Critical phenomena for dually weighted graphs and spanning forests via matrix models

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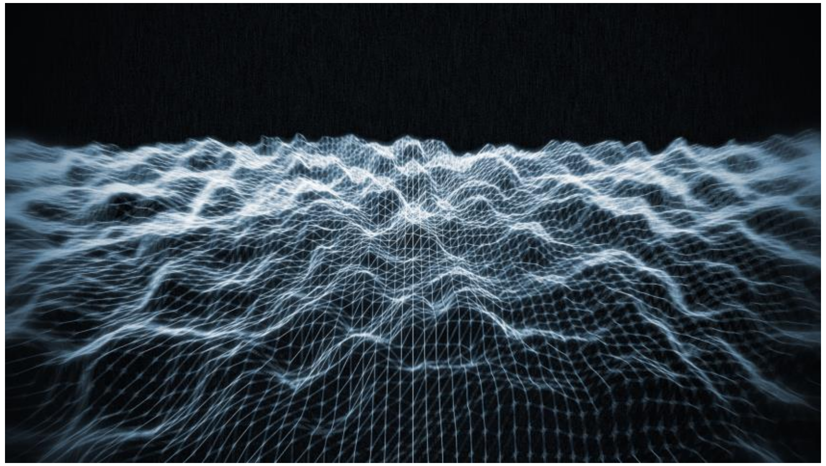
based on

[arXiv:2110.10104] with V. Kazakov

[to appear] with A. Gorsky, V. Kazakov, V. Mishnyakov

Matrix models provide a useful description for 2d quantum gravity

Matrix integral $\longrightarrow$ sum over graphs $\xrightarrow[\text{large N, continuum limit}]{}$ sum over 2d surfaces = 2d gravity path integral



Rich subject, but some questions remain unanswered

Some motivation:

- How to control the geometry?
  Could we get AdS or JT gravity? Saad, Shenker, Stanford ‘19 Witten, Mirzakhani

- Much of existing work focuses on isolated models
  Can we study flows?

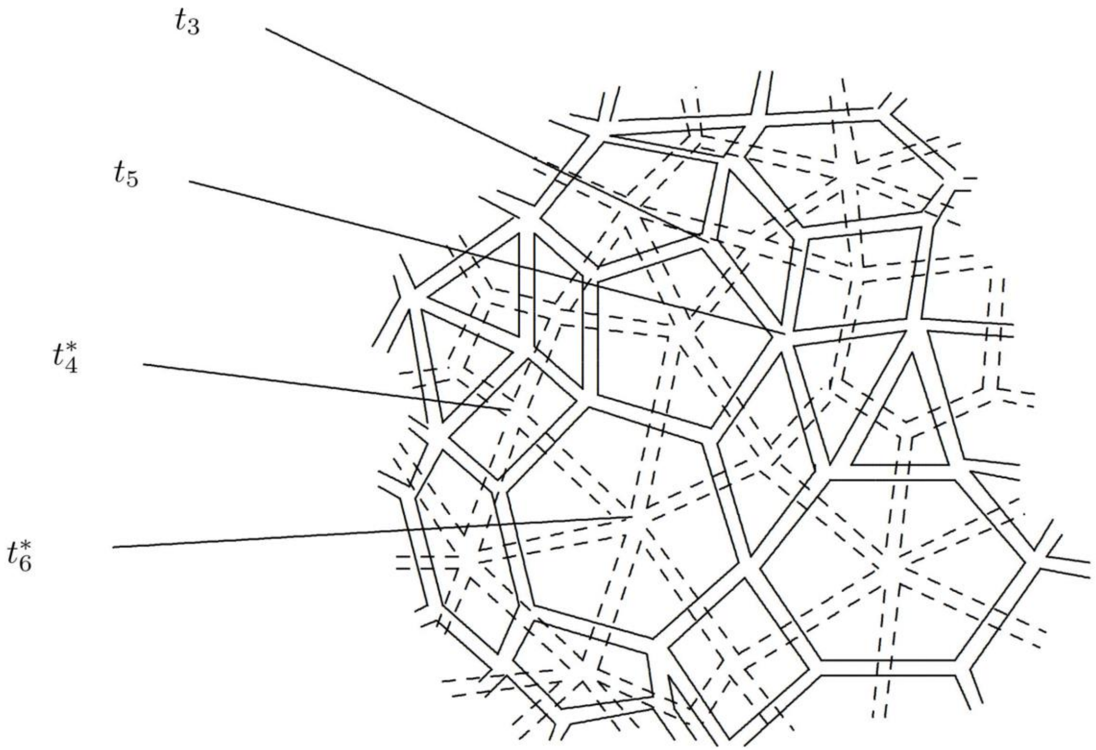
This talk:

- Counting dually weighted graphs
  and R^2 gravity Kazakov, FLM 21

- Counting spanning trees
  and flow from c=0 to c=-2 Gorsky, Kazakov, FLM, Mishnyakov
  to appear

PART 1

Dually weighted graphs



Dually weighted graphs (DWG) partition function

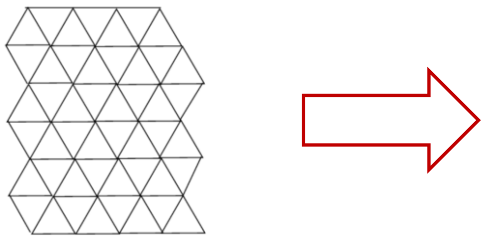
$$Z_N(t,t^*) = \sum_G N^{2-2g_G} \prod_{v_q^*, v_q \in G} t_q{}^{\# v_q}\, t_q^{*\,\# v_q^*}$$

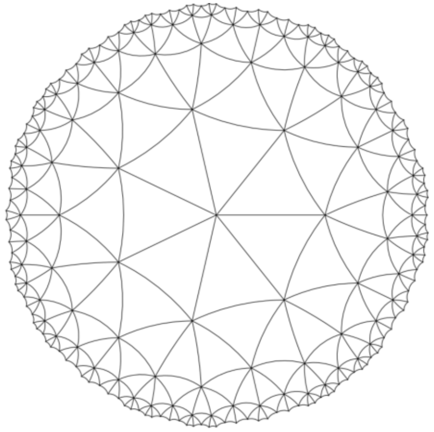
- counting of graphs, polyhedra, tilings

  Related works in mathematics [Kazarian, Zograf, Zorich, …]

- 2d quantum gravity

Dual weights control
local curvature (R^2 coupling)





Lattice model of AdS_2 ?

http://www.geom.at/fade2d/html/

continuum limit gives
sum over surfaces

Kazakov ’85 David ‘85
Kazakov, Kostov, Migdal ‘85

Kazakov, Staudacher, Wynter ‘96

DWG matrix model

Das, Dhar, Sentgupta, Wadya '90
Kazakov, Staudacher, Wynter '95

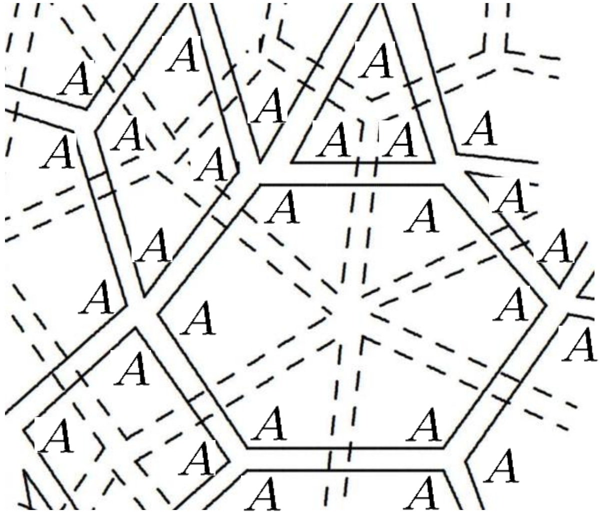
We study a refined version of usual 1-matrix model

$$Z_N(t,t^*) = \int \mathcal{D}M \exp N \ \mathrm{tr}\left(-\frac{1}{2}M^2 + \sum_{q=1}^{Q+1}\frac{1}{q}t_q(AM)^q\right)$$

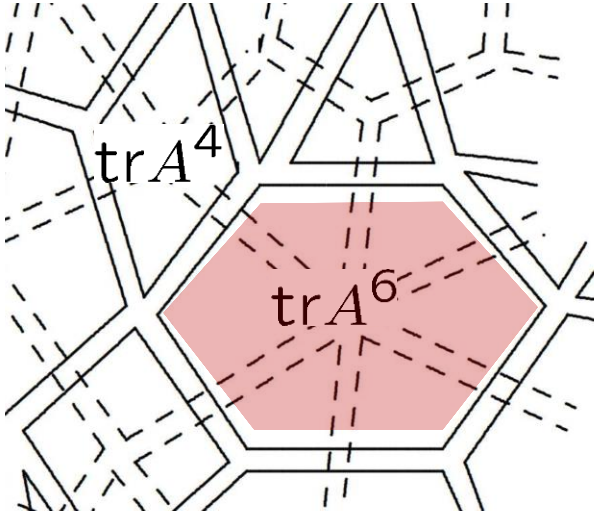
n-vertex $\rightarrow t_n$

n-face $\rightarrow t_n{}^*$

Dually weighted graphs (DWG)



$$t_n^* = \frac{1}{N}\mathrm{tr}A^n$$



Dual couplings $\longrightarrow$ better control the geometry of graphs / surfaces

DWG matrix model

Das, Dhar, Sentgupta, Wadya '90
Kazakov, Staudacher, Wynter '95

$$Z_N(t,t^*) = \int \mathcal{D}M \exp N \, \mathrm{tr} \left(-\frac{1}{2}M^2 + \sum_{q=1}^{Q+1} \frac{1}{q} t_q (AM)^q \right)$$

Goal: compute observables like $\langle \mathrm{Tr} M^n \rangle$ and $\langle \mathrm{Tr}(AM)^n \rangle$, explore continuum limit

Solving the matrix model

Kazakov, Staudacher, Wynter ’95, ‘96

DWG Matrix Model and sum over Young tableaux

Di Francesco, Itzykson '93
Kazakov, Staudacher, Wynter '95

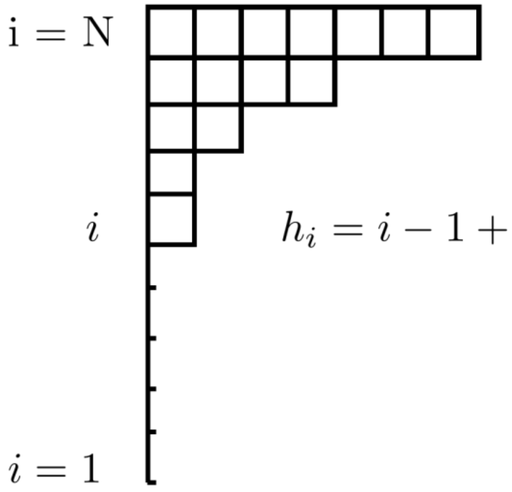
$$Z_N(t,t^*) = \int (d^{N^2}M)\, \exp\left(-\frac{1}{2}\mathrm{tr}M^2 + \sum_{q=1}^{Q+1}\frac{1}{q}t_q\,\mathrm{tr}(AM)^q\right)$$

Cannot reduce to eigenvalues like usual MM, instead use characters

$$\exp\mathrm{tr}\left(\sum_{q=1}^{Q+1}\frac{1}{q}t_q(AM)^q\right) = \sum_{\{h\}}\chi_{\{h\}}[t]\;\chi_{\{h\}}(MA) \stackrel{\Omega\, int}{\Longrightarrow} \sum_{\{h\}}\frac{1}{\mathrm{dim}_h}\chi_{\{h\}}[t]\;\chi_{\{h\}}[t^*]\chi_{\{h\}}(M) \qquad t_q^* = \frac{1}{N}\mathrm{tr}A^q$$

sum over Young tableaux

angular integration $\quad MA \to M\Omega^\dagger A\Omega\,, \qquad \Omega\in SU(N)$



$$h_i = i - 1 + \#\ \text{boxes in row}\ i$$

Schur character:

$$\chi_{\{h\}}[t] = \det_{1\le j,k\le N} P_{h_j-j+1}[t]\,,$$

$$\sum_{n=1}^{\infty} z^n P_n[t] = \exp\left(\sum_{m=1}^{\infty} z^m t_m\right)$$

What remains is a Gaussian integral of character

DWG Matrix Model and sum over Young tableaux

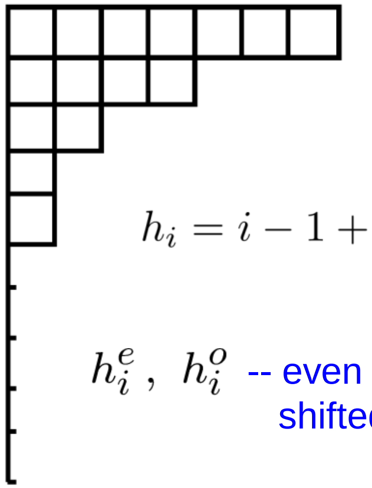
Di Francesco, Itzykson '93
Kazakov, Staudacher, Wynter '95

see also Kostov, Staudacher, Wynter 97

Result for partition function:

$$Z_N(t,t^*) = c(N) \sum_{\{h^e,h^o\}} \frac{\prod_i (h_i^e - 1)!!h_i^o!!}{\prod_{i,j}(h_i^e - h_j^o)} \chi_{\{h\}}[t]\, \chi_{\{h\}}[t^*]$$

$\mathrm{i} = \mathrm{N}$



i

$h_i = i - 1 + \#\ \text{boxes in row}\ i$

$h_i^e\ ,\ h_i^o$ -- even and odd
shifted highest weights

$i = 1$

Large Young tableaux dominate at large N

Saddle point gives a single Young tableau

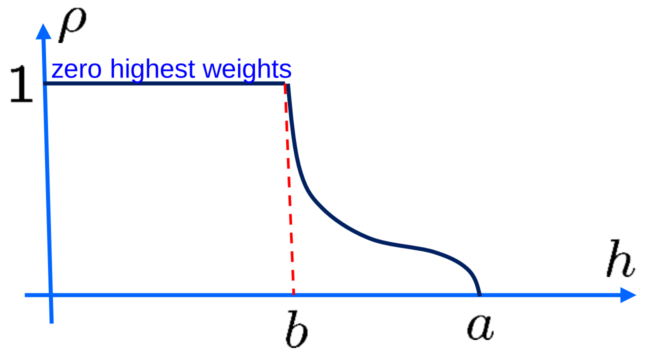
$$h_i \sim N$$

$$\chi_{\{h\}} \sim e^{N^2(\dots)}$$

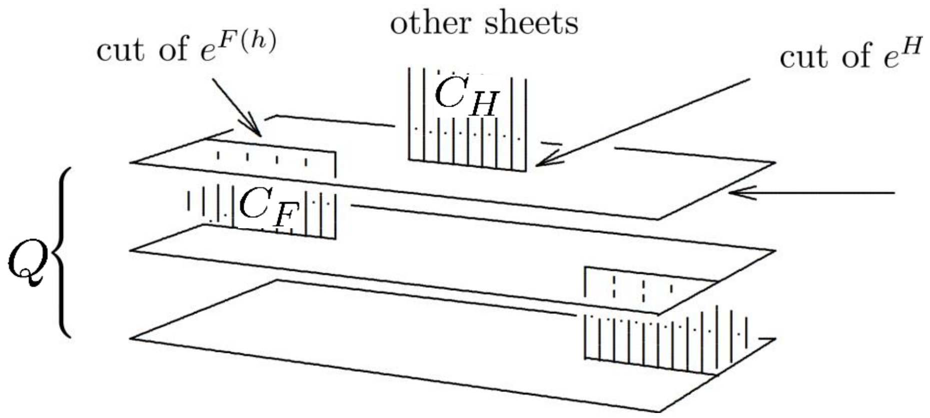
Algebraic curve of DWG

At large N we parameterize the dominating Young tableau by some continuous shape

$$H(h) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{h - h_k/N} \qquad H(h) = \int_0^a \frac{\rho(s)ds}{h-s}$$



It can be found from a nontrivial Riemann-Hilbert problem



$$he^{-H(h)} = \frac{(-1)^{Q-1}}{t_Q}\prod_{q=1}^{Q} g_q(h)$$

Product over Q sheets of Riemann surface of $g(h)$

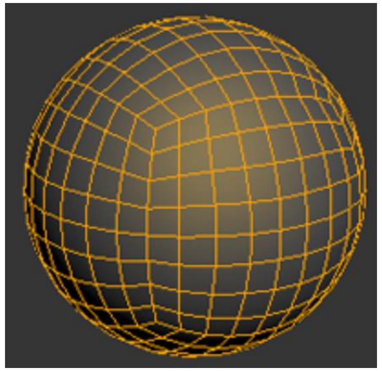
$$h = \sum_{q=1}^{Q+1} t_q\, g^q + \sum_{k=0}^{\infty}\langle\frac{\mathrm{tr}}{N}(MA)^k\rangle\, g^{-k}$$

This gives (in principle) solution of the model

Special case: quadrangulations

Disc Quadrangulations

Particular case: counting quadrangulations of sphere and disc



Here we impose :

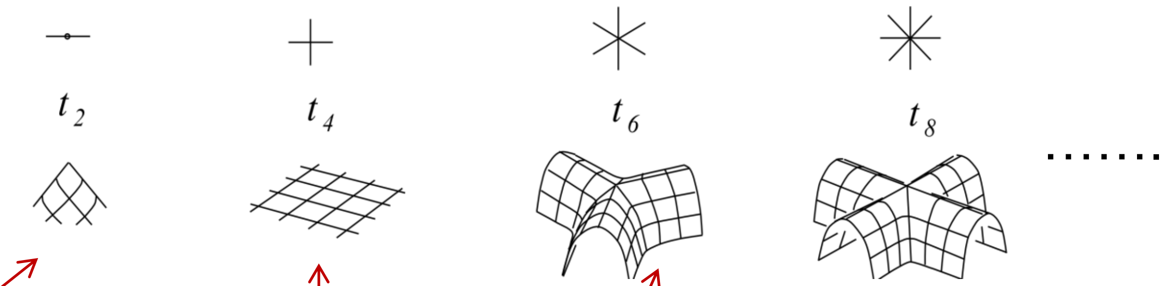
$$t_q^* = \lambda\, \delta_{q,4}$$

Special choice of weights makes the model tractable:

$$t_2 = \lambda\, \beta^2, \qquad t_{2n} = \lambda$$

Kazakov, Staudacher, Wynter '96

Build surface from:



positive curvature

flat

negative curvature

Overall curvature must be balanced (sphere topology)

Graph combinatorics gives

$$Z \propto \sum_G \lambda^F \beta^{2(\#v_2 - 4)}$$

λ controls area

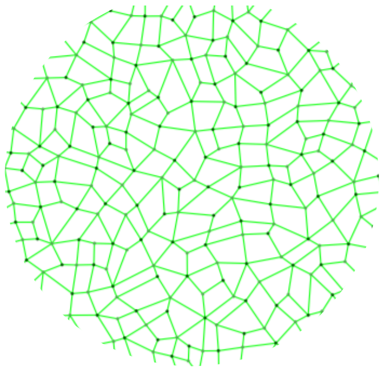
β controls curvature

Z was computed by [Kazakov, Staudacher, Wynter 96] Corresponds to sphere topology

We will compute two resolvents, i.e. disc partition functions

Type 1 $$\mathcal{W}(g) = \langle \frac{\text{tr}}{N} \frac{1}{g - MA} \rangle$$

Type 2 $$W(g) = \langle \frac{\text{tr}}{N} \frac{1}{g - M} \rangle$$



Disc topology, two types of boundary

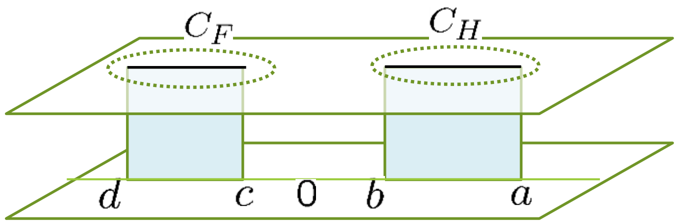
$$W(g) = \sum_{n \geq 0} \frac{1}{g^{n+1}} \frac{1}{N} \langle \text{tr} M^n \rangle$$

n = length of boundary

Saddle point equation

We have a 2-cut problem

$$h-1=\frac{\lambda}{\beta^2}\left((\beta^2-1)g^2+\frac{g^2}{1-\beta^2g^2}\right)+\mathcal{O}(1/g)$$



Elliptic Riemann surface of $2F+H+\log h$

Saddle point equation: $2F+\cancel{H}+\log h=0\,,\qquad h\in[b,a]$

From algebraic curve: $2\cancel{F}+H+\ln h=\log[\lambda(\beta^2-1)]\,,\qquad h\in[d,c]$

Solved by elliptic integrals

$$H(h)+2F(h)=\log\frac{(h-c)^2}{h(h-b)}+r(h)\left[\oint_{C_H}\frac{dh}{2\pi i}\frac{\log\frac{(s-b)}{(s-c)^2}}{(h-s)r(s)}+\oint_{C_F}\frac{dh}{2\pi i}\frac{\log\frac{\lambda(\beta^2-1)(s-b)}{(s-c)^2}}{(h-s)r(s)}\right]$$

$$r(x)=\sqrt{(a-x)(x-b)(x-c)(x-d)}$$

Solution of Riemann-Hilbert problem

We get density in terms of theta-functions   Kazakov, Staudacher, Wynter '96

$$\rho(h) = \frac{u}{K} - \frac{i}{\pi}\ \ln\left[\frac{\theta_4(\frac{\pi}{2K}(u-iv),q)}{\theta_4(\frac{\pi}{2K}(u+iv),q)}\right]$$

$$u = \mathrm{sn}^{-1}\sqrt{\frac{(a-h)(b-d)}{(h-d)(a-b)}}$$

Then we find H(h)
etc

$$H(h) = \int_b^a \frac{\rho(s)ds}{h-s}$$

(already not a standard special function)

Parameters are fixed from
consistency requirements
(e.g. asymptotics)

$$\mathrm{sn}^2(v,m') = \sqrt{\frac{a-c}{a-d}} \qquad m = \frac{(a-b)(c-d)}{(a-c)(b-d)}$$

$$v = -K' - \frac{K}{\pi}\log(\lambda(1-\beta^2)) \qquad \dots$$

Asymptotics of Large Area & Boundary

Search for universal regimes
when the details of discretization do not matter

Quadrangulations become continuous 2d geometries

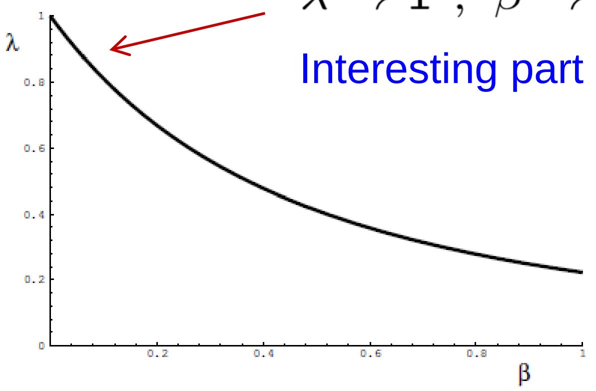
Asymptotics of Large Area & Boundary

Kazakov, Staudacher, Wynter '96

$$Z \propto \sum_G \lambda^F \beta^{2(\#v_2-4)}$$

Critical regime: density has zero slope at endpoint

Gives $\lambda_c(\beta)$



$\lambda \to 1\ ,\ \beta \to 0$

Interesting part (the rest gives pure gravity)

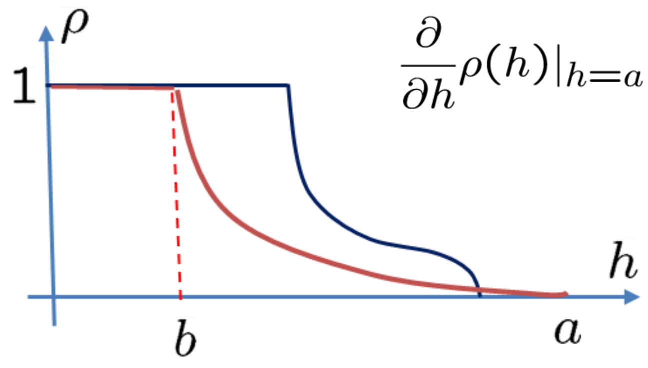
elliptic modulus

$$\lambda \simeq q = e^{-\pi\tau} \qquad \tau \to 0$$

Parametrise via $x = \dfrac{\sqrt{2}(1-\lambda)}{\pi\beta}$

$$V = \frac{1}{\sqrt{2}}(x - \sqrt{x^2-1})$$

finite ratio



$$\frac{\partial}{\partial h}\rho(h)|_{h=a} = 0$$

Three key regimes:

- All orders in τ (exp precision)
- Flattening: 1st order in τ, $V =$ finite
  Small curvature, various n-vertices
- Almost flat: 1st order in τ with $V \to 0$
  Small curvature, only 2- and 4- vertices

Solution in exp. approximation

Kazakov, FLM

$\mathcal{O}(e^{-\frac{\pi}{\tau}})\,, \quad \tau \to 0$ We need to compute $H(h) = \int_b^a \frac{\rho(s)ds}{h-s}$

Density simplifies $\rho = -\frac{i}{\pi}\log\left[-\frac{\theta_1\left(\frac{\pi}{2K}(u-i(v+K')),q\right)}{\theta_1\left(\frac{\pi}{2K}(u+i(v+K'),q\right)}\right] \simeq \frac{4\tau}{\pi^2}u(v+\pi/2) - \frac{i}{\pi}\log\left(\frac{\cosh(u-iv)}{\cosh(u+iv)}\right)$

We get integrals of the kind $\int_0^\infty dw\left(\frac{\sinh w}{\cosh w + \sinh u} + \ldots\right)\log\frac{\cosh(w+iv)}{\cosh(w+iv)}$ $\longrightarrow$ Lots of polylogs

Long calculation gives result in parametric form $h = b + e^{\tau(4v^2/\pi - \pi)}\frac{\sin^2(2v)}{\sin^2(v)-\sin^2(r)}\,, \quad g^4 = \frac{1}{h-b}\,e^{-\frac{i}{\pi}F(r)}$

where

$$\begin{aligned} F \;=\; & -\frac{i}{\pi}\tau(2v+\pi)\left(-2r^2-2\pi r+2v^2+\pi^2\right) - \left(2(r-v)(r-3v)-2\pi r+\frac{5\pi^2}{6}\right) \\ & + i(2v-2r-\pi)\log\left(1-e^{2i(r-v)}\right) + i(2(r+v)+\pi)\log\left(1-e^{2i(r+v)}\right) - \frac{i}{6\pi}2\pi\log\left(1+e^{2iv}\right) \\ & + \mathrm{Li}_2\left(e^{-2i(r-v)}\right) + \mathrm{Li}_2\left(e^{2i(r+v)}\right) + 2\,\mathrm{Li}_2\left(1-e^{2iv}\right) + 2\,\mathrm{Li}_2\left(1+e^{2iv}\right) \end{aligned}$$

[thanks to Omer Gurdogan
for help with polylogs]

Parameters from couplings:

$$\tau = -\frac{\log[\lambda(1-\beta^2)]}{2v+\pi} \qquad v = V\beta$$

$$\lambda \simeq (\lambda(1-\beta^2))^{1-\frac{2v}{\pi}}\left[\left(1-\log[\lambda(1-\beta^2)]\left(\frac{2v}{\pi}+1\right)\right)\sin(2v) + \pi(\frac{2v}{\pi}+1)\cos(2v)\right]$$

Solution in exp. approximation

$$h = b + e^{\tau(4v^2/\pi-\pi)}\frac{\sin^2(2v)}{\sin^2(v)-\sin^2(r)}$$
$$g^4 = \frac{1}{h-b}\,e^{-\frac{i}{\pi}F(r)}$$

$$h = \sum_{q=1}^{Q+1} t_q\, g^q + \sum_{k=0}^{\infty}\langle\frac{\mathrm{tr}}{N}(MA)^k\rangle\, g^{-k}$$

A rich phase structure to be explored

Describes large 2d surfaces with strong fluctuations of curvature

We can compare $\langle \mathrm{Tr} M^2\rangle$ with exact result

[Kazakov, Staudacher, Wynter 96]

$$\langle \mathrm{Tr}M^2\rangle = \frac{\xi^2}{4\lambda}\Big[\frac{Kk'^2\mathrm{sn}^2}{8\pi\ \mathrm{cn}\ \mathrm{dn}}(1-k'^2\mathrm{sn}^4) - \frac{k'^2\mathrm{sn}^2}{4} - \frac{1}{2\pi}(K\,\Xi - \frac{\pi}{2})(\mathrm{dn}^2 + k'^2\mathrm{cn}^2)$$
$$+\frac{3\,\Upsilon\,\Phi}{8\pi\ \mathrm{sn}^2} - \frac{\Phi^2}{2\pi^2\mathrm{sn}^2} - \frac{K\,\Upsilon}{4\pi\ \mathrm{sn}}(1-\mathrm{dn}^2) + \frac{\Phi\,\Xi}{2\pi\ \mathrm{sn}} + \frac{\Omega\,\Phi}{\pi\ \mathrm{sn}} - \frac{\Omega^2}{2}\Big]$$

Expand our solution to 5th order in tau, find pefect match!

$$\langle \mathrm{Tr}M^2\rangle = 1 - \frac{8\beta^5V^3(6V^2-5)}{15\pi} + \dots$$

Flattening regime

$$\beta \sim v \sim (1-\lambda) \sim \tau \to 0$$

Curvature is suppressed, Young tableau almost empty

Resolvent = disc partition function: $\mathcal{W}(g) = \frac{1}{g} + \frac{1}{g^3}\langle \frac{\mathrm{tr}}{N}(AM)^2\rangle + \frac{1}{g^5}\langle \frac{\mathrm{tr}}{N}(AM)^4\rangle + \frac{1}{g^7}\langle \frac{\mathrm{tr}}{N}(AM)^6\rangle + \ldots$

We get

Kazakov, FLM

$$\frac{\mathcal{W}}{\beta^3} = \frac{\left(\frac{4V^3}{\pi} + \frac{2V}{\pi}\right)\arcsin(y)^2}{y^2} + \left(\frac{4V^3 + 2V}{y^2} + \frac{16V^3\sqrt{1-y^2}}{\pi y^3}\right)\arcsin(y)$$
$$+ \frac{8V^3\left(\sqrt{1-y^2}-1\right)}{y^3} + \frac{4V^3}{3\pi} - \frac{16V^3}{\pi y^2} - \frac{2V}{y} - \frac{2V}{\pi}$$

where $y = \frac{2\beta V}{g^2}$

Interpolates between pure 2d QG and almost flat case

Pure gravity limit

Kazakov, FLM

$$\lambda_c \simeq 1 - \frac{\pi\beta}{\sqrt{2}}$$

Small bulk and boundary cosmological constants

$$\Lambda = 2\sqrt{2}\frac{\lambda_c - \lambda}{\pi\beta} \to 0\,, \qquad \eta = 1 - \frac{\sqrt{2}\beta}{g^2} \to 0 \quad \text{with} \quad z = \frac{\sqrt{\Lambda}}{\eta} = \text{fixed}$$

We get

$$\frac{\mathcal{W}}{\beta^3} = \underbrace{\frac{-28 - 18\pi + 9\pi^2}{3\sqrt{2}\pi} + \frac{\sqrt{2}\left(-12 - 7\pi + 3\pi^2\right)\eta}{\pi}}_{\text{non-universal terms}} - \frac{8}{3}\underbrace{\eta^{3/2}(z-2)\sqrt{z+1}}_{\text{universal part}} + O\left(\eta^2\right)$$

Matches known prediction! Kazakov '88

Almost flat limit

Only 2- and 4-vertices

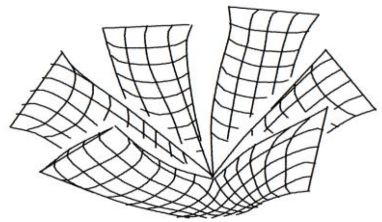
$$\beta \to 0\ ,\ V \to 0$$

$$\frac{\mathcal{W}}{\beta^3} = \frac{2V}{\pi y^2}\left[(\arcsin y)^2 - y^2 + \pi(\arcsin y - y)\right]$$

Kazakov, Staudacher, Wynter '96

$$y = \frac{t_2}{2\pi\tau\, g}$$

Describes trees



2nd resolvent: $$W(P) = \frac{2\xi}{T} + \tau\frac{T\xi}{\pi(T^2 - 4\xi^2)}\left(T^2 + \frac{8\pi^2}{T^2}\xi^2 - \frac{\arcsin^2\xi}{\xi^2}\right)$$

Kazakov, FLM

$$\xi = \frac{T\,(P - \sqrt{P^2-4})}{4}$$

$$T = \frac{t_2}{\tau} \to \text{fixed}$$

Near criticality we find again a tree-like behavior :

$$W(g) - \frac{g - \sqrt{g^2-4}}{2} \simeq t\frac{\pi^2-4}{4\pi} + t\sqrt{2}\sqrt{1-\xi} + \ldots$$

$$T \sim g \to \infty$$

$$\xi = \frac{T}{2g} = \frac{t}{2g\tau} \to \text{fixed}$$

Interpretation: $\beta \neq 0$ controls local curvature fluctuations

Produces R^2 term in QG action

$$S_{\rm QG} = \int_{\mathcal{M}} \sqrt{g}\,(\tilde{\Lambda} + \frac{\log N}{2\pi} R + \frac{1}{\tilde{\beta}} R^2) \quad + \text{boundary terms}$$

- Curvature strongly fluctuates at distance $\tilde{\Lambda}^{1/2} \gg \tilde{\beta}$ $\longrightarrow$ QG regime
- Curvature is suppressed at distance $\tilde{\Lambda}^{1/2} \ll \tilde{\beta}$ $\longrightarrow$ "almost flat" regime

Our solution interpolates between the two regimes

We see no phase transition, in agreement with expectations from [Kazakov, Staudacher, Wynter '96]

Part 1 – summary & prospects

We count the number of dually weighted graphs, topology of sphere, disc (& more?)
Compute 2 disc partition functions, observe universality in 2d QG regime

Future:

- Explore our exponential solution and its phase structure
- General spectral curve of DWG, integrability/tau function
- Jackiw-Teitelboim gravity? May need more involved observables / develop character methods

PART 2

Critical phenomena in the forest

[Gorsky, Kazakov, FLM, Mishnyakov
to appear]

Forests on random graphs

Key properties of a graph are encoded in its Laplacian matrix $\Delta = D - A$

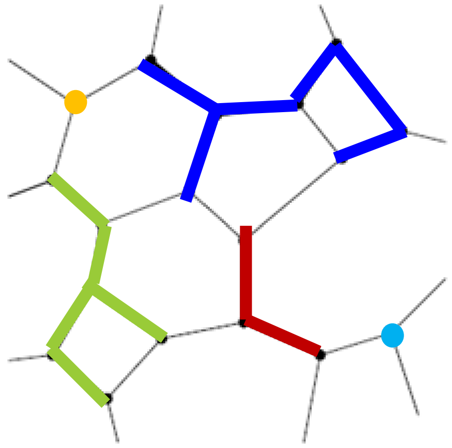
vertex degrees

adjacency matrix

Matrix tree theorem: $\det{}'\Delta = \#$ of spanning trees

Generalization: matrix forest theorem [Chelnokov, Kelmans] [David, Duplantier]

$$\det(\Delta + M^2) = \sum_{F=(F_1 \dots F_l) \in G} M^{2l} \prod_{i=1}^{l} |F_i|$$



We study the partition function:

$$Z = \sum_{\text{graphs } G} \lambda^{|G|} \det(\Delta + M^2) = \sum_{F=(F_1 \dots F_l) \in G} \lambda^{|G|} M^{2l} \prod_{i=1}^{l} |F_i|$$

Sum over random graphs with triple vertices

We count weighted rooted spanning forests

Related work:

[Caracciolo, Jacobsen, Saleur, Sokal 04]

[Caracciolo, Sportiello 09]

[Bondesan, Caracciolo, Sportiello 16]

$$Z = \sum_{\text{graphs } G} \lambda^{|G|} \det(\Delta + M^2)$$

Many reformulations!

- Counting rooted forests
- Massive spinless fermions interacting with 2d gravity
- Hermitian 1-matrix model with nontrivial potential
- O(n) / loop model [Kostov, Staudacher]

Parisi-Sourlas reformulation via fermions

$$Z = \sum_G N^{2-2g} \lambda^{|G|} \int \prod_{i \in G} d^2\theta_i \prod_{<ij> \in G} e^{-(\bar\theta_i - \bar\theta_j)(\theta_i - \theta_j)} \prod_{i \in G} e^{-\frac{q}{2} M^2 \bar\theta_i \theta_i}$$

Massive spinless fermions on random graphs

To get sum over graphs we introduce NxN supermatrix $\Phi(\theta) = \phi + \bar\theta\psi + \theta\bar\psi + \bar\theta\,\theta\epsilon$

$$Z = \int d^{N^2}\phi\, d^{2N^2}\psi\, d^{2N^2}\epsilon\; e^{NS(\Phi)} \qquad S(\Phi) = \mathrm{tr} \int d^2\theta \left(-\frac{1}{2}\Phi^2(\theta) - \frac{1}{2}\partial_\theta \Phi(\theta) \partial_{\bar\theta} \Phi(\theta) + \frac{\lambda}{3} e^{-\frac{3}{2} M^2 \bar\theta\theta} \left[\Phi(\theta)\right]^3 \right)$$

Feynman graphs give our partition function!

Notice that for unrooted trees one finds massive fermions + 4-fermion interaction
Instead we have rooted trees and no interaction

[Bondesan, Caracciolo, Sportiello 16]

The matrix model

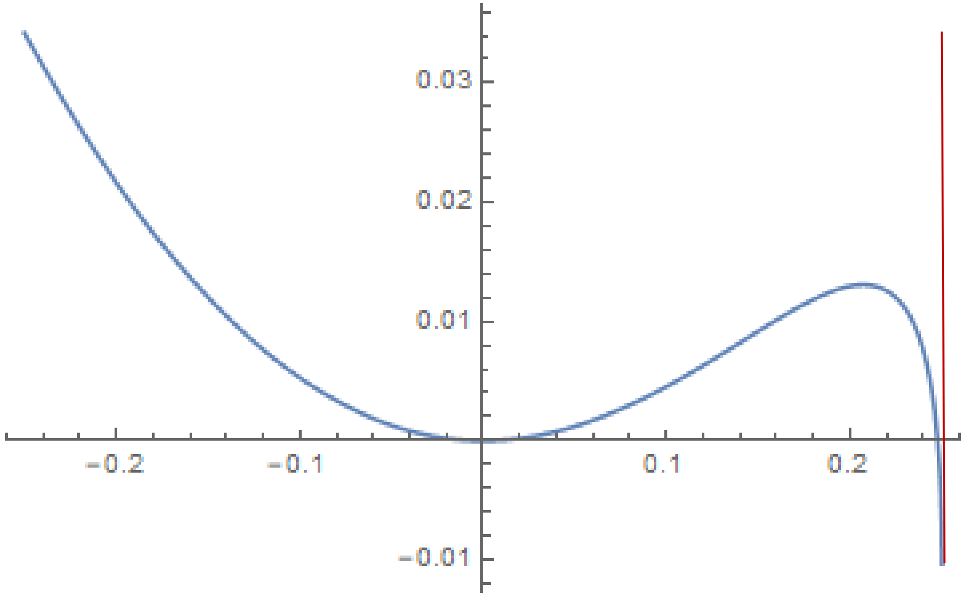
Next we integrate out the fermions
and get an Hermitian 1-matrix model

$$Z = \int d^{N^2} X e^{N S_X}$$

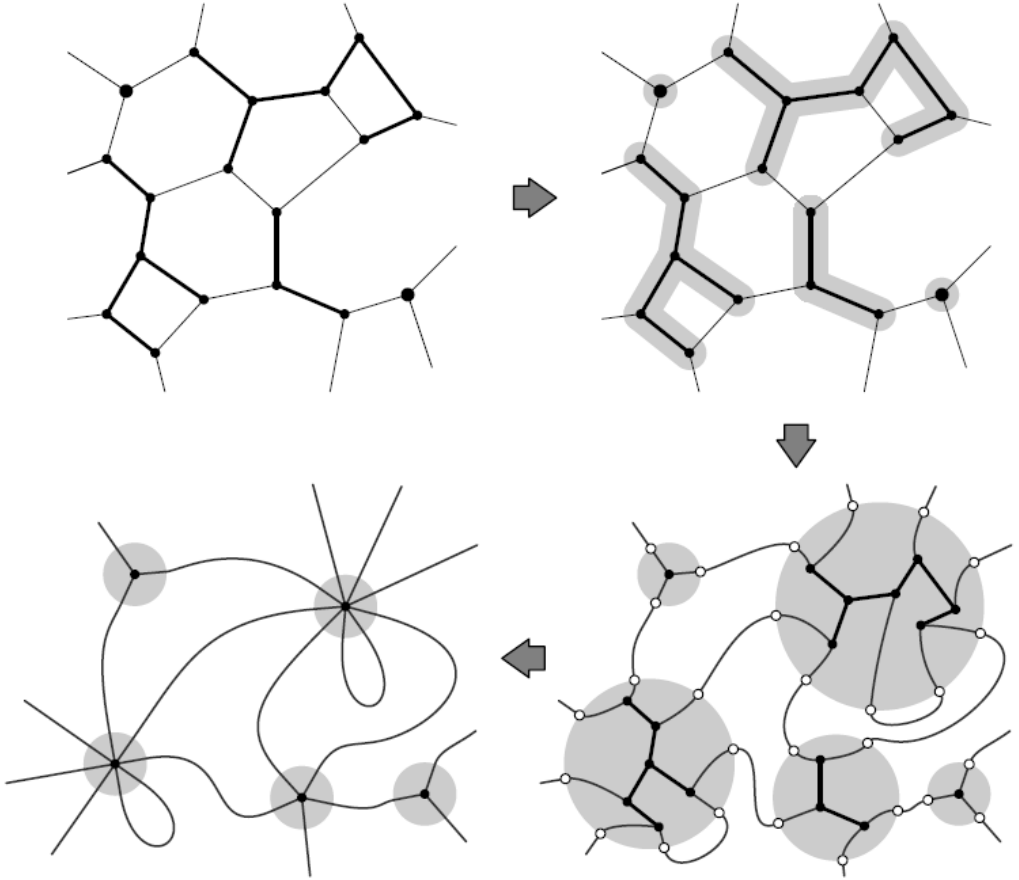
$$S_X = \mathrm{tr}\left[-\frac{1}{2} X^2 + \frac{\lambda M^2}{2} \left(\frac{1 - \sqrt{1 - 4\lambda X}}{2\lambda} \right)^3 \right]$$

generating function for trees
i.e. Catalan numbers

Potential:
two extrema + wall



Matrix model – combinatoric derivation



picture from
[Bondesan, Cracciolo, Sportiello 16]
see also [Bouttier, Di Francesco, Guitter]

Another way to derive the same matrix model

Extension of [Bondesan, Caracciolo, Sportiello 16]
who found the potential for unrooted trees:

$$\tilde{V}(z) = \frac{1}{12z^2}\left(-6z^2 + (1-4z)^{3/2} + 6z - 1\right)$$
$$= \sum_{n=1}^{\infty} \tilde{c}_n z^{n+2}, \qquad \text{where } \tilde{c}_n = \frac{(2n)!}{n!(n+2)!}.$$

For our rooted trees we find:

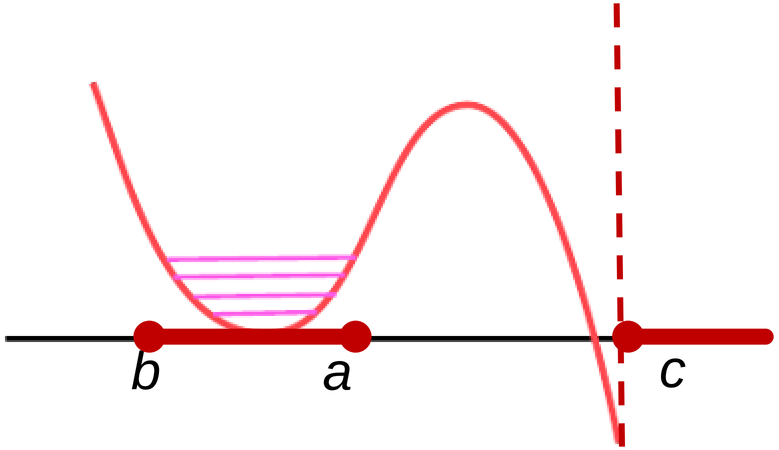
$$V = \frac{\lambda}{3}\left(\frac{1-\sqrt{1-4\lambda X}}{2\lambda}\right)^3 = \lambda\partial_\lambda\left(X^2\tilde{V}(\lambda X)\right)$$

i.e. the same potential as from fermion realization !

Solution of the model

One-cut solution: $$G(x)=\int_b^a \frac{\rho(y)dy}{x-y}$$

Solved in elliptic functions



$$G(x)=\frac{9}{4\pi}\sqrt{c}M^2\sqrt{x-a}\sqrt{x-b}\left(\frac{2(x-2c)\Pi\left(\frac{a-b}{x-b}\middle| m\right)}{\sqrt{c-b}(x-b)}-\frac{2K(m)}{\sqrt{c-b}}\right)$$
$$+\frac{1}{2}\left(-\sqrt{x-a}\sqrt{x-b}+9cM^2+x\right)$$

$$c=\frac{1}{4\lambda}\qquad B^2=c-b\,,\ m=\frac{a-b}{c-b}$$

Consistency with asymptotics fixes endpoints

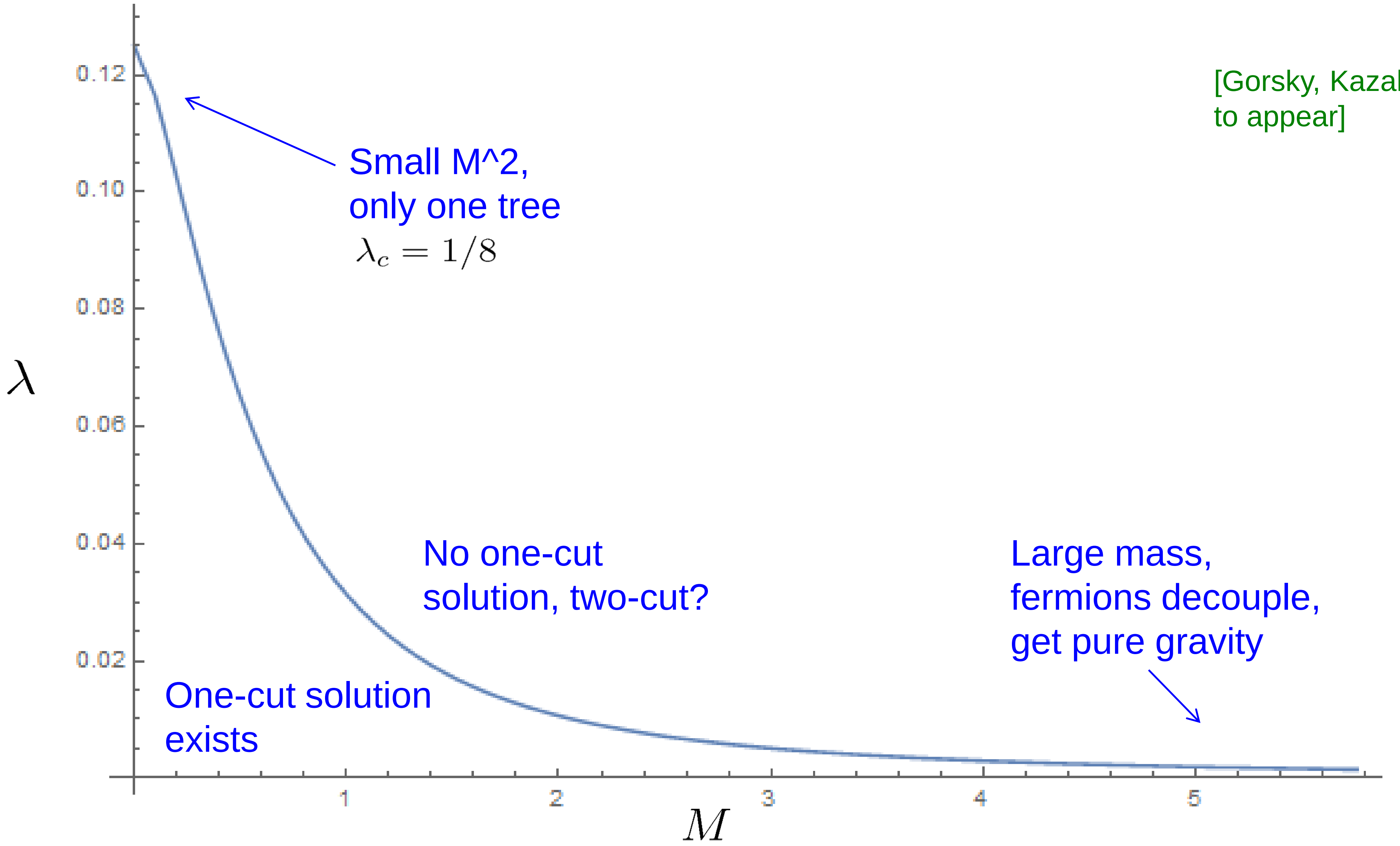
$$\pi B^3(m-2)-18B^2\sqrt{c}M^2E+2\pi Bc\left(9M^2+1\right)-18c^{3/2}M^2K=0$$

$$3\pi B^4m^2-36B^3\sqrt{c}M^2((m-2)E-2(m-1)K)+108Bc^{3/2}M^2((m-2)K+2E)-48\pi=0$$

Merging of solutions $\longrightarrow$ critical curve $$9\sqrt{c}M^2E(m)\left(c-B^2(m-1)\right)+(m-1)\left(\pi B^3m+9\sqrt{c}M^2\left(B^2+c\right)K(m)\right)=0$$

Critical curve

[Gorsky, Kazakov, FLM, Mishnyakov to appear]



Limiting regimes

$M \to \infty:$ Heavy fermions decouple $\qquad \lambda_{eff} = M^2 \lambda$

We reproduce pure gravity (c=0) prediction $(\lambda_{eff})^2_{crit} = \frac{1}{12\sqrt{3}}$

[Brezin, Itzyckson, Parisi, Zuber 78]
[Kazakov, Migdal, Kostov 85]

$M \to 0:$ Only 1 tree in the forest

Resolvent becomes Gaussian $G(x) \sim \sqrt{x^2-4}$

$$\det{}'(\Delta) \sim \frac{d}{dM^2}\det(\Delta + M^2)\Big|_{M=0} \sim \frac{d}{dM^2} Z\Big|_{M=0} = \langle \phi^3(x) \rangle \sim \int \sqrt{x^2-4}(1-\sqrt{1-4\lambda x})^3$$

When branch points collide we reproduce the (c=-2) behavior $\lambda_{crit} = 1/8$

[David; Kazakov, Kostov, Migdal]

Our model interpolates between these

Can we get more details on the flow from $c=0$ to $c=-2$?

Near-critical behavior

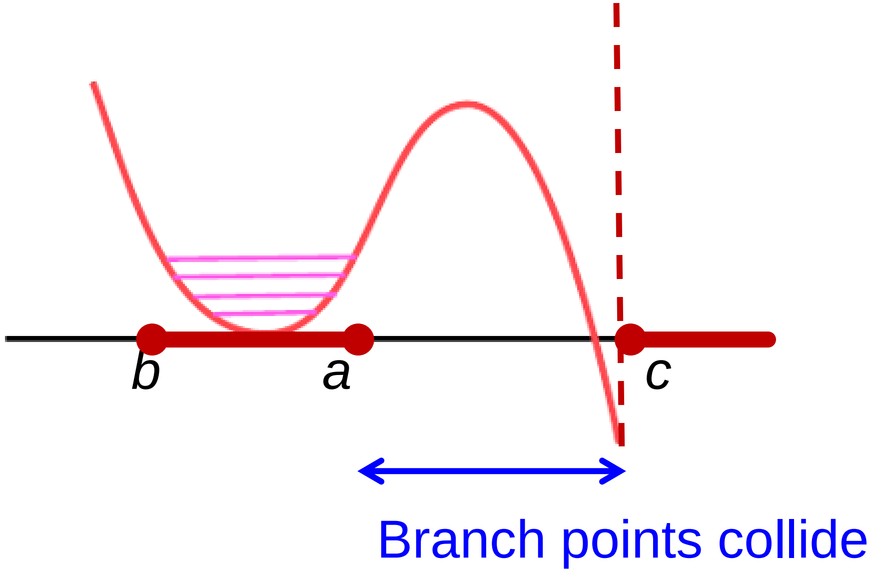
We look for nontrivial double scaling limits

Expand near one corner
of the critical line

$$M \to 0\ ,\ \ \lambda \to 1/8$$

Then $m \to 1$ and elliptic functions blow up

$$t = \frac{2\sqrt{2}\pi}{9}\frac{1-m}{M^2} = \text{fixed}$$



Good probe of criticality: natural observables like

$$\phi = 1 - \sqrt{1 - 4\lambda X}$$

$$\langle \mathrm{tr}\ \phi \rangle = \int_b^a dx \rho(x)\phi(x)$$

$$\rho(x) = \sqrt{(a-x)(x-b)}\left(\frac{9\sqrt{c}M^2K}{2\pi^2\sqrt{c-b}} + \frac{1}{2\pi}\right)$$
$$-\frac{9\sqrt{c}M^2(x-2c)}{2\pi^2\sqrt{c-b}}\sqrt{\frac{a-x}{x-b}}\left(K - \Pi\left(\frac{x-b}{c-b}, m\right)\right)$$

Can compute exacty

$$\langle\phi\rangle = \frac{c^3 y}{2\pi^2}\left[\left(M^2\left(6K^2(m-1)y + 6\pi K(m-1)y^2 + 3E\left(\pi\left(6-(m-2)y^2\right) - 4Ky\left(my^2+m-y^2-2\right)\right)\right.\right.\right.$$
$$\left.+9\pi K(m-2) + 6E^2 y\left((m-2)y^2-3\right)\right)$$
$$\left.\left.+\frac{1}{60}\pi y^3\left(16K(m-2)(m-1)y - 32E\left(m^2-m+1\right)y + 15\pi m^2\right)\right)\right]$$

$$y = c/\sqrt{B}$$

Critical behavior: 1pt function

$$M \to 0 \ , \ m \to 1 \qquad t = \frac{2\sqrt{2}\pi}{9}\frac{1-m}{M^2} = \text{fixed}$$

We find new universal scaling function:

$$\langle \mathrm{tr}\,\phi\rangle_{sing} \simeq 2(t - \log(M^2 t))(t^2 - 2t) + 3t^2 - 2t^3$$

where

$$t - \log(M^2 t) = C\frac{\lambda - 1/8}{M^2}$$

Lambert function

- c=-2 (one tree) limit

  $t \to \infty$

  $$\langle \mathrm{tr}\,\phi\rangle_{sing} \simeq (\lambda - 1/8)^2 \log(\lambda - 1/8)$$

  matches
  [David] [Kazakov, Kostov, Migdal]

- c=0 (pure gravity) limit

  $$t \to t_c = \frac{9}{32\sqrt{2}\pi}$$

  $$\langle \mathrm{tr}\,\phi\rangle_{sing} \simeq (\lambda - \lambda_c)^{3/2}$$

  (nontrivial cancellation
  kills ½ power)

  matches
  [Kazakov, Kostov, David, …]

Perfect interpolation

Critical behavior: resolvent

$$H(r) = \int_b^a \frac{\rho(y)dy}{r-\phi(y)}$$

The integral seems hard
but we can compute the singular part

$$H_{sing} = \frac{72R\sqrt{R^2-t}\left(\log\left(R-\sqrt{R^2-t}\right)-\log\sqrt{t}\right)+9\log^2\left(\frac{R-\sqrt{R^2-t}}{\sqrt{R^2-t}+R}\right)}{4\pi^2}$$

$$r = c + M\frac{3\ 2^{3/4}}{\sqrt{\pi}}R$$

- c=-2 (one tree) limit: $H_{sing} \simeq (\lambda - 1/8)\zeta\sqrt{\zeta^2-1}\log(\sqrt{\zeta^2-1}+\zeta)$ matches [Kostov]

$$t\to\infty\ ,\quad \zeta = \frac{R}{\sqrt{t}} = \text{fixed}$$

- c=0 (pure gravity) limit $\frac{1}{M}H_{sing}^{(c=0)} \simeq c_1 + c_2\frac{\delta(X+1)}{X} + c_3\frac{\delta^{3/2}}{X^{3/2}}(X-2)\sqrt{X+1}+\dots$

$$t = 1+\delta\ ,\ R\to 1\ ,\quad X = C\frac{R}{\sqrt{\lambda-\lambda_c}}$$

matches [Kazakov] [David]

Thus we observe a smooth flow between very different regimes: $c=-2$ and $c=0$

Universal results written
in terms of Lambert function, origin - ?

(Hurwitz numbers?)

Future

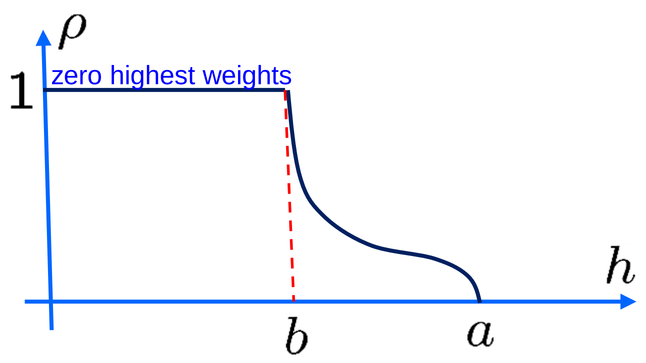
- Nonperturbative effects, i.e. instantons
- Multicut solutions
- Resolvent for X vs phi – Dirichlet/Neumann b.c.?
- Generalization to Dirac (spinful) fermions

- Derive/generalize via Liouville
- Other flows? c=1, c=-1?
- Interpolation between models on fixed and random lattices?

Characters at large N

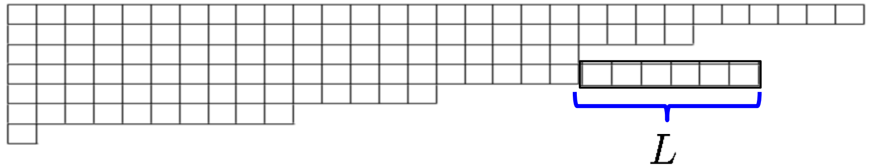
Vershik '77
Douglas, Kazakov '93
Kazakov, Wynter '94
Kazakov, Staudacher, Wynter '95, '96

$$H(h) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{h-h_k/N} \qquad\qquad H(h)=\int_0^a \frac{\rho(s)ds}{h-s}$$



We use the identity for Schur polynomials

$$\boxed{t_L = \sum_{k=1}^{N}\frac{\chi_{\{h+L\delta_k\}}[t]}{\chi_{\{h\}}[t]}} = \sum_{k=1}^{N}\frac{\mathrm{dim}(h+L\delta_k)}{\mathrm{dim}(h)}e^{\log\frac{\chi_{\{h+L\delta_k\}}}{\mathrm{dim}(h+L\delta_k)}-\log\frac{\chi_{\{h\}}}{\mathrm{dim}(h)}} = \sum_{k=1}^{N}\prod_{j(\neq k)}\left(1+\frac{L}{h_j-h_k}\right)e^{L\,F(h_k)}$$



where $F(h_k) = \partial_{h_k}\log\frac{\chi_{\{h\}}}{\mathrm{dim}(h)}$

So we get $t_L = \oint_{C_H}\frac{dh}{2\pi i}[g(h)]^{-L} = \oint\frac{dg}{2\pi i}g^{-L-1}h(g)$

$$g(h) = e^{-H(h)-F(h)}$$

Similarly $\langle \mathrm{tr}(MA)^L\rangle = \oint_{C_H}\frac{dg}{2\pi i}\,g^{L-1}h(g)$

Some motivation

- Matrix models offer another way to understand nonperturbative physics, complementary to AdS/CFT integrability

- Potential links to JT gravity in AdS and its dual matrix model/topological recursion

Saad, Shenker, Stanford '19

- New integrability structures to be uncovered

‘Flattening’ regime – type 2 resolvent

$$W(g) = \frac{1}{g} + \frac{1}{g^3}\langle\frac{\mathrm{tr}}{N}M^2\rangle + \frac{1}{g^5}\langle\frac{\mathrm{tr}}{N}M^4\rangle + \frac{1}{g^7}\langle\frac{\mathrm{tr}}{N}M^6\rangle + \dots$$

$$\beta \sim v \sim (1-\lambda) \sim \tau \to 0$$

We get from inversion formula

$$\frac{W(g) - \frac{L}{2D}}{\beta} =$$

$$= \frac{16D^3}{L^2(4D^2 - L^2)}\left[L^2\frac{\sin^{-1}(\xi)^2 - \xi^2}{4\pi\xi} + \frac{\xi^3 + 12\sqrt{1-\xi^2}\sin^{-1}(\xi) - 12\xi + 3\xi\sin^{-1}(\xi)^2}{6\pi}\right]$$

Kazakov, FLM

$$\xi = \frac{1}{\sqrt{2}}\left(x - \sqrt{x^2-1}\right)L\,, \qquad L = D\left(g - \sqrt{g^2-4}\right)$$

sums up boundary “trees”

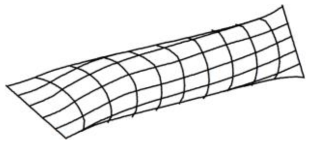
Highly similar to previous resolvent though not exactly equal

Related by a simple but formal replacement (meaning to be understood)

Asymptotics of large area & boundary

$$Z \propto \sum_G \lambda^F \beta^{2(\# v_2 - 4)}$$

Simplest case : only 4 2-vertices -- Dedekind function

$$\sim \sum$$  $$= \frac{1}{8}\frac{\partial^2}{\partial q^2} \log \left[\prod_{n=1}^{\infty} (1 - q^{2n}) \right]$$

Large area: elliptic modulus = 1 $\lambda = q \to 1$

What happens when we add other vertices?

Recent motivation – Jackiw-Teitelboim (JT) gravity

Saad, Shenker, Stanford '19
Witten, Mirzakhani

Gravity theory in AdS_2, fascinating dualities with matrix models as well as AdS_2/CFT_1

Speculative hope – study it using DWG