Supersymmetric gauge theories, quivers and Painlevé equations

Andrei Marshakov

Center for Advanced Studies, Skoltech

New Mathematical Methods in Solvable Models and Gauge/String Dualities

Varna, August 2022

Introduction:

joint with M.Bershtein, P.Gavrylenko and M.Semenyakin

New mathematical methods:

- SUSY gauge theories and integrable systems ($>$ 25 years);
- Quivers: cluster algebras ($>$ 20 years);
- Cluster integrable systems ($>$ 10 years);
- ...
- Painlevé equations ($>$ 100 years);

Relation with gauge/string duality: partition function $Z = \mathcal{T}$ as a tau-function ...

Here:

- (Dual) Nekrasov instanton partition function (Fourier-transformed 2d conformal blocks);
- 5d SYM on $\mathbb{R}^4 \times S^1_R$ (topological strings on non-compact CY?);
- (q-difference) isomondromic tau-function (Painlevé etc).

Introduction:

Solution to:

- Integrable system (classical or quantum: $\Omega_{\epsilon_1\epsilon_2}$-background);
- on a *Poisson cluster* variety (relativistic/group or 5d);
- Non-autonomous version: (q-difference!) Painlevé equations (q-isomonodromic deformations?);

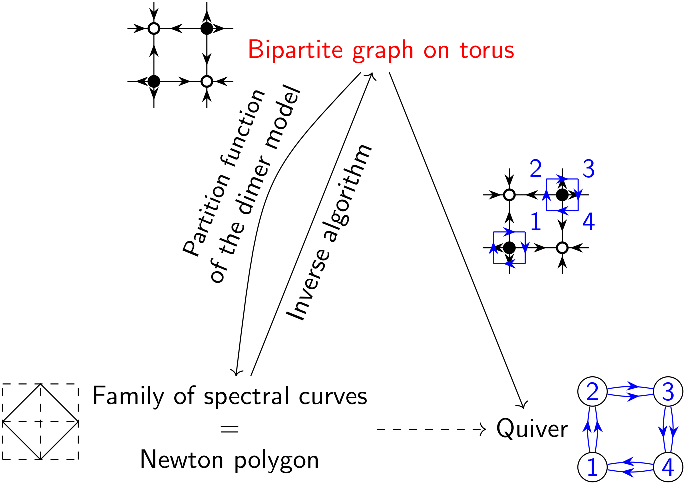
Parameters: $(q, p \sim e^{\hbar}) \sim (e^{R\epsilon_1}, e^{R\epsilon_2})$ ...

Effective description

- GK integrable systems on *Poisson cluster* varieties (with their Hamiltonian or cluster?) reductions;
- Quivers and their mutations : discrete flows & cluster (symmetry!) group $\mathcal{G}_{\mathcal{Q}}$;
- ....

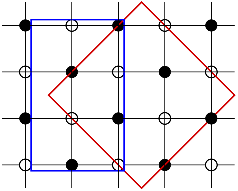
- Almost completed: $SU(2)$ gauge group with $N_f < 8$ (5d!) and q-Painlevé family;
- Flat-connections and Fock-Goncharov or 4d story ... (Ruijsenaars, DAHA?);
- Relation with BPS-quivers etc.

Integrable system: GK



Dimers on bipartite graphs

$\Gamma \subset \mathbb{T}^2$



- Domains of square lattice: $2 \times N$ (Toda) and $N \times M$ 'fence-net' XXZ-type spin chain;
- Triangle NP: hexagonal GK graphs;

Dimers $\mapsto$ loops: $\partial D = \sum \bullet - \sum \circ$, so that

$$D - D_0 = \partial F + \gamma \quad (\in H_1(\mathbb{T}^2))$$

GK construction

- Dimer partition function defines $\mathcal{C} \subset \mathbb{C}^\times \times \mathbb{C}^\times$:

$$f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b}(x) = 0$$

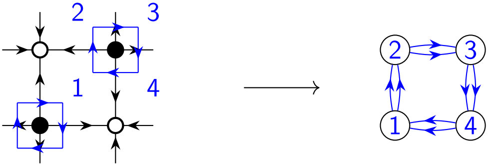
  instead of $\det(\mu + g(\lambda)) = 0$ (with $g(\lambda) \in \widehat{G}^\natural$);
- $\Delta \subset \mathbb{Z}^2 \subset \mathbb{R}^2$: convex NP (up to $SA(2,\mathbb{Z}) = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$);
- Δ with $\frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu}$: SW data for 5d SYM (when known!).

GK duality

- Fat graph structures – duality of faces (on $\mathbb{T}^2$) and zig-zag paths (dual surface Σ);
- Zig-zag paths on $\mathbb{T}^2 \simeq$ boundaries of faces on $\Sigma \simeq$ boundaries of Δ;
- Intersection form $\langle \bullet, \bullet \rangle_\Sigma$ on $H_1(\Sigma)$: Poisson quiver $\mathcal{Q}$ with face variables at vertices $\{x | \prod_f x_f = 1\}$;
- Intersection form $\langle \bullet, \bullet \rangle_{\mathbb{T}^2}$ on $H_1(\mathbb{T}^2)$: zig-zag quiver $\mathcal{Q}_\zeta$ with zig-zag's at vertices $\sum \zeta = 0$.

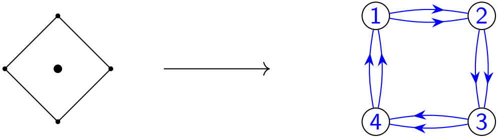
GK duality

E.g.



defines the bracket

$$\{x_i, x_{i+1}\}_{\mathcal{Q}} = 2x_i x_{i+1}, \qquad i = 1, \ldots, 4$$



- $\sum \zeta = 0$, $\#(i \to j) = \langle \zeta_i, \zeta_j \rangle_{\mathbb{T}} = \zeta_i \times \zeta_j$;
- rank $(\mathcal{Q}_\zeta\ ) = 2$, $\varpi = \frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu}$ for $(\lambda, \mu) \in H^1(\mathbb{T}^2)$.

Sometimes – self-duality (Painlevé)!

Quiver mutations

Mutations
of the graph:

$$\mu_j : \epsilon_{ik} \mapsto -\epsilon_{ik}, \text{ if } i = j \text{ or } k = j, \quad \epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}|\epsilon_{jk}| + \epsilon_{jk}|\epsilon_{ij}|}{2} \quad \text{otherwise,}$$

and x-variables:

$$\mu_j : \quad x_j \to \frac{1}{x_j}, \qquad x_i \to x_i \left(1 + x_j^{\mathrm{sgn}(\epsilon_{ij})}\right)^{\epsilon_{ij}}, i \neq j$$

Poisson quivers

Poisson map:

$$\{x'_i, x'_k\}_{\mathcal{Q}'} = \epsilon'_{ik} x'_i x'_k$$

in addition to gluing, forgetting etc

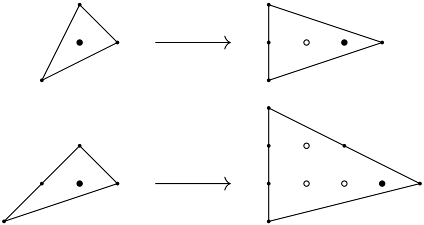
- At 4-valent vertices (squares): a spider move of bipartite graph on base $\mathbb{T}^2$;
- At higher vertices (e.g. hexagons): pushes out of GK construction with $\mathbb{T}^2$.

ζ-quiver mutations

$$\mu_k(\zeta_i) = \begin{cases} \zeta_i + [\varepsilon_{ik}]_+ \zeta_k, & i \neq k \\ -\zeta_i, & i = k \end{cases}$$

Result of ζ-quiver mutation (Gaiotto transform):

- Another NP with $g = g_0$;
- NP with $g > g_0$ with special coefficients, examples:



Cluster integrable system

- Boundary coefficients $\{f_{a,b}(x)|(a,b) \in \partial\Delta\}$ are Casimir functions;
- Their number is $B-2 = B-3+1$, with an extra $q = \prod_f x_f$,
  $B = \#$ boundary segments $= \#$ zig-zag paths on $\Gamma \subset \mathbb{T}^2$;
- $\{\vec{H}(x)\}$ are (normalized!) coefficients of dimer partition function $\{f_{a,b}(x)\}$, corresponding to *internal* $a,b) \in \Delta \setminus \partial\Delta$;
- $\{H_I, H_J\}_{\mathcal{Q}}=0$, $I, J = 1, \ldots, g$ (r-matrix bracket from group theory).

Cluster integrable system

Integrability: Pick's formula

$$\dim\mathcal{X} = 2\mathrm{Area}(\Delta) = B - 2 + 2g$$

Alternatively $V - E + F = 0$ for $\Gamma \subset \mathbb{T}^2$, and $V - E + B = 2 - 2g$ for $\Gamma \subset \Sigma$, hence

$$F = E - V = B - 2 + 2g$$

In GK construction $q = \prod_f x_f = 1$, breaking $q \neq 1$ is *deautonomization*.

Discrete flow: example

For $q = 1$ the flow $T = (12)(34) \circ \mu_3 \circ \mu_1 : \mathcal{Q} \mapsto \mathcal{Q}$

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 \left(\frac{1 + x_3}{1 + x_1^{-1}} \right)^2, x_1^{-1}, x_4 \left(\frac{1 + x_1}{1 + x_3^{-1}} \right)^2, x_3^{-1} \right)$$

or

$$T : (x_1, x_2, {\color{red} z}, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, {\color{red} qz}, q \right) \underset{q=1}{=} \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, z, q \right)$$

preserves the Hamiltonian $\mathcal{H} = \sqrt{x_1 x_2} + \frac{1}{\sqrt{x_1 x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$.

Deautonomization: Painlevé

Let $x_1 x_2 x_3 x_4 = q \neq 1$

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, qz, q \right)$$

Consider z as "time" $T : x(z) \mapsto x(qz)$, then $x_1 = x(z)$, $x_2 = x^{-1}(q^{-1}z)$, satisfy

$$x(qz)x(q^{-1}z) = \left(\frac{x(z) + z}{x(z) + 1} \right)^2$$

or q-Painlevé III_3 equation.

Discrete integrability

- MCG $\mathcal{G}_\mathcal{Q} : \mathcal{Q} \mapsto \mathcal{Q}$, generated by quiver mutations (and permutations);
- $\mathcal{G}_\mathcal{Q} \supset \mathcal{G}_\Delta = \mathbb{Z}^\# \oplus \text{finite}$ – Abelian group of discrete flows;
- $\mathcal{G}_\mathcal{Q} \supset \widehat{W}$, affine Weyl group, when non-Abelian;

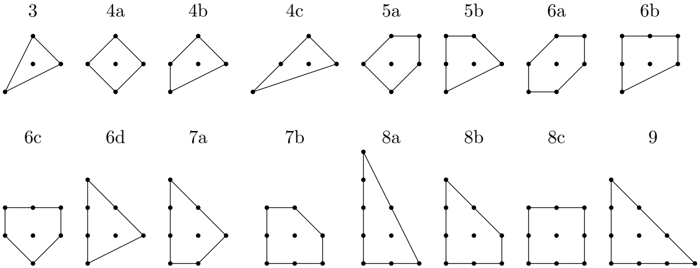
At $q = 1$

- $\{\vec{H}\}$ are cluster functions, invariant wrt $\mathcal{G}_\mathcal{Q}$;
- Only $\widehat{W} \subset W \curvearrowright \mathcal{C}$, while $\mathcal{G}_\Delta \curvearrowright \mathrm{Pic}(\mathcal{C})$;
- W extends to global symmetry of 5d theory in UV (?!)

At $q \neq 1$ $\mathcal{G}_\mathcal{Q}$ is a symmetry-group of a non-autonomous system ...

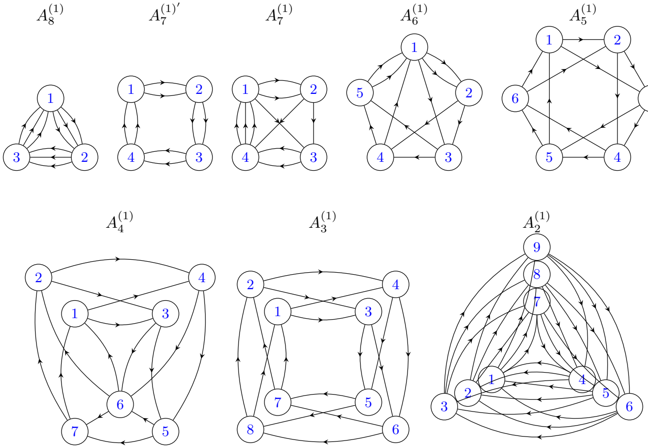
Painlevé NP

with a single internal point and $3 \leq B \leq 9$ boundary points:



Here $\mathcal{C}$: $f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is obviously a torus $g = 1$.

Painlevé quivers



Notations: Sakai classification

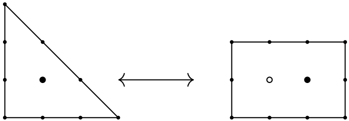
$$\frac{A_0^{(1)}}{E_8^{(1)}} \to \frac{A_1^{(1)}}{E_7^{(1)}} \to \frac{A_2^{(1)}}{E_6^{(1)}} \to \frac{A_3^{(1)}}{E_5^{(1)}} \to \frac{A_4^{(1)}}{E_4^{(1)}} \to \frac{A_5^{(1)}}{E_3^{(1)}} \to \frac{A_6^{(1)}}{E_2^{(1)}} \begin{array}{l} \nearrow \quad \frac{A_7^{(1)}}{\tilde{A}_1^{1}} \to \frac{A_8^{(1)}}{E_0^{(1)}} \\ \searrow \quad \frac{A_7^{(1)'}}{E_1^{(1)}} \end{array}$$

by (surface type)/(symmetry group)

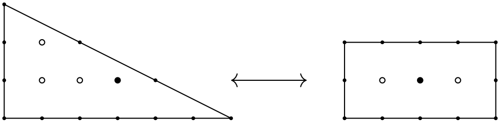
- $\mathcal{G}_{\mathcal{Q}} \supset \widehat{W}(E_{\#}^{(1)})$;
- $\widehat{W}(E_0^{(1)}) = \mathbb{Z}/3\mathbb{Z}$;
- From $E_1^{(1)} = A_1^{(1)}$ till $E_5^{(1)} = D_5^{(1)}$ q-Painlevé with well-defined 4d limit (from PIII to PVI);
- Higher $E_7^{(1)}$ and $E_8^{(1)}$ *do not* have corresponding (naive) $g = 1$ triangles.

Extension for Painlevé

$$g_r = g - \sum_{i=1}^{N_r} \frac{h_i(h_i-1)}{2} d_i$$

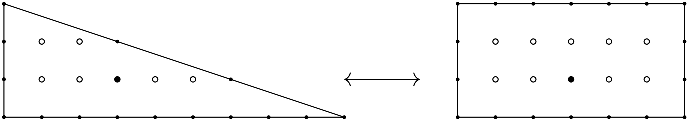


$E_6^{(1)}$: $B = 9$, $g = 1$ versus reduced $B = 10$, $g = 2$, $h = 2$



$E_7^{(1)}$: reduced $B = 12$, $g = 4$, $h = 3$ versus double-reduced $B = 12$, $g = 3$, $h_1 = h_2 = 2$

Extension for Painlevé



$E_8^{(1)}$: double-reduced $B = 15$, $g = 7$, $h_1 = h_2 = 3$ versus triple-reduced $B = 18$, $g = 10$, $h_1 = h_2 = 3$, $h_3 = 2$, $d_3 = 3$.

- Reductions of higher-rank gauge theories;
- Flavor symmetry restored from discrete symmetry of an integrable/Painlevé system.

GK-reductions

Theorem (Conjecture):

A NP with side of length $d \cdot h$ and fixed vertex at a distance h gives rise to a *cluster* reduction of corresponding GK system by fixing $(h-1)$ original Casimir functions, and imposing $\frac{h(h-1)}{2}$ Hamiltonian constraints, which reduces the dimension of the (Poisson) phase space by

$$\left(h-1+2\frac{h(h-1)}{2}\right)d=(h^2-1)d$$

Actual (smooth-) genus reduction

$$g_r = g - \frac{h(h-1)}{2}d$$

New class (extended-GK) of cluster integrable systems.

GK-reductions

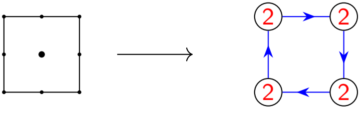
- Hamiltonian/Poisson reduction: $\{\mathtt{C}_1, \ldots, \mathtt{C}_{h-1}\}$ and $\{\mathtt{H}_{ij} | 1 \leq i < j \leq h\}$ so that
$$\mathcal{R}: \ \mathtt{C}_i = \mathtt{H}_{i,i+1} = 1, \quad \mathtt{H}_{ij} = 0$$
- Poisson (better – quantum!) algebra is isomorphic to the Sevostyanov algebra.

- $\mathcal{X}//\mathcal{R}$ has a structure of cluster Poisson variety, $\mathcal{Q} \mapsto \mathcal{Q}_{\mathcal{R}}$ (mutation classes) by Poisson maps;
- No clear transformation for the bipartite graph: beyong the GK construction on $\mathbb{T}^2$.

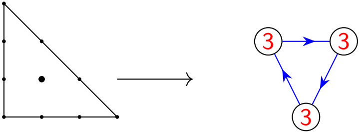
Local example: hexagonal lattice $2l \times h$ gives cluster structure on $Gr(h, l) = Gr(l-h, l)$; $h > l$ realized in 4d or FG (flat connections) story ...

Global examples: q-Painlevé ...

Higher Painlevé systems



q-PVI case, with $E_5^{(1)} = D_5^{(1)}$ symmetry, and limit to 4d.



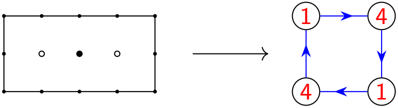
last case with $E_6^{(1)}$ symmetry $g = 1$ NP within $\mathcal{Q}_\zeta$-mutation class.

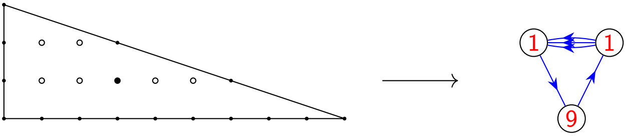
Notation: circle with red number $\equiv$ a group of identical vertices.

Higher Painlevé cases

Theorem (Conjecture):

q-Painlevé $E_7^{(1)}$- and $E_8^{(1)}$-symmetry can be realized as deautonomization of the GK-reduced cluster integrable systems.





... since there are no corresponding NP with (naive) $g = 1$.

q-Painlevé $E_7^{(1)}$

- Reduction from higher-rank or quiver 5d gauge theories;
- Counting:

$$2 \cdot \mathrm{Area} - 2(h^2 - 1) = 16 - 2 \cdot 3 = 10$$

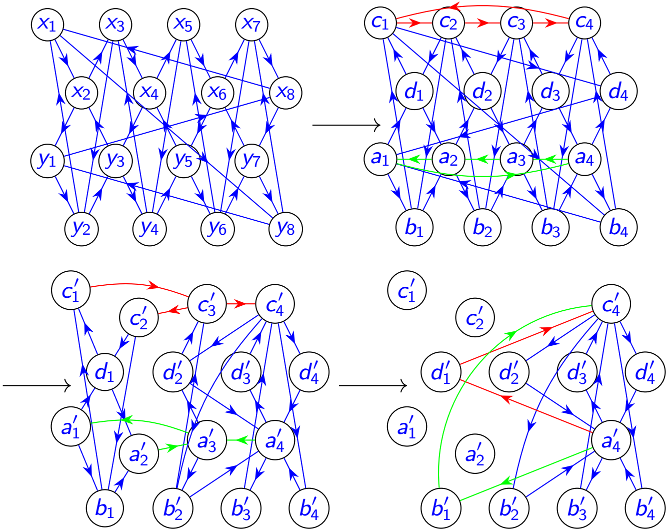
$$B - 2 = 2(N + M) - 2 = 10, \qquad 2(h-1) = 2$$

- Result:
$$H = \frac{1}{\sqrt{x_0x_1x_2x_3x_4x_5x_6x_7x_8}}(1 + x_0^2x_1x_2x_3x_4x_5^3x_6x_7x_8((1 + x_6)(1 + x_8) + x_7(1 + x_6 + (1 + x_6 + x_0x_6)x_8)) + x_5(1 + x_6 + x_8 + x_6x_8 + x_0x_6x_8 + x_7(1 + x_6 + x_0x_6 + (1 + x_0 + x_6 + x_0^2(1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4)x_6 + x_0(2 + x_1 + x_2 + x_3 + x_4)x_6)x_8)) + x_0^2x_5^2x_6x_7x_8((x_1 + x_2)x_3x_4 + x_1x_2(x_3 + x_4 + x_3x_4(2 + x_6 + x_7 + x_8))))$$

$$q = x_0^2x_1x_2x_3x_4x_5^2x_6x_7x_8x_9$$

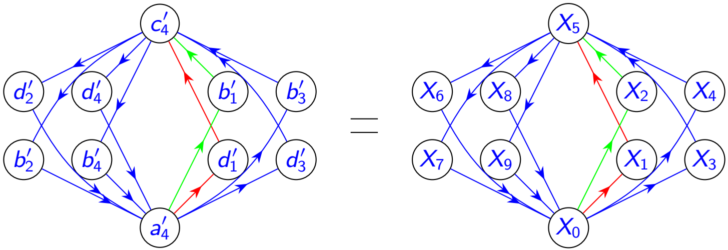
– a non-GK case ...

Cluster reduction



q-Painlevé $E_7^{(1)}$

Results in:



generates $W(E_7^{(1)})$:

$$\sigma_1 = s_{12}, \quad \sigma_2 = s_{23}, \quad \sigma_3 = s_{34}$$

$$\sigma_5 = s_{67}, \quad \sigma_6 = s_{78}, \quad \sigma_7 = s_{89}$$

$$\sigma_4 = \mu_4 \mu_6 s_{46}, \quad \sigma_0 = \mu_5 \mu_0 s_{50}$$

Further perspectives

- Similar reduction results in Painlevé $E_8^{(1)}$, to be completed;
- Towards 4d or the 'Fock-Goncharov framework' – moduli spaces of flat connections: cluster integrable system arise after reduction (Ruijsenaars, reduced spin chains ...);
- Relation with other non-GK cluster cases (BCD - series, reflection equations, Schrader-Shapiro, ...);
- ... etc

What else it gives for YM and strings?