HIGHER-POINT CORRELATORS:

Toda Bootstrap for Null Polygons & the Stampedes

Varna - August 2022

based on [*E. Olivucci, P.Vieira;* 2111.12131]

PLAN OF THE TALK

- *Introduction: 1/2-BPS correlators.*
- *Light-cone limit & factorisations.*
- *Null polygons: conjecture.*

- *Stampedes & Symbols.*
- *Q/A*

WHAT WE DEAL WITH

- *N=4 SYM* theory (conformal, gauge, supersymmetric)
- Large-N limit (t'Hooft): $g^2_{YM} N_c \sim g^2$ (planar)

1/2-BPS OPERATORS

- $O_k(x, y) = \mathrm{Tr}\left(y \cdot \Phi\right)(x)\,; \quad y_I, \Phi_I, \quad \text{for } I = 1,\ldots,6.$
- $y \cdot y = 0 \implies$ symmetric traceless tensor SO(6)

Protected 2-point function: $\gamma(g^2) = 0$

- **n-point functions** $\langle O_1(x_1, y_1) O_2(x_2, y_2) \cdots O_n(x_n, y_n) \rangle\,, \quad n \geq 4$

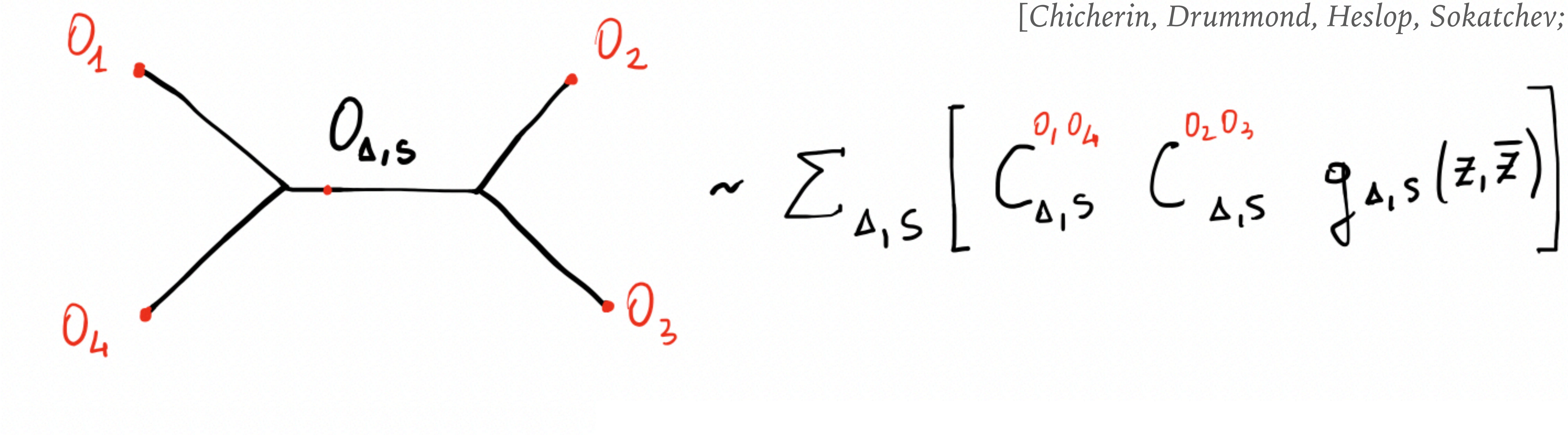
1/2-BPS OPERATORS

- Protected 2-pt function $\Delta(g^2) = \Delta(g^2=0) = L$
- Protected 3-pt functions $C^{\circ\circ\circ} = C^{\circ\circ\circ}(g^2=0)$

4-POINT FUNCTION

- $\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = R(x_{ij}^2) \times G(z,\bar{z})$,
- $\dfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad \dfrac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$
- Studied at three-loops.

[*Chicherin, Drummond, Heslop, Sokatchev;* 1512.02926]



$x_{14}^{\mu} \to 0 \;\wedge\; x_{23}^{\mu} \to 0$ *Operator Product Expansion*

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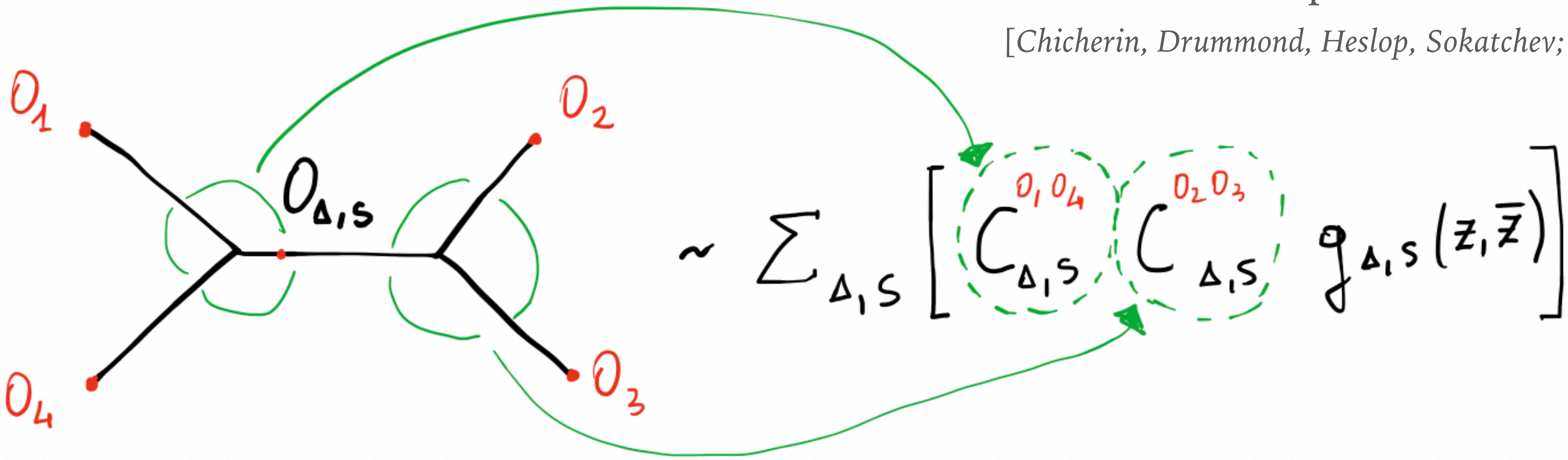


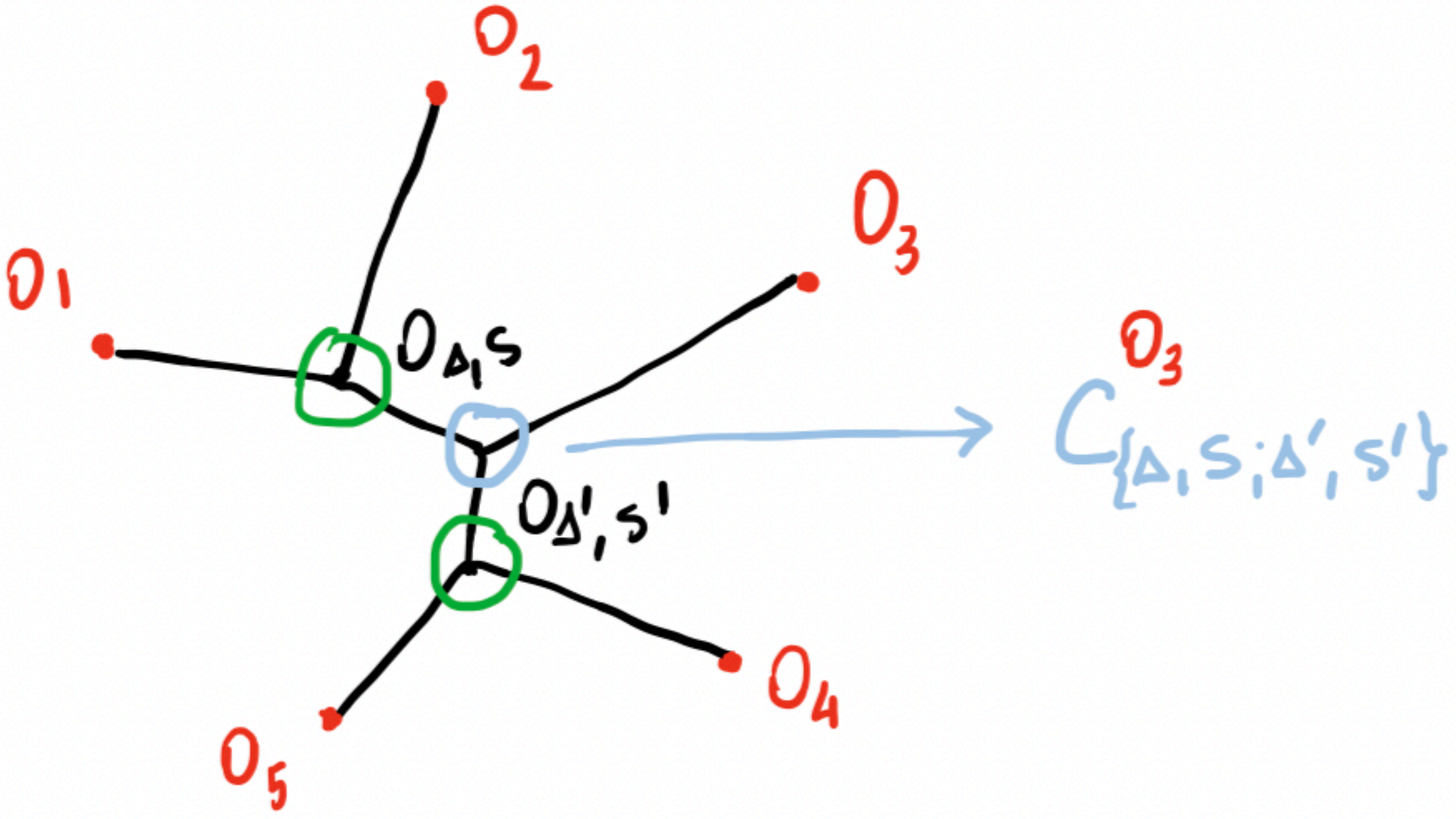
$x_{14}^\mu \to 0 \;\wedge\; x_{23}^\mu \to 0$ *Operator Product Expansion*

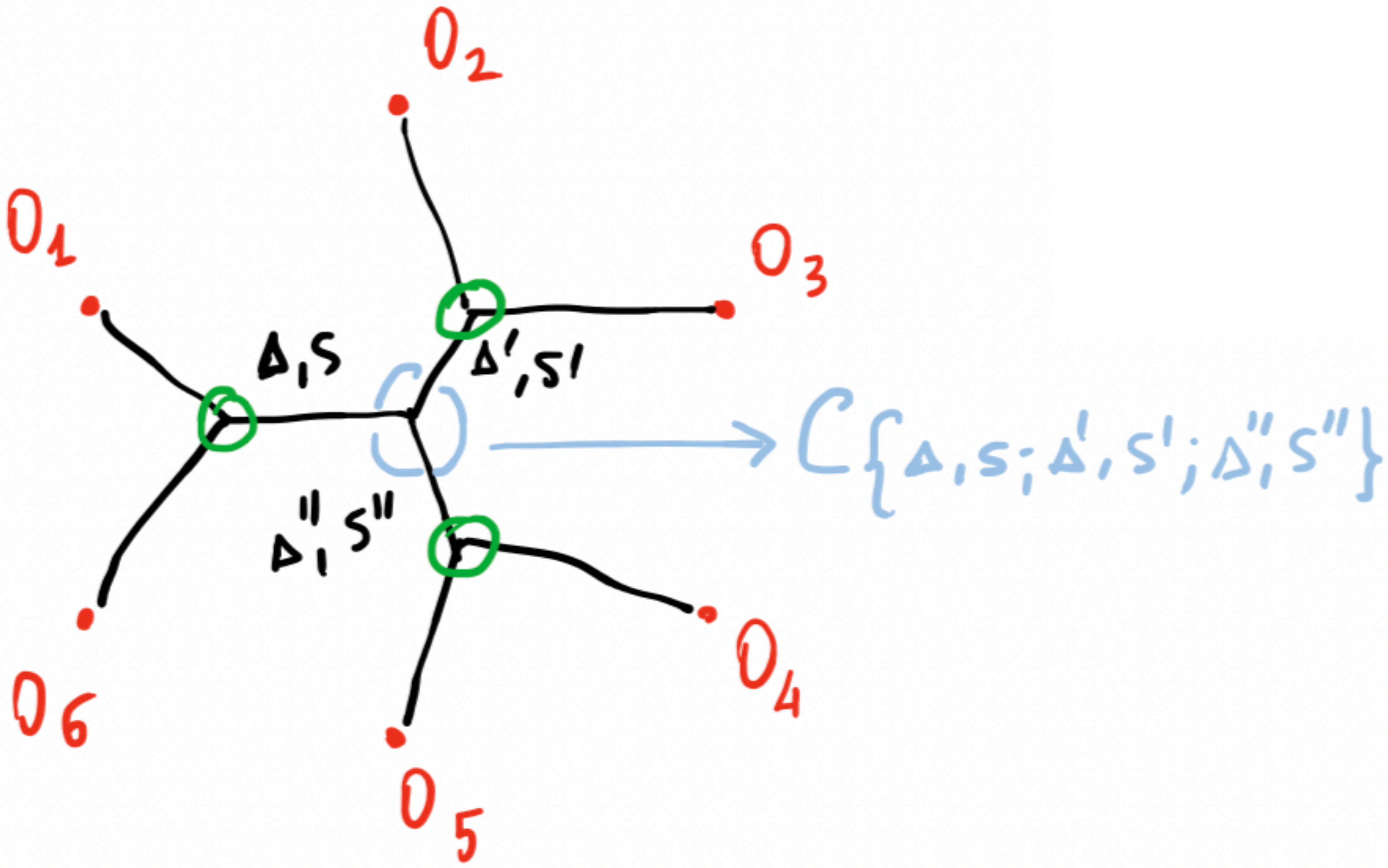
HIGHER-POINT FUNCTIONS

[*Bercini, Goncalves, Homrich, Vieira*]

[*Fortin, Ma, Prilepina, Skiba*]

[*Buric, Lacroix, Mann, Quintavalle, Schomerus*]

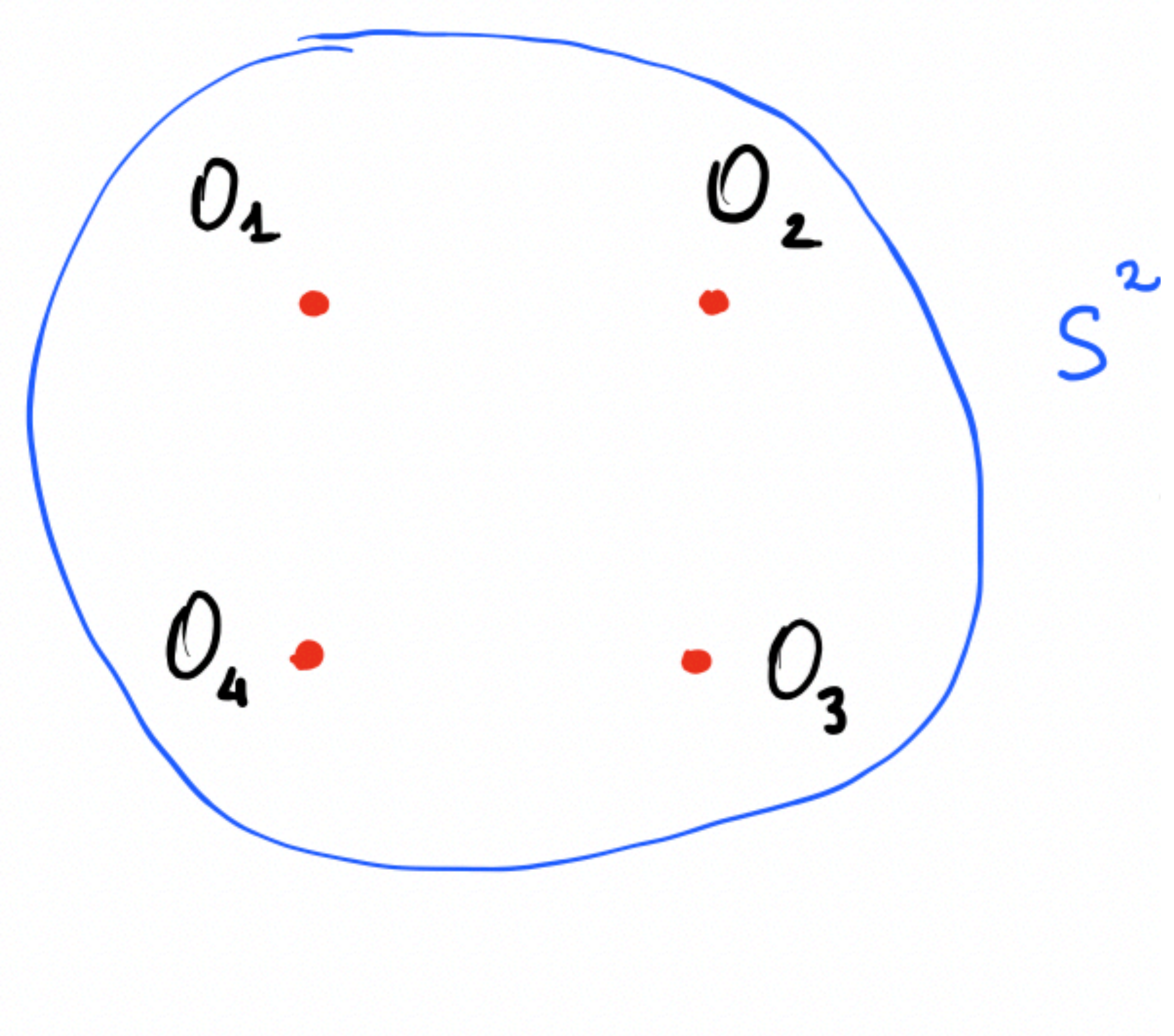




*Multiple OPE for 5-pt and 6-pt functions: "**Snowflake**" channel*

4-POINTS: DISK CORRELATORS

- Planar correlators: expansion in Feynman diagrams on S^2
- Disk correlators appear when: $S^2 \sim \Delta \times \Delta$



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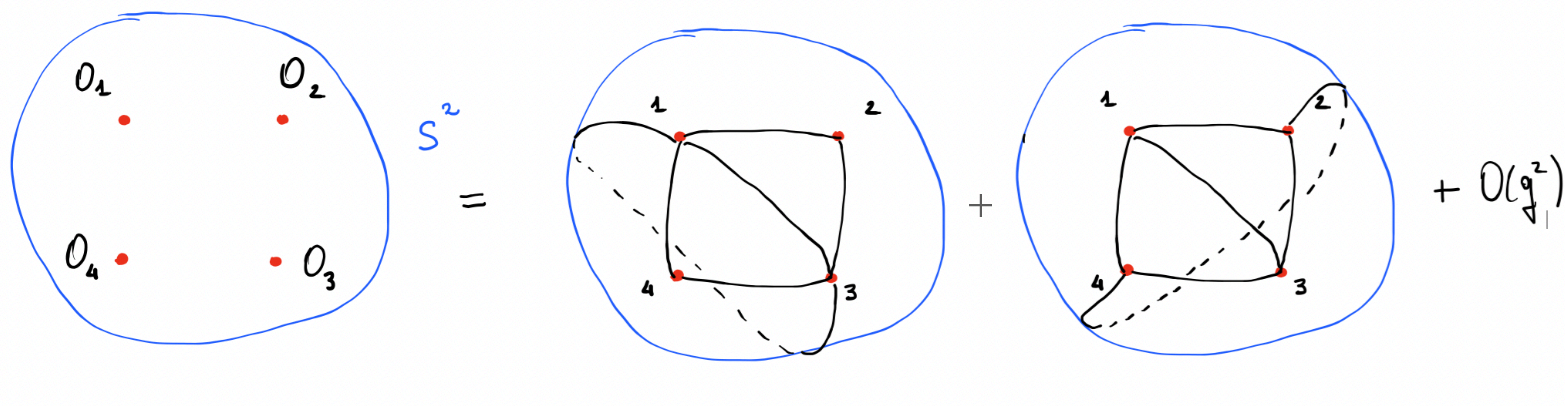
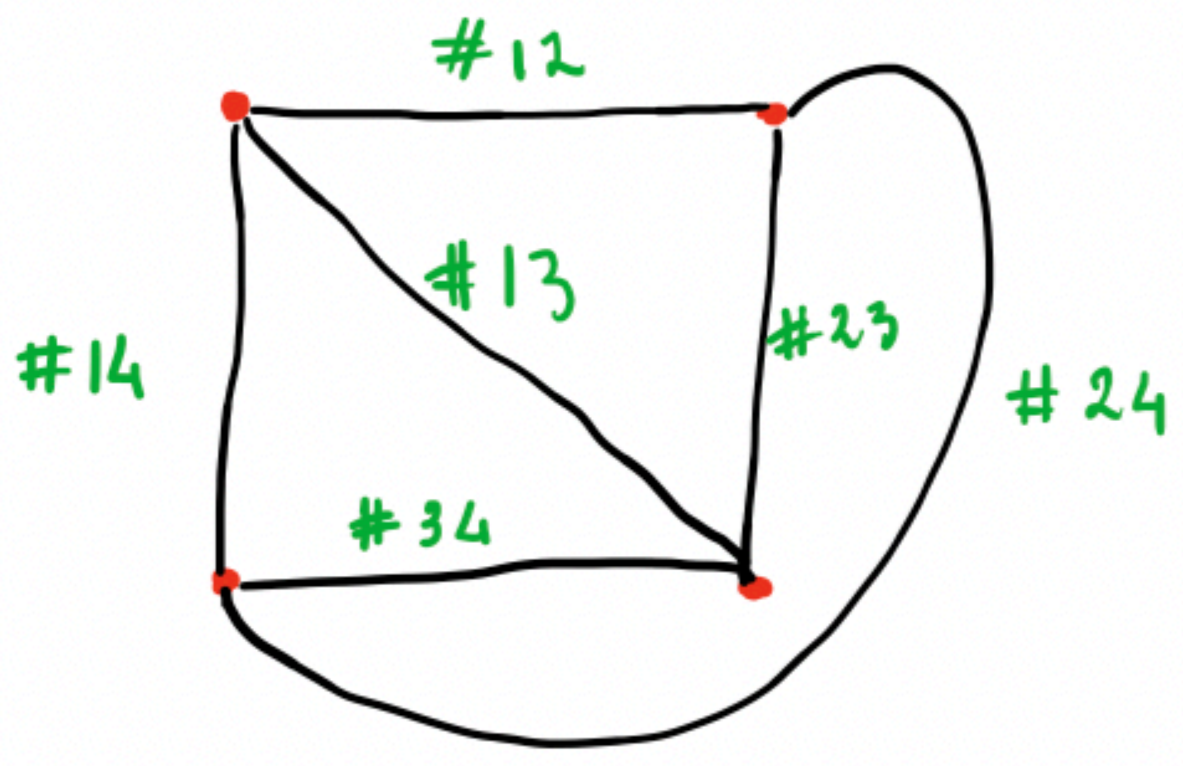


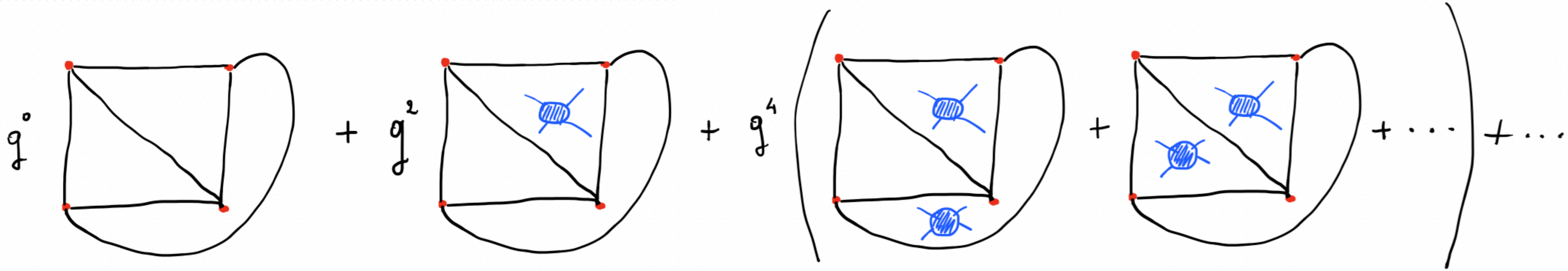
*Tree level: planar "**Skeletons**"*

TREE-LEVEL

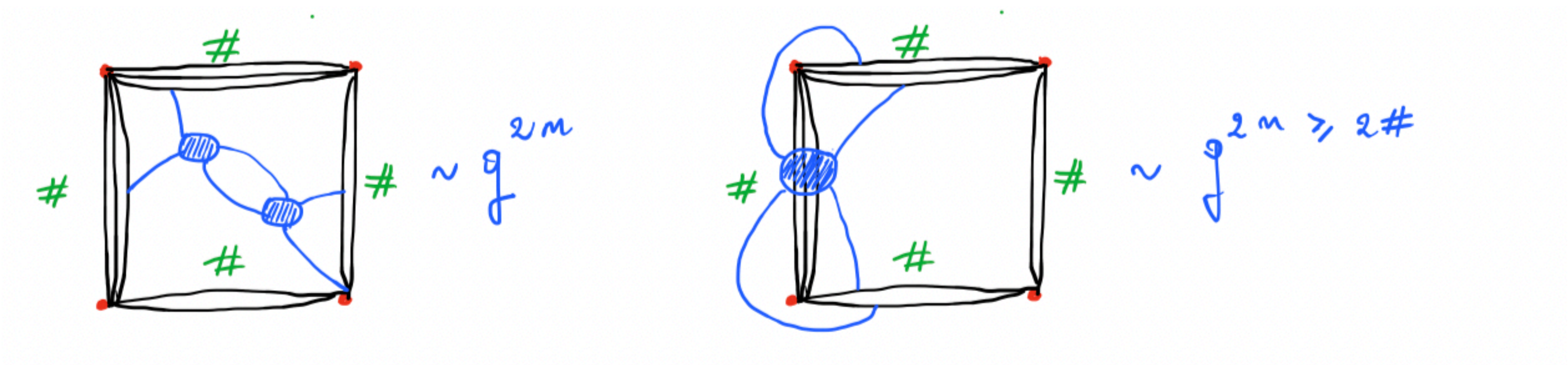


$$\sim \frac{(y_1 \cdot y_2)^{\#12}}{(x_{12})^{2\,\#12}}$$

PERTURBATION THEORY



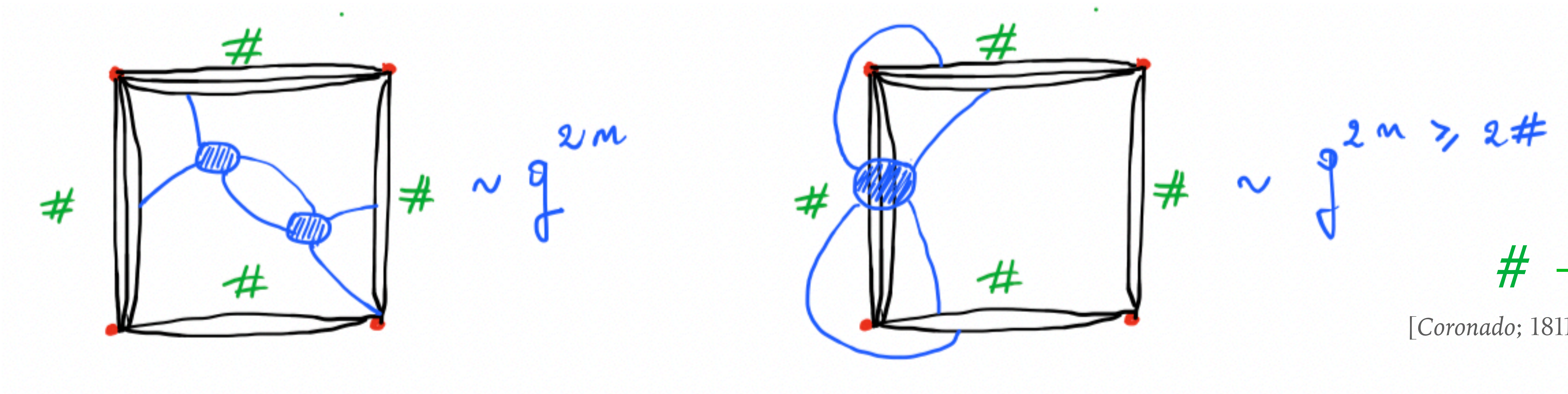
DISK CORRELATORS: LARGE CHARGE



Quantum corrections: insertion of vertices ***inside*** *the disk* ***versus*** *insertions* ***across the edges.***

- Vertices **inside/outside** the square frame: do not couple the two **disks.**
- Interactions that traverse the edges: coupling of two **disks** into a **sphere.**

DISK CORRELATORS: LARGE CHARGE



$\# \to \infty$

[Coronado; 1811.00467, 1811.03282]

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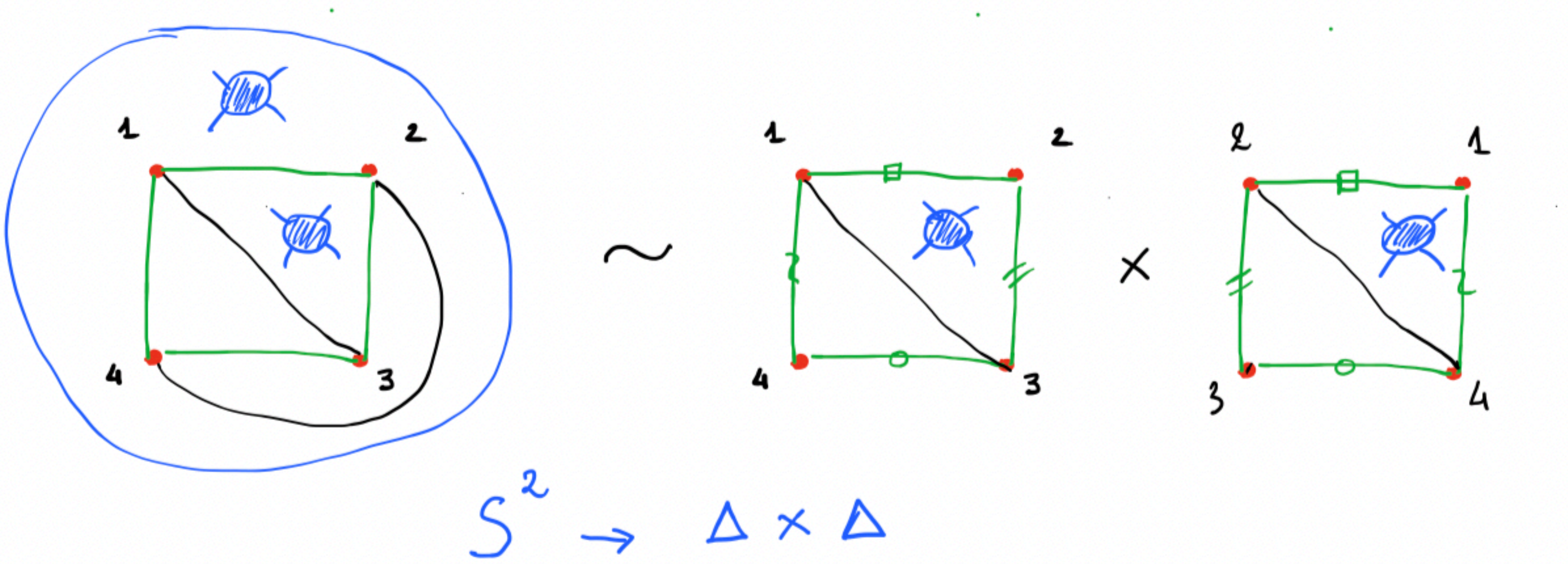
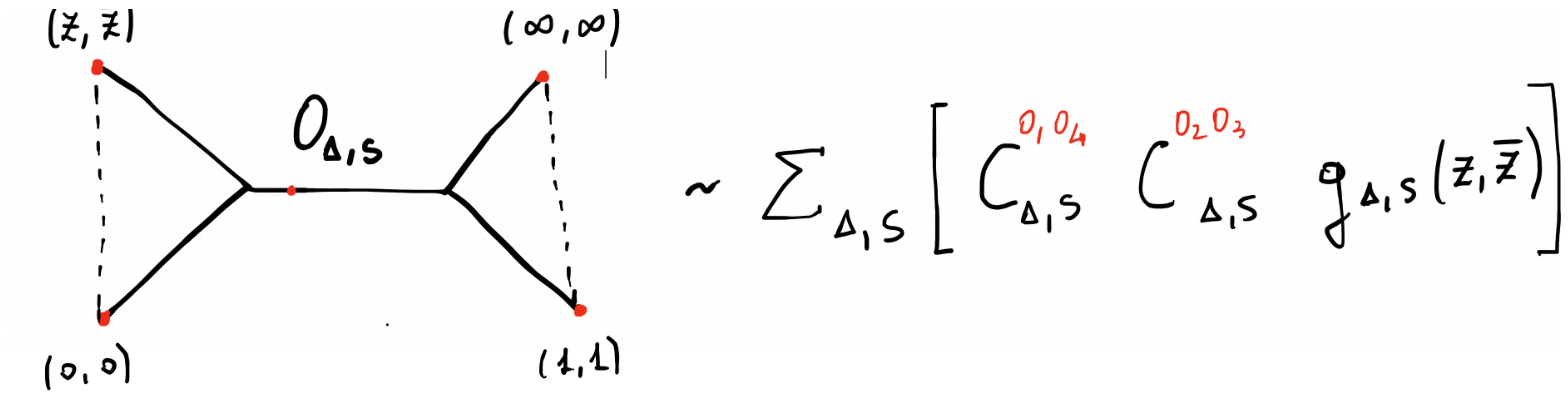


*Planar correlator = **disk** correlator x **disk** correlator*

DISK CORRELATORS: LIGHT-CONE LIMIT

- Light-cone contour: $x_{i,i+1}^2 \to 0\,,\;\; x_i \neq x_{i+1}$
- Logarithmic divergence: $g_{\Delta,S}(z,\bar{z}) = e^{-\log(z\bar{z})\Delta(g^2)} \times F(\Delta(g^2), S, z, \bar{z})$



- *First* Double-scaling limit: **leading logs** $g^2, z \to 0,\;\; (g^2 \log z) < \infty$

$$g_{\Delta,S}(0,\infty) \sim e^{-\log(z) g^2 \gamma_1} \times F(\Delta(0), S, 0, \bar{z})$$

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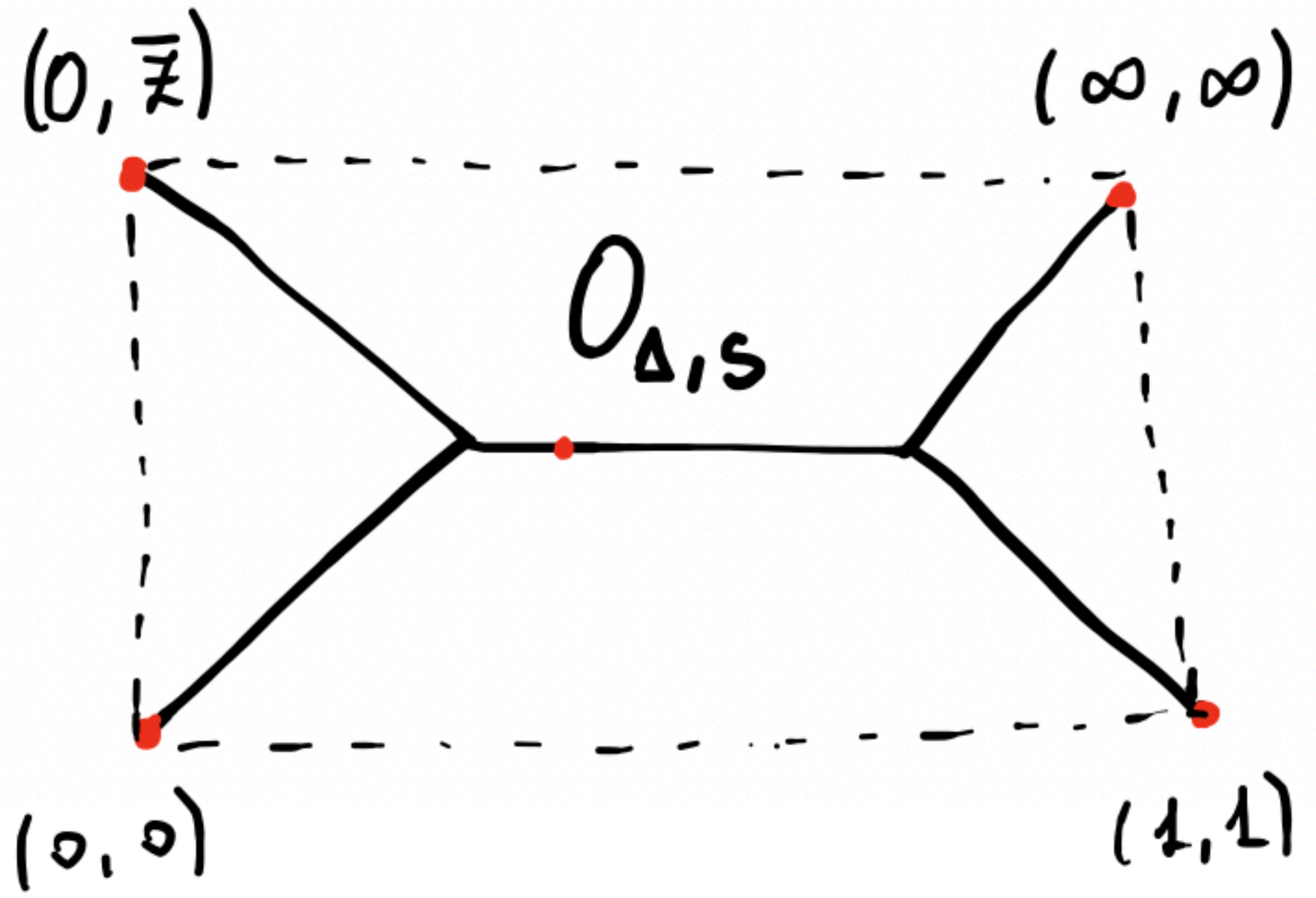

$$\sim \sum_{\Delta,S} \left[C^{O_1 O_4}_{\Delta,S} \; C^{O_2 O_3}_{\Delta,S} \; g_{\Delta,S}(z,\bar{z}) \right]$$

- *Second* Double-scaling limit : $g^2 \log z \log \bar{z} < \infty$

$$g_{\Delta,S}(0,\infty) \sim e^{-\log(z) g^2 \gamma_1} \times F(\Delta(0), S, 0, \bar{z})$$

$$s^{2n} = g^{2n} \log^n z \log^n \bar{z}$$

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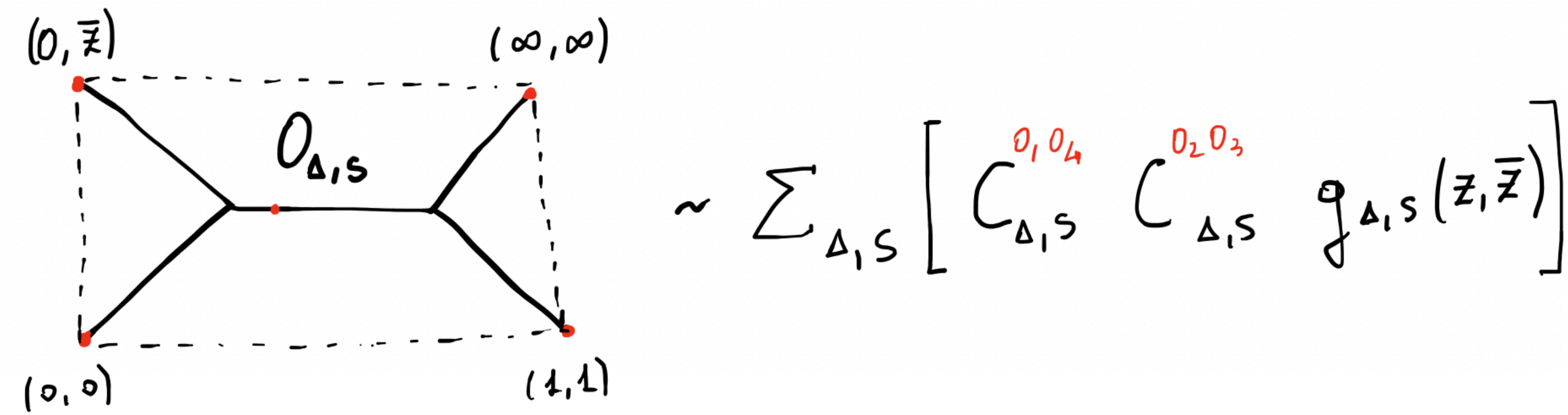

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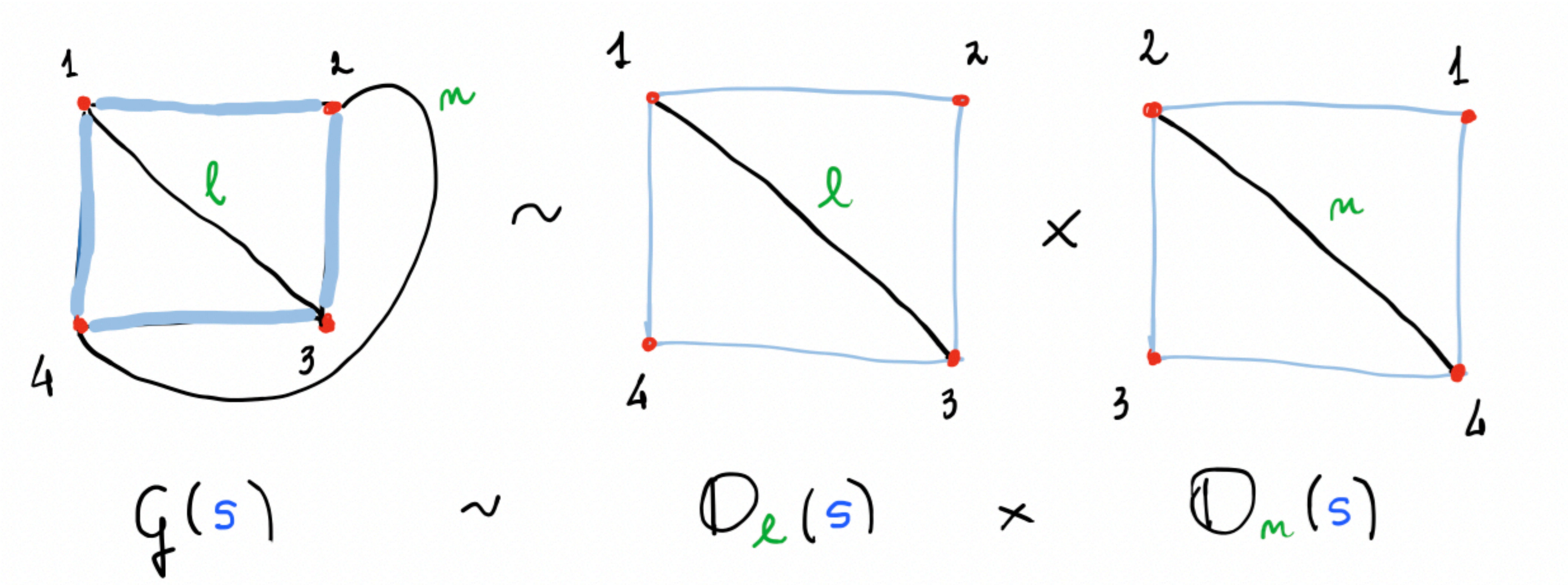
All-loop result from 1-loop dynamic!

NULL POLYGONS



Light-cone frame double-scaling limit: factorization into ***Null Polygons***

NULL SQUARES (AKA NULL OCTAGONS)

- Light-cone double-scaling limit: ***null squares***

$$\mathbb{O}_n(s) = e^{-s^2} \times \tau_n(s)$$

- Toda equation

$$\left(s\frac{d}{ds}\right)^2 \log \tau_n = s^2 \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}$$

- Boundary conditions

$$\tau_0(s) = 1\,,\, \tau_1(s) = I_0(2s)$$

$$\tau_n(2s) = \det_{1\leq i,j\leq n} I_{i-j}(2s)$$

[*Belitsky, Korchemsky;* 2006.01831]

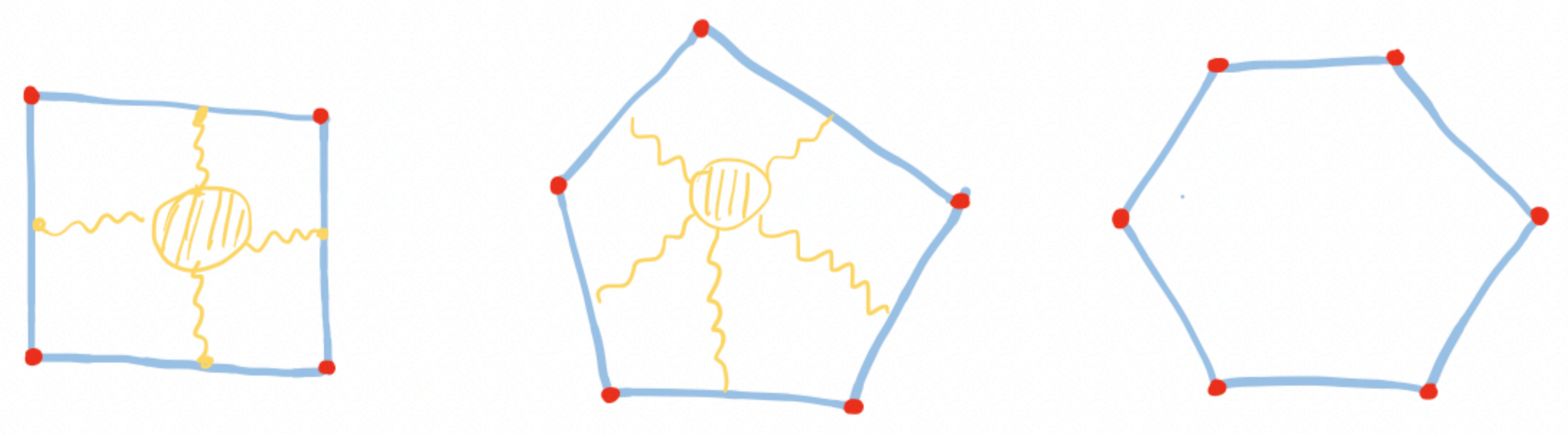
NULL POLYGONAL WL / NULL POLYGONS

$$\lim_{x^2_{i,i+1}\to 0} \frac{G_n}{G_n^{\rm tree}} = W_\epsilon^{\rm adj}[C_n] = \frac{1}{N_c^2-1}\langle 0|\,{\rm Tr}_{\rm adj}\,{\rm P}\exp\left(ig\oint_{C_n} dx\cdot A(x)\right)|0\rangle$$

[*Alday, Korchemsky, Eden, Maldacena, Sokatchev*; 1007.3243]

[*Bercini, Goncalves, Vieira*; 2008.10407]

- Polygonal Wilson Loop: null-limit factorization $\langle W_{\rm adj}\rangle \sim \langle W_{fund}\rangle^2$



- $20'$ correlators in double-scaling LC limit $\langle {\rm Tr}\left(y_1\cdot\Phi\right)^2(x_1)\ldots{\rm Tr}\left(y_n\cdot\Phi\right)^2(x_n)\rangle$

$$F(t_1,\ldots,t_n) = e^{-t_1^2-t_2^2+\ldots-t_n^2}$$

$$t_i^2 = g^2\log(x^2_{i,i-1})\log(x^2_{i,i+1})$$

NULL POLYGONAL WL / NULL POLYGONS

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$$\langle O(1)\cdots O(n)\rangle \longrightarrow e^{-\frac{\Gamma^{\rm adj}_{\rm cusp}}{4}\sum_{i=1}^n \log\frac{x^2_{i-1,i}}{x^2_{i-1,i+1}}\log\frac{x^2_{i,i+1}}{x^2_{i-1,i+1}}}$$

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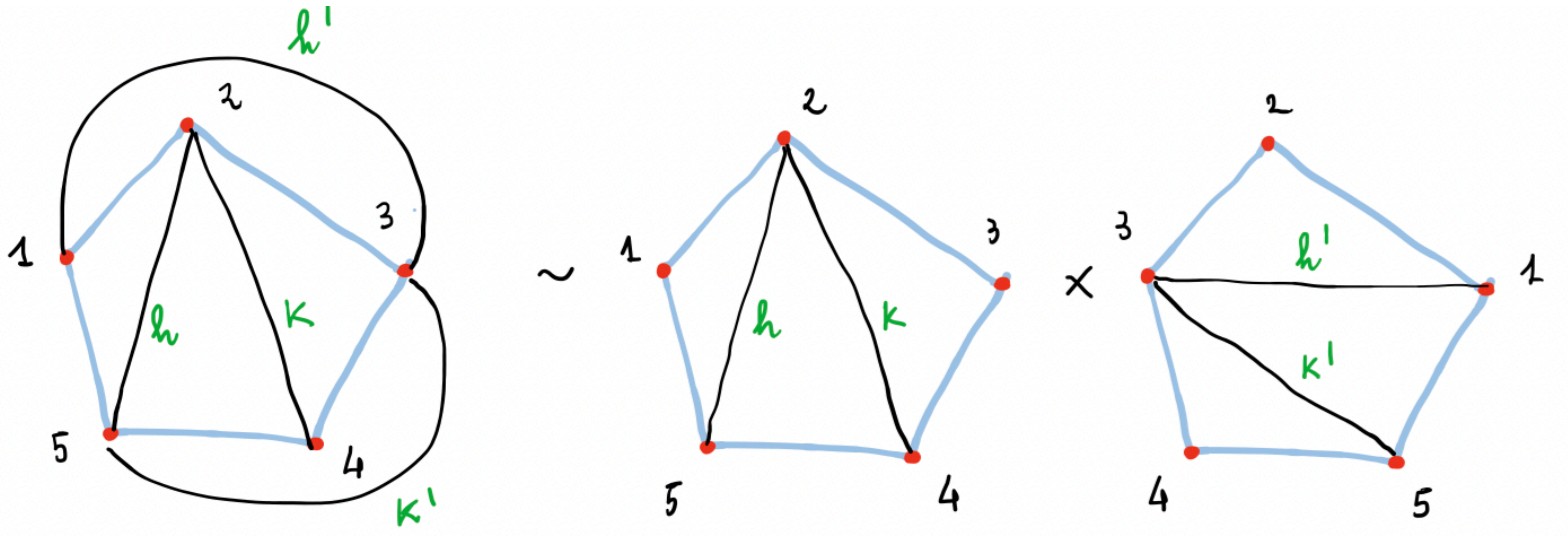
NULL PENTAGONS



$$g(t_1, t_2, t_3, t_4, t_5) \sim \mathbb{P}_{hk}(t_i) \times \mathbb{P}_{h'k'}(t_i)$$

Factorization of 5-point correlators into ***Null Pentagons***

NULL PENTAGONS

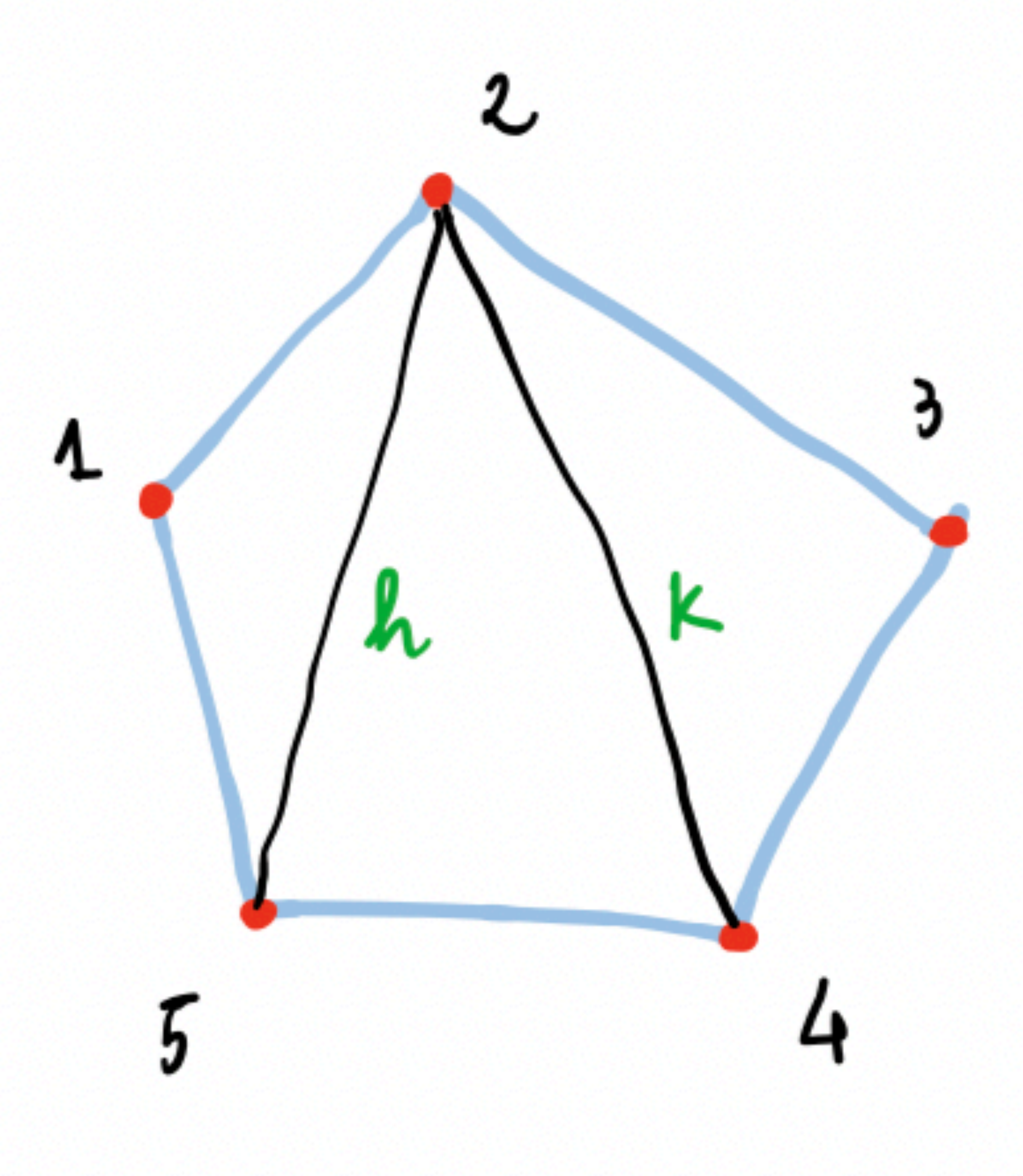
- Light-cone double-scaling limit: ***null pentagons***

$$\mathbb{P}_{h,k}(t_j) = e^{-\sum_{i=1}^{5} t_i^2} \times \mathbf{P}_{h,k}(t_1, \ldots, t_5)$$

- Toda equations

$$\left(t_1\partial_{t_1} + t_2\partial_{t_2}\right)\left(t_1\partial_{t_1} + t_5\partial_{t_5}\right)\log \mathbf{P}_{h,k} = t_1^2 \frac{\mathbf{P}_{h+1,k}\mathbf{P}_{h-1,k}}{\mathbf{P}_{h,k}^2}$$

$$\left(t_3\partial_{t_3} + t_2\partial_{t_2}\right)\left(t_3\partial_{t_3} + t_4\partial_{t_4}\right)\log \mathbf{P}_{h,k} = t_3^2 \frac{\mathbf{P}_{h,k+1}\mathbf{P}_{h,k-1}}{\mathbf{P}_{h,k}^2}$$



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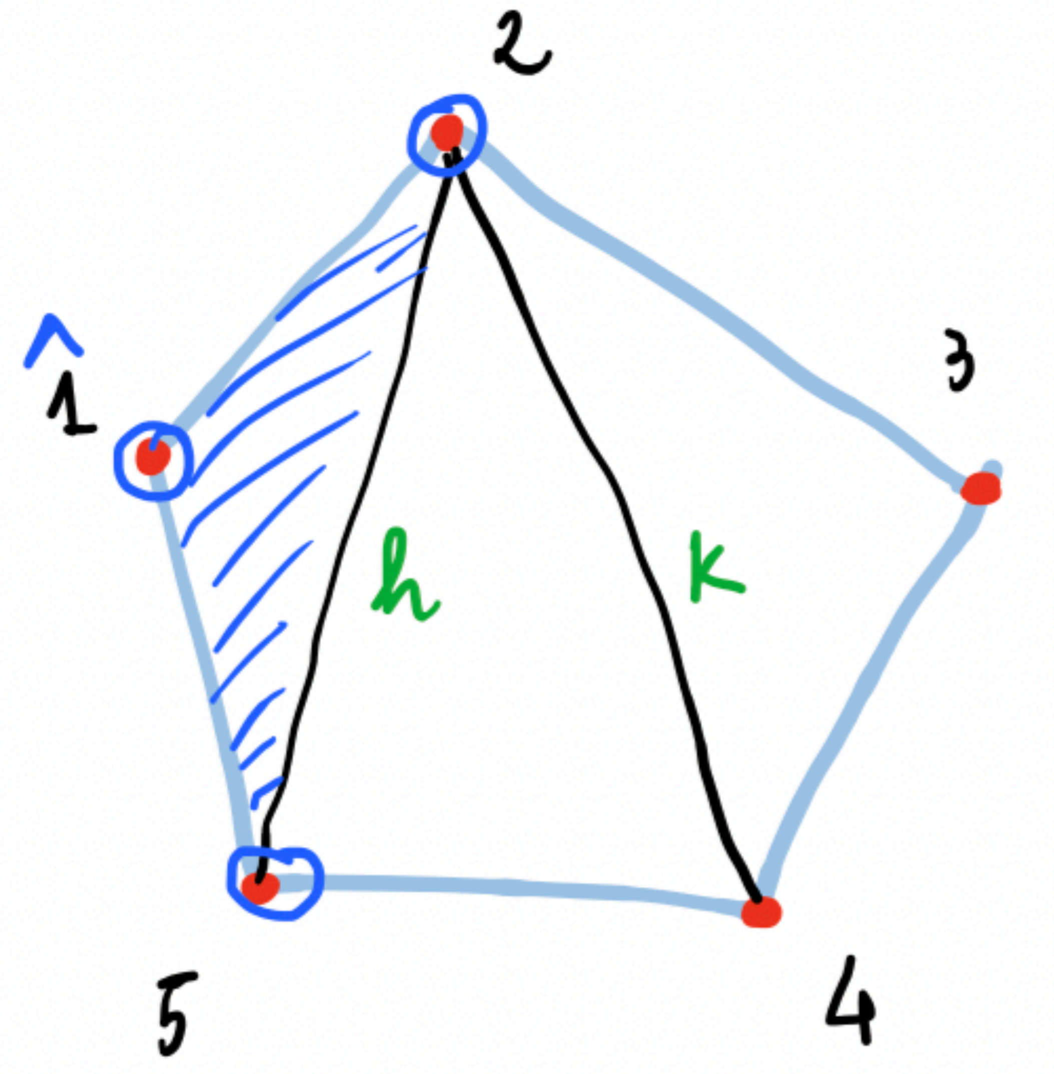

NULL PENTAGONS

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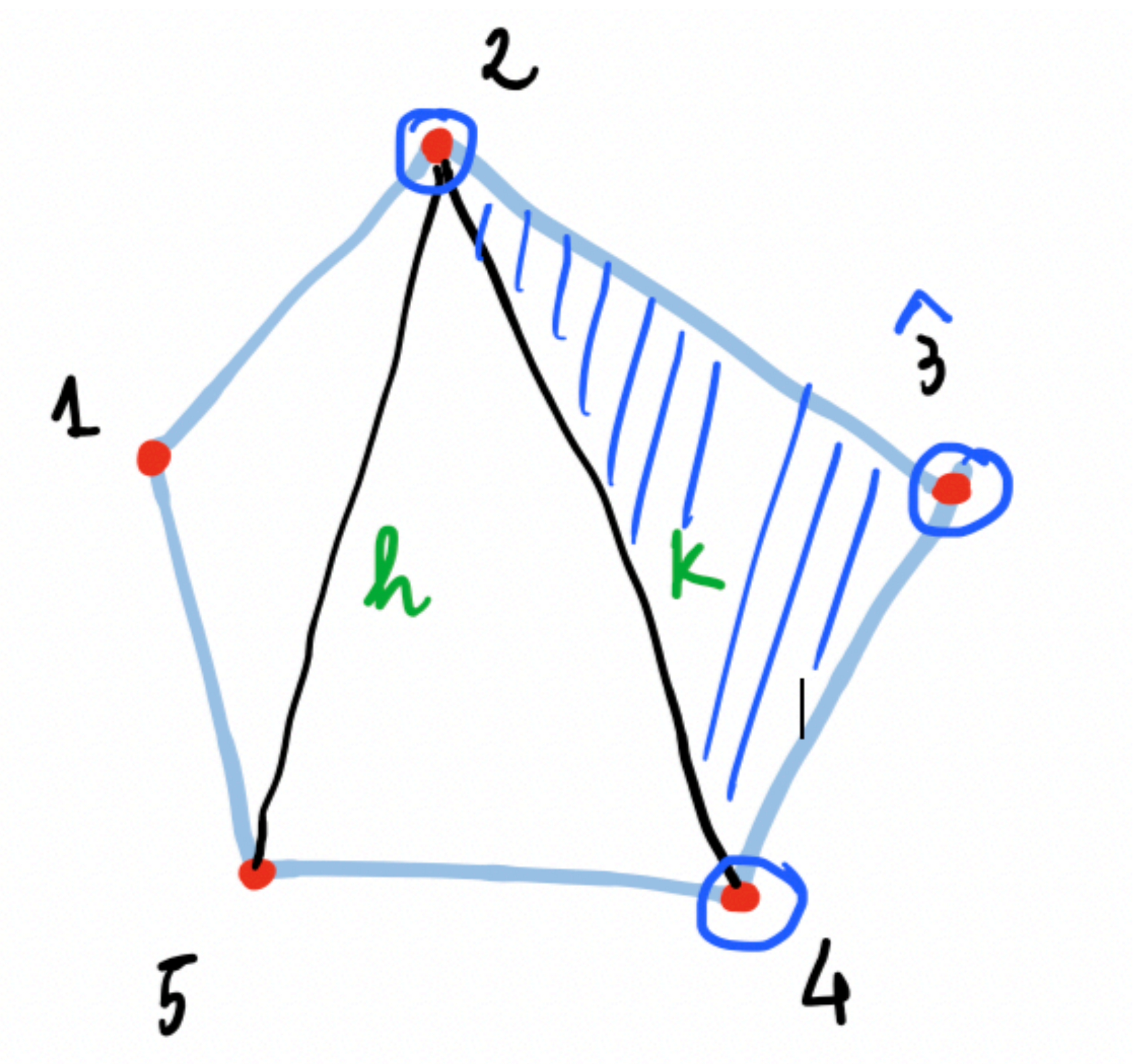

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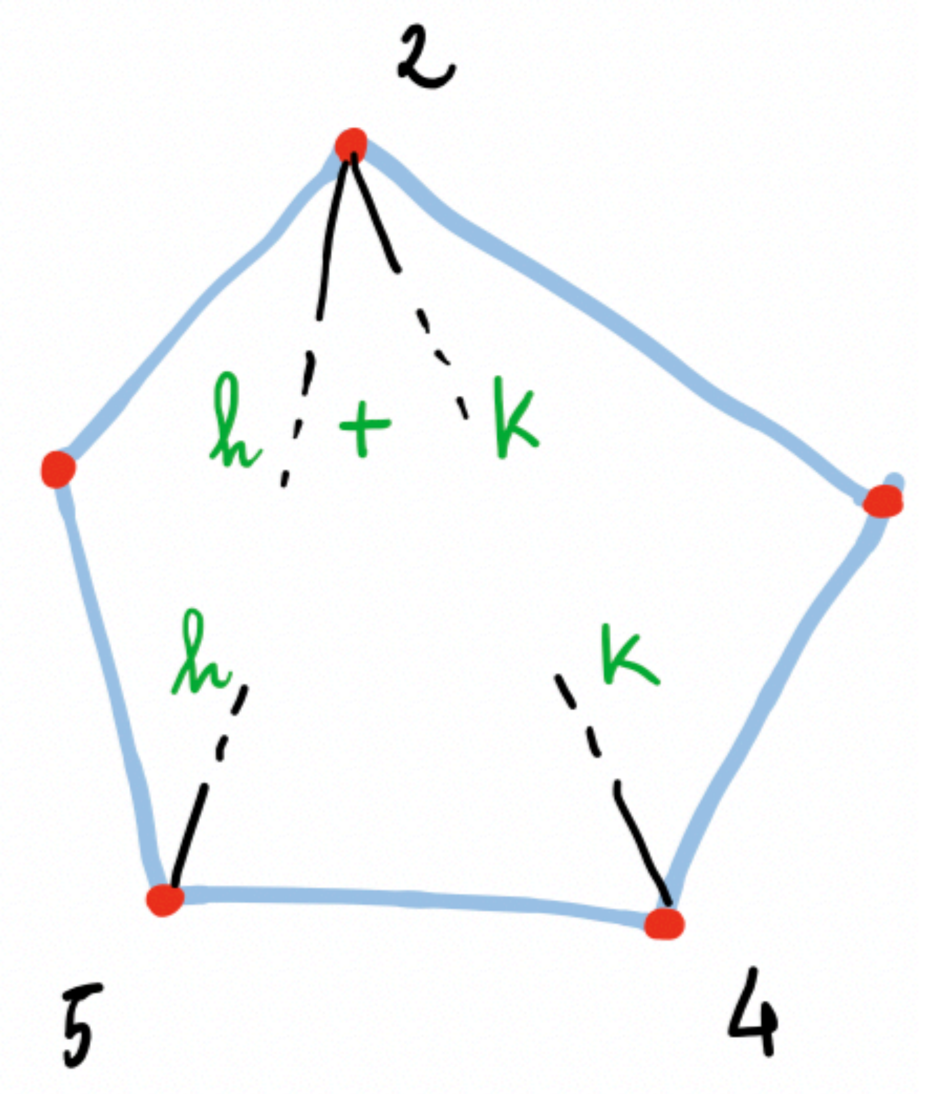
- Boundary conditions

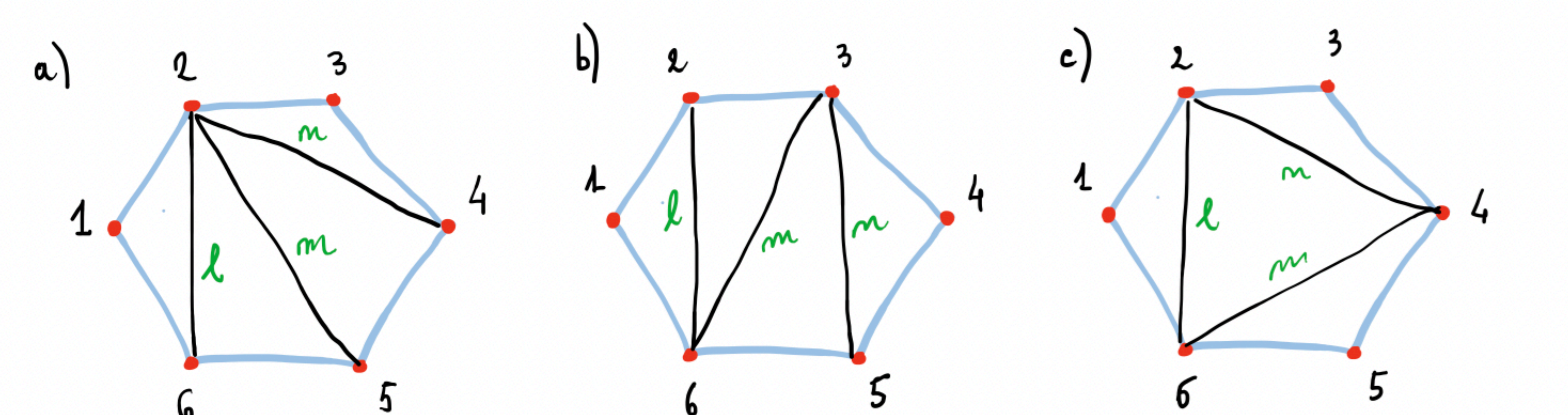
$$\mathbf{P}_{h+k}(0,t_2,0,t_4,t_5) = \tau_{h+k}(t_2) \times \tau_k(t_4) \times \tau_h(t_5)$$



NULL HEXAGONS

- Light-cone double-scaling limit: ***null hexagons***

$$\mathbb{H}_{l,m,n}(t_j) = e^{-\sum_{i=1}^{6} t_i^2} \times \mathbf{H}_{l,m,n}(t_1, \ldots, t_6)$$



*All three possible **planar** configuration of bridges inside a null hexagon.*

NULL HEXAGONS

- Light-cone double-scaling limit: ***null hexagons***

$$\mathbb{H}_{l,m,n}(t_j) = e^{-\sum_{i=1}^{6} t_i^2} \times \mathbf{H}_{l,m,n}(t_1, \ldots, t_6)$$

- Differential equations ?

- Boundary conditions ?

NULL HEXAGONS

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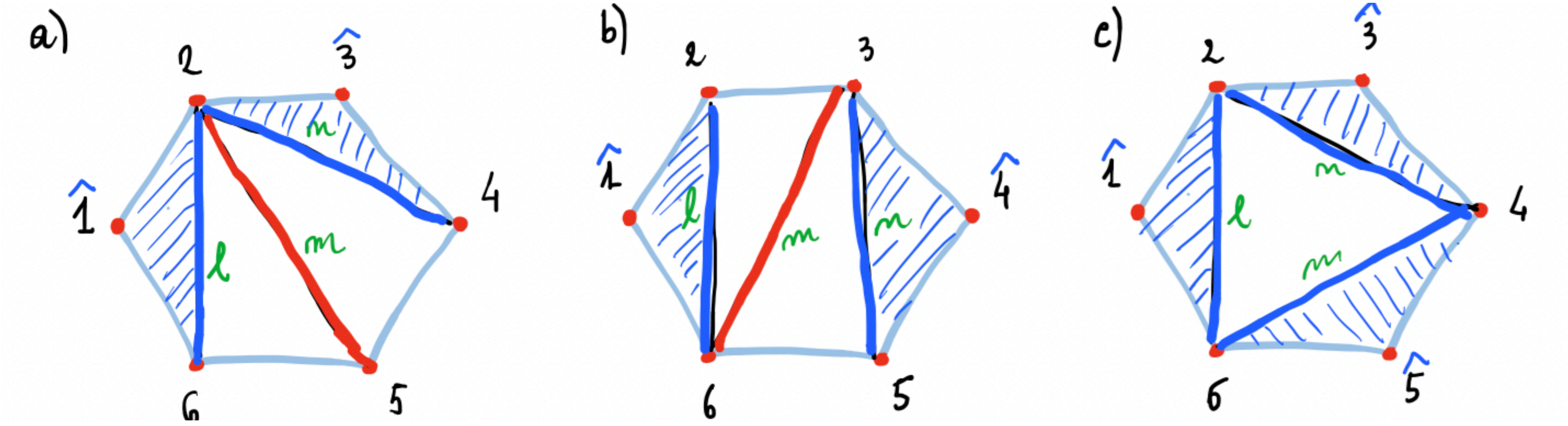

Each blue triangle: *one* **Toda** equation in the vertex variables .

NULL HEXAGONS

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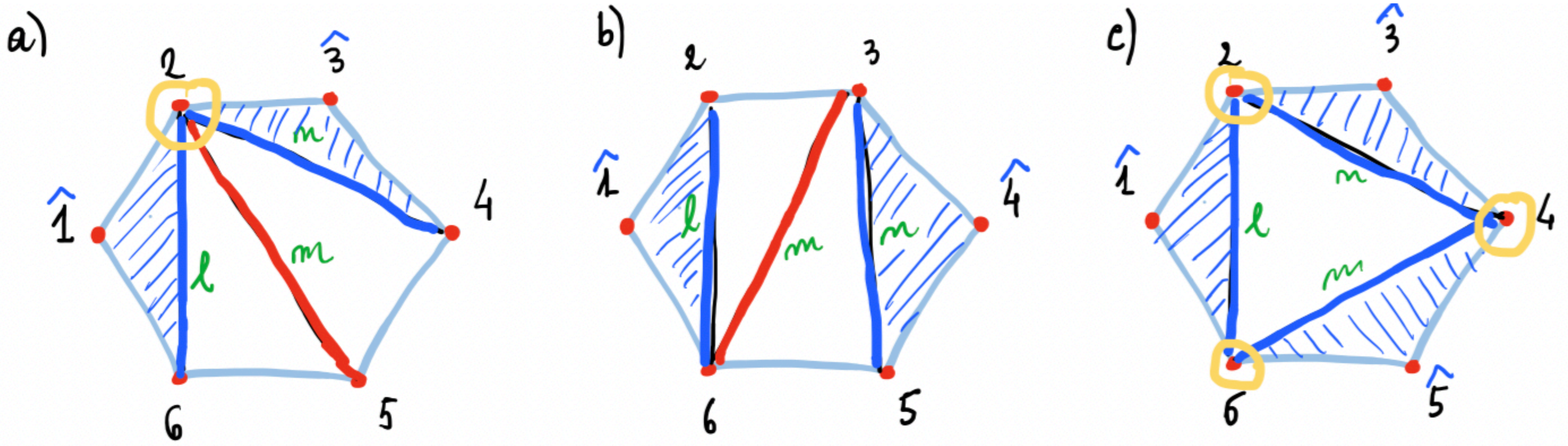
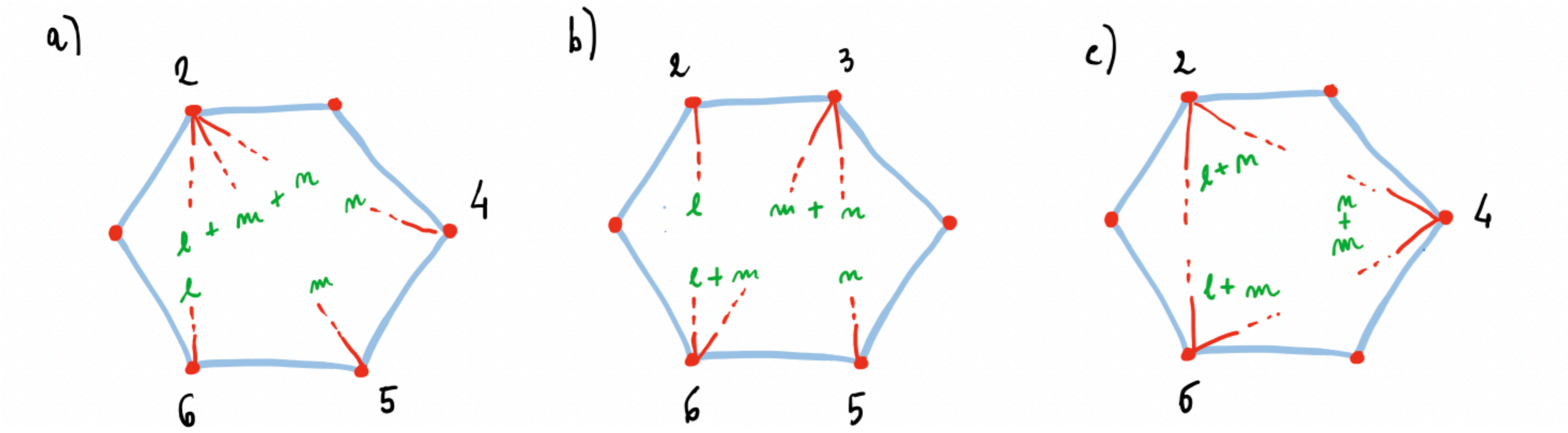


Coupled *vs* ***decoupled*** *Toda equations*

NULL HEXAGONS

- Boundary conditions



a) $\tau_{l+m+n}(t_2)\,\tau_l(t_6)\,\tau_m(t_5)\,\tau_n(t_4)$ b) $\tau_{l+m}(t_6)\,\tau_{m+n}(t_3)\,\tau_l(t_2)\,\tau_n(t_5)$ c) $\tau_{l+m}(t_6)\,\tau_{m+n}(t_3)\,\tau_l(t_2)\,\tau_n(t_5)$

NULL N-GONS: GENERAL CONJECTURE

- Light-cone double-scaling limit: ***null n-gons***

$$\mathbb{X}_{l_1,l_2,\ldots}(t_j) = e^{-\sum_{i=1}^{n} t_i^2} \times \mathbf{X}_{l_1,l_2,\ldots}(t_1, \ldots, t_n)$$

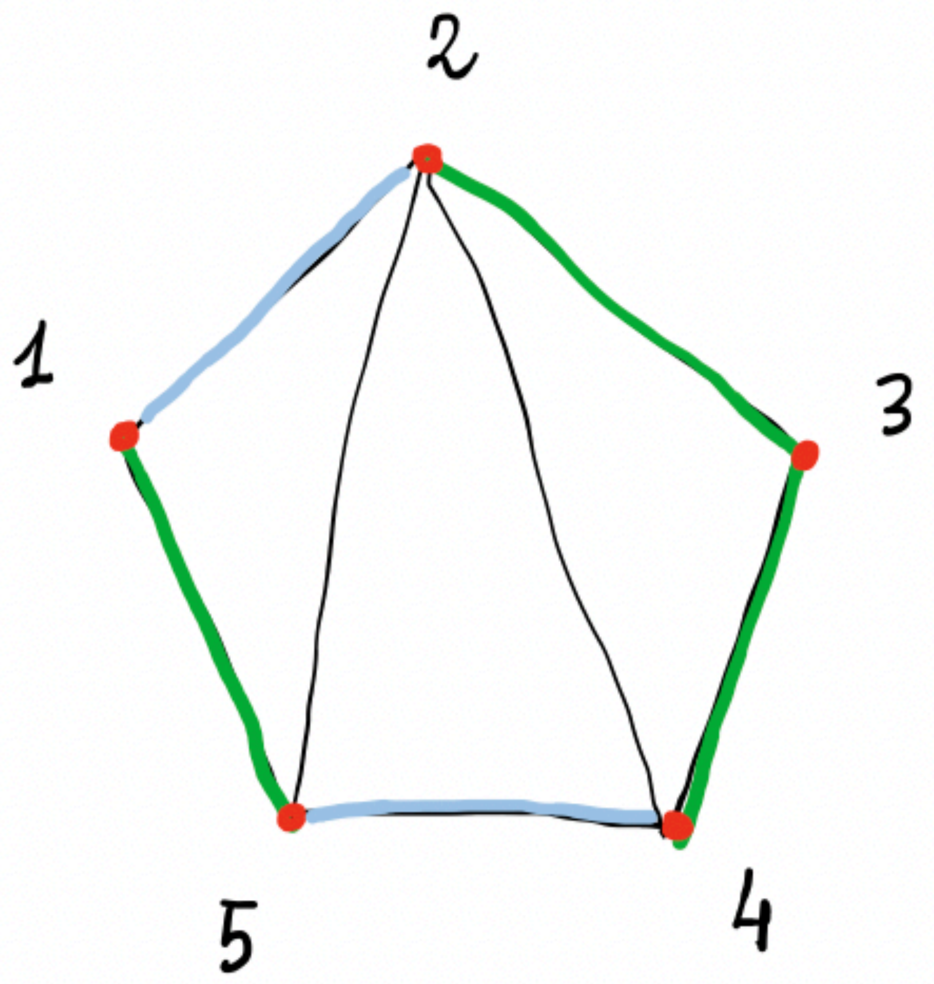
- Each ***blue triangle*** = one Toda equation (2*d Toda equation* a.k.a. *Toda Field Theory*)
- Each ***vertex*** of the polygons = boundary condition $\tau_{\text{\#of emitted lines}}(t_{cusp})$

- Toda equations + boundary condition = ***any-loop*** solution (iterative algorithm).

- **n-point functions** $\langle O_1(x_1, y_1) O_2(x_2, y_2) \cdots O_n(x_n, y_n) \rangle\ , \quad n \geq 4$

NEXT STEPS:

- Mixed factorization limit: ***large charge*** + ***null edges***



$$\sim \sum_{m,n} g^{2(n+m)} \log^n(x_{12}^2) \log^m(x_{45}^2)\, \Phi_{m,n}(u_1,u_2,u_3)$$

- Relation to null polygonal WL in presence of ***bridges.***
- Derivation of null polygons from ***Integrability*** (hexagon formalism).
- Validity in general gauge CFT and (**even**) in QFT with zero ***one-loop beta-function***.

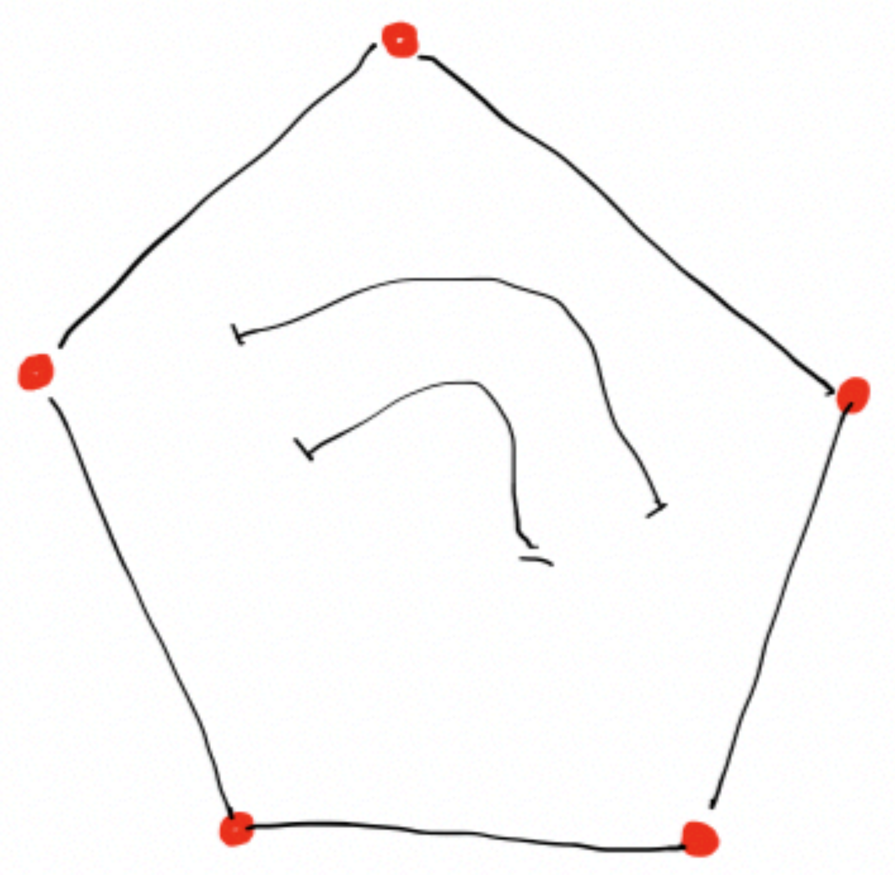
MORE NEXT STEPS:

- Sub-leading logs $t_i^2 \to t_i^2 + g^2\, s_i^2$: Toda bootstrap holds for ***null squares.***

[E.O., Vieira; 2111.12131]

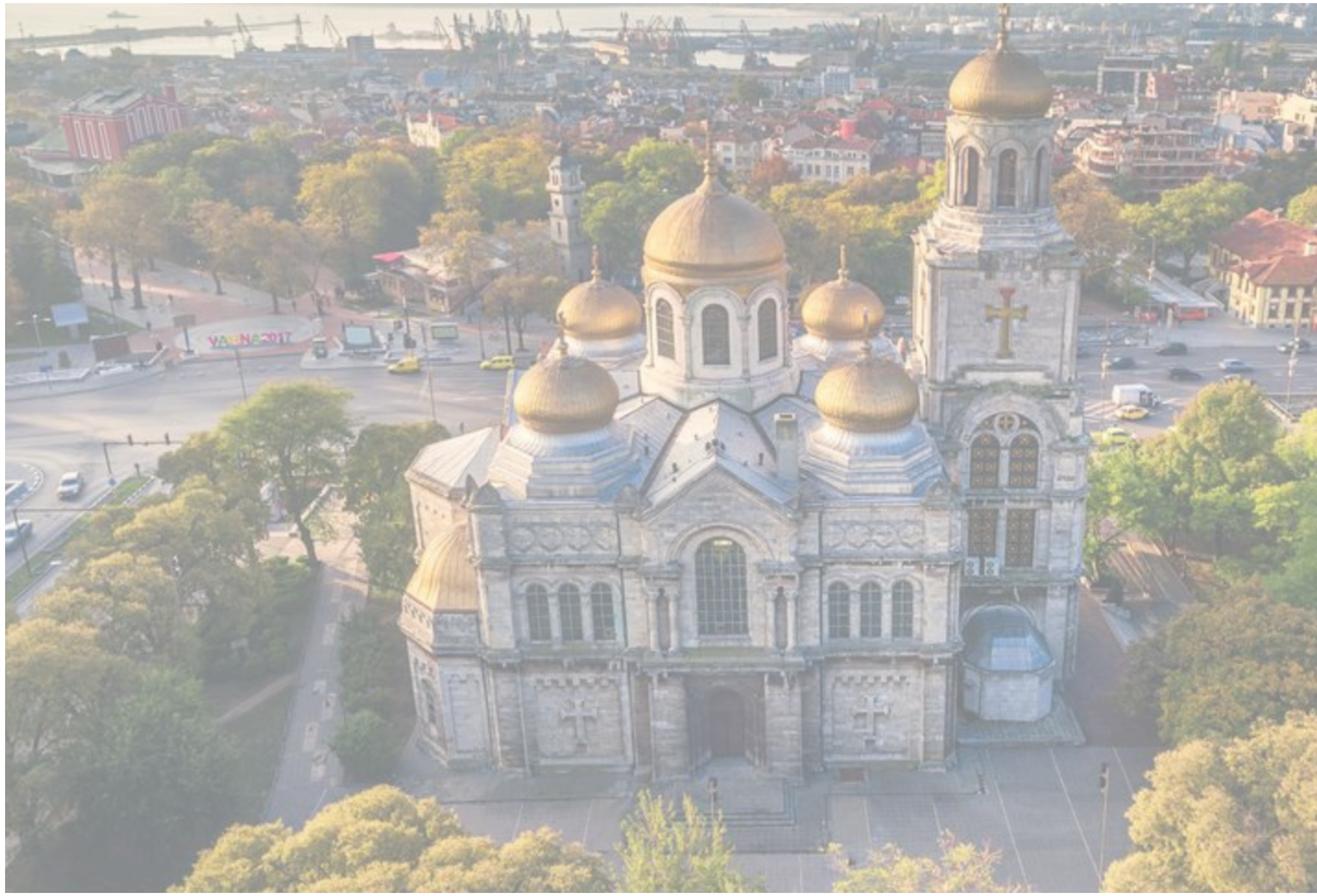
- Finite # of colours: ***non-planar corrections*** to disk correlators.

[Bargheer, Caetano, Fleury Komatsu, Vieira; 1711.05326]



Благодаря за вниманието!

Thanks for your attention



TODA EQUATION

$$\left(s\frac{d}{ds}\right)^2 \log \tau_n = s^2 \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},$$

$$\frac{\tau_{n+1}}{\tau_n} = \exp\left(q_n(t) - 2nt\right), \;\; s = e^t .$$

$$\ddot{q}_n(t) = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}$$

[*M. Toda,* J. Phys. Soc. Jpn., **22** (2): 431–436; *1967*]

TODA FIELD THEORY

$$\left(t_1\partial_{t_1} + t_2\partial_{t_2}\right)\left(t_1\partial_{t_1} + t_5\partial_{t_5}\right) \log \mathbf{P}_{h,k} = t_1^2 \frac{\mathbf{P}_{h+1,k}\mathbf{P}_{h-1,k}}{\mathbf{P}_{h,k}^2}$$

$$\frac{\mathbf{P}_{h+1,k}}{\mathbf{P}_{h,k}} = \exp\left(p_{h,k} - 2ns_1\right), \;\; t_j = e^{s_j} .$$

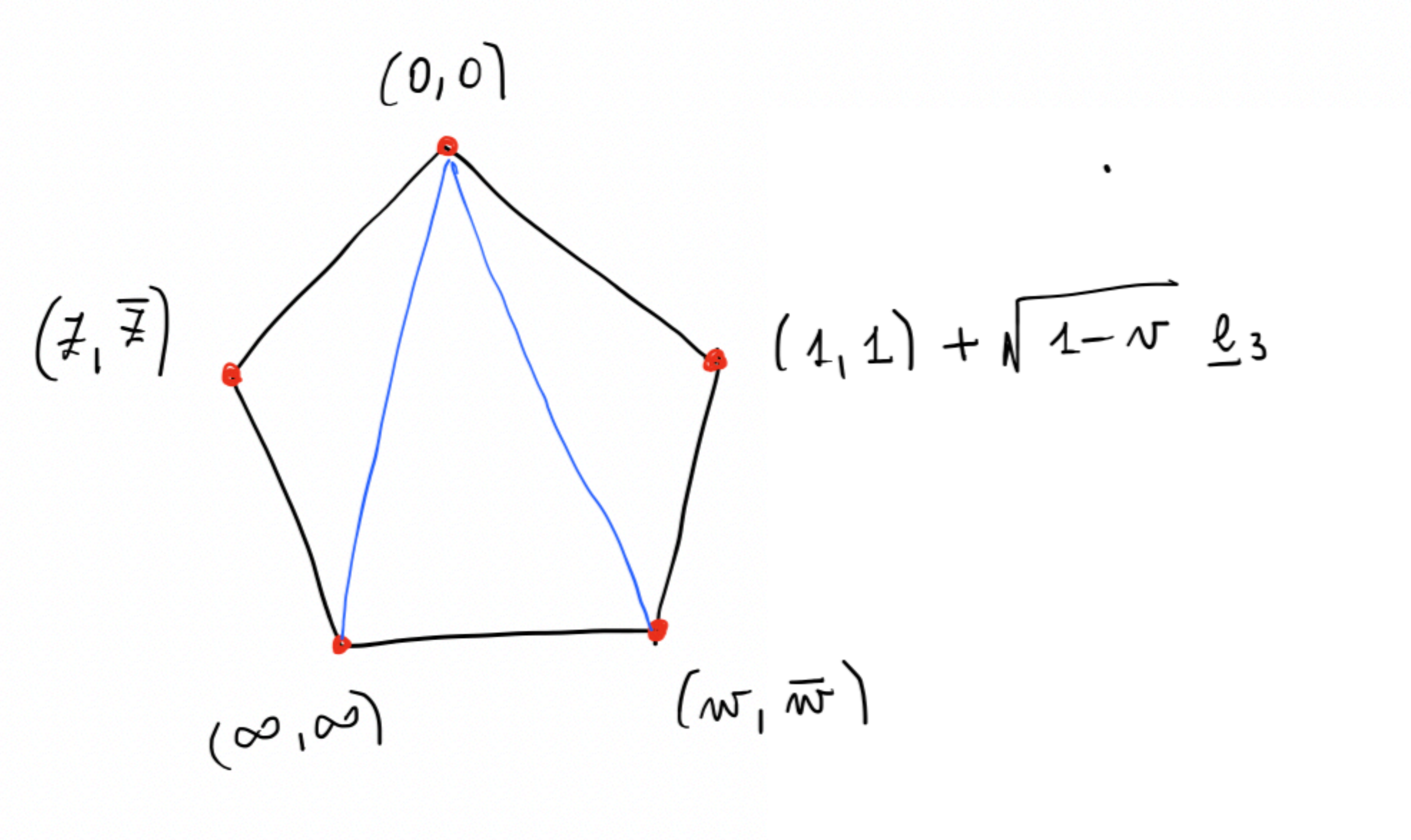
$$x = 2(s_1 + s_2), \;\; y = 2(s_1 + s_5) .$$

$$\partial_x \partial_y p_{h,k}(x,y) = e^{p_{h+1,k}-p_{h,k}} - e^{p_{h,k}-p_{h-1,k}}$$

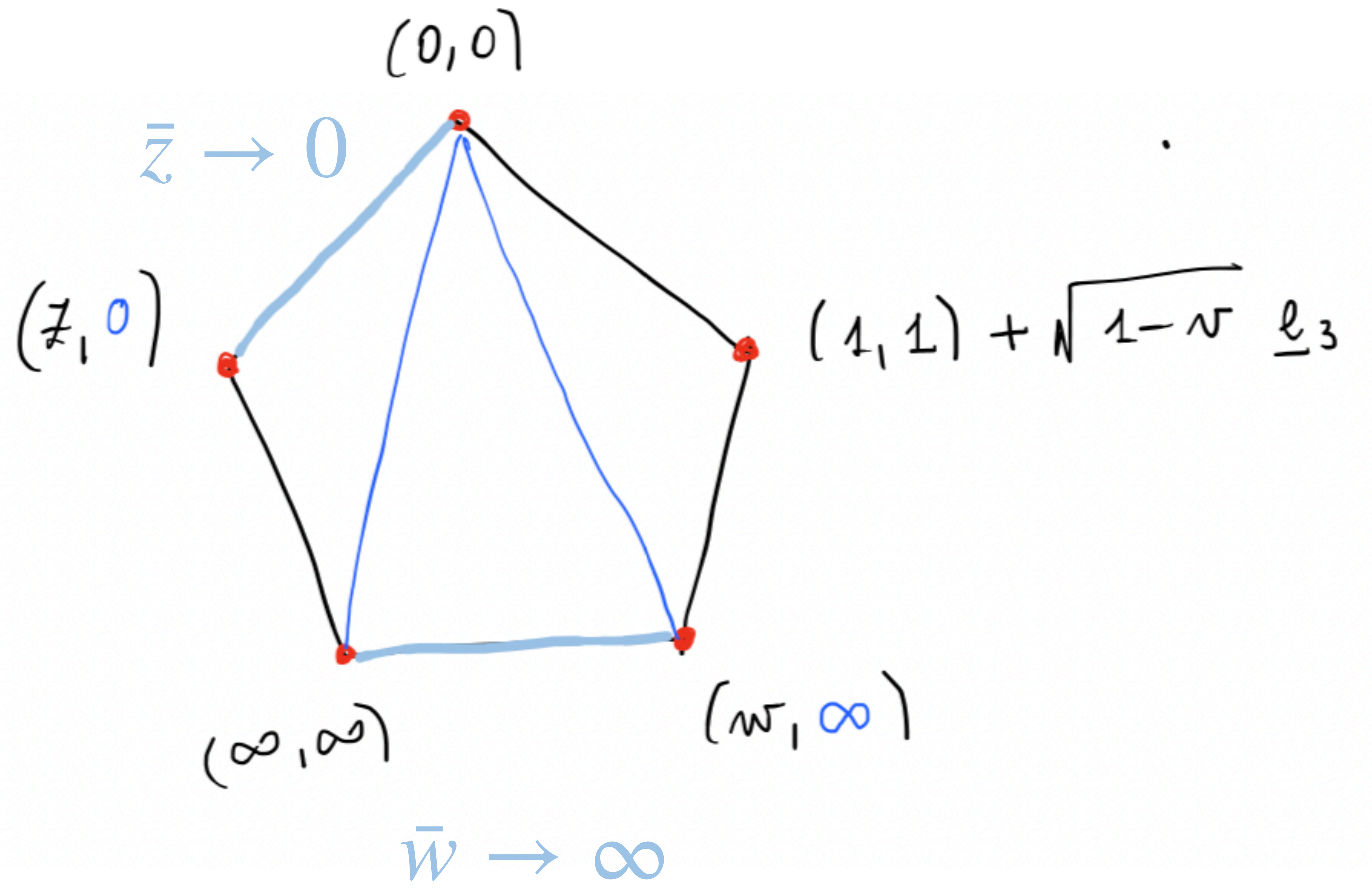
[*A.V. Mikhailov;* JETP Lett. 30 (1979) 414]

5-POINT: PERTURBATIVE DATA AKA THE STAMPEDE

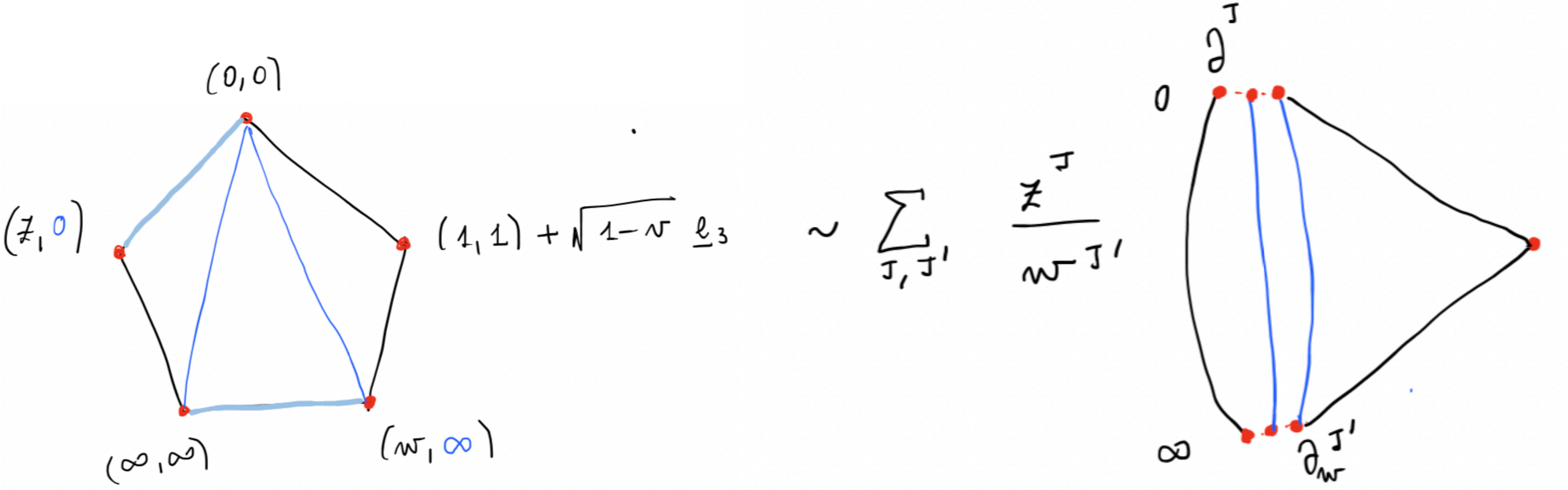
[E.O., Vieira; 2111.12131]



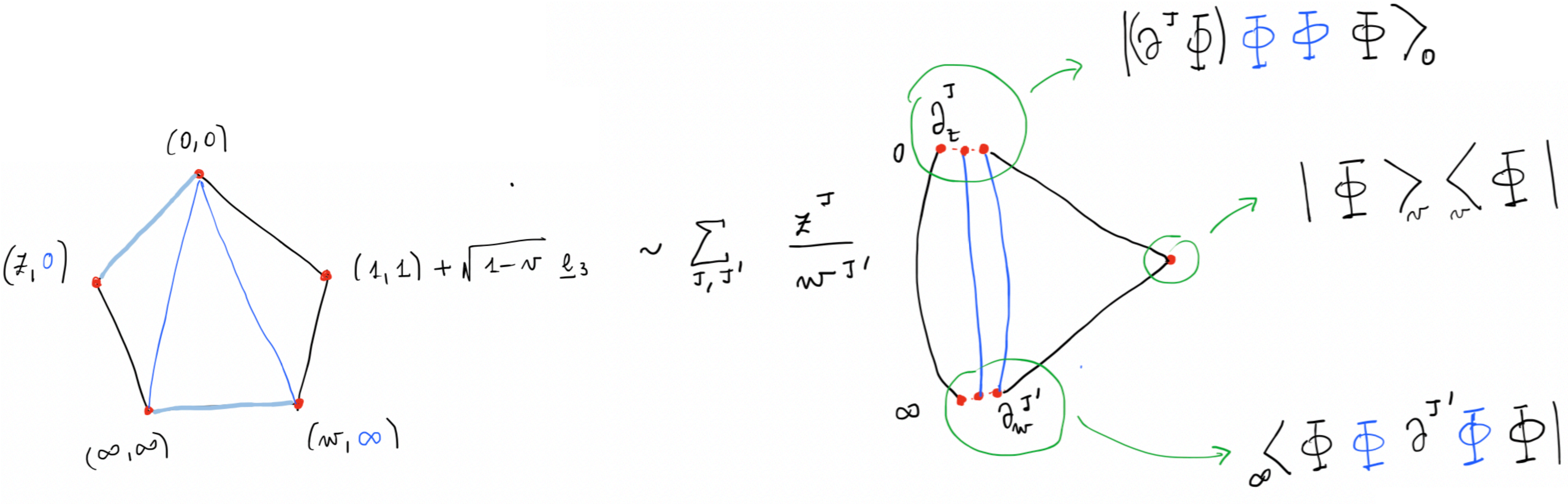
STEP 1: START WITH 2 LIGHT-LIKE EDGES



STEP 2: TAYLOR EXPANSION ON THE LIGHT-CONE

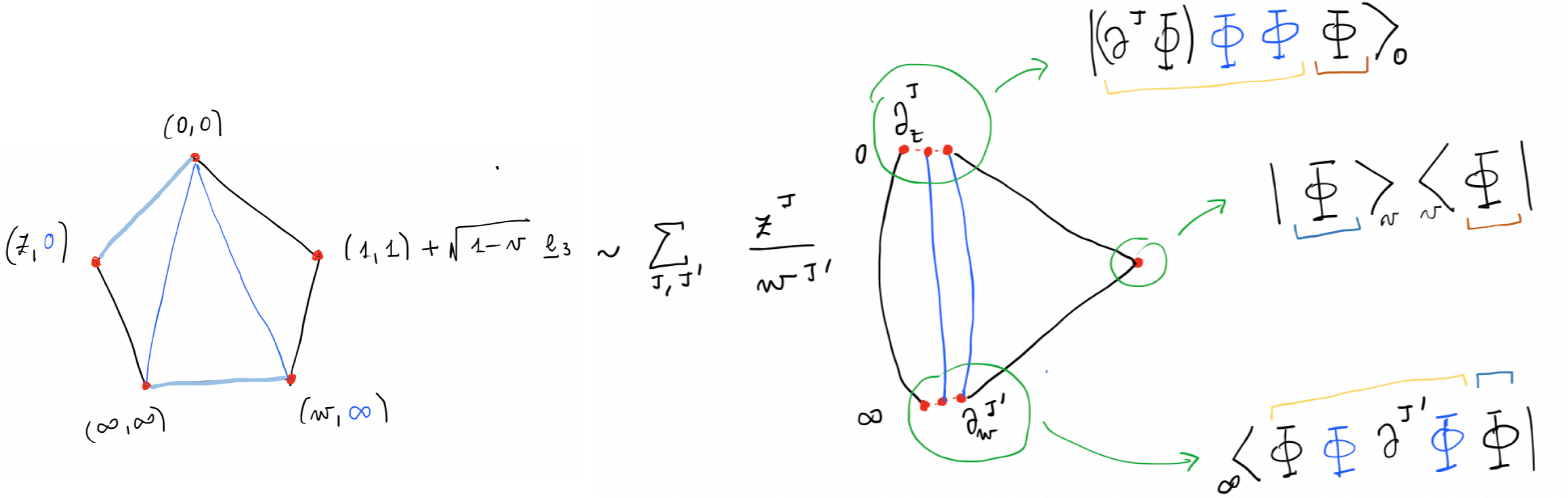


STEP 2: TAYLOR EXPANSION ON THE LIGHT-CONE



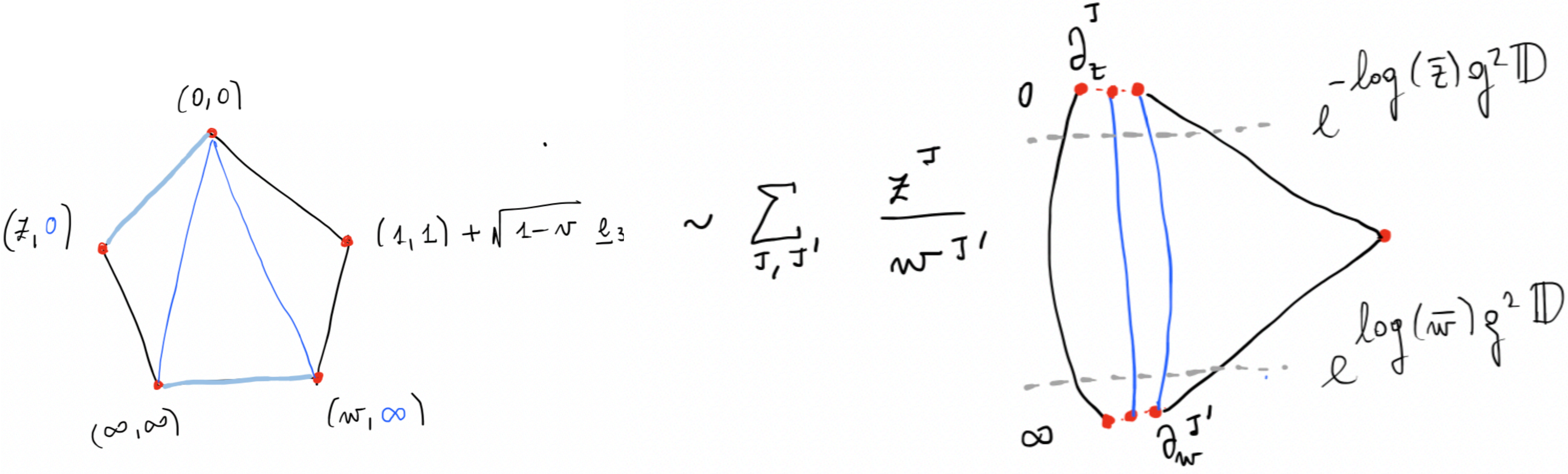
Open spin-chain states. ***Excitations*** = ***derivatives***

STEP 3: TAYLOR EXPANSION ON THE LIGHT-CONE



Open spin-chain states. ***Excitations = derivatives.***
Contractions of fields ***highlighted.***

STEP 3: STATES EVOLUTION (STAMPEDEING)



STEP 3: STATES EVOLUTION (STAMPEDEING)

$$\mathbb{D}\,|\,\partial^J\Phi\,\partial^{J'}\Phi\rangle = (S(J)+S(J'))\,|\,\partial^J\Phi\,\partial^{J'}\Phi\rangle - \sum_k \frac{1}{k}\,|\,\partial^{J-k}\,\Phi\partial^{J'+k}\Phi\rangle$$

[Beisert; 0307015]

$$_\infty\langle bottom;J'\,|\,e^{g^2\log\bar{w}\,\mathbb{D}}\left(\,|\Phi\rangle\langle\Phi|\otimes 1\right)e^{-g^2\log\bar{z}\,\mathbb{D}}\,|\,top;J'\rangle_0 \equiv (stampede)_{J,J'}$$

STEP 4: SYMBOL ANSATZ (BOOTSTRAP IN ACTION)

$$ {}_{\infty}\langle bottom; J' | \, e^{g^2 \log \bar{w}\, \mathbb{D}} \left(|\Phi\rangle\langle\Phi| \otimes 1 \right) e^{-g^2 \log \bar{z}\, \mathbb{D}} \, | top; J' \rangle_0 \equiv (stampede)_{J,J'} $$

$$ \sum_{J,J'} \frac{z^J}{w^{J'}} (stampede)_{J,J'} = 1+ $$

$$ +g^2 \left[\log \bar{z} F^{(1)}_{1,0}(z,w,v) + \log \bar{w} F^{(2)}_{0,1}(z,w,v) \right] + $$

$$ +g^4 \left[\log^2 \bar{z} \bar{F}^{(2)}_{2,0}(z,w,v) + \log \bar{z} \log \bar{w} F^{(2)}_{1,1}(z,w,v) + \log^2 \bar{w} F^{(2)}_{0,2}(z,w,v) \right] + \ldots $$

STEP 4: SYMBOL ANSATZ (BOOTSTRAP IN ACTION)

$$ {}_{\infty}\langle bottom; J' \,|\, e^{g^2 \log \bar{w}\, \mathbb{D}} \left(|\Phi\rangle\langle\Phi| \otimes 1 \right) e^{-g^2 \log \bar{z}\, \mathbb{D}} \,|\, top; J' \rangle_0 \equiv (stampede)_{J,J'} $$

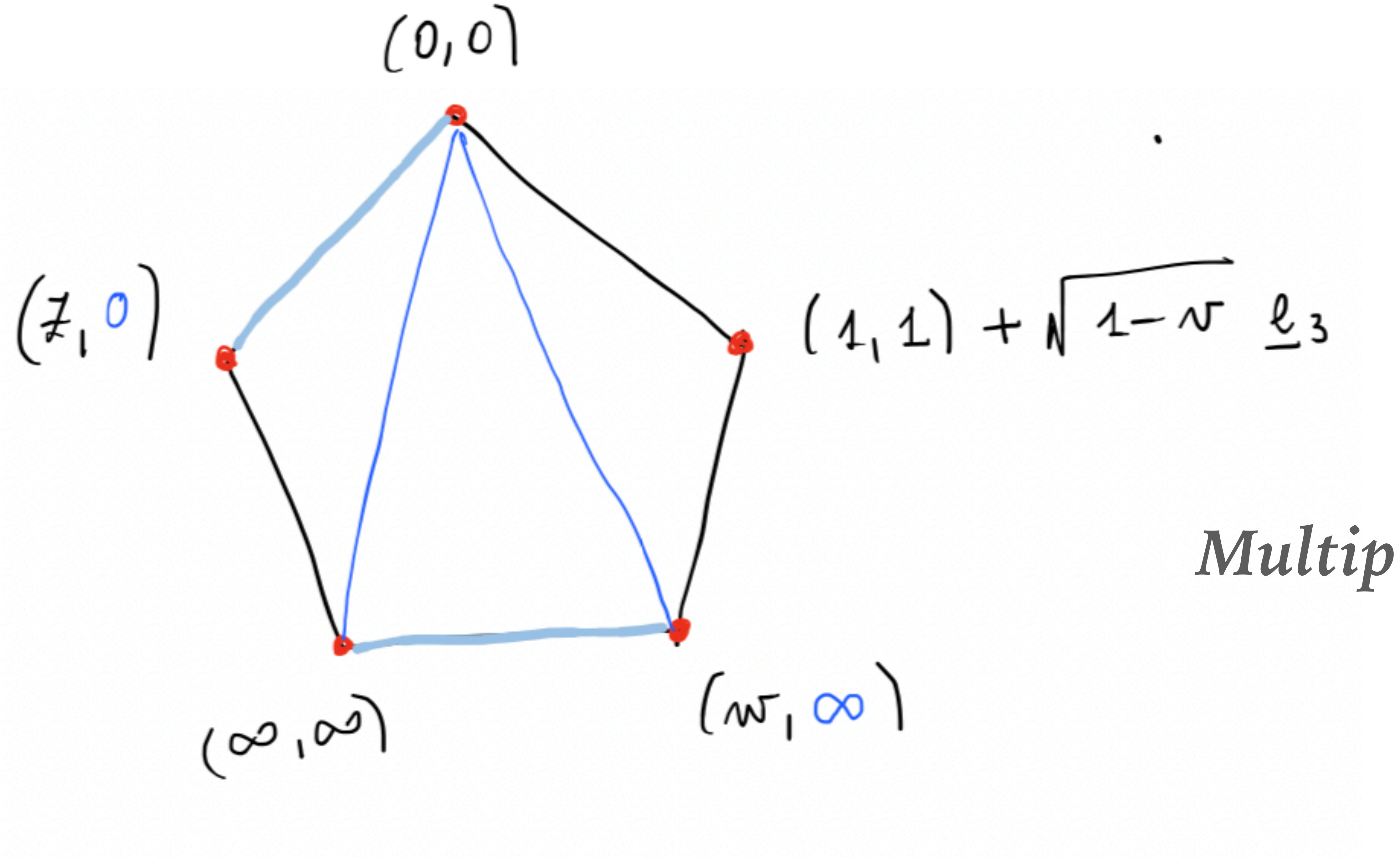
$$ F^{(m+n)}_{m,n}(z,w,v) = a_1 \otimes \cdots \otimes a_{m+n} $$

***Pure function** via length-$(m+n)$ symbol.*

$$ a_j \in \{z, w, v, 1-z, 1-w, 1-v\} $$

Symbol language = Goncharov's Polylogs

STEP 5: NULL-LIMIT DATA



$$\bar{z} \to 0, \bar{w} \to \infty$$

$$z \to \infty, w \to 0, v \to 1$$

Multiple channels = ***Multiple choice*** *of first 2 null edges*

$$g^{2(m+n)} \log^m \bar{z} \log^n \bar{w} \times F^{(m+n)}_{m,n}(z, w, v) \to L_{m,n}(t_1, t_2, t_4, t_5)$$