D-dimensional Conformal Field Theories with anomalous dimensions as Dual Resonance Models

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dedicated to Ivan Todorov on the occasion
of his 75th birthday

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Question: What part of the structure of conformal field theories is D-independent?

Starting point: For any $D \geq 2$, n-point correlation functions are given by functions of the same number $\frac{1}{2}n(n-3)$ of independent anharmonic ratios ω_i.

Let $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, $i < j = 1...n$.

Special case $D > 2, n = 4$: $\omega_1\omega_2\omega_3 = 1$,

$$\omega_1 = \frac{\mathbf{x}_{12}^2\mathbf{x}_{34}^2}{\mathbf{x}_{13}^2\mathbf{x}_{24}^2}, \quad \omega_2 = \frac{\mathbf{x}_{14}^2\mathbf{x}_{23}^2}{\mathbf{x}_{12}^2\mathbf{x}_{34}^2}, \quad \omega_3 = \frac{\mathbf{x}_{13}^2\mathbf{x}_{24}^2}{\mathbf{x}_{14}^2\mathbf{x}_{23}^2}$$

Consider Euklidean Green functions of a CFT or correlation functions of commuting Euklidean fields

$$\begin{aligned} G_{i_4,...,i_1}(\mathbf{x}_4, ..., \mathbf{x}_1) &= < \varphi^{i_4}(\mathbf{x}_4), ..., \varphi^{i_1}(\mathbf{x}_1) > \\ &= \prod_{i<j} (\mathbf{x}_{ij}^2)^{-\delta_{ij}^0} F_{i_4...i_1}(\omega_1, \omega_2, \omega_3) \end{aligned}$$

$\delta_{ij}^0 = \delta_{ji}^0$ depend on the dimensions d_j of the fields ϕ^{i_j}: $\sum_j \delta_{ij}^0 = d_i$. Euklidean$\leftrightarrow$Minkowski $\mathbf{x}_{ij}^2 = -x_{ij}^2$

Locality or crossing symmetry

Commutativity of Euklidean fields $\varphi^i(x)$ implies symmetry of correlation functions

$$G_{i_4,...,i_1}(\mathbf{x}_4,...,\mathbf{x}_1) = G_{i_{\pi 4},...,i_{\pi 1}}(\mathbf{x}_{\pi 4},...,\mathbf{x}_{\pi 1})$$

for all permutations π of $1...n = 4$. Permutations $\pi : i \leftrightarrow j$ act on harmonic ratios via $\omega \mapsto \pi\omega$, as follows

$$(ij) = (12) \text{ or } (34): \ \omega_1 \mapsto \omega_2^{-1}, \ \omega_2 \mapsto \omega_1^{-1}, \omega_3 \mapsto \omega_3^{-1}$$
$$(ij) = (13) \text{ or } (24): \ \omega_1 \mapsto \omega_3^{-1}, \ \omega_2 \mapsto \omega_2^{-1}, \omega_3 \mapsto \omega_1^{-1}$$
$$(ij) = (14) \text{ or } (23): \ \omega_1 \mapsto \omega_1^{-1}, \ \omega_2 \mapsto \omega_3^{-1}, \omega_3 \mapsto \omega_2^{-1}$$

Therefore,locality is equivalent to symmetry properties of $F_{i_4,...,i_1}(\omega)$ as

$$F_{i_4,...,i_1}(\omega) = F_{i_{\pi 4},...,i_{\pi 1}}(\pi\omega)$$

Operator product expansions (OPE)

In CFT in Minkowski space (or on its ∞-sheeted covering $\mathcal{M}_D \simeq S^{D-1} \times \mathbf{R}$), partially summed OPE converge on the vacuum Ω,

$$\phi^i(-\frac{1}{2}x)\phi^j(\frac{1}{2}x)\Omega =$$
$$= \sum_k \sum_a g^{ij}_{k,a} \tilde{Q}^a(\chi_k, p; \chi_j, -\frac{1}{2}x, \chi_i, \frac{1}{2}x)\phi^k(z)\Omega|_{z=0}$$

$p = -i\nabla_z$, with *kinematically determined* coefficients Q^a and coupling constants $g^{ij}_{k,a}$. $\chi_k = [l_k, d_k]$ indicate Lorentz spin l_k and dimension d_k of field ϕ^k. k-summation is over all nonderivative fields.

The CFT is completely determined by knowledge of coupling constants $g^{ij}_{k,a}$ and spin and dimension l_k, d_k of all nonderivative fields ϕ^k. Consistency=locality of 4-point functions. OPE imply positivity (unitarity) if all fields ϕ^k have positive 2-point functions.

Main theme of this talk: Extract some D-independent structural properties of CFT from OPE, and clarify what remains D-dependent, by use of

Mellin representation of correlation functions

Remember the Mellin representation of functions $f(x)$ of real variables $x > 0$: $f(x) = (2\pi i)^{-1} \int_{-i\infty}^{\infty} ds \tilde{f}(s) x^{-s}$

Inserting Mellin representation of $F_{i_4,...,i_1}(\omega_1, \omega_2, \omega_3)$ in independent (Euklidean) harmonic ratios, e.g. ω_1, ω_2,

$$G_{i_4,...,i_1}(\mathbf{x}_4, ..., \mathbf{x}_1) = (2\pi i)^{-2} \int d^2\delta M_{i_4,...,i_1}(\{\delta_{ij}\}) \prod_{i>j} \Gamma(\delta_{ij})(\mathbf{x}_{ij}^2)^{-\delta_{ij}}$$

Integration is over the 2-dimensional surface of imaginary $\delta_{ij} = \delta_{ji}$, $1 \leq i \neq j \leq 4$ subject to $\sum_j \delta_{ij} = d_i$.

And similarly for scalar n-point functions $G_{i_n,...,i_1}(x_n, ..., x_1)$, with $\frac{1}{2}n(n-3)$ integrations.

n-point Wightman functions, time ordered Green functions and Euklidean Green functions from the same Mellin amplitude

Let $m = \frac{1}{2}n(n-3)$. In Euklidean space

$$G_{i_n,...,i_1}(\mathbf{x}_n, ..., \mathbf{x}_1) = (2\pi i)^{-m} \int d^m\delta M_{i_n,...,i_1}(\{\delta_{ij}\}) \prod_{i>j} \Gamma(\delta_{ij})(\mathbf{x}_{ij}^2)^{-\delta_{ij}}$$

In Minkowski space

$$\begin{aligned} < \Omega, \phi^{i_n}(x_n)...\phi^{i_1}(x_1)\Omega > &= (2\pi i)^{-m} \int d^m\delta M_{i_n,...,i_1}(\{\delta_{ij}\}) \\ &\prod_{i>j} \Gamma(\delta_{ij})(-x_{ij}^2 + i\epsilon x_{ij}^0)^{-\delta_{ij}} \\ < \Omega, T\{\phi^{i_n}(x_1)...\phi^{i_1}(x_1)\}\Omega > &= (2\pi i)^{-m} \int d^m\delta M_{i_n,...,i_1}(\{\delta_{ij}\}) \\ &\prod_{i>j} \Gamma(\delta_{ij})(-x_{ij}^2 + i\epsilon)^{-\delta_{ij}} \end{aligned}$$

Duality properties of Mellin amplitudes $M_{i_n,\dots,i_1}(\{\delta_{ij}\})$

From localiy: Mellin amplitudes are symmetric under permutations π of $1...n$,

$$M_{i_n,\dots,i_1}(\{\delta_{ij}\}) = M_{i_{\pi n},\dots,i_{\pi 1}}(\{\delta_{\pi i \pi j}\})$$

From OPE: Mellin amplitudes $M_{i_n,\dots,i_1}(\{\delta_{ij}\})$ are meromorphic functions of the (independent) variables δ_{ij}, with simple poles in single variables (e.g. δ_{12}), at positions which are determined by the twist $d_k - l_k$ of the fields ϕ^k in the OPE and whose residues are polynomials in the other independent variables .

More precise statements are made below, and compared to properties of dual resonance models.

Solution of the constraints $\sum_j \delta_{ij} = d_i$, and pole positions

Let p_i, $i = 1, ..., n$ be conserved D' dimensional "momenta" satisfying $p_i^2 = d_i$, and $\sum_i p_i = 0$

Then $\delta_{ij} = -p_i \cdot p_j$ satisfy the constraint $\sum_j \delta_{ij} = d_i$. (D' need not equal D).

Define Mandelstam variables:

$$s_{jl} = (p_j + p_l)^2 = d_j + d_l - 2\delta_{jl}$$

If the OPE

$$\phi^{i_j}(x_j)\phi^{i_l}(x_l)\Omega = ...\phi^k\Omega + ...$$

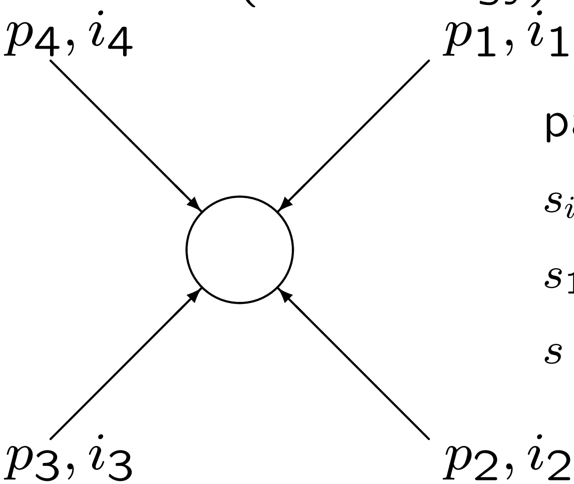
then the Mellin amplitude has a "leading" pole in δ_{jl} at position $s_{jl} = d_k - l_k$ and "satellite poles" at $s_{jl} = d_k - l_k + 2n$, $n = 1, 2, 3, ...$

The polynomial residues are of l_k-th order proportional to g_k^{jl}. They depend on l_k, n, differences of external dimensions incl. $d_j - d_l$, and on D.

Dual resonance models

Consider for instance scattering of 2 particles into 2 particles (spinless). The same analytic scattering amplitude $A(s,t,u)$ defines scattering in all 3 channels:

$12 \mapsto 34$ (c.m. energy)$^2 = s \geq \max\ m_1^2 + m_2^2, m_3^2 + m_4^2$
$13 \mapsto 24$ (c.m. energy)$^2 = t \geq \max\ m_1^2 + m_3^2, m_2^2 + m_4^2$
$14 \mapsto 23$ (c.m. energy)$^2 = u \geq \max\ m_1^2 + m_4^2, m_2^2 + m_3^2$



particles of types $i_1, ..., i_4$

$s_{ij} = (p_i + p_j)^2$

$s_{12} = s,\ s_{13} = t,\ s_{14} = u,\ i \neq j$

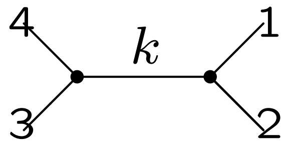
$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

Dual resonance models furnished meromorphic ("narrow resonance") approximations to $A(s,t,u)$ with simple poles in s, t, u with polynomial residues.

Duality

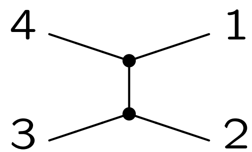
A resonance in the s-channel of type k with spin l_k and mass m_k can couple to (decay into) particles $i_1 + i_2$ with strength g_k^{12} and to $i_3 + i_4$ with strength g_k^{34}. The amplitude is the sum of their contributions

$$A(s,t,u) = \sum_k \bar{g}_k^{34} g_k^{12} \frac{P_{l_k}(\cos\theta_{12})}{s - m_k^2} = \sum_k$$



This is an equality of analytic functions, valid not only for $s \geq m_1^2 + m_2^2$... Using resonances in the u-channel

$$A(s,t,u) = \sum_k \bar{g}_k^{23} g_k^{14} \frac{P_{l_k}(\cos\theta_{14})}{u - m_k^2} = \sum_k$$



Duality: Both sums are equal (θ_{12} =polynomial in t or u) & similar for t-channel. P_l depend on D.

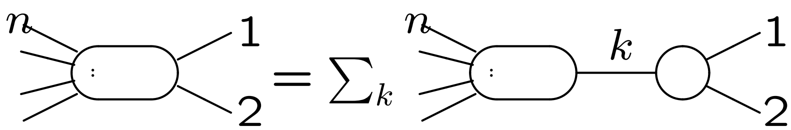
Methods exist to construct dual amplitudes

Factorization properties

Consider first dependence on particle types $i_4, ..., i_1$ of the amplitudes for scattering $i_1 + i_2 \mapsto i_3 + i_4$
$A(s,t,u) = A_{i_4,...,i_1}(\{s_{ij}\})$
Duality guarantees symmetry under permutations π

$$A_{i_4,...,i_1}(\{s_{ij}\}) = A_{i_{\pi 4},,...,\mathrm{l}_{pi_1}}(\{s_{\pi i \pi j}\}).$$

The contribution of a s-channel resonance is the product of a factorizing expression $\bar{g}_k^{34} g_k^{12}$ which carries the dependence on $i_4, ..., i_1$, times a kinematically determined factor. More generally , for $2 \mapsto n-2$ particles



Contribution of resonance k factorizes into 3-point amplitude $\times$ $(n-1)$-point amplitude. And similarly for $m \mapsto n$ particles.

Comparison with properties of Mellin amplitudes $M_{i_4,\ldots,i_1}(\{s_{ij}\})$

Correspondence : $s_{ij} = d_i + d_j - 2\delta_{ij}$, $m_k^2 = d_k - l_k$.

- The meromorphy properties are the same. There are simple poles in individual variables s_{ij}
- The positions of the poles are the same, $s_{ij} = m_k^2$ (indep. of m_i, m_j) if there is a field ϕ^k with spin l_k and dimension d_k in the OPE of $\phi^i\phi^j$. In addition there are satellite poles at $s_{ij} = m_k^2 + 2n$, $n = 1, 2, \ldots$.
- The poles come with polynomial residues P_{l_k} of degree l_k which are related to zonal spherical functions. They depend on D, n and differences of dimensions like $d_i - d_j$ and are not identically the same as in dual resonance models.
- The residues of the leading poles factorize. The residues of the satellite poles are determined by the residues of the leading poles.

Remark 1: m_k is the lowest possible energy of particle k. d_k is the lowest possible conformal energy (Eigenvalue of conformal Hamiltonian H) in irreps. $\mathcal{H}^{[l_k,d_k]}$ spanned by $\phi^k\Omega$, for scalar fields ϕ^k

Remark 2: OPE are valid for full disconnected Green functions. In this talk I ignore technicalities associated with the presence of disconnected parts. They are pure powers in ω_i.

The analog of Regge trajectories

In dual resonance models, the particles lie on Regge trajectories K, with masses

$$m_k^2 = \alpha^K(l_k)$$

In the simples models the trajectories are linear

$$\alpha^K = \alpha_0 + \alpha' l_k \qquad l_k \text{ in steps of 2.}$$

In soluble models of CFT there are trajectories

$$d_k = \alpha_0 + l_k + \sigma_k$$

$\sigma_k =$ anomalous part of the dimension, increases with l_k to a limit 2Δ or to ∞,

$$m_k^2 \equiv d_k - l_k = \alpha_0 + \sigma_k$$

If the anomalous part σ_k of the dimensions were 0, poles would fall on top of each other.

approximately linear rising trajectories of slope 0

CFT models with an expansion parameter

In CFT, fields of dimension $d < \frac{D}{2}$ are called fundamental fields. The existence of fundamental fields does not destroy duality: shadow pole at $D-d$ absent

I. ϕ^3-theory in $D = 6 + \epsilon$ dimensions: OPE schematic

$$\phi\phi\Omega = (\phi + \sum_{l=2,4,\ldots} \phi_{\mu_1\ldots\mu_l})\Omega$$

fundamental field ϕ has dimension $d = \frac{D-2}{2} + \Delta$,

$$\Delta = \frac{1}{18}\epsilon + \ldots$$

fields $\phi_{\mu_1\ldots\mu_l}$ have dimensions $d_l = D - 2 + l + \sigma_l$,

$$\sigma_l = 2\Delta - \frac{4}{3(s+2)(s+1)}\epsilon + \ldots., \quad d_0 = D - d$$

hence $d_l = 2d + l -$binding energy, binding energy $\mapsto 0$, as $l \mapsto \infty$. Interpret $2d =$energy of constituents.

Poles at $s_{12} = D - 2 + \sigma_l$ have a limit point at $s_{12} = 2d$

II. $\mathcal{N} = 4$ SUSY Yang Mills theory in 4 dimensions

$$\sigma_l \sim \gamma \ln l \text{ as } l \mapsto \infty, \qquad \gamma = \text{cusp anomaly}$$

Poles at $s_{12} = D - 2 + \sigma_l$ have no limit point as $l \mapsto \infty$

Interpretation: constituent fields have ∞ dimension.

Dimensional Reduction & Appearence of AdS

Conformal group $G = SO(D,2)$ of Minkowski space is not simply connected, because the maximal compact subgroup $K = SO(D) \times SO(2)$ is not simply connected. Its universal covering is $SO(D) \times \mathbf{R}$.

The Hilbert space $\mathcal{H}$ of a CFT carries a unitary representation of $\tilde{G}$ =universal covering of G, center $\mathbf{Z}_2 \times \mathbf{Z}$

Space time $\mathcal{M}_D$ must be a homogeneous space $\tilde{G}/H$ possibilities with $H \supseteq$ max parabolic subgroup $P \subset \tilde{G}$: (P contains the rotation group $U \simeq Spin(D-1) \subset \tilde{K}$).

1. $H = P$: $\mathcal{M}_D = \tilde{K}/U \simeq S^{D-1} \times \mathbf{R}$
   $\mathcal{M}$ admits a $\tilde{G}$-invariant causal ordering
   (hyperbolic space)
   fields ϕ^k can have anomalous dimensions.

2. $H = P \times \Gamma$: $\mathcal{M}_D$=compactified Minkowski space
   $\mathcal{M}$ has closed timelike curves. ($\Gamma \supset \mathbf{Z}$ discrete)

Manifestly conformal covariant formalism

Coordinatize points x on (two fold cover) of compactified Minkowski space by rays of lightlike vectors $\xi = (\xi_0 ... \xi_{D-1}, \xi_{D+1}, \xi_{D+2})$ in $D+2$ dimensions, $\xi \sim \lambda\xi$, $\lambda > 0$. harmonic ratios: $\omega_1 = \xi^1 \cdot \xi^2 \; \xi^3 \cdot \xi^4 / \xi^1 \cdot \xi^3 \; \xi^2 \cdot \xi^4$

$$\xi_0^2 - \xi_1^2 - ... - \xi_{D-1}^2 - \xi_{D+1}^2 + \xi_{D+1}^2 = 0.$$

$x_\mu = \xi_\mu / \kappa \qquad \kappa = \xi_{D+1} + \xi_{D+2}$ for $\mu = 0 ... D-1$. Elements of $SO(D,2)$ act on ξ as pseudorotations.

Orbits after dimensional reduction: Restrict G to $SO(D-1,2)$, and correspondingly for its covering $\tilde{G}$.

$\mathcal{M}_D$ decomposes into orbits

$$\mathcal{M}_D = AdS_D \cup \mathcal{M}_{D-1} \cup AdS_D$$

AdS_D=univ cover of D-dimensional Anti de Sitter space
$\mathcal{M}_{D-1}$ is the common boundary of the two AdS spaces:

$$\mathcal{M}_D \supset \mathcal{M}_{D-1} = \{x^{D-1} = 0\} = \{\xi_{D-1} = 0\}$$

why? ξ_{D-1} is $SO(D-1,2)$-invariant. Distinguish $\xi_{D-1} < 0$, $\xi_{D-1} = 0$, $\xi_{D-1} > 0$. Scale to $\xi_{D-1} = -1$, $\xi_{D-1} + 0, \xi_{D-1} = 1$. If $\xi_{D-1} \neq 0$ then

$$\xi_0^2 - \xi_1^2 - ... - \xi_{D-2}^2 - \xi_{D+1}^2 + \xi_{D+1}^2 = 1.$$

after scaling. This is Anti de Sitter space.

On compactified Minkowski space, ξ and $-\xi$ are identified, therefore the two AdS-spaces are identified.

Dimensional reduction of CFT.

In the manifestly covariant formalism, Wightman functions (WF)

$$W_{i_n,\ldots,i_1}(\xi^n,\ldots,\xi^1) = \kappa_n^{-d_n}\ldots\kappa_1^{-d_1} < \Omega\phi^{i_n}(x_n)\ldots\phi^{i_1}(x_1)\Omega >$$

are multivalued functions of ξ_i. The Mellin representation becomes

$$W_{i_n\ldots i_1}(\xi^n,\ldots,\xi^1) = (2\pi i)^{-m}\int d^m\delta\; M_{i_n\ldots i_1}(\{\delta_{ij}\})(2\xi^i\cdot\xi^j)^{-\delta_{ij}}$$

$m = \frac{1}{2}n(n-3)$. The restriction to $\xi_{D-1} = 0$ exists, is invariant under the restricted conformal group, and is given by identically the same formula with the understanding that $\xi^i_{D-1} = 0$. Hence
The dimensionally reduced CFT has the same Mellin amplitude M.

Dimensional induction of CFT's

Idea: Construct the dimensional reduction of a 3-dimensional CFT as a 2-dimensional CFT, compute its Mellin amplitude, and use it to write down the correlation functions of the 3-dimensional theory.

Multiplets of fields: Conformal OPE involve non-derivative fields. But not all derivatives of fields in 3 dimensions (evaluated at $x_{D-1} = 0$) are derivatives in 2 dimensions. Ordinary derivatives $\partial_{D-1} ... \partial_{D-1} \phi^k(x)|_{x_{D-1}=0}$ do not transform right, but

Use the Bargmann-Todorov homogeneous differential operator D_A on the cone $\xi^2 = 0$. Get field multiplets

$$\phi^k_{,n} = D_{D-1} ... D_{D-1} \phi^i(\xi)|_{\xi_{D-1}=0}$$

The missing generators J_{AB}, $A = D - 1$ of $SO(D, 2)$ act as generators of "internal" symmetry. It can map $\phi \mapsto D_{D-1}\phi$. The 2-dimensional stress tensor T can be adjoined: WF with T's from WF without T's

Mellin amplitudes for CFT from string theory?

How are correlation functions related? educated Guess:

$$\delta(\sum p_i) M(\{-p_i \cdot p_j\}) = \langle \int dV \;\; e^{i \sum_i p_{i\mu} X^\mu(\sigma_i, \tau_i)} \rangle \qquad (1)$$

for bosonic string. dV is a (2d- conformal) invariant volume element which integrates over$(n-3)$ of the arguments (σ_i, τ_i), the others can be arbitrarily (distinct) prescribed.
(this is an old device, cf. Veneziano 1970 Erice)

Conjecture: This fulfills physical requirements if the state spaces of the string theory and the CFT can be identified. In the Maldacena AdS/CFT correspondence, such an identification is known.