac{u_2}{u_1} \\ e^{\tau} &= -\frac{1}{u_1} \\ a &= -u_2 \end{aligned}$$

- Relationship between the Bruhat and Iwasawa decompositions of $SO_0(2,1)$:

$$\begin{bmatrix} 1-\frac{x^2}{2} & -x & -\frac{x^2}{2} \\ x & 1 & x \\ \frac{x^2}{2} & x & 1+\frac{x^2}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} u_1 & u_2 & 0 \\ -u_1 & u_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{ch}(\tau) & 0 & \mathrm{sh}(\tau) \\ 0 & 1 & 0 \\ \mathrm{sh}(\tau) & 0 & \mathrm{ch}(\tau) \end{bmatrix} \begin{bmatrix} 1-\frac{a^2}{2} & a & \frac{a^2}{2} \\ -a & 1 & a \\ -\frac{a^2}{2} & a & 1+\frac{a^2}{2} \end{bmatrix}$$

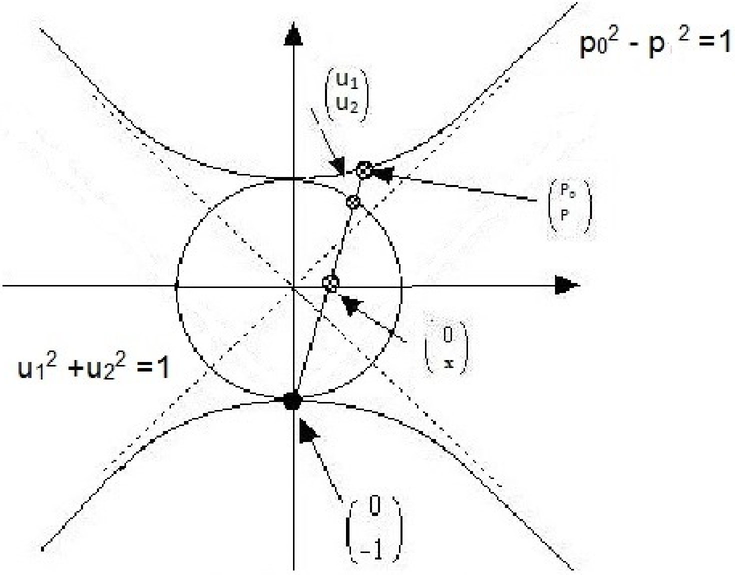
$$\Longrightarrow$$

$$x = \frac{u_2}{1+u_1}.$$

$$a = \frac{u_2}{2}$$

$$e^{\tau} = \frac{2}{1+u_1}.$$

- Stereographic projections showing relationship between various decompositions



Generalized Principal Series of $SO_0(1,2)$

- Characters of A: A character of A is a homomorphism $\chi : A \to \mathbb{C}$. Let A^* be the space of all characters.

  For $\sigma \in \mathbb{C}$ let $\chi_\sigma \in A^* \ni \chi_\sigma(a(\tau)) = e^{-(\sigma+1/2)\tau}$

- Irreducible Representations of M: $\rho(I) = (+1)$ $(M = \{I\})$. (So $\rho = 1$.)

- Representations of the Inducing Subgroup

  Consider $1 \otimes \chi_\sigma : MA \to \mathbb{C}$ and extend this mapping to a mapping from the parabolic subgroup $P = MAN$ to $\mathbb{C}$ by requiring that it act trivially on N: for $p = ma(\tau)n \in P$

  $$1 \otimes \chi_\sigma \otimes 1 : P \to \mathbb{C} \ni (1 \otimes \chi_\sigma \otimes 1)(p) = \chi_\sigma(a(\tau)).$$

- Generalized Principal Series

  $$\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1) = \{\mathcal{F} \in C^\infty(G) | \mathcal{F}(gp) = (1 \otimes \chi_\sigma \otimes 1)(p)\mathcal{F}(g), g \in G, p \in P\}$$
  $$= \{\mathcal{F} \in C^\infty(G) |\ \mathcal{F}(gp) = e^{-(\sigma+1/2)\tau}\mathcal{F}(g), g \in G, p = ma(\tau)n \in P\}.$$

  G acts in this space by left translation: $\mathcal{F}(g) \quad \xrightarrow{g_1} \quad \mathcal{F}(g_1 g)$.

Parallelizations of $\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1)$ associated with the decompositions

- $L^{(\sigma)}(G/P)$ is the line bundle over $u \in G/P$ whose elements are equivalence classes of ordered pairs $(u, s) \in G \times \mathbb{C}$ with respect to the equivalence relation $(p = ma(\tau)n \in P)$:

$$(u, s) \sim (up, \rho(m)\chi_\sigma(a(\tau))s).$$

- Let $C^\infty(G/P, L^{(\sigma)})$ be the space of all C^∞ sections of $L^{(\sigma)}(G/P)$. The Iwasawa decomposition gives us the isomorphism

$$C^\infty(G/P, L^{(\sigma)}) \ \ni \ \mathbf{\Psi} \longrightarrow \phi \ \in C^\infty(K/M).$$

This defines a parallelization of the bundle $L^{(\sigma)}(G/P)$, which we call the "spherical parallelization."

Action of Induced Representation on $C^\infty(K/M)$

We have the following formula for the action of the representation $\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1)$ on $C^\infty(K/M)$: (spherical parallelization)

$$\pi^\sigma(g)\phi(u) = e^{-(\sigma+1/2)\tau}\phi(g^{-1}u) = e^{-(\sigma+1/2)\tau}\phi(g^{-1}u)$$

where

$$u \in K/M \;\stackrel{\sim}{=}\; K,$$

and

$$g^{-1}k(u) = k(u')a(\tau)n,$$

with

$$g^{-1}u := u'.$$

Other parallelizations and their relationship to the spherical parallization

The Hannabuss/Sekiguchi and Bruhat decompositions lead to:

- "Hyperbolic parallelization"

$$\tilde{\phi}(\hat{p}_0, \hat{p}) = \left(\frac{1}{u_1}\right)^{-(\sigma+1/2)} \phi(u) \qquad (u_0 > 0)$$

$$\tilde{\phi}(-\hat{p}_0, \hat{p}) = \left(\frac{1}{-u_1}\right)^{-(\sigma+1/2)} \phi(u) \qquad (u_0 < 0)$$

- "Flat parallelization"

$$\psi(x) = \left(\frac{2}{1+u_1}\right)^{-(\sigma+1/2)} \phi(u)$$

Some Results about the action of $\mathfrak{g} = so(2,1)$ in the Representations $\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1)$

Let $L_{ij}f = d\pi^\sigma(\mathbf{L}_{ij})f := \frac{d\pi^\sigma(e^{t\mathbf{L}_{ij}})}{dt}\Big|_{t=0} f$ for f a C^∞ function on one of the spaces defined by the any one of the three parallelizations. The results are summarized in the Table.

TABLE. Presentations of the infinitesmal generators in various parallelizations.($S = x\partial_x$, and $w = \sigma + \frac{1}{2}$)

generator	spherical	flat	hyperbolic
$\mathbf{L}_{-1,1}$	$\sin\theta\partial_\theta + w\cos\theta$	$(S+w)$	$\mathrm{sh}\beta\partial_\beta + w\mathrm{ch}\beta$
$\mathbf{L}_{0,1}$	$\cos\theta\partial_\theta - w\sin\theta$	$\frac{1}{2}(1-x^2)\partial_x + x(S+w)$	$-\partial_\beta$
$\mathbf{L}_{-1,0}$	$-\partial_\theta$	$\frac{1}{2}(1+x^2)\partial_x - x(S+w)$	$\mathrm{ch}\beta\partial_\beta - w\mathrm{sh}\beta$

The Generator $\mathbf{P}_0$:

From the Table we obtain (with $x = \frac{1}{2}x_0$) for the action of the three basic generators of $SO(2,1)$ in the representation $d\pi^\sigma(so(2,1))$:

$$d\pi^\sigma(\mathbf{L}_{-10}) = -\left(1 - \frac{x_0^2}{4}\right)\partial_0 - \frac{1}{2}x_0(S+w) = -\partial_0 - \frac{1}{4}x_0^2\partial_0 - \frac{w}{2}x_0 \quad (1a)$$

$$d\pi^\sigma(\mathbf{L}_{01}) = -\left(1 + \frac{x_0^2}{4}\right)\partial_0 + \frac{1}{2}x_0(S+w) = -\partial_0 + \frac{1}{4}x_0^2\partial_0 + \frac{w}{2}x_0 \quad (1b)$$

$$d\pi^\sigma(\mathbf{L}_{-1,1}) = (S+w) = (x_0\partial_0 + w) \quad (1c)$$

From these expressions we see that

$$\partial_0 = -\frac{1}{2}\left[d\pi^\sigma(\mathbf{L}_{-10}) + d\pi^\sigma(\mathbf{L}_{01})\right] \quad (2)$$

So define the abstract translation generator $\mathbf{P}_0$ to be:

$$\mathbf{P}_0 := -\frac{1}{2}(\mathbf{L}_{-10} + \mathbf{L}_{01}) \quad (3)$$

The Generator $\mathbf{Q}_0$:

Similarly, based on Eq. (1c) we define $\mathbf{Q}_0$ as follows. Replace w by an indeterminate Y and take as the defining equation for the position operator $\mathbf{Q}_0$ the following:

$$\mathbf{L}_{-1,1} = \mathbf{Q}_0 \mathbf{P}_0 + Y \cdot \mathbf{I} \tag{4}$$

where $\mathbf{I}$ is the identity operator in the abstract enveloping algebra of $so(2,1)$. Solving this equation for $\mathbf{Q}_0$ (on the left!) we take:

$$\mathbf{Q}_0 := -2(\mathbf{L}_{-1,1} - Y \cdot \mathbf{I}) \left(\frac{\mathbf{I}}{\mathbf{L}_{-10} + \mathbf{L}_{01}} \right). \tag{5}$$

Consistency Condition: Y must be such that

$$-\mathbf{C}_2 = \mathbf{L}_{01}^2 + \mathbf{L}_{-1,1}^2 - \mathbf{L}_{-1,0}^2 = (Y^2 - Y) \cdot \mathbf{I}. \tag{6}$$

Proposition I: Let $\mathfrak{Loc}(U(so(2,1); Y)$ be a commutative algebraic extension (in Y) of a localization of $U(so(2,1))$ for which

$$\mathbf{Q}_0 := -2(\mathbf{L}_{-1,1} - Y \cdot \mathbf{I}) \left(\frac{\mathbf{I}}{\mathbf{L}_{-10} + \mathbf{L}_{01}} \right)$$

and

$$\mathbf{P}_0 := -\frac{1}{2}(\mathbf{L}_{-10} + \mathbf{L}_{01}).$$

Then we have

$$[\mathbf{P}_0, \mathbf{Q}_0] = \mathbf{I}.$$

Proposition II: Let $*$ denote the usual star structure on $U(so(2,1))$. Extend this $*$ structure to one on $\mathfrak{Loc}(U(so(2,1); Y)$ for which

$$Y^\dagger + Y = \mathbf{I}.$$

Then

$$\mathbf{Q}_0^\dagger = \mathbf{Q}_0.$$

Main Theorem:

Let Y commutes with all elements of $U(so(2,1))$ and be such that

$$\mathbf{L}_{01}^2 + \mathbf{L}_{-1,1}^2 - \mathbf{L}_{-1,0}^2 = (Y^2 - Y) \cdot \mathbf{I}.$$

We define $\tau : \mathbb{R}[x_0; \partial_0] \to \mathfrak{Loc}(U(so(2,1); Y)$ so that

$$\tau(\partial_0) = \mathbf{P}_0 \quad ; \quad \tau(x_0) = \mathbf{Q}_0$$

and extend this mapping by linearity to all of $\mathbb{R}[x_0; \partial_0]$. Then τ is a homomorphism from $\mathbb{R}[x_0; \partial_0]$ into $\mathfrak{Loc}(U(so(2,1); Y)$ which preserves the $*$ structure on $\mathbb{R}[x_0; \partial_0]$ provided

$$Y^\dagger + Y = \mathbf{I}.$$

Proof: The homomorphism property follows from the Proposition 1. By preserves the $*$ structure we mean

$$\tau(x_0)^\dagger = \tau(x_0^\dagger) \quad , \quad \tau(\partial_0)^\dagger = \tau(\partial_0^\dagger) \ .$$

These are an immediate consequence of Proposition 2.

Applications:

In order to construct representations of $\mathbb{R}[x_0; \partial_0]$ out of representations of $U(so(2,1))$ by using our just established results, we need the following result:

Lemma: Suppose $f : R \longrightarrow R_1$ is a ring homomorphism and Q is a left (resp. right) quotient ring of R with respect to S. If $f(s)$ is a unit in R_1 for every $s \in S$, then there exists a (unique) ring homomorphism $g : Q \longrightarrow R_1$ which extends f.

Roughly speaking, this criterion implies for our case that the action of $\mathbf{P}_0$ in a given representation $d\pi$ of $U(so(2,1))$ must be invertible. For Hilbert space representations this means that zero should lie in the resolvent set of $d\pi(\mathbf{P}_0)$.

Likely candidates for such representations are highest or lowest weight representations. The reason for this is because:

$d\pi(\mathbf{P}_0)$ is a positive, self adjoint operator $\iff$ $d\pi(so(2,1))$ is an (infinitesimally) unitarizable lowest weight representation of $so(2,1)$.

Unitary Irreducible Representations of $SO_0(2,1)$

- x x trivial representation
- x ✓ principal series: $\mathrm{Ind}_P^G(1 \otimes \chi_{\sigma=i\rho} \otimes 1)$ $(0 < \rho < \infty)$
- ✓x D_0^+ and D_0^-: $V_K = V^+ \oplus V^- \subset \mathrm{Ind}_P^G(1 \otimes \chi_{\sigma=-\frac{1}{2}} \otimes 1)$ (V^+ positive & V^- negative energy). $U(\mathfrak{g})$ acts on the quotient module V_K/Y_0 via the quotient action of $\mathfrak{g}$ on V_K/Y_0 and $D_0^+ = V^+/Y_0$, $D_0^- = V^-/Y_0$.

  Furthermore
  $$0 \to Y_0 \xrightarrow{\iota} V_K \xrightarrow{\pi} V_K/Y_0 \to 0$$
  is short exact sequence of $\mathfrak{g}$ equivariant $U(\mathfrak{g})$ module homomorphisms.
- ✓x discrete series: D_ℓ^+ and D_ℓ^- $(\ell = 1, 2, 3, ...)$: (*ditto*)
- ? x complementary series: $\mathrm{Ind}_P^G(1 \otimes \chi_{\sigma=-c} \otimes 1)$ $(0 < c \le \frac{1}{2})$

- In order to find a representation with two ✓s, i.e. it satisfies all conditions of the Main Theorem and also of the Lemma, we found it necessary to go to a covering space of $SO_0(2,1)$.
- Theorem (Ørsted, Segal (IES)): Let $\bar{\bar{G}}$ denote the four-fold cover $G = SO_0(2,1)$ and consider the representation of $\bar{\bar{G}}$ induced from $\bar{P}$ of weight $w = \frac{1}{2}$ $(\sigma = 0)$. This representation is equivalent to the direct sum of two irreducible, unitary positive and negative energy irreducible representations of $\bar{\bar{G}}$.
- Proposition: Let $\mathcal{H}^+_{1/2}$ and $\mathcal{H}^-_{1/2}$ respectively denote the positive and negative energy subspaces of the Hilbert space associated with the representation in the above Theorem. Then on either $\mathcal{H}^+_{1/2}$ or $\mathcal{H}^-_{1/2}$ $\mathbf{P}_0$ acts as a skew-symmetric operator and $\mathbf{Q}_0$ acts as a well-defined symmetric operator.
- We have on $\mathcal{H}^+_{1/2}$ or $\mathcal{H}^-_{1/2}$ representations of $\mathbb{R}[x_0, \partial_0]$ in which x_0 and ∂_0 act, respectively, as symmetric and skew-symmetric operators.

Concluding Remarks

- We also have results for $U_q(so(2,1))$ i.e. we construct representations of $\mathbb{R}[x_0, \partial_0]$ from $U_q(so(2,1))$ modules.

- Almost everything generalizes to $SO(2,q)$ and we have similar results for representations i.e. representations for which x_i and ∂_j act as symmetric and skew-symmetric operators.

- We have analogous but somewhat different results for $SO(1,n)$.

- Generalizations of Eqs.(3) and (5) to $SO(p,q)$ $(p > 1)$.

- What happens under contraction of group representations e.g. when $\mathfrak{g} \overset{c\to\infty}{\longrightarrow} \mathfrak{h}$, what happens to the representations of $\mathbb{R}[x_0, \partial_0]$?