String integrability of defect CFT and dynamical reflection matrices

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New mathematical methods in solvable models and gauge/string dualities
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Section 1

Introduction

De facto boundaries and defects

Boundaries and defects go hand in hand with real-world physical systems... They are simply a manifestation of imperfection: impurities, domain walls, boundaries, interfaces and defects separate regions with different properties and break many of the underlying symmetries. Here are some examples:

Formal aspects

- Holography & the AdS/CFT correspondence
- String theory
- Quantum entanglement

Applied aspects

- Statistical systems near surfaces
- Topological materials such as graphene and topological insulators
- Out-of-equilibrium systems and quantum quenches

Naturally, boundaries and defects permeate all branches of physics, from high-energy and particle physics, to condensed matter, statistical, even gravity and mathematical physics.

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- The study of higher codimension defects and using them to bootstrap both the original and the boundary CFT forms part of the so-called Boundary Conformal Bootstrap program (initiated by Liendo, Rastelli, van Rees, 2012).
- The boundaries can either host new degrees of freedom or just provide BCs for the bulk fields

$$S = S_d + S_{d-1},$$

thus giving rise to boundary and defect CFTs (BCFTs & dCFTs)...

The AdS/CFT correspondence

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The prototype of holographic dualities is the $\text{AdS}_5/\text{CFT}_4$ correspondence:

$$\mathcal{N} = 4,\ \mathfrak{su}(N) \text{ super Yang-Mills theory in 4d} \ \Leftrightarrow\ \text{Type IIB superstring theory on } \text{AdS}_5 \times \text{S}^5$$

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There also exists an $\text{AdS}_4/\text{CFT}_3$ duality... reading, for $k^5\gg N$:

$\mathcal{N}=6,\ U(N)_k\times\hat{U}(N)_{-k}$ super Chern-Simons theory in 3d with Chern-Simons levels $\pm k\in\mathbb{Z}$ $\Leftrightarrow$ Type IIA string theory on $\text{AdS}_4\times\mathbb{CP}^3$ with N units of flux in AdS_4 and k units in $\mathbb{CP}^3$

(Aharony-Bergman-Jafferis-Maldacena, 2008)

IIA/ABJM is quantum integrable in the planar limit $k,N\to\infty$, $\lambda\equiv g_{\text{CS}}^2 N=\text{const}$. ($g_{\text{CS}}^2\equiv 1/k$).

$\mathcal{N} = 4$, super Yang-Mills theory

$\mathcal{N} = 4$, super Yang-Mills (SYM) theory is a 4-dimensional superconformal gauge theory:

$$\mathcal{L}_{\mathcal{N}=4} = \frac{2}{g_{\text{YM}}^2} \cdot \text{tr}\Big\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(D_\mu \varphi_i\right)^2 + i\, \bar{\psi}_\alpha \not{D} \psi_\alpha + \frac{1}{4} \left[\varphi_i, \varphi_j\right]^2 + \\ + \sum_{i=1}^{3} G^i_{\alpha\beta} \bar{\psi}_\alpha \left[\varphi_i, \psi_\beta\right] + \sum_{i=4}^{6} G^i_{\alpha\beta} \bar{\psi}_\alpha \gamma_5 \left[\varphi_i, \psi_\beta\right] \Big\}$$

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- Half-BPS boundary conditions were studied by Gaiotto-Witten (2008)...

ABJM theory

ABJM theory is a 3-dimensional superconformal gauge theory:

$$\mathcal{L}_{\text{ABJM}} = \frac{k}{4\pi} \cdot \Big[\epsilon^{\mu\nu\rho}\text{tr}\Big\{A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho - \hat{A}_\mu \partial_\nu \hat{A}_\rho - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho\Big\} - \text{tr}\Big\{ \left(D_\mu Y_B\right)^\dagger D^\mu Y_B + \\ + i\psi_B^\dagger \not{D} \psi_B \Big\} - V_{\text{ferm}} - V_{\text{bos}}\Big], \text{ where } B = 1, \ldots, 4,$$

where the potential contains mixed quartic and sextic bosonic terms which read

$$V_{\text{ferm}} = \frac{i}{2}\text{tr}\Big\{ Y_A^\dagger Y_A \psi_B^\dagger \psi_B - Y_A Y_A^\dagger \psi_B \psi_B^\dagger + 2 Y_A Y_B^\dagger \psi_A \psi_B^\dagger - 2 Y_A^\dagger Y_B \psi_A^\dagger \psi_B - \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon^{ABCD} Y_A \psi_B^\dagger Y_C \psi_D^\dagger \Big\}$$

$$V_{\text{bos}} = -\frac{1}{12}\text{tr}\Big\{ Y_A Y_A^\dagger Y_B Y_B^\dagger Y_C Y_C^\dagger + Y_A^\dagger Y_A Y_B^\dagger Y_B Y_C^\dagger Y_C + 4 Y_A Y_B^\dagger Y_C Y_A^\dagger Y_B Y_C^\dagger - 6 Y_A Y_B^\dagger Y_B Y_A^\dagger Y_C Y_C^\dagger \Big\}.$$

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- Solution of the spectral problem via QSC... (Cavaglià-Fioravanti-Gromov-Tateo, 2014)

Section 2

String integrability in AdS/CFT

Superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathrm{AdS}_4 \times \mathbb{CP}^3$

- The formulation of type IIB Green-Schwarz superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is based on the observation that the target space is a product of cosets that spans the bosonic section of a supercoset:

$$\mathrm{AdS}_5 \times \mathrm{S}^5 = \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)} \subseteq \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)},$$

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- Similarly, the target space of type IIA Green-Schwarz superstring theory on $\text{AdS}_4 \times \mathbb{CP}^3$ is a product of cosets and the bosonic section of a supercoset:

$$\text{AdS}_4 \times \mathbb{CP}^3 = \frac{SO(3,2)}{SO(3,1)} \times \frac{SO(6)}{U(3)} \subseteq \frac{OSP(2,2|6)}{SO(3,1) \times U(3)}$$

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- Both of the above supercosets are semi-symmetric spaces... semi-symmetric spaces are known to give rise to integrable nonlinear sigma models that describe Green-Schwarz superstrings...

Green-Schwarz σ-models in semi-symmetric superspaces

Green-Schwarz superstrings in semi-symmetric spaces are described by nonlinear σ-models with action:

$$S = -\frac{T}{2}\int \text{str}\left[J^{(2)} \wedge \star J^{(2)} + J^{(1)} \wedge J^{(3)}\right], \qquad T \equiv \frac{\ell^2}{2\pi\alpha'}.$$

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Semi-symmetric superspaces are endowed with a $\mathbb{Z}_4$ grading. Their elements afford a $\mathbb{Z}_4$ decomposition:

$$J \equiv \mathfrak{g}^{-1} d\mathfrak{g} = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}, \qquad \Omega\left[J^{(n)}\right] = i^n J^{(n)},$$

where J is the (moving-frame) current, $\mathfrak{g}$ is an element of the superspace and Ω a $\mathbb{Z}_4$ automorphism:

$$\Omega(M) = -\mathcal{K} M^{\text{st}} \mathcal{K}^{-1}, \quad \mathcal{K}_{\text{AdS}_5\times\text{S}^5} = \begin{bmatrix} \gamma_{13} & 0 \\ 0 & \gamma_{13} \end{bmatrix}, \; M \in \mathfrak{sl}(4|4)$$

$$\mathcal{K}_{\text{AdS}_4\times\mathbb{CP}^3} = \begin{bmatrix} K_4 & 0 \\ 0 & -K_6 \end{bmatrix}, \quad K_4 \equiv \gamma_{12}, \; K_6 \equiv \mathrm{I}_3 \otimes (i\sigma_2), \; M \in OSP(2,2|6).$$

Parameter matching relates the string tension T (and the AdS radius ℓ) to the 't Hooft coupling λ:

$$T_{\text{AdS}_5\times\text{S}^5} = \frac{\sqrt{\lambda}}{2\pi}, \qquad T_{\text{AdS}_4\times\mathbb{CP}^3} = \sqrt{\frac{\lambda}{2}}.$$

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$$S = -\frac{T}{2}\int \text{str}\left[J^{(2)} \wedge \star J^{(2)} + J^{(1)} \wedge J^{(3)}\right], \qquad T \equiv \frac{\ell^2}{2\pi\alpha'}.$$

The current J has a vanishing curvature, i.e.

$$dJ + J \wedge J = 0.$$

The equations of motion that follow from the superstring action afford a Lax representation:

$$dL + L \wedge L = 0,$$

where the Lax connection L is given by

$$L(\mathrm{x}) = J^{(0)} + \frac{\mathrm{x}^2+1}{\mathrm{x}^2-1} J^{(2)} - \frac{2\mathrm{x}}{\mathrm{x}^2-1} \star J^{(2)} + \sqrt{\frac{\mathrm{x}+1}{\mathrm{x}-1}} J^{(1)} + \sqrt{\frac{\mathrm{x}-1}{\mathrm{x}+1}} J^{(3)},$$

and x is the spectral parameter.

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Defining the fixed-frame current (or left current)

$$j \equiv \mathfrak{g} J \mathfrak{g}^{-1} = d\mathfrak{g}\, \mathfrak{g}^{-1} = j^{(0)} + j^{(1)} + j^{(2)} + j^{(3)}, \qquad j^{(n)} \equiv \mathfrak{g} J^{(n)} \mathfrak{g}^{-1}$$

the corresponding Lax connection assumes the following form:

$$a(\mathrm{x}) = \frac{2}{\mathrm{x}^2 - 1}\left(j^{(2)} - \mathrm{x} \star j^{(2)}\right) + (\mathrm{z} - 1)\, j^{(1)} + \left(\frac{1}{\mathrm{z}} - 1\right) j^{(3)}, \qquad \mathrm{z} \equiv \sqrt{\frac{\mathrm{x}+1}{\mathrm{x}-1}}.$$

(Bena, Polchinski, Roiban, 2003)

The Green-Schwarz σ-model is classically integrable... Note also the fixed-frame flatness conditions:

$$dj - j \wedge j = 0, \qquad da + a \wedge a = 0.$$

Integrability of superstrings on $AdS_4 \times \mathbb{CP}^3$

The notion of classical Liouville integrability is typically extended to $(1 + 1$ dimensional) field theory systems by means of a monodromy matrix:

$$\mathcal{M}(\sigma_1, \sigma_2, \tau; \mathrm{x}) \equiv \overrightarrow{\mathrm{P}} \exp \left(\int_{\sigma_1}^{\sigma_2} ds \, a_\sigma(s, \tau; \mathrm{x}) \right).$$

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$$\mathcal{M}(\mathrm{x}) = \mathbb{1} - \frac{2}{\mathrm{x}}\int_0^{2\pi} ds\, j_\tau^{(2)} + \frac{2}{\mathrm{x}^2}\left[\int_0^{2\pi} ds\, j_\sigma^{(2)} + 2\int_0^{2\pi}\int_0^{s} ds ds'\, j_\tau^{(2)'} j_\tau^{(2)}\right] - \ldots = \exp\left[2\sum_{r=0}^{\infty}\left(-\frac{1}{\mathrm{x}}\right)^{r+1} Q_r\right]$$

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All the charges of the MT σ-model Poisson-commute (Mikhailov, Schäfer-Nameki, 2007; Magro, 2008).

Section 3

Brane integrability in AdS/CFT

Integrable holographic dCFTs

All these methods can used to study boundaries and defects...

- *Holographic* dCFTs were first realized in the context of the AdS/CFT correspondence by Karch-Randall (2001), in an attempt to provide an explicit realization of gravity localization on an AdS_4 brane.

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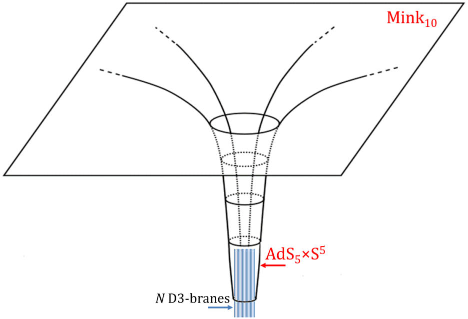
- *Supersymmetric localization* methods were introduced by Komatsu-Wang (2020). They permit to carry out precision tests of the planar AdS/dCFT duality beyond perturbation theory.

To set the stage for the study of string BCs, let us briefly review the Karch-Randall system...

The D3-D5 probe-brane system

Type IIB string theory on $AdS_5 \times S^5$ is encountered very close to a system of N coincident D3-branes:



The D3-branes extend along x_1, x_2, x_3...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	●	●	●	●						

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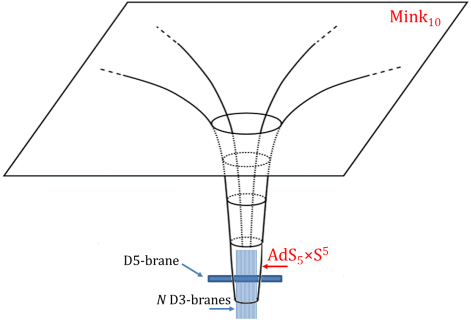

Now insert a single (probe) D5-brane at $x_3 = x_7 = x_8 = x_9 = 0$...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
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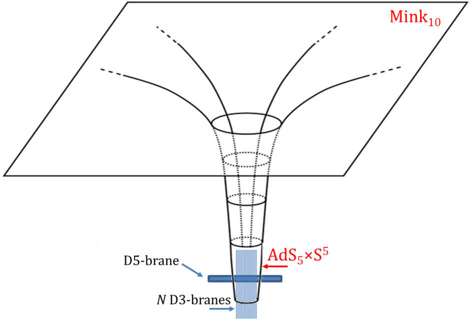

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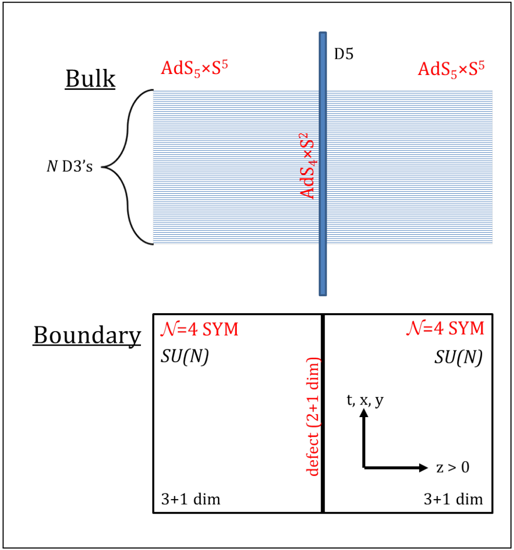
The dual defect SCFT



- In the bulk, the D3-D5 system describes IIB string theory on $AdS_5 \times S^5$ bisected by a D5 brane with worldvolume geometry $AdS_4 \times S^2$.
- The dual field theory is still $SU(N)$, $\mathcal{N} = 4$ SYM in $3+1$ dimensions, that interacts with a CFT living on the $2+1$ dimensional defect:
$$S = S_{\mathcal{N}=4} + S_{2+1}.$$
(DeWolfe, Freedman, Ooguri, 2001)
- Due to the presence of the defect, the total bosonic symmetry of the system is reduced from $SO(4,2) \times SO(6)$ to $SO(3,2) \times SO(3) \times SO(3)$.
- The corresponding superalgebra $\mathfrak{psu}(2,2|4)$ becomes $\mathfrak{osp}(4|4)$.

D5-brane embedding

There are also k units of magnetic flux through the S^2... forcing k D3-branes to end on the D5-brane...

$$\int_{S^2} \frac{F}{2\pi} = k, \qquad F = \frac{k}{4} \cdot \sum_{a,b,c=4}^{6} \varepsilon_{abc}\, x_a\, dx_b \wedge dx_c .$$

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The geometry of the D5-brane in $AdS_5 \times S^5$ will still be $AdS_4 \times S^2$... Its embedding will be described by:

$$x_3 = \kappa \cdot z, \qquad \kappa \equiv \frac{\pi k}{\sqrt{\lambda}} \equiv \tan \alpha .$$

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where the metric of $AdS_5 \times S^5$ is written as:

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2\right) + \ell^2 d\Omega_5^2, \qquad z \equiv \frac{1}{r}$$

and the 2-sphere is parameterized by:

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2 = \sum_{a=4}^{6} dx_a\, dx_a, \qquad \sum_{a=4}^{6} x_a\, x_a = 1, \qquad x_7 = x_8 = x_9 = 0.$$

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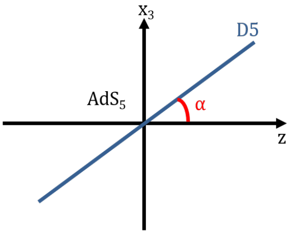
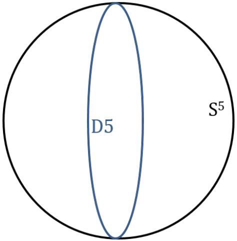
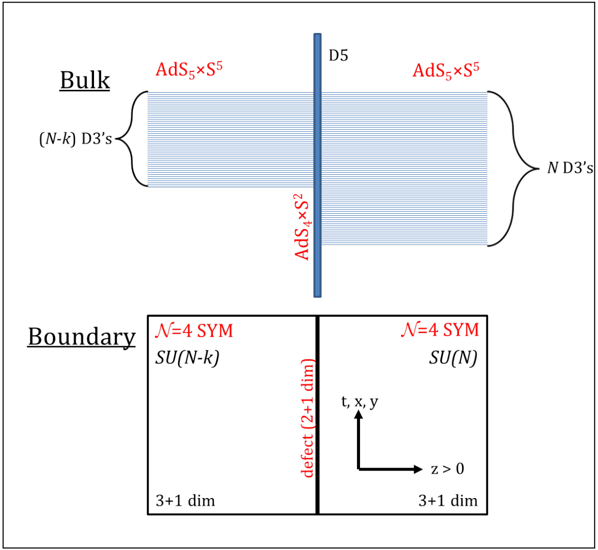




The D3-D5 dSCFT



- On the dual SCFT side, the gauge group $SU(N) \times SU(N)$ breaks to $SU(N-k) \times SU(N)$.
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  (Karch-Randall, 2001b)

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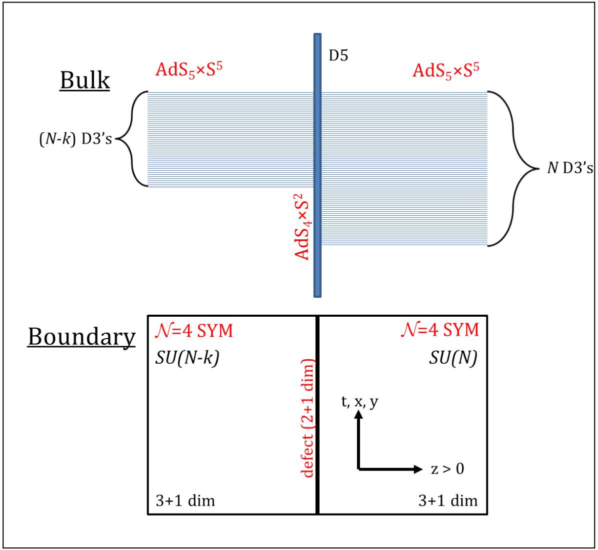

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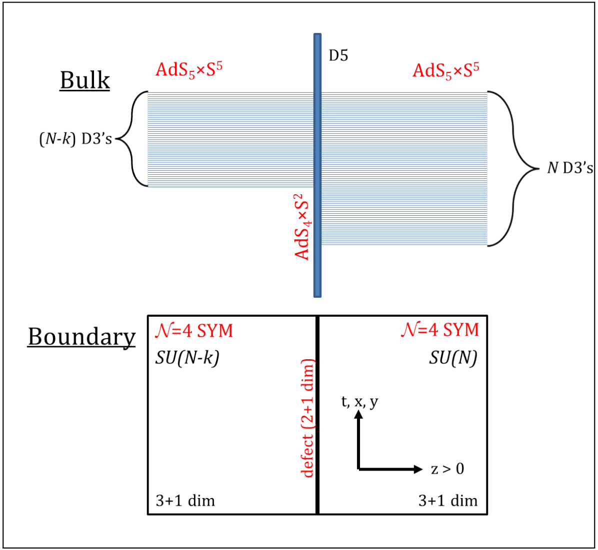

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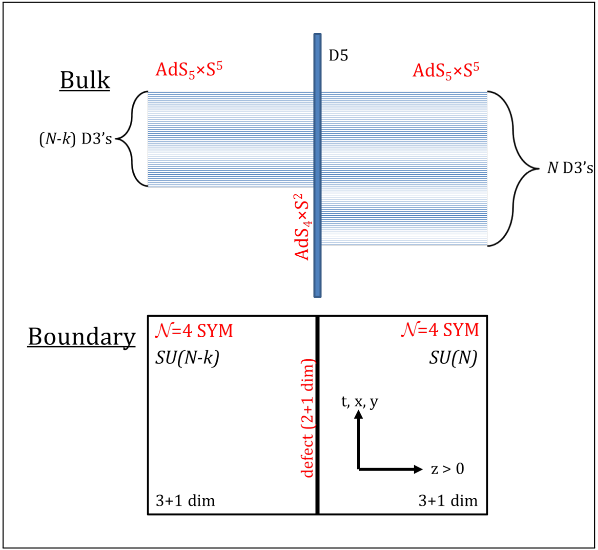

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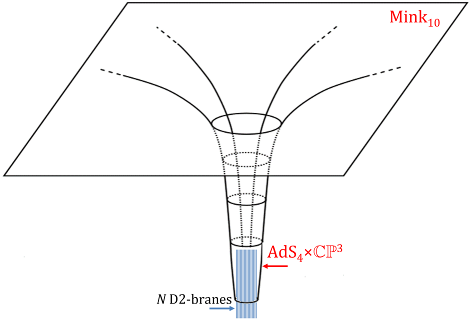

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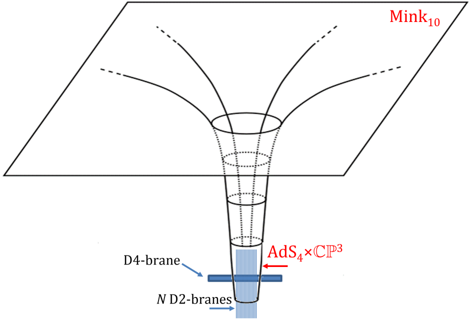

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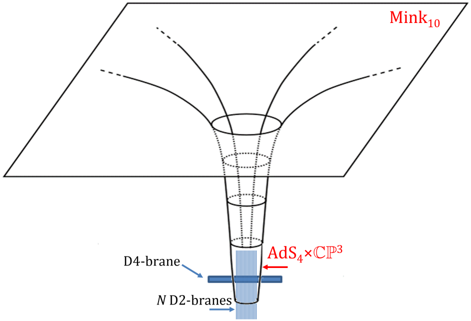

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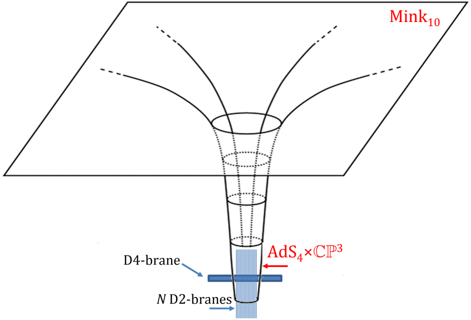

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$$ds^2 = \frac{\ell^2}{z^2}\left(-dx_0^2 + dx_1^2 + dx_2^2 + dz^2\right) + 4\ell^2\Bigg[d\xi^2 + \cos^2\xi\,\sin^2\xi\left(d\psi + \frac{1}{2}\cos\theta_1\,d\phi_1 - \frac{1}{2}\cos\theta_2\,d\phi_2\right)^2 + \\ + \frac{1}{4}\cos^2\xi\left(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2\right) + \frac{1}{4}\sin^2\xi\left(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2\right)\Bigg],$$

where $\xi \in [0, \pi/2)\,,\ \theta_{1,2} \in [0, \pi]\,,\ \phi_{1,2} \in [0, 2\pi)\,,\ \psi \in [-2\pi, 2\pi]$.

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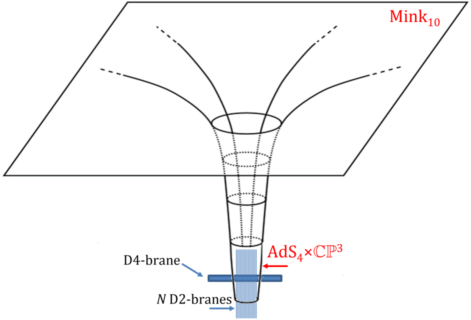

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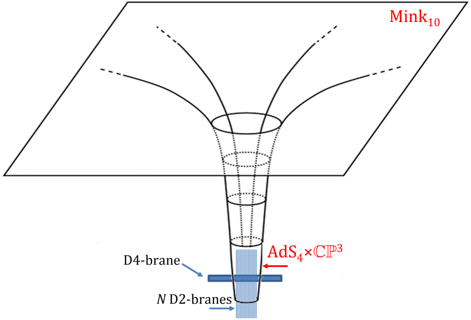
$$\left. + \frac{1}{4}\cos^2\xi\left(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2\right) + \frac{1}{4}\sin^2\xi\left(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2\right)\right],$$

where $\xi \in [0, \pi/2)$, $\theta_{1,2} \in [0, \pi]$, $\phi_{1,2} \in [0, 2\pi)$, $\psi \in [-2\pi, 2\pi]$.

The D2-D4 probe-brane system

IIA string theory on $AdS_4 \times \mathbb{CP}^3$ is encountered very close to a system of N coincident D2-branes:



The probe D4-brane lies at $x_1 = \xi = \theta_2 = \phi_2 = \psi = 0$... its geometry will be $AdS_3 \times \mathbb{CP}^1$...

$$ds^2 = \frac{\ell^2}{z^2}\left(-dx_0^2 + dx_2^2 + dz^2\right) + \ell^2\left(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2\right).$$

Note that $\mathbb{CP}^1$ is just a 2-sphere: $ds^2_{\mathbb{CP}^1} = \ell^2\left(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2\right) = \sum_{i=4}^6 dx_i\, dx_i, \quad \sum_{i=4}^6 x_i\, x_i = \ell^2.$

D4-brane embedding

The brane geometry is also supported by Q units of magnetic flux through $\mathbb{CP}^1$...

$$F = \ell^2\, Q\, d\cos\theta_1 \wedge d\phi_1 = -\ell^2\, Q\, \sin\theta_1\, d\theta_1\, d\phi_1 = dA.$$

The flux forces exactly $q \equiv \sqrt{2\lambda}\, Q$ of the D2-branes to terminate on one side of the D4-brane...

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The $AdS_3 \times \mathbb{CP}^1 \subset AdS_4 \times \mathbb{CP}^3$ embedding of the probe D4-brane is described by the set of equations:

$$x_2 \stackrel{!}{=} Q \cdot z \qquad \& \qquad \xi \stackrel{!}{=} 0, \qquad \theta_2, \, \phi_2, \, \psi \stackrel{!}{=} \text{constant}.$$

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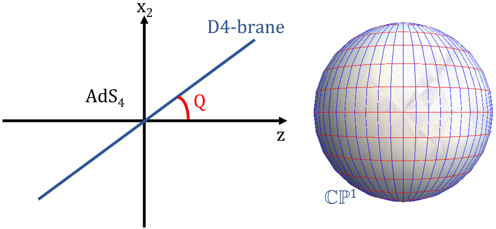
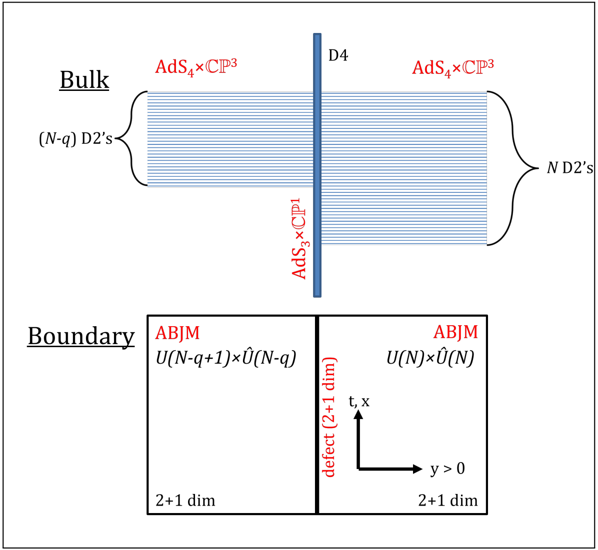


The D2-D4 dSCFT



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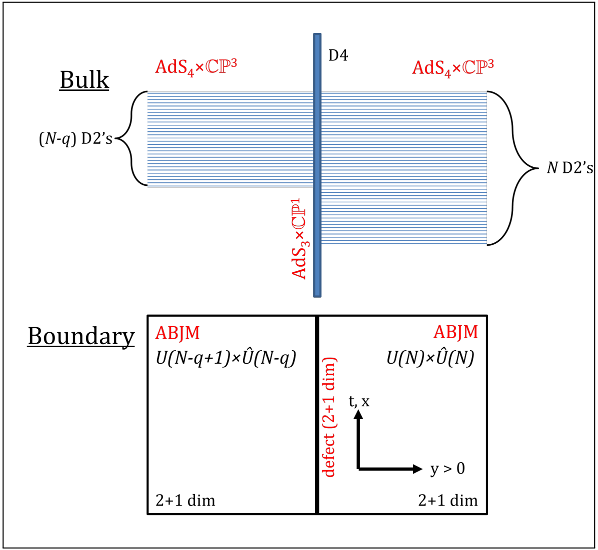

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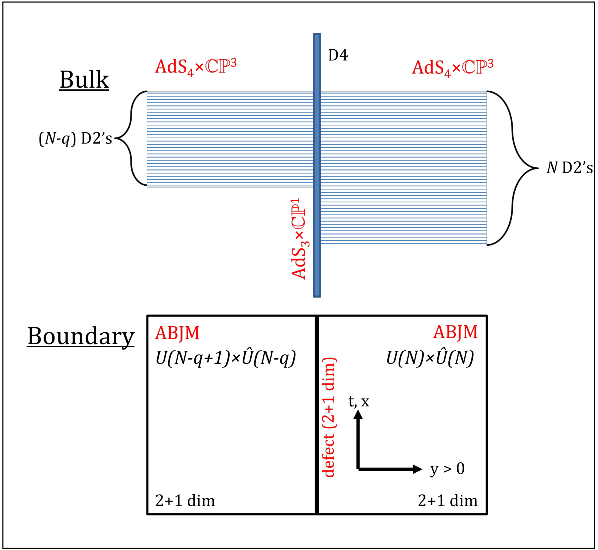

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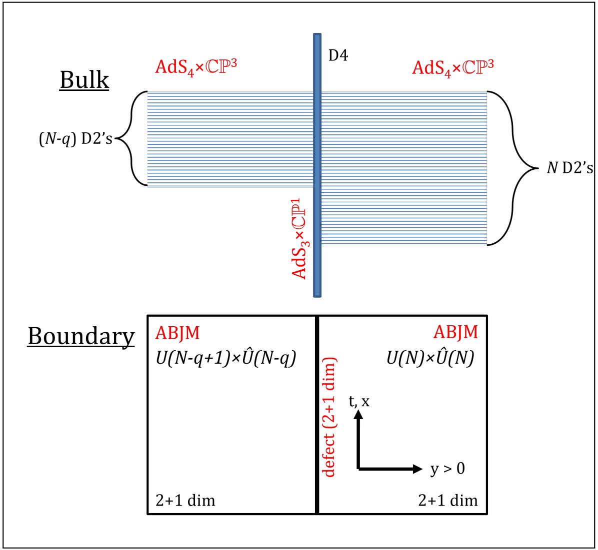

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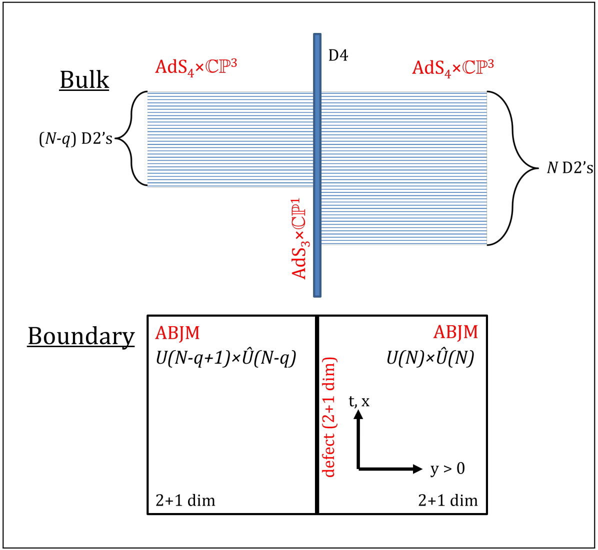

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- Closed strings in $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ are integrable but how about open strings? (String) BCs may break integrability...
- Dekel-Oz (2011) proved classical integrability of the D3-D5 & D2-D4 systems in the zero-flux case...

Subsection 2

D-brane integrability

String boundary conditions

In the absence of fermions, the superstring action reduces to the string Polyakov action,

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\, G_{mn}\gamma^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n + \int d\tau \left[A_i \dot{X}_i\right]_{\sigma=0},$$

on either $\mathrm{AdS}_5 \times \mathrm{S}^5$ or $\mathrm{AdS}_4 \times \mathbb{CP}^3$. Commonly, the brane cuts the string worldsheet at a constant-σ section while the variations of the initial (τ_1) and final (τ_2) string states vanish:

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By varying the Polyakov action it is easy to derive the string boundary conditions:

$$\acute{X}_i - 2\pi\alpha' F_{ij}\dot{X}_j \stackrel{!}{=} 0 \qquad \text{(longitudinal: Neumann-Dirichlet)}$$

$$\dot{X}_a \stackrel{!}{=} 0 \qquad \text{(transverse: Dirichlet).}$$

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By varying the Polyakov action it is easy to derive the string boundary conditions, finding in particular:

AdS$_5$: $\acute{x}_{0,1,2} \stackrel{!}{=} \acute{z} + \kappa\, \acute{x}_3 \stackrel{!}{=} 0$ (Neumann)

$\dot{x}_3 - \kappa\, \dot{z} \stackrel{!}{=} 0$ (Dirichlet)

S^5: $\acute{x}_4 \stackrel{!}{=} \kappa\,(x_5\, \dot{x}_6 - x_6\, \dot{x}_5)$ (Neumann-Dirichlet)

$\acute{x}_5 \stackrel{!}{=} \kappa\,(x_6\, \dot{x}_4 - x_4\, \dot{x}_6)$

$\acute{x}_6 \stackrel{!}{=} \kappa\,(x_4\, \dot{x}_5 - x_5\, \dot{x}_4)$

$\dot{x}_7 \stackrel{!}{=} \dot{x}_8 \stackrel{!}{=} \dot{x}_9 \stackrel{!}{=} 0$ (Dirichlet).

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AdS_4

$$\acute{x}_{0,1} \stackrel{!}{=} \acute{z} + Q\,\acute{x}_2 \stackrel{!}{=} 0 \quad \text{(Neumann)}$$
$$\dot{x}_2 - Q\,\dot{z} \stackrel{!}{=} 0 \quad \text{(Dirichlet)}$$

$\mathbb{CP}^3$

$$\acute{\theta}_1 + \tilde{Q}\sin\theta_1\dot{\phi}_1 \stackrel{!}{=} \sin\theta_1\acute{\phi}_1 - \tilde{Q}\,\dot{\theta}_1 \stackrel{!}{=} 0 \quad \text{(Neumann-Dirichlet)}$$
$$\dot{\xi} \stackrel{!}{=} \dot{\theta}_2 \stackrel{!}{=} \dot{\phi}_2 \stackrel{!}{=} \dot{\psi} \stackrel{!}{=} 0 \quad \text{(Dirichlet)},$$

where we have defined $\tilde{Q} \equiv q/\lambda$.

Integrable boundary conditions

Is the probe D-brane integrable?

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Is the probe D-brane integrable? To be able to answer this question we need to set up a formalism that takes into account the boundary at $\sigma = 0$. The double-row monodromy matrix (Sklyanin, 1987),

$$\mathcal{T}(\tau; \mathrm{x}) = \mathcal{M}^{\mathrm{st}}(\tau; -\mathrm{x}) \cdot \mathbb{U}(\tau; \mathrm{x}) \cdot \mathcal{M}(\tau; \mathrm{x}),$$

is formed by joining two monodromy matrices through a reflection matrix $\mathbb{U}$ which generally depends on both the spectral parameter x and the string embedding coordinates (x, z).

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is formed by joining two monodromy matrices through a reflection matrix $\mathbb{U}$ which generally depends on both the spectral parameter x and the string embedding coordinates (x, z). Taking the derivative,

$$\dot{\mathcal{T}}(\mathrm{x}) = \mathcal{M}^{\mathsf{st}}(-\mathrm{x}) \cdot \Big(\dot{\mathbb{U}}(\mathrm{x}) - a_\tau^{\mathsf{st}}(-\mathrm{x}) \cdot \mathbb{U}(\mathrm{x}) - \mathbb{U}(\mathrm{x}) \cdot a_\tau(\mathrm{x})\Big) \cdot \mathcal{M}(\mathrm{x}),$$

we conclude that the corresponding transfer matrix will be conserved if

$$\dot{\mathbb{U}}(\mathrm{x}) \overset{!}{=} a_\tau^{\mathsf{st}}(-\mathrm{x}) \cdot \mathbb{U}(\mathrm{x}) + \mathbb{U}(\mathrm{x}) \cdot a_\tau(\mathrm{x}),$$

where $a(\mathrm{x})$ stands for the flat fixed frame connection

$$a\,(\mathrm{x}) = \frac{2}{\mathrm{x}^2 - 1}\left(j^{(2)} - \mathrm{x} \star j^{(2)}\right) + (\mathrm{z} - 1)\, j^{(1)} + \left(\frac{1}{\mathrm{z}} - 1\right) j^{(3)}, \qquad \mathrm{z} \equiv \sqrt{\frac{\mathrm{x}+1}{\mathrm{x}-1}}.$$

Integrable boundary conditions

In terms of the current j, the D-brane integrability condition becomes:

$$\dot{\mathbb{U}} \stackrel{!}{=} \frac{2}{\mathrm{x}^2-1} \cdot \left\{ j_\tau^{(2)\,\mathrm{st}}\,\mathbb{U} + \mathbb{U}\,j_\tau^{(2)} \right\} + \frac{2\mathrm{x}}{\mathrm{x}^2-1} \cdot \left\{ j_\sigma^{(2)\,\mathrm{st}}\,\mathbb{U} - \mathbb{U}\,j_\sigma^{(2)} \right\} + \\ + (\mathrm{z}-1) \cdot \left\{ \mathbb{U}\,j_\tau^{(1)} + j_\tau^{(3)\,\mathrm{st}}\,\mathbb{U} \right\} + \left(\frac{1}{\mathrm{z}} - 1 \right) \cdot \left\{ \mathbb{U}\,j_\tau^{(3)} + j_\tau^{(1)\,\mathrm{st}}\,\mathbb{U} \right\},$$

so that if the reflection matrix $\mathbb{U}$ is a constant matrix that does not depend on x and τ,

$$j_\tau^{(2)\,\mathrm{st}}\,\mathbb{U} + \mathbb{U}\,j_\tau^{(2)} \stackrel{!}{=} j_\sigma^{(2)\,\mathrm{st}}\,\mathbb{U} - \mathbb{U}\,j_\sigma^{(2)} \stackrel{!}{=} 0, \qquad \mathbb{U}\,j_\tau^{(1)} + j_\tau^{(3)\,\mathrm{st}}\,\mathbb{U} \stackrel{!}{=} \mathbb{U}\,j_\tau^{(3)} + j_\tau^{(1)\,\mathrm{st}}\,\mathbb{U} \stackrel{!}{=} 0$$

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$$+(\mathrm{z}-1)\cdot\left\{\mathbb{U}\,j_\tau^{(1)}+j_\tau^{(3)\,\mathrm{st}}\,\mathbb{U}\right\}+\left(\frac{1}{\mathrm{z}}-1\right)\cdot\left\{\mathbb{U}\,j_\tau^{(3)}+j_\tau^{(1)\,\mathrm{st}}\,\mathbb{U}\right\},$$

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In the bosonic case, the fermionic currents $j^{(1,3)}$ vanish, supertransposes get replaced by transposes...

$$\dot{\mathbb{U}} \stackrel{!}{=} \frac{2}{\mathrm{x}^2-1}\cdot\left\{j_\tau^{(2)\,\mathrm{t}}\,\mathbb{U}+\mathbb{U}\,j_\tau^{(2)}\right\}+\frac{2\mathrm{x}}{\mathrm{x}^2-1}\cdot\left\{j_\sigma^{(2)\,\mathrm{t}}\,\mathbb{U}-\mathbb{U}\,j_\sigma^{(2)}\right\},$$

while the reflection matrix $\mathbb{U}$ is block diagonal, with one block for AdS and another for either S^5 or $\mathbb{CP}^3$:

$$\mathbb{U}=\mathcal{K}\cdot\begin{bmatrix}\mathbb{U}_{\mathrm{AdS}} & 0\\ 0 & \mathbb{U}_{\mathrm{S}}\end{bmatrix}, \qquad \mathbb{U}=\begin{bmatrix}\mathbb{U}_{\mathrm{AdS}} & 0\\ 0 & \mathbb{U}_{\mathbb{CP}}\end{bmatrix}.$$

Integrable boundary conditions

In terms of the current j, the D-brane integrability condition becomes:

$$\dot{\mathbb{U}} \overset{!}{=} \frac{2}{\mathrm{x}^2-1}\cdot\left\{j_\tau^{(2)\,\mathrm{st}}\,\mathbb{U}+\mathbb{U}\,j_\tau^{(2)}\right\}+\frac{2\mathrm{x}}{\mathrm{x}^2-1}\cdot\left\{j_\sigma^{(2)\,\mathrm{st}}\,\mathbb{U}-\mathbb{U}\,j_\sigma^{(2)}\right\}+$$
$$+(\mathrm{z}-1)\cdot\left\{\mathbb{U}\,j_\tau^{(1)}+j_\tau^{(3)\,\mathrm{st}}\,\mathbb{U}\right\}+\left(\frac{1}{\mathrm{z}}-1\right)\cdot\left\{\mathbb{U}\,j_\tau^{(3)}+j_\tau^{(1)\,\mathrm{st}}\,\mathbb{U}\right\},$$

so that if the reflection matrix $\mathbb{U}$ is a constant matrix that does not depend on x and τ,

$$j_\tau^{(2)\,\mathrm{st}}\,\mathbb{U}+\mathbb{U}\,j_\tau^{(2)} \overset{!}{=} j_\sigma^{(2)\,\mathrm{st}}\,\mathbb{U}-\mathbb{U}\,j_\sigma^{(2)} \overset{!}{=} 0, \qquad \mathbb{U}\,j_\tau^{(1)}+j_\tau^{(3)\,\mathrm{st}}\,\mathbb{U} \overset{!}{=} \mathbb{U}\,j_\tau^{(3)}+j_\tau^{(1)\,\mathrm{st}}\,\mathbb{U} \overset{!}{=} 0$$ (Dekel-Oz, 2011).

In the bosonic case, the fermionic currents $j^{(1,3)}$ vanish, supertransposes get replaced by transposes...

$$\dot{\mathbb{U}} \overset{!}{=} \frac{2}{\mathrm{x}^2-1}\cdot\left\{j_\tau^{(2)\,\mathrm{t}}\,\mathbb{U}+\mathbb{U}\,j_\tau^{(2)}\right\}+\frac{2\mathrm{x}}{\mathrm{x}^2-1}\cdot\left\{j_\sigma^{(2)\,\mathrm{t}}\,\mathbb{U}-\mathbb{U}\,j_\sigma^{(2)}\right\},$$

while the reflection matrix $\mathbb{U}$ is block diagonal, with one block for AdS and another for either S^5 or $\mathbb{CP}^3$:

$$\mathbb{U}=\mathcal{K}\cdot\begin{bmatrix}\mathbb{U}_{\mathrm{AdS}} & 0\\ 0 & \mathbb{U}_{\mathrm{S}}\end{bmatrix}, \qquad \mathbb{U}=\begin{bmatrix}\mathbb{U}_{\mathrm{AdS}} & 0\\ 0 & \mathbb{U}_{\mathbb{CP}}\end{bmatrix}.$$

A given set of BCs is integrable, if a reflection matrix $\mathbb{U}$ can be found that satisfies the above condition.

Integrable D5-brane in AdS_5

The coset representative of AdS_5 can be expressed in terms of the standard conformal generators:

$$\mathfrak{g}_{\text{AdS}} = \mathrm{e}^{P_\mu x^\mu} z^D,$$

where $\mu = 0, \ldots, 3$ and the 4-dimensional conformal group dimensions is spanned by

$$D = \frac{\gamma_4}{2}, \qquad P_\mu = \Pi_+ \gamma_\mu, \qquad K_\mu = \Pi_- \gamma_\mu, \qquad L_{\mu\nu} = \gamma_{\mu\nu}, \qquad \Pi_\pm = \frac{1 \pm \gamma_4}{2}.$$

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The corresponding fixed-frame current is

$$j^{(2)}_{\text{AdS}} = \frac{1}{2z^2} \left[2\left(zdz + xdx\right)\left(D - xP\right) + \left(z^2 + x^2\right) Pdx + Kdx + L_{\mu\nu} x^\mu dx^\nu\right],$$

while the reflection matrix that satisfies the D-brane integrability condition is dynamical:

$$\mathbb{U}_{\text{AdS}} = \gamma_3 + \frac{2\kappa}{\mathrm{x}^2 + 1} \cdot \frac{x^\mu \gamma_\mu - \Pi_+ - (x^2 + z^2)\Pi_-}{z}.$$

(Zarembo-GL, 2021)

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The coset parametrization of S^5 is

$$\mathfrak{g}_S = n_6 + i\gamma_a n_a, \qquad a = 1, \ldots, 5$$

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$$n_6 = \cos\frac{\theta}{2}, \qquad n_a = m_a \sin\frac{\theta}{2}, \qquad m_a^2 = 1.$$

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$$j_S^{(2)} = i(2n_6^2 - 1)n_6 dn_a\gamma_a - i(2n_6^2 + 1)dn_6 n_a\gamma_a - 2n_6^2 n_a dn_b\gamma_{ab}$$

and the integrable reflection matrix is

$$\mathbb{U}_S = \gamma_{45} + \frac{2\kappa\,\mathrm{x}}{\mathrm{x}^2 + 1} \cdot \frac{n_i\gamma_i}{n_6}, \qquad i = 1, 2, 3,$$

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similar to the D5-brane result GL-Zarembo (2021)... For $Q = 0$ we obtain the Dekel-Oz (2011) result...

Integrable D4-brane in $\mathbb{CP}^3$

The coset parametrization of $\mathbb{CP}^3$ is

$$\mathfrak{g}_{\mathbb{CP}} = e^{-R_8\psi} e^{T_3\phi_1} e^{T_4\left(\theta_1+\frac{\pi}{2}\right)} e^{R_3\phi_2} e^{R_4\left(\theta_2+\frac{\pi}{2}\right)} e^{2T_6\xi}$$

where $R_1, \ldots, R_9$ are the graded-0 generators of $\mathfrak{so}(6)$ with respect to its $\mathfrak{u}(3)$ subalgebra, and $T_1, \ldots, T_6$ are the graded-2 generators...

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$$j_\tau^{(2)} \stackrel{!}{=} T_3\dot{\phi}_1 \sin^2\theta_1 + T_4\left(\dot{\theta}_1\cos\phi_1 - \dot{\phi}_1\sin\theta_1\cos\theta_1\sin\phi_1\right) - \frac{1}{4}(R_7+R_9)\left(2\dot{\theta}_1\sin\phi_1 + \dot{\phi}_1\sin 2\theta_1\cos\phi_1\right)$$

$$j_\sigma^{(2)} \stackrel{!}{=} -\tilde{Q}\cdot\frac{d}{d\tau}\left[T_3\cos\theta_1 + T_4\sin\theta_1\sin\phi_1 + \frac{1}{2}(R_7+R_9)\sin\theta_1\cos\phi_1 - R_8\right] + f(\theta_1,\theta_2,\phi_1,\phi_2,\psi)\cdot\xi',$$

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Благодаря!