Commutative Poisson algebras from deformations of noncommutative algebras and non-Abelian Hamiltonian systems

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Outline

Motivating example: A simple system on the quantum plane

The quantum plane: $\mathcal{A}_q = \mathbb{C}(q)\langle x, y\rangle / \langle yx - qxy\rangle$. Hamiltonian:

$$\hat{H} = (y - qx)^2 = y^2 - q(q+1)xy + q^2x^2 \in \mathcal{A}_q$$

the Heisenberg equation

$$\frac{dx}{dt} = \frac{1}{q^2-1}[\hat{H}, x] = xy^2 - qx^2y, \quad \frac{dy}{dt} = \frac{1}{q^2-1}[\hat{H}, y] = \cdots$$

In the classical limit $q = 1 + \nu \to 1$, $\mathcal{A}_q \to \mathbb{C}[x, y]$, $(\nu = i\hbar)$

$$\{a, b\} = \lim_{\nu \to 0} \frac{1}{\nu}[a, b] = (a_y b_x - b_y a_x)xy, \quad a, b \in \mathbb{C}[x, y].$$

$$\frac{dx}{dt} = \frac{1}{q^2-1}[\hat{H}, x] \to xy^2 - x^2y = \{H, x\}, \quad H = \frac{1}{2}(y - x)^2$$

In the limit: $q \to -1$, $\mathcal{A}_q \to \mathbb{C}\langle x, y\rangle / \langle yx + xy\rangle$

$$\frac{dx}{dt} = \frac{1}{q^2-1}[\hat{H}, x] \to xy^2 + x^2y \overset{?!}{=} \{\mathbf{H}; x\}, \quad \mathbf{H} = ?!$$

Volterra type hierarchy

Volterra type system on the algebra $\mathcal{A}_q := \mathbb{C}(q)\langle x_i\,;\, i\in\mathbb{Z}\rangle/J_q$

$$J_q = \langle x_{i+1}x_i - (-1)^i q\, x_i x_{i+1},\ x_i x_j + x_j x_i\,;\ i,j\in\mathbb{Z},\ |i-j|\neq 1\rangle.$$

$$\frac{dx_\ell}{dt_2} = \frac{1}{q^2-1}[\hat{H}_2, x_\ell] = x_\ell x_{\ell+1}^2 - x_{\ell-1}^2 x_\ell + x_\ell^2 x_{\ell+1} - x_{\ell-2}x_{\ell-1}x_\ell + x_\ell x_{\ell+1}x_{\ell+2} - x_{\ell-1}x_\ell^2$$

$$\hat{H}_2 = \sum_{k\in\mathbb{Z}}\left(x_k^2 + (1+(-1)^k q)x_k x_{k+1}\right)$$

There is a quantum hierarchy of symmetries (SC,AVM,JPW):

$$\frac{dx_\ell}{dt_{2m}} = \frac{1}{q^{2m}-1}[\hat{H}_{2m}, x_\ell], \quad [\hat{H}_{2m}, \hat{H}_{2n}] = 0.$$

There is a well defined limit $q\to 1$:

$$\mathcal{A}_q \to \mathcal{A} = \mathbb{C}\langle \ldots, x_{-1}, x_0, x_1, \ldots\rangle/\langle x_{i+1}x_i - (-1)^i x_i x_{i+1},\ x_i x_j + x_j x_i\,;\ |i-j|>1\rangle.$$

What Poisson structure, Hamiltonian derivations, if any, correspond to this limit? Can we present this hierarchy on $\mathcal{A}$ in the Hamiltonian form:

$$\frac{dx_\ell}{dt_{2m}} = \{\mathbf{H}_{2m}; x_\ell\}, \quad \{\mathbf{H}_{2m}, \mathbf{H}_{2n}\} = 0\,?$$

Definition

Let $\mathcal{A}$ be any (unitary) associative algebra over a commutative ring R. A skew-symmetric R-bilinear map $\{\cdot\,,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be a Poisson bracket on $\mathcal{A}$ when it satisfies the Jacobi and Leibniz identities: for all $a, b, c \in \mathcal{A}$,

$$
\begin{array}{llr}
(1) & \{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0\ , & \text{(Jacobi identity)},\\
(2) & \{a,bc\} = \{a,b\}\,c + b\,\{a,c\}\ , & \text{(Leibniz identity)}.
\end{array}
$$

$(\mathcal{A}, \{\cdot\,,\cdot\})$ or $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ is then said to be a Poisson algebra (over R). When $\mathcal{A}$ is commutative one says that the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\})$ is commutative.

Any associative algebra $\mathcal{A}$ has a natural Poisson bracket, given by the commutator $\{a, b\} := [a, b]$. Thus, $(\mathcal{A}, [\cdot, \cdot])$ is a Poisson algebra.

Some noncommutative history remarks

1998: Farkas and Letzter proved that for any prime Poisson algebra $\mathcal{A}$, which is not commutative, the Poisson bracket must be the commutator in $\mathcal{A}$, up to an appropriate scalar factor.

2004: Van den Bergh introduced double Poisson bracket $\{\{\cdot,\cdot\}\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfying a modified skew-symmetry condition and modified Jacobi identity, and defined a double Poisson algebra.

2005: Croawley-Boevey studied non-commutative Poisson structures
2007: Croawley-Boevey, Etingof and Ginzburg introduced double derivations and Hamiltonian reductions

1998: Olver and Sokolov, 2000: Olver and Wang introduced and studied Hamiltonian structure of integrable PDEs on free associative algebras

2000: AVM and Sokolov introduced and studied Hamiltonian structure of ODEs on free associative algebras and “Poisson brackets” on $\mathcal{A}^\natural = \mathcal{A}/[\mathcal{A},\mathcal{A}]$.

2019: De Sole, Kac, Valeri and Wakimoto introduced Local and Non-local Multiplicative Poisson Vertex Algebras.

Kontsevich, Efimovskaya, Wolf, Chalykh, Fairon, Casati, Wang, ...

[yet unfinished paper]: Reshetikhin with Liashyk and Sechin.

Formal deformation $(\mathcal{A}[[\nu]], \star)$ of and associative algebra $\mathcal{A}$

Let $\mathcal{A}$ be any associative algebra over R.
$\mathcal{A}[[\nu]]$, the $R[[\nu]]$-module of formal power series in ν. Any element $A \in \mathcal{A}[[\nu]]$ can be written in a unique way as

$$A = a_0 + \nu a_1 + \nu^2 a_2 + \cdots, \qquad a_i \in \mathcal{A}.$$

Definition

Suppose that $\mathcal{A}[[\nu]]$ is equipped with the structure of an associative algebra over $R[[\nu]]$, with product denoted by $\star$. Then $(\mathcal{A}[[\nu]], \star)$, or simply $\mathcal{A}[[\nu]]$, is said to be a (formal) deformation of $\mathcal{A}$ if for any $a, b \in \mathcal{A}$,
$a \star b = ab + \mathcal{O}(\nu)$, i.e., $a \star b - ab \in \nu\mathcal{A}[[\nu]]$.

The commutator in $\mathcal{A}[[\nu]]$ is denoted by $[A, B]_\star := A \star B - B \star A$.

The R-bilinear maps $(\cdot, \cdot)_i, \{\cdot\,, \cdot\}_i : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are defined by

$$a \star b = ab + \nu(a, b)_1 + \nu^2(a, b)_2 + \cdots, \qquad a, b \in \mathcal{A} \subset \mathcal{A}[[\nu]]$$
$$[a, b]_\star = [a, b] + \nu\{a, b\}_1 + \nu^2\{a, b\}_2 + \cdots, \qquad \{a, b\}_i = (a, b)_i - (b, a)_i.$$

Poisson subalgebra, ideal and quotient algebra

- When $(\mathcal{A}, \{\cdot\,,\cdot\})$ is a Poisson algebra and $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ which is also a Lie subalgebra of $(\mathcal{A}, \{\cdot\,,\cdot\})$, then $(\mathcal{B}, \{\cdot\,,\cdot\})$ is a Poisson algebra; we say that $(\mathcal{B}, \{\cdot\,,\cdot\})$ is a Poisson subalgebra of $\mathcal{A}$.
- Similarly, if $\mathcal{I}$ is an ideal of $\mathcal{A}$ which is also a Lie ideal of $(\mathcal{A}, \{\cdot\,,\cdot\})$ then $\mathcal{I}$ is a Poisson ideal of $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ is a Poisson algebra; we say that it is a quotient Poisson algebra of $\mathcal{A}$.

Example: Let $\mathcal{A}[[\nu]]$ be a deformation of a commutative R-algebra $\mathcal{A}$.

$(\mathcal{A}[[\nu]], [\cdot,\cdot]_\star)$ is the corresponding natural Poisson algebra.

$\Big[\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\Big]_\star \subset \nu\mathcal{A}[[\nu]]$. Thus $(\mathcal{A}[[\nu]], [\cdot,\cdot]_\nu)$ is a Poisson algebra, where the bracket $[A,B]_\nu = \frac{1}{\nu}[A,B]_\star$ is well defined.

The ideal $\nu\mathcal{A}[[\nu]] \subset (\mathcal{A}[[\nu]], [\cdot,\cdot]_\nu)$ is also a Lie ideal:

$$\Big[\nu\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\Big]_\nu = \Big[\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\Big]_\star \subset \nu\mathcal{A}[[\nu]].$$

Thus $\mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]] = \mathcal{A}$ is a Poisson algebra with the Poisson bracket

$$\{a,b\} = \{a,b\}_1 \quad \left(= \lim_{\nu\to 0} \frac{a\star b - b\star a}{\nu}\right).$$

Deformation of a noncommutative algebra $\mathcal{A}$ and Poisson structures

Algebra $\mathcal{A}$ which is not necessarily commutative, and $Z(\mathcal{A})$ is its centre.
$\mathcal{A}[[\nu]]$ is a deformation of $\mathcal{A}$ and $(\mathcal{A}[[\nu]], [\cdot,\cdot]_\star)$ is its natural Poisson algebra.
Since $\left[\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\right]_\star \subset \mathcal{A}[[\nu]]$, and not $\nu\mathcal{A}[[\nu]]$, we cannot introduce $[\cdot,\cdot]_\nu$.

We define $\mathcal{H}_\nu = Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]]$. Quantum Hamiltonians live in $\mathcal{H}_\nu$.

We proved the following statements:

- $\mathcal{H}_\nu$ is a Poisson subalgebra of $(\mathcal{A}[[\nu]], [\cdot,\cdot]_\star)$, i.e. $[\mathcal{H}_\nu, \mathcal{H}_\nu]_\star \subset \mathcal{H}_\nu$.
- $(\mathcal{H}_\nu, [\cdot,\cdot]_\nu)$ is a Poisson algebra, i.e.
  $[\mathcal{H}_\nu, \mathcal{H}_\nu]_\star \subset \nu\mathcal{H}_\nu \Leftrightarrow \{Z(\mathcal{A}), Z(\mathcal{A})\}_1 \subset Z(\mathcal{A})$.
- $\nu^2\mathcal{A}[[\nu]]$ is a Poisson ideal of $(\mathcal{H}_\nu, [\cdot,\cdot]_\nu)$.
- $\mathcal{H}_\nu/\nu^2\mathcal{A}[[\nu]] \simeq Z(\mathcal{A}) \times \mathcal{A}$ is a (noncommutative) Poisson algebra.
- $\nu\mathcal{H}_\nu$ is a Poisson ideal of $(\mathcal{H}_\nu, [\cdot,\cdot]_\nu)$.
- $\mathcal{H}_\nu/\nu\mathcal{H}_\nu \simeq Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})}$ is a commutative Poisson algebra.

Commutative Poisson algebra $\Pi(\mathcal{A}) = Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})}$

$$A \in \mathcal{H}_\nu = Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]] \;\Rightarrow\; A = a_0 + \nu a_1 + \nu^2 a_2 + \cdots, \quad a_0 \in Z(\mathcal{A}),\; a_1, a_2, \ldots \in \mathcal{A}.$$

$$\pi_\Pi : \mathcal{H}_\nu/\nu\mathcal{H}_\nu \;\to\; \Pi(\mathcal{A}) := Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})} \quad \Big(\text{Canonical projection}\Big).$$

$$A \in \Pi(\mathcal{A}) \;\Rightarrow\; A = (a_0, a_1 + Z(\mathcal{A})) = (a_0, \overline{a_1}), \quad a_0 \in Z(\mathcal{A}),\; a_1 \in \mathcal{A}$$

Proposition

$(\Pi(\mathcal{A}), \{\cdot\,,\cdot\})$ is a commutative Poisson algebra with associative multiplication $\cdot$ and Poisson bracket $\{\cdot\,,\cdot\}$:

$$(a, \overline{a_1}) \cdot (b, \overline{b_1}) = \left(ab, \overline{ab_1 + a_1 b + (a,b)_1}\right),$$

$$\Big\{(a, \overline{a_1}), (b, \overline{b_1})\Big\} = \left(\{a,b\}_1, \overline{\{a,b\}_2 + \{a_1, b\}_1 + \{a, b_1\}_1 + [a_1, b_1]}\right),$$

for all $(a, \overline{a_1}), (b, \overline{b_1}) \in \Pi(\mathcal{A})$.

When $\mathcal{A}$ is commutative, $Z(\mathcal{A}) = \mathcal{A}$ and $\mathcal{H}_\nu = \mathcal{A}[[\nu]]$

$$\Pi(\mathcal{A}) \simeq \mathcal{H}_\nu/\nu\mathcal{H}_\nu \simeq \mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]] \simeq \mathcal{A}, \quad \{\cdot\,,\cdot\} = \{\cdot\,,\cdot\}_1 .$$

Heisenberg and Hamiltonian derivations

The Heisenberg derivation $\delta_{\hat{H}} : \mathcal{A}[[\nu]] \to \mathcal{A}[[\nu]]$:

$$\delta_{\hat{H}}(a) := \frac{1}{\nu}[\hat{H}, a]_\star$$

with $\hat{H} = H_0 + \nu H_1 + \nu^2 H_2 + \cdots \in \mathcal{A}[[\nu]]$ is well defined (admits a finite limit, as $\nu \to 0$, for any $a \in \mathcal{A}$) if and only if $H_0 \in Z(\mathcal{A})$.

Thus $\hat{H} \in \mathcal{H}_\nu$, and

$$\lim_{\nu \to 0} \delta_{\hat{H}}(a) = \lim_{\nu \to 0} \frac{1}{\nu}[H_0 + \nu H_1 + \nu^2 H_2 + \cdots , a]_\star = \{H_0, a\}_1 + [H_1, a].$$

For $\mathbf{H} = (H_0, \overline{H_1}) \in \Pi(\mathcal{A})$ we define a Hamiltonian derivation $\partial_{\mathbf{H}} : \mathcal{A} \to \mathcal{A}$

$$\partial_{\mathbf{H}}(a) = \{H_0, a\}_1 + [H_1, a].$$

We have shown that $\mathcal{A}$ is a Poisson module over $(\Pi(\mathcal{A}), \cdot, \{\cdot\,,\cdot\})$, with actions given for $(a, \overline{a_1}) \in \Pi(\mathcal{A})$ and $b \in \mathcal{A}$ by

$$(a, \overline{a_1}) \cdot b = b \cdot (a, \overline{a_1}) = ba = ab \ , \quad \{(a, \overline{a_1}); b\} = \{a, b\}_1 + [a_1, b] \ .$$

Definition

Let $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ be a Poisson algebra over R and let M be an R-module. Then M is said to be a $\mathcal{A}$-Poisson module (or Poisson module over $\mathcal{A}$ or over $(\mathcal{A}, \{\cdot\,,\cdot\})$) when M is both a $(\mathcal{A}, \cdot)$-bimodule and a $(\mathcal{A}, \{\cdot\,,\cdot\})$-Lie module, satisfying the following derivation properties: for all $a, b \in \mathcal{A}$ and $m \in M$,

$$\begin{aligned} \{a\,;b\cdot m\} &= \{a,b\}\cdot m + b\cdot\{a\,;m\}\ , \\ \{a\,;m\cdot b\} &= m\cdot\{a,b\} + \{a\,;m\}\cdot b\ , \\ \{a\cdot b\,;m\} &= a\cdot\{b\,;m\} + \{a\,;m\}\cdot b\ . \end{aligned}$$

In the above formulas, the three actions of $\mathcal{A}$ on M have been written $a\cdot m$, $m\cdot a$ and $\{a\,;m\}$ for $a \in \mathcal{A}$ and $m \in M$. In this notation, the fact that M is a $\mathcal{A}$-bimodule (respectively a $(\mathcal{A}, \{\cdot\,,\cdot\})$-Lie module), takes the form

$$a\cdot(b\cdot m) = (a\cdot b)\cdot m\ , \quad (m\cdot a)\cdot b = m\cdot(a\cdot b)\ , \quad a\cdot(m\cdot b) = (a\cdot m)\cdot b\ ,$$

$$\{\{a,b\}\ ;m\} = \{a\,;\{b\,;m\}\} - \{b\,;\{a\,;m\}\}\ ,$$

for $a, b \in \mathcal{A}$ and $m \in M$.

Non-Abelian Hamiltonian Equations

In our setting, let a Hamiltonian $\mathbf{H} = (H_0, \overline{H_1}) \in \Pi(\mathcal{A})$. Then the corresponding non-Abelian Hamiltonian equation on $\mathcal{A}$ is defined as

$$\frac{da}{dt} = \{\mathbf{H}\,;a\} = \{H_0, a\}_1 + [H_1, a].$$

Proposition

Suppose that $\mathbf{F} = (F_0, \overline{F_1}), \mathbf{G} = (G_0, \overline{G_1}) \in \Pi(\mathcal{A})$. Then

$$\partial_{\mathbf{F}}\partial_{\mathbf{G}} - \partial_{\mathbf{G}}\partial_{\mathbf{F}} = \partial_{\{\mathbf{F},\mathbf{G}\}}.$$

In particular, if $\mathbf{F}$ and $\mathbf{G}$ are in involution, $\{\mathbf{F}, \mathbf{G}\} = 0$, their associated derivations $\partial_{\mathbf{F}}$ and $\partial_{\mathbf{G}}$ of $\mathcal{A}$ commute.

We have two types of derivations:

$$\partial_{\mathbf{H}} = \{\mathbf{H}\,;\cdot\} \,:\, \mathcal{A} \to \mathcal{A}, \quad \text{and} \quad \partial'_{\mathbf{H}} = \{\mathbf{H}\,,\cdot\} \,:\, \Pi(\mathcal{A}) \to \Pi(\mathcal{A})$$

When $\mathcal{A}$ is commutative, $\partial_{\mathbf{H}}$ and $\partial'_{\mathbf{H}}$ are both derivations of $\mathcal{A}$ and $\partial_{\mathbf{H}} = \partial'_{\mathbf{H}}$. When $\mathcal{A}$ is not commutative, these derivations are defined on different algebras and none of the two determines the other one.

In the case of a commutative Poisson algebra $(\Pi(\mathcal{A}), \{\cdot\,,\cdot\})$ generated by the set $X_1, \ldots, X_M$, it is sufficient to find the Poisson brackets $\{X_i, X_j\}$. Then for any $P, Q \in \Pi(\mathcal{A})$

$$\{P, Q\} = \sum_{i,j=1}^{M} \frac{\partial P}{\partial X_i} \frac{\partial Q}{\partial X_j} \{X_i, X_j\} \ .$$

To compute Hamiltonian derivations, it is sufficient to find the table $\partial_{X_i}(y_k) = \{X_i\,;\, y_k\}$, where $y_1, \ldots$ are generators of the algebra $\mathcal{A}$.

Example: quantum plane at $q = -1 + \nu$

Quantum plane:

$$\mathcal{A}_q = \frac{\mathbb{C}(q)\langle x, y\rangle}{\langle yx - q\, xy\rangle}.$$

If $q^N \neq 1$ for some $N \in \mathbb{N}$, then $Z(\mathcal{A}_q) = \mathbb{C}$.

Quantum plane at $q = -1 + \nu$

$$\frac{\mathbb{C}[[\nu]]\langle x, y\rangle}{\langle yx - (\nu - 1)xy\rangle} \simeq \mathcal{A}[[\nu]] \ , \ \text{where } \mathcal{A} := \frac{\mathbb{C}\langle x, y\rangle}{\langle yx + xy\rangle} \ .$$

The product $\star$ on $\mathcal{A}[[\nu]]$ is defined by

$$y \star x = (\nu - 1)x \star y = (\nu - 1)xy,$$

$$a \star b =: ab :, \quad \text{for example } (xy^2) \star (x^3y + y^2) = (\nu - 1)^6 x^4 y^3 + xy^4 \ .$$

$\mathcal{A}$ is generated by x, y, satisfying the condition $yx = -xy$.
$Z(\mathcal{A})$ is generated by x^2, y^2.
$\mathcal{A}/Z(\mathcal{A})$ is generated as a $Z(\mathcal{A})$-module by $\overline{x}$, $\overline{y}$ and $\overline{xy}$.
$\Pi(\mathcal{A})$ is generated by 5 elements:

$$X = \left(x^2, \overline{0}\right) \ , \ Y = \left(y^2, \overline{0}\right) \ , \ U = (0, \overline{x}) \ , \ V = (0, \overline{y}) \ , \ W = (0, \overline{xy}) \ ,$$

Example: quantum plane at $q = -1 + \nu$

$\Pi(\mathcal{A}) \simeq \mathbb{C}[X, Y, U, V, W]/\langle U^2, V^2, W^2, UV, VW, UW\rangle.$

Poisson brackets between the generators of $\Pi(\mathcal{A})$:

$\{\cdot,\cdot\}$	X	Y	U	V	W
X	0	$4X \cdot Y$	0	$2X \cdot V$	$2X \cdot W$
Y	$-4X \cdot Y$	0	$-2U \cdot Y$	0	$-2Y \cdot W$
U	0	$2U \cdot Y$	0	$2W$	$2X \cdot V$
V	$-2X \cdot V$	0	$-2W$	0	$-2Y \cdot U$
W	$-2X \cdot W$	$2Y \cdot W$	$-2X \cdot V$	$2Y \cdot U$	0

Example: quantum plane at $q = -1 + \nu$

In the $\Pi(\mathcal{A})$-Poisson module $\mathcal{A}$ we have:

$\cdot$	x	y
X	x^3	x^2y
Y	xy^2	y^3
U	0	0
V	0	0
W	0	0

$\{\cdot\,;\,\cdot\}$	x	y
X	0	$2x^2y$
Y	$-2xy^2$	0
U	0	$2xy$
V	$-2xy$	0
W	$-2x^2y$	$2xy^2$

$$\hat{H} = -\frac{1}{2-\nu}(y^2 - \nu(-1+\nu))xy + (-1+\nu)^2x^2)$$

$$\mathbf{H} = -\frac{1}{2}(X + Y + W), \qquad \frac{dx}{dt} = \{\mathbf{H}; x\} = xy^2 + x^2y.$$

What is integrability of non-Abelian Hamiltonian systems?

In the commutative classical world, a Hamiltonian system in $\mathbb{R}^N$ is Liouville integrable if it has n functionally independent first integrals $H_1, \ldots H_n$ in involution $\{H_p, H_k\} = 0,\ 1 \leqslant p, k \leqslant n$ and $N - 2n$ functionally independent Casimir elements of the Poisson bracket.

In the corresponding quantum system, the integrability is often identified with the existence of n commuting and algebraically independent elements of the quantum algebra $\mathcal{A}_\hbar$ and $N - 2n$ generators of its center $Z(\mathcal{A}_\hbar)$.

It is evident that the central elements of $\mathcal{A}[[\nu]]$ give rise to Casimir elements within the Poisson algebra $\Pi(\mathcal{A})$. It would be logical to propose a definition of integrability for a non-Abelian Hamiltonian system, requiring the existence of n algebraically independent elements $\mathbf{H}_k \in \Pi(\mathcal{A})$ in involution $\{\mathbf{H}_p\,, \mathbf{H}_k\} = 0,\ 1 \leqslant p < k \leqslant n$, along with $N - 2n$ independent Casimir elements within the Poisson algebra $\Pi(\mathcal{A})$.

The problem of solutions for non-Abelian Hamiltonian systems is wide open.