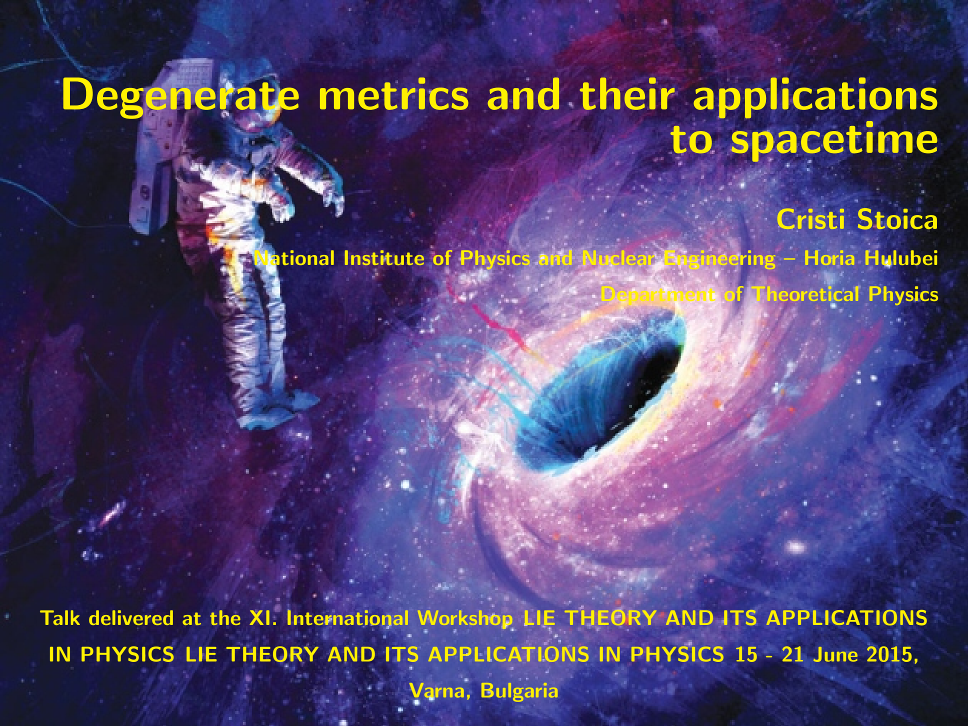

Degenerate metrics and their applications to spacetime

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Abstract

The Lie groups preserving degenerate quadratic forms appear in various contexts related to spacetime. The homogeneous Galilei group is the intersection of two such groups. The structure group of sub-Riemannian geometry and of singular semi-Riemannian geometry, as well as of some submanifolds of semi-Riemannian manifolds, is of this kind. Such groups are shown to replace the Lorentz group at a very large class of singularities in General Relativity. Also, these groups are shown to be fundamental in Kaluza-Klein theory and in gauge theory.

Degenerate quadratic forms

Let (V, q) be a vector space with a **quadratic form** q.
Let g be the symmetric bilinear form associated to q by polarization,

$$g(u, v) = \frac{1}{4}\left(q(u + v) - q(u - v)\right)$$

for any $u, v \in V$.
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The signature of q is (r, s, t) if g can be diagonalized to

$$g = \begin{pmatrix} -I_t & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & O_r \end{pmatrix}.$$

We denote by $O(t, s, r)$ the group of transformations of the vector space $\mathbb{R}^n$ which preserve this bilinear form.

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The metric g and the quadratic form q are called **degenerate** if $r > 0$.

Degenerate quadratic forms

Examples:

- The orthogonal group $O(n) = O(0, n, 0)$ preserves a non-degenerate form $\operatorname{diag}(1, \dots, 1)$.
- The Lorentz group $O(1, 3) = O(1, 3, 0)$ preserves the Lorentz metric $\operatorname{diag}(-1, 1, 1, 1)$, which is non-degenerate.
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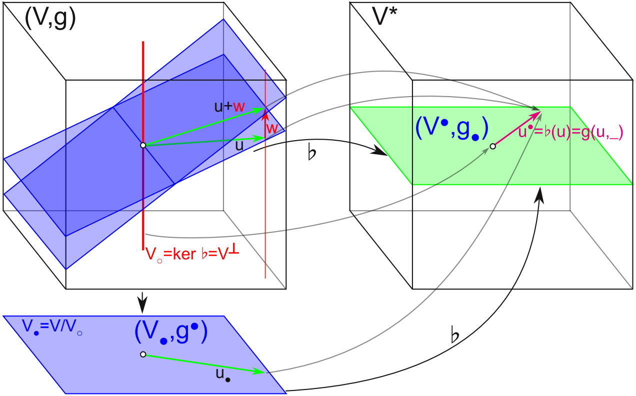
- The interesting cases will be in the following $O(t, s, r)$ with $r > 0$.



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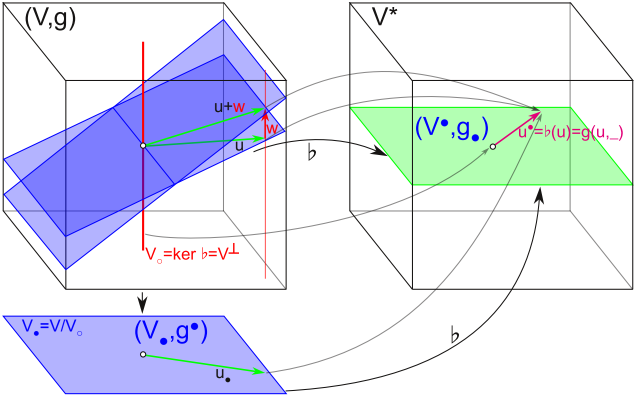
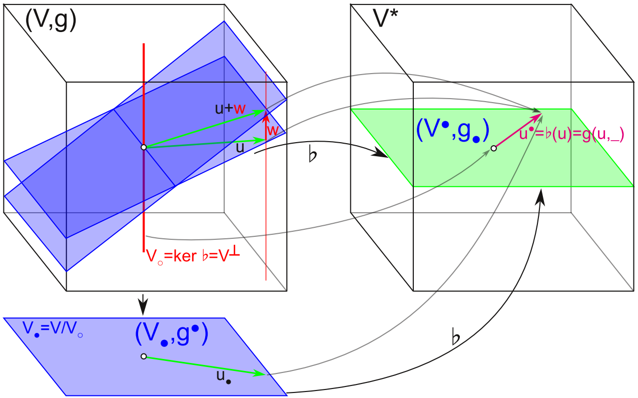


(V, g) is an inner product vector space. The morphism $\flat : V \to V^*$ is defined by $u \mapsto u^\bullet := \flat(u) = u^\flat = g(u, \_)$.



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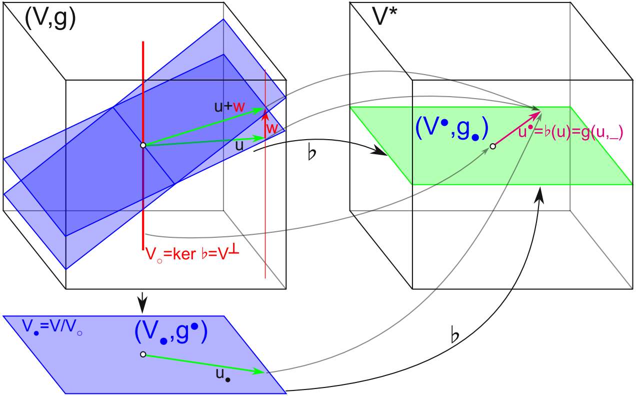

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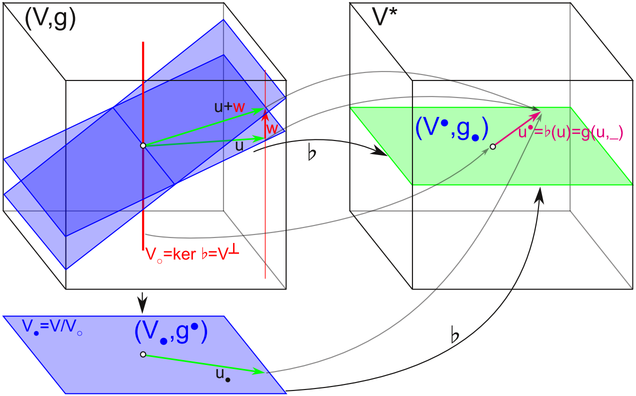

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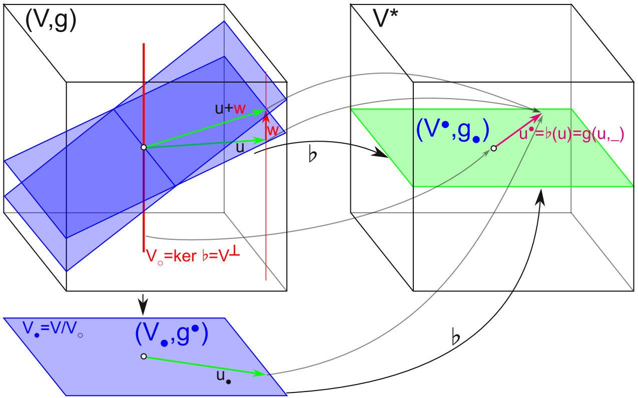

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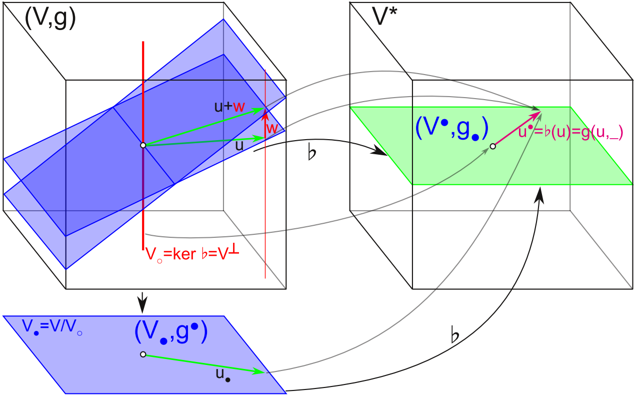

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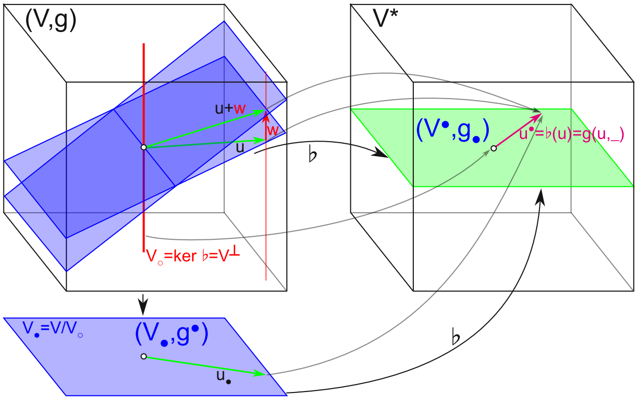

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and the **metric giving the durations**

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We can't combine these two metrics in a 4-dimensional metric, since $g_{\text{time}\,ij}$ is defined on the entire spacetime, while $g_{\text{space}\,ij}$ on a subspace.

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where v has the dimensions of a speed.

But this would make v a special, absolute speed, and this is not allowed in Galilean relativity (unlike in the relativistic spacetime).

Degenerate metrics in sub-Riemannian geometry

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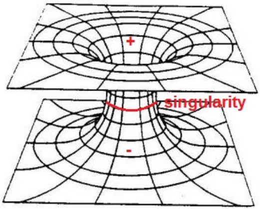
Its study was originated in

G. Vrănceanu. "Sur les espaces non holonomes". *C.R. Acad. Sci. Paris* 183 (1926);
G. Vrănceanu. "Studio geometrico dei sistemi anolonomi". *Annali di Matematica Pura ed Applicata* 6.1 (1929);
G. Vrănceanu. "Sur les trois points de vue dans l'étude des espaces non holonomes". *CR Acad. Sci. Paris* 188 (1929);
G. Vrănceanu. "Sur une théorie unitaire non holonome des champs physiques". *J. Phys. Radium* 7.12 (1936)
P.K. Rashevskii. "About connecting two points of complete nonholonomic space by admissible curve [Russian]". *Uch. Zapiski Ped. Inst. Libknexta* 2 (1938);
Wei-Liang Chow. "Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung". *Math. Ann* 117.1 (1939);
Lars Hörmander. "Hypoelliptic second order differential equations". *Acta Mathematica* 119.1 (1967);

Mikhael Gromov. *Carnot-Carathéodory spaces seen from within*. Springer, 1996

Degenerate metrics in singular semi-Riemannian geometry



Einstein disliked for long time singularities, and rejected the idea of black holes predicted by his theory. However, when he and Rosen used wormholes to explain the electric charge, they obtained a singularity. They mentioned the possibility that the infinities can be eliminated from the equations, without giving an invariant solution which makes geometric and physical sense.

A. Einstein and N. Rosen. "The Particle Problem in the General Theory of Relativity". *Phys. Rev.* 48.1 (1935)

Degenerate metrics in singular semi-Riemannian geometry

Spaces with degenerate metrics were studied by

Dan Barbilian. "Galileische Gruppen und quadratische Algebren". *Bull. Math. Soc. Roumaine Sci.* (1939);

G. C. Moisil. "Sur les géodésiques des espaces de Riemann singuliers". *Bull. Math. Soc. Roumaine Sci.* 42 (1940);

K. Strubecker. "Differentialgeometrie des isotropen Raumes. I. Theorie der Raumkurven". *Sitzungsber. Akad. Wiss. Wien, Math.-Naturw. Kl., Abt. IIa* 150 (1941);

K. Strubecker. "Differentialgeometrie des isotropen Raumes. II. Die Flächen konstanter Relativkrümmung $K = rt - s^2$". *Math. Z.* 47.1 (1942);

K. Strubecker. "Differentialgeometrie des isotropen Raumes. III. Flächentheori *Math. Z.* 48.1 (1942);

K. Strubecker. "Differentialgeometrie des isotropen Raumes. IV. Theorie der flächentreuen Abbildungen der Ebene". *Math. Z.* 50.1 (1944);

G. Vrănceanu. "Sur les invariants des espaces de Riemann singuliers". *Disqu. Math. Phys. București* 2 (1942)

Degenerate metrics in singular semi-Riemannian geometry

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D. Kupeli. "On Null Submanifolds in Spacetimes". *Geom. Dedicata* 23.1 (1987);

D. Kupeli. "Degenerate Manifolds". *Geom. Dedicata* 23.3 (1987);

D. Kupeli. "Degenerate Submanifolds in Semi-Riemannian geometry". *Geom. Dedicata* 24.3 (1987);

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The results from

O. C. Stoica. "On Singular Semi-Riemannian Manifolds". *Int. J. Geom. Methods Mod. Phys.* 11.5 (2014)

apply to changing signature, are invariant, and don't rely on a particular choice. The particular cases of Kupeli and Riemann are obtained.

Fiber bundles – metric on the base space

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If M is a semi-Riemannian manifold with metric g, then the total space E has a structure of a **singular semi-Riemannian manifold** $(E, \tilde{g})$, where the degenerate metric $\tilde{g}$ is uniquely defined as the pull-back of g,

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Let $V < TE$ be the *vertical bundle*, $V := \ker(\mathrm{d}\pi)$. Then, at every point $p \in E$, the vertical tangent space V_p is the radical of $\tilde{g}_p$. So we have

$$\ker(\mathrm{d}\pi) = \ker \tilde{g}.$$

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The metrics g and h are in the same relation as the metrics $g_{\text{time}\,ij}$ and g^{ij}_{space} in the Galilean spacetime, since the radical of one of them is the distribution on which the other one is defined.

Gauge theory and Kaluza-Klein theory

Let (E, M, π, F, G) be a *principal G-bundle*, where the typical fiber F is a *G-torsor* (hence is diffeomorphic with G and G acts freely and transitively on F), and $\mathfrak{g}$ is the Lie algebra of the group G.

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The metric $\tilde{g}$ induces a metric $\hat{g}$ on the horizontal distribution H.
From H, V and $\hat{g}$ one can recover $\tilde{g}$ as

$$\tilde{g}(X, Y) = \hat{g}(\pi_H X, \pi_H Y).$$

Gauge theory and Kaluza-Klein theory

The Kaluza-Klein theory can be seen now as combining the two metrics $\hat{g}$ on H and h on V in a metric on E, by

$$g^E(X, Y) = \hat{g}(\pi_H X, \pi_H Y) + h(\pi_V X, \pi_V Y).$$

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We can identify E at least locally with the product $E = M \times F$. Then to obtain the metric g^E on $M \times F$ we apply a transformation that leaves the fibers invariant, and projects the horizontal space H_p to the space $T_p M$,

$$S = \begin{pmatrix} I_4 & A \\ 0 & I_d \end{pmatrix}, \tag{1}$$

where $A = A_a^\mu$ is the connection determined by H, and $d = \dim G$.

Gauge theory and Kaluza-Klein theory

Then,

$$g_0^E = Sg^E S^T = \begin{pmatrix} g_{ab} + h_{\mu\nu} A_a^\mu A_b^\nu & h_{\mu\beta} A_a^\mu \\ h_{\alpha\nu} A_b^\nu & h_{\alpha\beta} \end{pmatrix}, \tag{2}$$

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where g_{ab} is the Lorentzian metric on M.

We recover thus the generalized Kaluza-Klein theory for an arbitrary non-abelian gauge group (

R Kerner. *Generalization of the Kaluza–Klein theory for an arbitrary non-abelian gauge group.* Tech. rep. Univ., Warsaw, 1968).

Gauge theory and Kaluza-Klein theory

To obtain the original Kaluza-Klein theory, which unifies gravity with electromagnetism, one takes $G = U(1)$ and $h = 1$:

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By imposing the condition that g_0^E satisfies the vacuum Einstein equation $\mathrm{Ric}(g^{(5)}) = 0$, *i.e.* that the five-dimenisonal manifold is Ricci flat, one obtains the **Einstein-Maxwell equations**, that is, the source-free Maxwell equations,

Gauge theory and Kaluza-Klein theory

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$$T_{ab} = \frac{1}{\mu_0}\left(F_{as}F_b{}^s - \frac{1}{4}F_{st}F^{st}g_{ab}\right) \tag{4}$$

sourced by the electromagnetic field.

Gauge theory and Kaluza-Klein theory

On the one hand, considering a metric h on V and identifying the metric on E with

$$g^E = \hat{g} + h$$

allows us to obtain from the vacuum $4 + d$-dimensional Einstein equation the source-free Einstein-Maxwell (for $G = U(1)$) and Einstein-Yang-Mills equations.

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On the other hand, it is natural to consider the metrics $\hat{g}$ and h as of different nature (as in the case of Galilean spacetime). This implies that the extra dimensions can't be detected by measuring distances along them, and this is in accord with the current experimental results without needing to make them compact to undetectable sizes.

Signature change in cosmology

In some cosmological models in General Relativity, the initial singularity of the Big Bang is replaced, by making the metric of the early Universe Riemannian. Such models, constructed in connection to the Hartle-Hawking no-boundary approach to Quantum Cosmology, assume that the metric was Riemannian, and it changed, becoming Lorentzian, when traversing a hypersurface, on which the metric becomes degenerate

A. D. Sakharov. "Cosmological Transitions with a Change in Metric Signature". *Sov. Phys. JETP* 60 (1984),
G. F. R. Ellis et al. "Change of Signature in Classical Relativity". *Classical Quantum Gravity* 9 (1992),
S. A. Hayward. "Signature Change in General Relativity". *Classical Quantum Gravity* 9 (1992),
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T. Dray, C. A. Manogue, and R. W. Tucker. "Particle production from signature change". *Gen. Relat. Grav.* 23.8 (1991);

T. Dray et al. "Gravity and Signature Change". *Gen. Relat. Grav.* 29.5 (1997).

Singular general relativity

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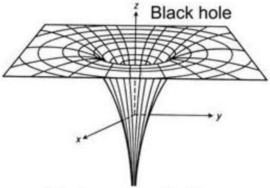
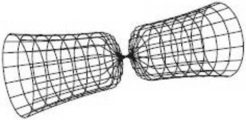
There are two types of singularities:



Malign singularity



Benign singularity

1. **Malign singularities**: some of the components $g_{ab} \to \infty$.
2. **Benign singularities**: g_{ab} are smooth and finite, but $\det g \to 0$.

Singular general relativity

The main problem with the singularities is that the mathematics normally used for general relativity breaks down.

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Connection:

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Curvature:

$$R^{d}{}_{abc} = \Gamma^{d}{}_{ac,b} - \Gamma^{d}{}_{ab,c} + \Gamma^{d}{}_{bs}\Gamma^{s}{}_{ac} - \Gamma^{d}{}_{cs}\Gamma^{s}{}_{ab}$$

Einstein tensor:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

$$R_{ab} = R^{s}{}_{asb}, \; R = g^{pq} R_{pq}$$

Singular general relativity

However, recently, I generalized the geometric theory of spacetime to include **benign singularities**

O. C. Stoica. "On Singular Semi-Riemannian Manifolds". *Int. J. Geom. Methods Mod. Phys.* 11.5 (2014);

O. C. Stoica. "Warped Products of Singular Semi-Riemannian Manifolds". *Arxiv preprint math.DG/1105.3404* (2011);

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For such singularities, there are finite geometric descriptions.

Moreover, the equations which previously gave infinite or undefined quantities, now can be rewritten in terms of finite quantities only.

Singular general relativity

Basically, what I did was to replace some geometric objects which become infinite at singularities, with others "synonymous" with them, but which remain finite:

Singular	**Non-Singular**	**When g is...**
$\Gamma^{c}{}_{ab}$ (2-nd)	Γ_{abc} (1-st)	smooth
$R^{d}{}_{abc}$	R_{abcd}	semi-regular
$R_{ab} = R^{s}{}_{asb}$	$R_{ab}\sqrt{\lvert\det g\rvert}^{W}$, $W \leq 2$	semi-regular
$R = g^{st}R_{st}$	$R\sqrt{\lvert\det g\rvert}^{W}$, $W \leq 2$	semi-regular
Ric	$\mathrm{Ric} \circ g$	quasi-regular
R	$Rg \circ g$	quasi-regular

The Friedmann-Lemaître-Robertson-Walker spacetime

In the formulation I proposed, the solutions given by Friedmann-Lemaître-Robertson-Walker extends beyond the Big-Bang singularity, and the geometric and physical quantities stay finite.

O. C. Stoica. "The Friedmann-Lemaître-Robertson-Walker Big Bang Singularities are Well Behaved". *Int. J. Theor. Phys.* (2015);
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and to a large class of Big-Bang solutions which remain finite, and satisfy in addition Penrose's **Weyl curvature hypothesis**

O. C. Stoica. "On the Weyl Curvature Hypothesis". *Ann. of Phys.* 338 (2013)

Singular general relativity

Einstein's equation is

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}.$$

and a generalized version I obtained is

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which works at a class of singularities too.
This is equation actually obtained when deriving Einstein's equation, but one should not divide by $\sqrt{-g}$, which at singularities becomes 0.

Black hole singularities

But the solution I used for the Big-Bang, where the singularities were benign, seemed not to apply to the black hole singularities, which are malign.

Black hole singularities

But the solution I used for the Big-Bang, where the singularities were benign, seemed not to apply to the black hole singularities, which are malign.

In particular, the Schwarzschild's solution has a singularity at $r = 0$, and an apparent singularity at $r = 2m$.

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2,$$

K. Schwarzschild. "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". *Sitzungsber. Preuss. Akad. d. Wiss.* (1916);
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in the coordinates (ξ, τ), $\begin{cases} r &= \tau^2 \\ t &= \xi\tau^4 \end{cases}$ becomes:

$$ds^2 = -\frac{4\tau^4}{2m - \tau^2} d\tau^2 + (2m - \tau^2)\tau^4 (4\xi d\tau + \tau d\xi)^2 + \tau^4 d\sigma^2$$

which is benign at $r = 0$.

Black hole singularities

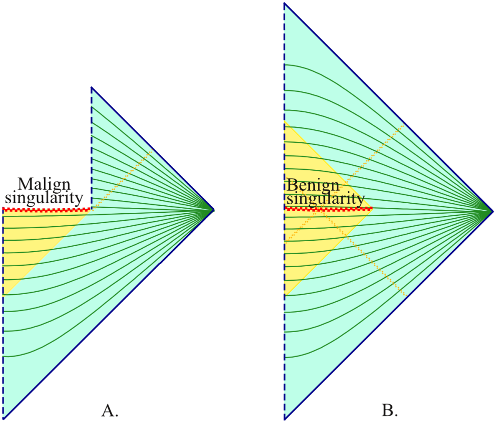
I did this for the Schwarzschild solution,

O. C. Stoica. "Schwarzschild Singularity is Semi-Regularizable". *Eur. Phys. J. Plus* 127.83 (7 2012)

but also for the other types of black holes

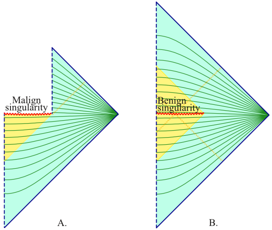
O. C. Stoica. "Analytic Reissner-Nordström Singularity". *Phys. Scr.* 85.5 (2012);
O. C. Stoica. "Kerr-Newman Solutions with Analytic Singularity and no Closed Timelike Curves". *U.P.B. Sci Bull. Series A* 77 (1 2015);
O. C. Stoica. "The Geometry of Black Hole Singularities". *Advances in High Energy Physics* 2014 (2014)

Evaporating Schwarzschild black hole and information loss



A. Standard evaporating black hole, whose singularity destroys the information.
B. Evaporating black hole extended through the singularity preserves information.

Evaporating Schwarzschild black hole and information loss



O. C. Stoica. "Schwarzschild Singularity is Semi-Regularizable". *Eur. Phys. J. Plus* 127.83 (7 2012);
O. C. Stoica. "Spacetimes with Singularities". *An. Șt. Univ. Ovidius Constanța* 20.2 (2012);
O. C. Stoica. "The geometry of singularities and the black hole information paradox". *Spacetime - Matter - Quantum Mechanics, Seventh International Workshop DICE2014* (2014)

Quantum gravity

There are two main reasons why it is said that general relativity should be replaced with something else:

- Singularities (infinities appear).
- Gravity couldn't be quantized in a generally acceptable way, because infinities appear (not the same infinities as at singularities).

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What if they also help in the quantum gravity problem?

Singular quantum gravity

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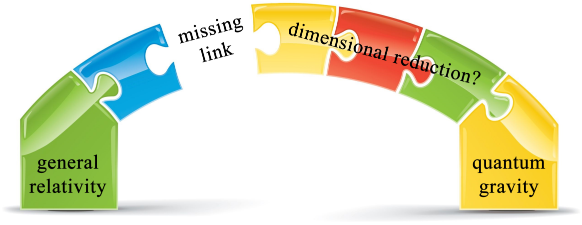
That's why many attempts to quantize gravity work if at small scales spacetime has fewer dimensions (**dimensional reduction** to < 4 dimension).



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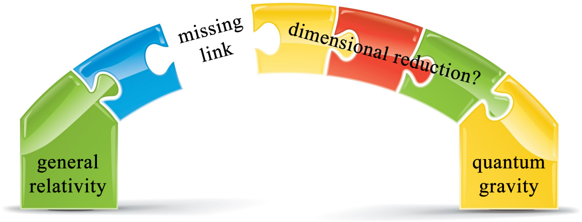


But usually the various sorts of dimensional reduction are introduced *ad hoc*, without justification.

Singular quantum gravity

Fractal universe

Calcagni, "Quantum field theory, gravity and cosmology in a fractal universe";
Calcagni, "Fractal universe and quantum gravity", based on a Lebesgue-Stieltjes measure or a fractional measure
Calcagni, "Geometry of fractional spaces", fractional calculus, and fractional action principles
El-Nabulsi, "A fractional action-like variational approach of some classical, quantum and geometrical dynamics";
El-Nabulsi and Torres, "Fractional actionlike variational problems";
Udriște and Opriș, "Euler-Lagrange-Hamilton dynamics with fractional action".

Singular quantum gravity

Topological dimensional reduction

Shirkov, "Coupling running through the looking-glass of dimensional reduction";
Fiziev and Shirkov, "Solutions of the Klein-Gordon equation on manifolds with variable geometry including dimensional reduction";
Fiziev, "Riemannian (1+d)-Dim Space-Time Manifolds with Nonstandard Topology which Admit Dimensional Reduction to Any Lower Dimension and Transformation of the Klein-Gordon Equation to the 1-Dim Schrödinger Like Equation";
Fiziev and Shirkov, "The (2+1)-dim Axial Universes – Solutions to the Einstein Equations, Dimensional Reduction Points, and Klein–Fock–Gordon Waves";
Shirkov, "Dream-land with Classic Higgs field, Dimensional Reduction and all that".

Singular quantum gravity

Other approaches

Vanishing Dimensions at LHC

Anchordoqui et al., "Vanishing dimensions and planar events at the LHC".

Dimensional reduction in Quantum Gravity

Carlip, "Lectures in (2+ 1)-dimensional gravity";

Carlip et al., "Spontaneous Dimensional Reduction in Short-Distance Quantum Gravity?";

Carlip, "The Small Scale Structure of Spacetime".

Asymptotic safety

Weinberg, "Ultraviolet divergences in quantum theories of gravitation."

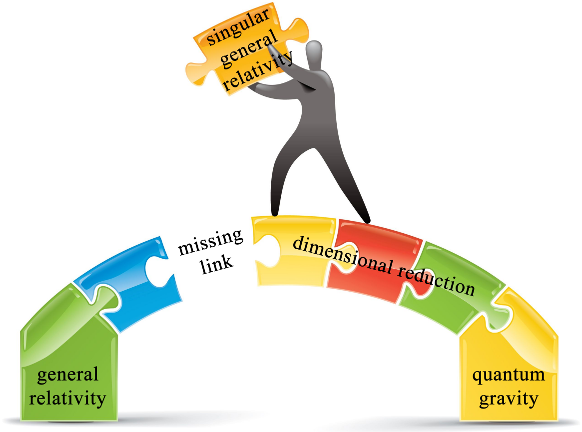
Causal dynamical triangulations

Ambjørn, Jurkiewicz, and Loll, "Nonperturbative Lorentzian path integral for gravity".

Hořava-Lifschitz gravity

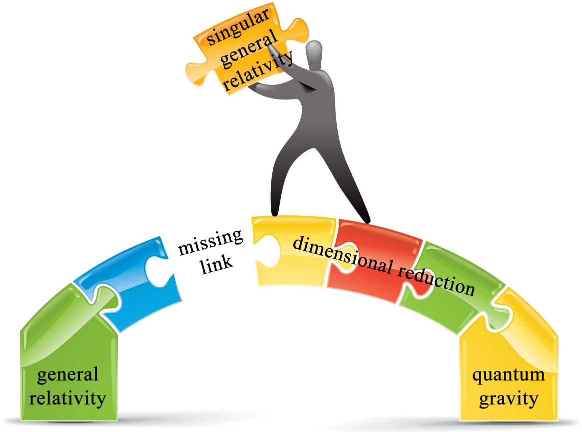
Hořava, "Quantum Gravity at a Lifshitz Point".

Singular quantum gravity



Fortunately, singularities lead automatically to the dimensional reduction postulated *ad hoc* in several different approaches to quantum gravity.

Singular quantum gravity



O. C. Stoica. "Metric dimensional reduction at singularities with implications to Quantum Gravity". *Ann. of Phys.* 347.C (2014)


Thank you!

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