Chern–Simons forms in Cartan geometries

Ben McKay

27-31 May 2024

Dogs and strawberries

Phd: weekends we took the dog for a walk

Dogs and strawberries

Phd: the dog could NEVER find wild strawberries

- My wife would push aside a leaf, say "Look!"

- My wife would push aside a leaf, say “Look!”

- THERE it is: a strawberry!

- My wife would push aside a leaf, say “Look!”

- THERE it is: a strawberry!
- Dogs jumps and eats.

Robert Bryant

Phd: weekdays

$$
(8)\quad\left\{
\begin{aligned}
\varpi_1' &= \varpi_3\varpi_2 + \frac{1}{3}\omega_3\varpi_7 - \frac{2}{3}\omega_4\varpi_5 + \frac{1}{3}\omega_5\varpi_6 + \omega_1\chi_1 + 2B_2\omega_1\omega_3 + B_3\omega_2\omega_3 \\
&\qquad + 2A_2\omega_1\omega_4 + 2A_3\omega_1\omega_5 + A_3\omega_2\omega_4 + A_4\omega_2\omega_5, \\
\varpi_2' &= \varpi_2(\varpi_1 - \varpi_4) - \omega_4\varpi_6 + \omega_1\chi_2 + B_4\omega_2\omega_3 + A_4\omega_2\omega_4 + A_5\omega_2\omega_5, \\
\varpi_3' &= \varpi_3(\varpi_4 - \varpi_1) - \omega_5\varpi_5 + \omega_2\chi_1 - B_1\omega_1\omega_3 - A_1\omega_1\omega_4 - A_2\omega_1\omega_5, \\
\varpi_4' &= \varpi_2\varpi_3 + \frac{1}{3}\omega_3\varpi_7 + \frac{1}{3}\omega_4\varpi_5 - \frac{2}{3}\omega_5\varpi_6 + \omega_2\chi_2 - B_2\omega_1\omega_3 - 2B_3\omega_2\omega_3 \\
&\qquad - A_2\omega_1\omega_4 - A_3\omega_1\omega_5 - 2A_3\omega_2\omega_4 - 2A_4\omega_2\omega_5, \\
\varpi_5' &= \varpi_1\varpi_5 + \varpi_3\varpi_6 - \omega_5\varpi_7 + \omega_3\chi_1 + \frac{9}{32}D_1\omega_1\omega_2 + \frac{9}{8}C_1\omega_1\omega_3 + \frac{9}{8}C_2\omega_2\omega_3 \\
&\qquad + A_2\omega_3\omega_4 + A_3\omega_3\omega_5 + \frac{3}{4}B_1\omega_1\omega_4 + \frac{3}{4}B_2(\omega_1\omega_5 + \omega_2\omega_4) + \frac{3}{4}B_3\omega_2\omega_5, \\
\varpi_6' &= \varpi_2\varpi_5 + \varpi_4\varpi_6 + \omega_4\varpi_7 + \omega_3\chi_2 + \frac{9}{32}D_2\omega_1\omega_2 + \frac{9}{8}C_2\omega_1\omega_3 + \frac{9}{8}C_3\omega_2\omega_3 \\
&\qquad - A_3\omega_3\omega_4 - A_4\omega_3\omega_5 + \frac{3}{4}B_2\omega_1\omega_4 + \frac{3}{4}B_3(\omega_1\omega_5 + \omega_2\omega_4) + \frac{3}{4}B_4\omega_2\omega_5, \\
\varpi_7' &= \frac{4}{3}\varpi_5\varpi_6 + (\varpi_1 + \varpi_4)\varpi_7 + \omega_4\chi_1 + \omega_5\chi_2 + \frac{9}{64}E\omega_1\omega_2 - \frac{3}{8}D_1\omega_1\omega_3 \\
&\qquad \cdots
\end{aligned}
\right.
$$

Robert Bryant

Phd: weekdays

$$
(8)\quad \left\{
\begin{aligned}
\varpi_1' &= \varpi_3\varpi_2 + \frac{1}{3}\omega_3\varpi_7 - \frac{2}{3}\omega_4\varpi_5 + \frac{1}{3}\omega_5\varpi_6 + \omega_1\chi_1 + 2\,\mathrm{B}_2\omega_1\omega_3 + \mathrm{B}_3\omega_2\omega_3 \\
&\qquad + 2\,\mathrm{A}_2\omega_1\omega_4 + 2\,\mathrm{A}_3\omega_1\omega_5 + \mathrm{A}_3\omega_2\omega_4 + \mathrm{A}_4\omega_2\omega_5, \\
\varpi_2' &= \varpi_2(\varpi_1 - \varpi_4) - \omega_4\varpi_6 + \omega_1\chi_2 + \mathrm{B}_4\omega_2\omega_3 + \mathrm{A}_4\omega_2\omega_4 + \mathrm{A}_5\omega_2\omega_5, \\
\varpi_3' &= \varpi_3(\varpi_4 - \varpi_1) - \omega_5\varpi_5 + \omega_2\chi_1 - \mathrm{B}_1\omega_1\omega_3 - \mathrm{A}_1\omega_1\omega_4 - \mathrm{A}_2\omega_1\omega_5, \\
\varpi_4' &= \varpi_2\varpi_3 + \frac{1}{3}\omega_3\varpi_7 + \frac{1}{3}\omega_4\varpi_5 - \frac{2}{3}\omega_5\varpi_6 + \omega_2\chi_2 - \mathrm{B}_2\omega_1\omega_3 - 2\,\mathrm{B}_3\omega_2\omega_3 \\
&\qquad - \mathrm{A}_2\omega_1\omega_4 - \mathrm{A}_3\omega_1\omega_5 - 2\,\mathrm{A}_3\omega_2\omega_4 - 2\,\mathrm{A}_4\omega_2\omega_5, \\
\varpi_5' &= \varpi_1\varpi_5 + \varpi_3\varpi_6 - \omega_5\varpi_7 + \omega_3\chi_1 + \frac{9}{32}\mathrm{D}_1\omega_1\omega_2 + \frac{9}{8}\mathrm{C}_1\omega_1\omega_3 + \frac{9}{8}\mathrm{C}_2\omega_2\omega_3 \\
&\qquad + \mathrm{A}_2\omega_3\omega_4 + \mathrm{A}_3\omega_3\omega_5 + \frac{3}{4}\mathrm{B}_1\omega_1\omega_4 + \frac{3}{4}\mathrm{B}_2(\omega_1\omega_5 + \omega_2\omega_4) + \frac{3}{4}\mathrm{B}_3\omega_2\omega_5, \\
\varpi_6' &= \varpi_2\varpi_5 + \varpi_4\varpi_6 + \omega_4\varpi_7 + \omega_3\chi_2 + \frac{9}{32}\mathrm{D}_2\omega_1\omega_2 + \frac{9}{8}\mathrm{C}_2\omega_1\omega_3 + \frac{9}{8}\mathrm{C}_3\omega_2\omega_3 \\
&\qquad - \mathrm{A}_3\omega_3\omega_4 - \mathrm{A}_4\omega_3\omega_5 + \frac{3}{4}\mathrm{B}_2\omega_1\omega_4 + \frac{3}{4}\mathrm{B}_3(\omega_1\omega_5 + \omega_2\omega_4) + \frac{3}{4}\mathrm{B}_4\omega_2\omega_5, \\
\varpi_7' &= \frac{4}{3}\varpi_5\varpi_6 + (\varpi_1 + \varpi_4)\varpi_7 + \omega_4\chi_1 + \omega_5\chi_2 + \frac{9}{64}\mathrm{E}\,\omega_1\omega_2 - \frac{3}{8}\mathrm{D}_1\omega_1\omega_3 \\
&\qquad - \frac{3}{8}\mathrm{D}_2\omega_2\omega_3 + 2\,\mathrm{A}_3\omega_4\omega_5 - \mathrm{B}_2\omega_3\omega_4 + \mathrm{B}_3\omega_3\omega_5.
\end{aligned}
\right.
$$

I see nothing.

Robert Bryant

Phd: weekdays

$$\begin{aligned}
\varpi_1' &= \varpi_3\varpi_2 + \frac{1}{3}\omega_3\varpi_7 - \frac{2}{3}\omega_4\varpi_5 + \frac{1}{3}\omega_5\varpi_6 + \omega_1\chi_1 + 2B_2\omega_1\omega_3 + B_3\omega_2\omega_3 \\
&\quad + 2A_2\omega_1\omega_4 + 2A_3\omega_1\omega_5 + A_3\omega_2\omega_4 + A_4\omega_2\omega_5, \\
\varpi_2' &= \varpi_2(\varpi_1 - \varpi_4) - \omega_4\varpi_6 + \omega_1\chi_2 + B_4\omega_2\omega_3 + A_4\omega_2\omega_4 + A_5\omega_2\omega_5, \\
\varpi_3' &= \varpi_3(\varpi_4 - \varpi_1) - \omega_5\varpi_5 + \omega_2\chi_1 - B_1\omega_1\omega_3 - A_1\omega_1\omega_4 - A_2\omega_1\omega_5, \\
\varpi_4' &= \varpi_2\varpi_3 + \frac{1}{3}\omega_3\varpi_7 + \frac{1}{3}\omega_4\varpi_5 - \frac{2}{3}\omega_5\varpi_6 + \omega_2\chi_2 - B_2\omega_1\omega_3 - 2B_3\omega_2\omega_3 \\
&\quad - A_2\omega_1\omega_4 - A_3\omega_1\omega_5 - 2A_3\omega_2\omega_4 - 2A_4\omega_2\omega_5,
\end{aligned}$$

Robert says “Look!”

Projective connections

$$d\omega^i + \omega^i_j \wedge \omega^j = \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l,$$

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k = \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l,$$

$$d\omega_i - \omega^j_i \wedge \omega_j = \frac{1}{2} K_{ikl} \omega^k \wedge \omega^l.$$

Projective connections: flat=projective space

$$
\begin{aligned}
d\omega^i + \omega^i_j \wedge \omega^j &= \cancel{\frac{1}{2} K^i_{kl}\omega^k \wedge \omega^l}, \\
d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= \cancel{\frac{1}{2} K^i_{jkl}\omega^k \wedge \omega^l}, \\
d\omega_i - \omega^j_i \wedge \omega_j &= \cancel{\frac{1}{2} K_{ikl}\omega^k \wedge \omega^l}.
\end{aligned}
$$

Projective connections: flat=projective space

$$\begin{aligned} d\omega^i + \omega^i_j \wedge \omega^j &= 0, \\ d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= 0, \\ d\omega_i - \omega^j_i \wedge \omega_j &= 0. \end{aligned}$$

Projective connections: flat=projective space

$$
\begin{aligned}
d\omega^i + \omega^i_j \wedge \omega^j &= 0, \\
d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= 0, \\
d\omega_i - \omega^j_i \wedge \omega_j &= 0.
\end{aligned}
$$

- Looks like an affine connection.

Projective connections: flat=projective space

$$
\begin{aligned}
d\omega^i + \omega^i_j \wedge \omega^j &= 0, \\
d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= 0, \\
d\omega_i - \omega^j_i \wedge \omega_j &= 0.
\end{aligned}
$$

- Looks like an affine connection.
- Can't be: this is a projective connection.

Projective connections: flat=projective space

$$
\begin{aligned}
d\omega^i + \omega^i_j \wedge \omega^j &= 0, \\
d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= 0, \\
d\omega_i - \omega^j_i \wedge \omega_j &= 0.
\end{aligned}
$$

What is this?

Complex geometries

$$\mathbb{C}$$

Fubini–Study metric

$$d\omega^i + \omega^i_j \wedge \omega^j = 0,$$

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j + (\delta^i_j \omega^{\bar{k}} + \delta^i_k \omega^{\bar{\jmath}}) \wedge \omega^k = 0,$$

Fubini–Study metric

$$d\omega^i + \omega^i_j \wedge \omega^j = 0,$$

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j + (\delta^i_j \omega^{\bar{k}} + \delta^i_k \omega^{\bar{\jmath}}) \wedge \omega^k = 0,$$

$$d\omega_i - \omega^j_i \wedge \omega_j = 0.$$

$$\omega_k \to -\omega^{\bar{k}}$$

$$
\begin{aligned}
d\omega^i + \omega^i_j \wedge \omega^j &= 0,\\
\boxed{d\omega^i_j + \omega^i_k \wedge \omega^k_j} + \boxed{(\delta^i_j \omega^{\bar{k}} + \delta^i_k \omega^{\bar{\jmath}}) \wedge \omega^k} &= 0,\\
d\omega_i - \omega^j_i \wedge \omega_j &= 0.
\end{aligned}
$$

The red box is curvature. Chern classes come from $(1,1)$-terms in curvature.

Back to projective space

- Forget Fubini–Study.

Back to projective space

- Forget Fubini–Study.
- $$\begin{aligned} d\omega^i + \omega^i_j \wedge \omega^j &= 0, \\ d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= 0, \\ d\omega_i - \omega^j_i \wedge \omega_j &= 0. \end{aligned}$$

Back to projective space

- Forget Fubini–Study.
- $$\begin{aligned} d\omega^i + \omega^i_j \wedge \omega^j &= 0, \\ d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= 0, \\ d\omega_i - \omega^j_i \wedge \omega_j &= 0. \end{aligned}$$
- PRETEND the red box is curvature.

Back to projective space

- Forget Fubini–Study.
- $$\begin{aligned} d\omega^i + \omega^i_j \wedge \omega^j &= 0, \\ \boxed{d\omega^i_j + \omega^i_k \wedge \omega^k_j} - \boxed{(\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k} &= 0, \\ d\omega_i - \omega^j_i \wedge \omega_j &= 0. \end{aligned}$$
- PRETEND the red box is curvature.
- PRETEND ω^k is $(1,0)$, ω_k is $(0,1)$.

Back to projective connections

$$
\begin{aligned}
d\omega^i + \omega^i_j \wedge \omega^j &= \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l, \\
d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l, \\
d\omega_i - \omega^j_i \wedge \omega_j &= \frac{1}{2} K_{ikl} \omega^k \wedge \omega^l.
\end{aligned}
$$

- Cartan geometry curvature terms?

Back to projective connections

$$\begin{aligned} d\omega^i + \omega^i_j \wedge \omega^j &= \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l, \\ d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l, \\ d\omega_i - \omega^j_i \wedge \omega_j &= \frac{1}{2} K_{ikl} \omega^k \wedge \omega^l. \end{aligned}$$

- Cartan geometry curvature terms?
- They are $(2, 0)$; can't count.

Back to projective connections

$$
\begin{aligned}
d\omega^i + \omega^i_j \wedge \omega^j &= \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l, \\
d\omega^i_j + \omega^i_k \wedge \omega^k_j - (\delta^i_j \omega_k + \delta^i_k \omega_j) \wedge \omega^k &= \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l, \\
d\omega_i - \omega^j_i \wedge \omega_j &= \frac{1}{2} K_{ikl} \omega^k \wedge \omega^l.
\end{aligned}
$$

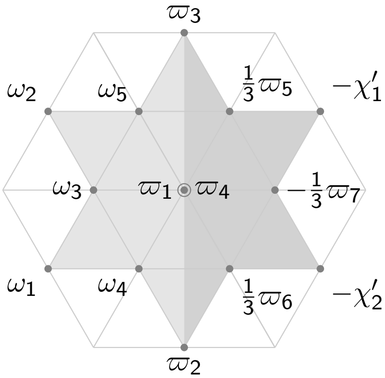
- Cartan geometry curvature terms?
- They are $(2, 0)$; can't count.
- Ignore them!

Slovák $\to$ Dolbeault

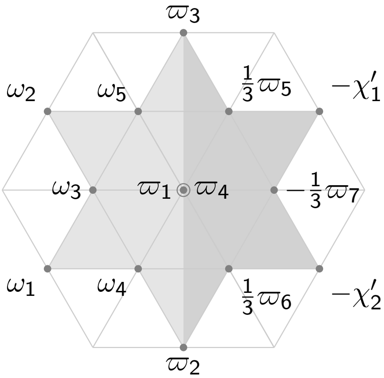
Theorem

Every complex manifold with a holomorphic projective connection satisfies all equations on Chern classes, in Dolbeault cohomology, satisfied by $\mathbb{CP}^n$.

Holomorphic parabolic geometries

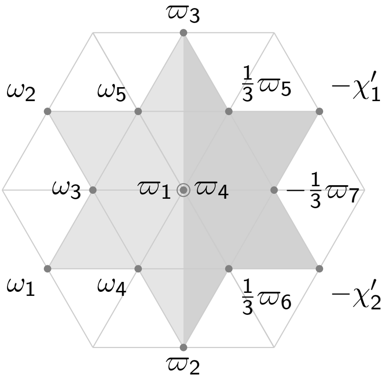
- 

Holomorphic parabolic geometries

- 

- Slovák cohomology: negative coroots $(1, 0)$, positive $(0, 1)$

Holomorphic parabolic geometries

- 

- Slovák cohomology: negative coroots $(1, 0)$, positive $(0, 1)$
- Slovák Chern classes come from $(1, 1)$-terms

Look!

$$
\left\{
\begin{aligned}
\varpi'_1 &= \varpi_3\varpi_2 + \frac{1}{3}\omega_3\varpi_7 - \frac{2}{3}\omega_4\varpi_5 + \frac{1}{3}\omega_5\varpi_6 + \omega_1\chi_1 + 2B_2\omega_1\omega_3 + B_3\omega_2\omega_3 \\
&\qquad + 2A_2\omega_1\omega_4 + 2A_3\omega_1\omega_5 + A_3\omega_2\omega_4 + A_4\omega_2\omega_5, \\
\varpi'_2 &= \varpi_2(\varpi_1 - \varpi_4) - \omega_4\varpi_6 + \omega_1\chi_2 + B_4\omega_2\omega_3 + A_4\omega_2\omega_4 + A_5\omega_2\omega_5, \\
\varpi'_3 &= \varpi_3(\varpi_4 - \varpi_1) - \omega_5\varpi_5 + \omega_2\chi_1 - B_1\omega_1\omega_3 - A_1\omega_1\omega_4 - A_2\omega_1\omega_5, \\
\varpi'_4 &= \varpi_2\varpi_3 + \frac{1}{3}\omega_3\varpi_7 + \frac{1}{3}\omega_4\varpi_5 - \frac{2}{3}\omega_5\varpi_6 + \omega_2\chi_2 - B_2\omega_1\omega_3 - 2B_3\omega_2\omega_3 \\
&\qquad - A_2\omega_1\omega_4 - A_3\omega_1\omega_5 - 2A_3\omega_2\omega_4 - 2A_4\omega_2\omega_5,
\end{aligned}
\right.
$$

Slovák $\to$ Dolbeault

Theorem

Every complex manifold with a holomorphic ~~projective connection~~ parabolic geometry satisfies all equations on Chern classes, in Dolbeault cohomology, satisfied by its model flag variety.

2,3,5

Corollary

If a complex manifold admits a holomorphic $2,3,5$*-distribution,*

$$0 = 5^5 c_5 - 3 c_1^5.$$

▶

Corollary

If a complex manifold admits a holomorphic $2,3,5$*-distribution, Chern-Simons form*

$$5^5 c_5 - 3 c_1^5$$

is exact in Slovák.

▶

2,3,5

$$T_{5^5c_5-3c_1^5} = \left(\frac{5i}{2\pi}\right)^5 \frac{1}{3^3}$$
$$(2\omega_1 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 \wedge \varpi_1 \wedge \varpi_5 \wedge \varpi_6 \wedge \varpi_7 \wedge \chi_1$$
$$+ \ldots 10 \text{ more terms} \cdots +$$
$$- 15\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \varpi_4 \wedge \varpi_6 \wedge \varpi_7 \wedge \chi_1 \wedge \chi_2)$$

is closed.

$$\Psi = \left(\frac{5i}{2\pi}\right)^5 \frac{2^2}{3}$$

$$(7\omega_2 \wedge \omega_4 \wedge \varpi_1 \wedge \varpi_2 \wedge \varpi_3 \wedge \varpi_5 \wedge \varpi_6 \wedge \varpi_7$$

$$+ 2\omega_1 \wedge \omega_5 \wedge \varpi_1 \wedge \varpi_2 \wedge \varpi_3 \wedge \varpi_5 \wedge \varpi_6 \wedge \varpi_7$$

$$- 5\omega_2 \wedge \omega_5 \wedge \varpi_1 \wedge \varpi_2 \wedge \varpi_4 \wedge \varpi_5 \wedge \varpi_6 \wedge \varpi_7$$

$$+ \ldots 26 \text{ more terms} \cdots +$$

$$+ 6\omega_1 \wedge \omega_2 \wedge \varpi_1 \wedge \varpi_3 \wedge \varpi_4 \wedge \varpi_6 \wedge \varpi_7 \wedge \chi_2)$$

$$d\Psi = T_{5^5 c_5 - 3c_1^5}.$$

Theorem

Every complex manifold with a holomorphic parabolic geometry satisfies all equations on Chern and Chern–Simons classes, in Dolbeault cohomology, satisfied by its model flag variety.

Thanks

- Characteristic forms of complex Cartan geometries, arXiv:0704.2555
- Characteristic forms of complex Cartan geometries II, arXiv:2201.05038
- Characteristic forms of complex Cartan geometries III: G-structures, arXiv:2206.04495