Boundaries and defect lines in 2d conformal field theories

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(Rational) conformal field theory

Euclidean CFT on the Riemann sphere

two commuting Virasoro algebras

$$T(z) = \sum_n L_{-n} z^{n-2}\,,\ \bar{T}(\bar{z}) = \sum_n \bar{L}_{-n} \bar{z}^{n-2}$$

central extension of the algebra generating the 2d conformal transformations - arbitrary analytic coordinate transformations $z \to \epsilon(z)\,, \bar{z} \to \bar{\epsilon}(\bar{z})$;

$z = x^1 + ix^2, \bar{z} = x^1 - ix^2$ - extend to $\mathcal{C}^2$ - cover Minkowski case as well

Steps:

- **Chiral data** – specify the **chiral algebra** $\mathbb{A}$ - Vir, or some extension, affine KM algebra, $\{J^a(z)\}$ etc., finite set $\mathcal{I}$ of irreps for a fixed value of the central charge, $\mathcal{V}_i = \mathcal{V}_i(h, c)\,,\ i \in \mathcal{I}$,

h.w. reps - primary field - lowest value h of L_0, $[L_0, L_{-n}] = nL_{-n}$, descendants, $h + n$

- characters

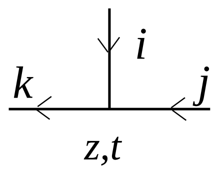
$$\chi_i(q) = \mathrm{tr}_{\mathcal{V}_i} q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_{n=0}^{\infty} \mathrm{mult}_{\mathcal{V}_i}(n) q^{h+n}$$

as functions of $q = e^{2\pi i \tau}$, Im $\tau > 0$ span a finite dim unitary rep of the modular group of the torus, $SL(2; \mathbf{Z})$.

• infinite symmetry - restricts the possible 3-point couplings

chiral vertex operators (CVO)

$$\phi^k_{i,j;t}(z):\ \mathcal{V}_i \otimes \mathcal{V}_j \to \mathcal{V}_k\,, \quad z \in \mathbb{C}$$



basis label $t = 1, 2, \ldots \mathcal{N}_{ij}{}^k$ = dim of vector space of CVO **fusion rule** multiplicities $\mathcal{N}_{ij}{}^k$,

$$\mathcal{V}_i \star \mathcal{V}_j = \oplus_k \mathcal{N}_{ij}{}^k\, \mathcal{V}_k$$

fusion algebra - assoc., commut., identity, $\mathcal{N}_{ij}^1 = \delta_{ij^*}$

$$\mathcal{N}_i \mathcal{N}_j = \sum_k \mathcal{N}_{ij}^k \mathcal{N}_k$$

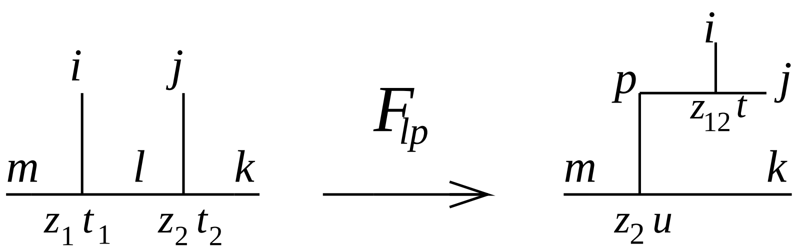
Verlinde formula:

$$\mathcal{N}_{ij}{}^k = \sum_{\ell \in \mathcal{I}} \frac{S_{il}\, S_{jl}\, S^*_{kl}}{S_{1l}} \ \in\ \mathbb{N},$$

the (symmetric) modular matrices $S_{ij} = S^{(k)}_{ij}$: representing the generator $\tau \to -1/\tau$

$n-$point correlators, multivalued functions

• **duality** matr. - braiding B and fusing F



F define the OPE of the chiral vertex operators, associativity - pentagon eqn

$$\sum FFF = FF$$

B - hexagon eqn,
Moore-Seiberg torus identity for the 2-point chiral correlator - expresses the 1-point mod matrix $S_{ij}(p)$ in terms of F $\Rightarrow$ implies Verlinde formula;

Spectral data - 2d theory

- Spectrum organized by irreps of **two copies** of $\mathbb{A}$:

$$\mathcal{H}_P = \oplus_{j,\bar{j}\in\mathcal{I}} Z_{j\,\bar{j}}\, \mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}}$$

$Z_{j\,\bar{j}}$: multiplicities of $(j, \bar{j})$, determined by a consistency condition:
the torus partition function

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}} q^{L_0 - c/24} q^{*\bar{L}_0 - c/24} = \sum_{j,\bar{j}\in\mathcal{I}} Z_{j\bar{j}}\ \chi_j(\tau)\, \chi_{\bar{j}}(\tau)^*$$

Z must be a modular invariant function of the modular parameter τ, $MZM^* = Z$, with $Z_{11} = 1$.
For a fixed central charge there might be several solutions - with different physical operator content.

- Physical fields, chiral factorisation of correlators

$$\Phi_{(i,\bar{i})}(z,\bar{z}) = \sum_{k,\bar{k},t,\bar{t}} d_{(i,\bar{i})(j,\bar{j})}^{(k,\bar{k});t,\bar{t}}\ \phi_{i,j;t}^{k}(z) \otimes \phi_{\bar{i},\bar{j};\bar{t}}^{\bar{k}}(\bar{z})$$

OPE coeffs $d_{(i,\bar{i})(j,\bar{j})}^{(k,\bar{k});t,\bar{t}}$ restricted by locality requirements (symmetry)

$$\sum d\,d\ F\,\bar{F} = d\,d$$

In general - "diagonal" solution $Z_{j,\bar{j}} = \delta_{j,\bar{j}}$,
relative OPE coeffs in the non-diagonal cases - computed only for $sl(2)$ since F known - quantum group 6j symbols (pentagon eqn solved in terms of basic ${}_4\phi_3$ hypergeometric function)

This spectral data – encoded in **graphs** G,
ADE Dynkin diagrams classify mod inv partition functions in the $\widehat{sl}(2)_k$ related conformal theories [CIZ], [Pasquier], higher rank generalisations [K, DiFZ]

the sets of A-D-E exponents parametrise the scalar fields at any value of the level $k = h - 2$
"Exponents" : diagonal part of the spectrum

$$\mathcal{E} = \{j \in \mathcal{I} | j = \bar{j}, Z_{jj} \neq 0\}$$

counted with a multiplicity Z_{jj} ,

• parametrise the eigenvalues $\gamma^j = \frac{S_{2j}}{S_{1j}}$ of the adjacency matrices $G_{ab} = 2\delta_{ab} - C_{ab}$

$$G_{ab} = \sum_{j \in \mathcal{E}} \frac{S_{2j}}{S_{1j}} \psi_a^j \psi_b^{j*}$$

$a \in \mathcal{V}$ - nodes of the graph, ψ - unitary (eigenvector) matrix

in the diagonal case G identified with the fundamental fusion matrix $\mathcal{N}_2$ (here integrable reps $1 \leq r \leq k + 1$);

using that the ratios $\frac{S_{ij}}{S_{1j}}$ furnish 1- dim reps of Verlinde fusion algebra, and the unitarity of the eigenvector matrix ψ_a^j - one generates n_{ja}^b, $n_2 = G$

$$n_i \, n_j = \sum_k \mathcal{N}_{ij}^k \, n_k \, , \quad i, j \in \mathcal{I} \ (*)$$

a matrix representation of the Verlinde fusion algebra with integer valued matrix elements.

• strc constants of **Pasquier algebra** coincide with relative (the same Coxeter number $h = k + 2$) scalar field OPE coeff

$$d_{(i,i)(j,j)}^{(k,k)} = M_{(i;\alpha)\,(j,\beta)}^{(k,\gamma)} = \sum_{a \in \mathcal{V}} \frac{\psi_a^{(i;\alpha)}\,\psi_a^{(j,\beta)}\,\psi_a^{(k,\gamma)\,*}}{\psi_a^1} \qquad (^{**})$$

$(i,i) \equiv (i;\alpha) \in \mathcal{E}\,,\ \alpha = 1,\ldots,Z_{ii}$

note a closely related paper [**Todorov, Rehren, Stanev**]

• **Boundary CFT** – an alternative "chiral" approach in which these graphs and the related algebraic structures become manifest

• "physical" interpretation of the sets $\mathcal{V}$ of graph nodes parametrise conformally invariant boundary conditions - matrix representation (*) recovered

• formula for OPE coeffs (**) derived

CFT in the presence of boundaries

• should restrict to conformal transformations which preserve the boundaries;
(upper) half plane H_+ with the real axis as a boundary - real analytic coordinate transformations, $\epsilon(z) = \bar{\epsilon}(\bar{z})$ for real $z = \bar{z}$)
only **one copy** of the chiral algebra

$$T(z) = \bar{T}(\bar{z})$$

the only algebra acts on the primary fileds as a *sum* of the two differential operators in z and $\bar{z}$ e.g. L^H_{-1} acts as $\partial_x = \partial_z + \partial_{\bar{z}}$;
non-vanishing 1-point function

$$\langle 0|\phi_{h,\bar{h}}(z,\bar{z})|0\rangle = \frac{C_h}{(z-\bar{z})^{2h}}\delta_{h,\bar{h}} = \frac{C_h}{(2y)^{2h}}\delta_{h,\bar{h}}\,, \quad \text{Im } z > 0$$

Can also expect physical fields living on the boundary, separating different boundary conditions.

extended to theories with additional symmetry, like the WZW models - the "gluing" of the right and left currents - can be done up to some automorphism ω so that on the real line $J^a = \omega(\bar{J}^a)$.

CFT on a cylinder - strip $L \times T$, $w = w + T$, boundaries labelled by a, b

• The partition function $Z_{b|a}$ is now linear in the characters since the space of states

$$\mathcal{H}_{b|a} = \oplus_{j\in\mathcal{I}} n_{ib}{}^{a}\, \mathcal{V}_i$$

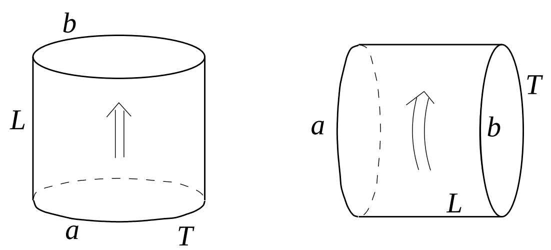
decomposes into representations of the only chiral algebra

$$Z_{b|a} = \mathrm{Tr}_{\mathcal{H}_{ba}} e^{-TH_{ba}} = \mathrm{Tr}_{\mathcal{H}_{ba}} q^{L_0 - \frac{c}{24}} = \sum_{i\in\mathcal{I}} n_{ia}{}^{b} \chi_i(\tau) \ \ ,$$

$$q = e^{-\pi \frac{T}{L}} = e^{2\pi i \tau}\,, \qquad \tau = i\frac{T}{2L} - \text{pure imaginary}$$

• the partition function $Z_{b|a}$ represents a periodic time evolution $e^{-TH_{ba}}$, on the cylinder, whence the trace, with boundary conditions a, b.

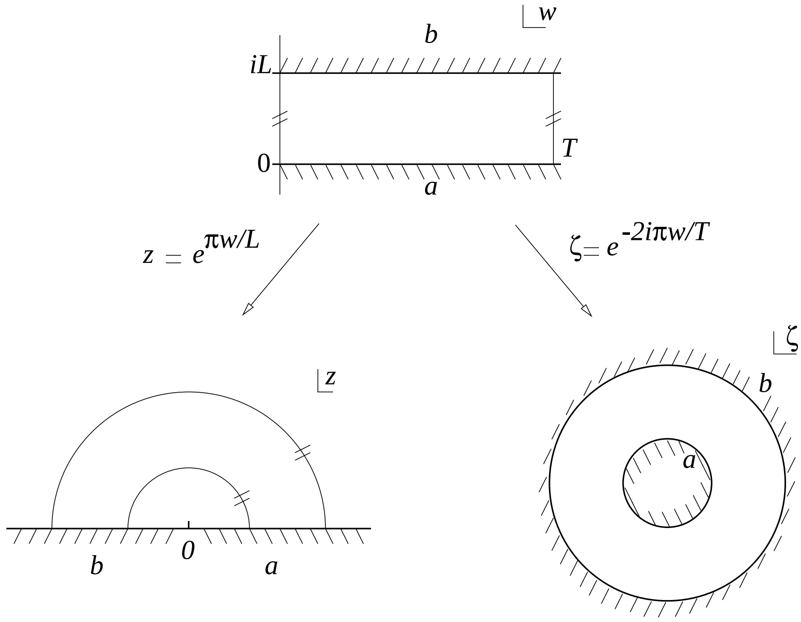
• We can compute the same partition function in a different alternative way,



(a) (b)

namely, as a matrix element of the evolution operator $e^{-L\, H^{(cyl)}}$ between some boundary states $|a\rangle$ and $\langle b|$, yet to be determined.

$$Z_{b|a}(\tau) = \langle b| e^{-LH^{(\mathrm{cyl})}} |a\rangle = \langle b| e^{\frac{-\pi i}{\tau}(L_0^P + \overline{L}_0^P - \frac{c}{12})} |a\rangle$$



Hamiltonians - generators of translations in the two directions - the two maps $w \to \zeta$ and $w \to z$ convert these generators into zero Vir modes on the plane, or the half-plane respectively; shifts by $\sim c$ due to inhomogeneous transformation of the stress tensor T, Schwartz derivative terms.

• The two alternative ways of computation (related by a modular transformation) - consistency condition on the multiplicities ${n_{ib}}^a \in \mathbf{Z}_{\geq 0}$, **Cardy eqn**, adaptation of a string theory argument;

boundary states $|a\rangle$ determined as solutions of the gluing condition

$$(L_n - \bar{L}_{-n})|a\rangle \ = 0$$

solutions - linear combinations

$$|a\rangle = \sum_{j\in\mathcal{E}} \frac{\psi_a{}^j}{\sqrt{S_{1j}}}\ |j\rangle\!\rangle$$

Ishibashi states (diagonal spectrum on the plane)

$$|j\rangle\!\rangle = \sum_N |j,N\rangle \otimes |j,N\rangle$$

one computes the trace on the Ishibashi states

$$\langle\!\langle k,\alpha|(\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|j,\beta\rangle\!\rangle = \delta_{kj}\delta_{\alpha\beta}\chi_j(-1/\tau)$$

$$\Rightarrow n_{ia}{}^b = \sum_{(j,\alpha)\in\mathcal{E}} \frac{S_{ij}}{S_{1j}}\ \psi_a^{(j,\alpha)}\psi_b^{(j,\alpha)\,*} = n_{i^*b}{}^a$$

ψ - unitary - **complete set of bound. conds**

$$n_{1a}{}^b = \delta_{ab}\,, \quad n_{j^*} = n_j^T$$

The multiplicities $\{n_i = n_{ia}^b\,,\ i\in\mathcal{I}\}$, form a representation of the Verlinde fusion algebra

$$n_i\, n_j = \sum_{k\in\mathcal{I}} \mathcal{N}_{ij}{}^k\, n_k \qquad\qquad (*)$$

Thus the classification of a complete set of conformal boundary conds is reduced to the classification of the **NIMreps**, the non-negative integer valued matrix representations of the Verlinde algebra.

Associate graph G, $a,b\in\mathcal{V}$ (collection of graphs), the vertices of which parametrise the conf. inv. bound. conds. Diagonal case: $n_i = \mathcal{N}_i\,, \mathcal{I} = \mathcal{V} = \mathcal{E}$

• In sl(2) related RCFT – reduces to the classification of the symmetric, irred, non-negative integer valued matrices of spectrum $|\gamma^j| < 2$, with the tadpole graph $A_{2n}/\mathbf{Z}_2$ discarded; ⇒ parallels the ADE classification of the corresponding modular invariants;

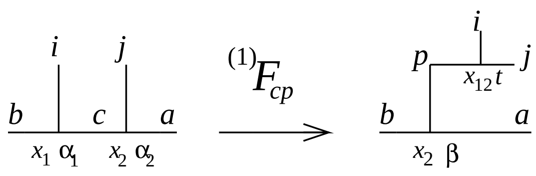
• Many examples solved;
still open: e.g., the general case of conformal embeddings, nead branching coeffs. n_{j1}^a.

Fields and relations in the boundary CFT

Boundary conditions (a, b) created by insertions of fields on the boundary (boundary fields)

$${}^a\Psi^b_{j;\beta}(x),\ \beta = 1, 2, \ldots n^a_{jb},\ \ x = \text{Re } z,\ z \in H^+.$$

OPE of boundary fields - new "3j symbols", ${}^{(1)}F$,



• Associativity of the product ⇒ **pentagon identity**

$$\sum F\ {}^{(1)}F\ {}^{(1)}F = {}^{(1)}F\ {}^{(1)}F \qquad (P1)$$

where F are the $6j$-symbols. In the diagonal case F and ${}^{(1)}F$ are identified (possibly up to a gauge), recovering the conventional pentagon identity

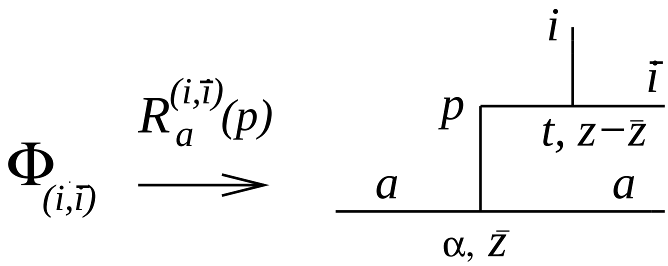
$$\sum F\ F\ F = F\ F \qquad (P0)$$

Thus the constants ${}^{(1)}F$ - determine the 3-point boundary field function.

Bulk fields

Half-plane bulk fields $\Phi^H_{(j,\bar{j})}(z,\bar{z})$ - compositions of chiral vertex operators at points z and $\bar{z}$ obtained by reflection in the real axis

• for small distance $z-\bar{z}=2i\,y\approx 0$, i.e., approaching the boundary - decomposes into boundary fields



new const, $R=\sum C\ {}^{(1)}F$ – the bulk-boundary **reflection coeffs**

$$\Phi^H_{(i,\bar{i})}(z,\bar{z})|_a =$$

$$\sum_{p\,,a\,,\alpha\,,t} R^{(i,\bar{i};t)}_{(a;\alpha)}(p)\ \langle p|\phi^p_{i\,\bar{i}^*;t}(2iy)|\bar{i}^*\rangle\ \ {}^a\Psi^a_{p,\alpha}(x) + \ \ldots$$

which fields appear appear dictated by the multiplicities $N_{i\bar{i}}{}^{p} n_{pa}{}^{b}$

• comparing different decompositions of correlators with bulk and boundary fields related by braiding - two bulk-boundary eqs for the constants R [Cardy-Lewellen]

• one of these eqs - from the 2-point bulk correlator

$$\Phi_{(i,\bar{i})}(z_1,\bar{z}_1)\Phi_{(j,\bar{j})}(z_2,\bar{z}_2)$$

written as a linear combination of 4-point chiral blocks:

- either use the OPE expansion for small distances $z_{12}, \bar{z}_{12}$ and then decompose to boundary fields

- or, take both fields close to the boundary $z_i - \bar{z}_i \to 0$

- the two representations are related by braid transformation B, $\to$ get a quadratic relation for the constants $R_a^{(i,i^*)}(p)$

$$R\ R \sim \sum d\, F\, R$$

The special choice $p = 1$ for $R_a^{(i,i^*)}(1) \sim \frac{\psi_a^i}{\psi_a^1}$ in this eqn corresponds in terms of the 4-point correlators correlator to the contribution in the leading order of the identity boundary field
$\Rightarrow$ - one recovers the Pasquier algebra formula for the scalar bulk 3-point OPE coeffs.

$$\frac{\psi_a^i}{\psi_a^1}\,\frac{\psi_a^j}{\psi_a^1} = \sum_k \; d_{(i,i)(j,j)}^{(k,k)}\,\frac{\psi_a^k}{\psi_a^1}$$

$R_a^{(i,i^*)}(1) \sim \frac{\psi_a^i}{\psi_a^1}$ - 1-dim reps of Pasquier algebra

this derivation - first in the example D_5

[Pradisi,Sagnotti, Stanev]

• On the other hand the second bulk-boundary eqn - linear in $R_a^{(i,i^*)}(p)$ - in the diagonal case yields an expression in terms of the 6j symbols F [Runkel];

one observes that actually this expression is proportional to the 1-point mod. matrix

$R_a^{(i,i^*)}(p) \sim S_{ai}(p)$

and that the CL eqn itself is equivalent to the Moore-Seiberg torus identity; the second CL eqn is a consequence

⇒ no new constants in the diagonal boundary theory, all the data provided by the usual chiral CFT formulation.

• Boundary fields can be defined as linear combinations of standard CVO - tensoring the CVO with an intertwining operator

$$P_{ab,cb}^{k,\alpha;j,\gamma} = |e_{ab}^{k,\alpha}\rangle\langle\, e_{cb}^{j,\gamma}|\,, \qquad V^j \to V^k$$

in auxiliary finite dim spaces, of dim $m_j = \sum_{a,b} n_{ja}{}^b$

$$ {}^a\Psi^c_{i,\beta;I}(z) \;=\; \sum_{j,k,t}\, \sum_{b,\alpha,\gamma} {}^{(1)}F_{ck} \begin{bmatrix} i & j \\ a & b \end{bmatrix}^{\alpha\, t}_{\beta\, \gamma} \phi^k_{ij,t;I}(z) \otimes P_{ab,cb}^{k,\alpha;j,\gamma}$$

reps of Ocneanu "graph quantum symmetry" (or weak C^*-Hopf algebra [BSz]), $\mathcal{A}$, associated to any solution for $\{n_i\,, i\,\in\,\mathcal{I}\}$; ${}^{(1)}F$ and F - the 3j and 6j symbols; together with its dual algebra $\widehat{\mathcal{A}}$ structure.

States ${}^a\Psi^c_{j,\beta}(0)\,|0\rangle \otimes |e_{cc}^1\rangle = \phi^j_{j1}(0)|0\rangle \otimes |e_{ac}^{j,\beta}\rangle =: |j\,,\beta\rangle$

Half-plane bulk fields - compositions of such generalised CVO.

• Up to now - we have exploited essentially the scalar field spectrum

Question: do these algebraic structures, e.g., the determination of the OPE coeffs generalise to the non-trivial (integer) spin fields described by the modular invariants?

another motivation - physical interpretation of the dual structure of Ocneanu DTA – based on graphs generalising the ADE diagrams, with a set of vertices $\tilde{\mathcal{V}} \ni x, |\tilde{\mathcal{V}}| = \sum_{i,j} Z_{ij}^2$.

One is led to construct torus partition functions but with the periodic boundary conditions modified by the insertion of operators X_x - **topological defects** ("twists, defect lines, seams") along non contractible cycles [VB-Zuber],

by definition:

$$[L_n, X] = [\bar{L}_n, X] = 0$$

Vir operators - generators of infinitesimal diffeomorphisms, this condition ensures that each operator X is invariant under a distorsion of the line to which it is attached.

The solution

$$X_x = \sum_{j\bar{j},\alpha,\alpha'} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')}}{\sqrt{S_{1j}S_{1\bar{j}}}} P^{(j,\bar{j};\alpha,\alpha')}$$

linear combinations of projectors

$$P^{(j,\bar{j};\alpha,\alpha')} = \sum_{\mathbf{n},\bar{\mathbf{n}}} (|j,\mathbf{n}\rangle \otimes |\bar{j},\bar{\mathbf{n}}\rangle)^{(\alpha)} (\langle j,\mathbf{n}| \otimes \langle \bar{j},\bar{\mathbf{n}}|)^{(\alpha')}$$

$$(\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha')} \to (\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha)} \quad \alpha,\alpha' = 1,\cdots, Z_{j\bar{j}}$$

$P^{(j,\bar{j};\alpha,\alpha')}$ play the rôle of Ishibashi states,

"exponents" $\tilde{\mathcal{E}} = \{j,\bar{j},\alpha,\alpha'\}$, $|\tilde{\mathcal{E}}| = \sum_{j,\bar{j}} Z^2_{j\bar{j}}$

$$X_{x=1} = \mathrm{Id} \Rightarrow \Psi_1^{(j,\bar{j};\alpha,\alpha')} = \sqrt{S_{1j} S_{1\bar{j}}}\ \delta_{\alpha\alpha'}$$

Partition function in the presence of operators X computed in two ways, analogous to the derivation of Cardy eq, now on a "double" cylinder $T \times 2L$ with identified ends.

First the trace of the translation operator in the "space" direction –

$$Z_{x|y} = \mathrm{tr}\,(X_x^+ X_y\ e^{-2LH}) = \mathrm{tr}_{\mathcal{H}_P}(X_x^+ X_y\ \tilde{q}^{L_0 - c/24}\, \tilde{q}^{\bar{L}_0 - c/24})\ ,$$

using that

$$\mathrm{tr}_{\mathcal{H}_P}(P^{(j,\bar{j};\alpha,\alpha')} \tilde{q}^{L_0 - c/24}\, \tilde{q}^{\bar{L}_0 - c/24}) = \chi_j(\tilde{q})\, \chi_{\bar{j}}(\tilde{q})\ \delta_{\alpha\alpha'}$$

Next, mapping $w \to z = e^{\frac{\pi w}{L}}$, compute the trace of the time evolution operator in the Hilbert space

$$\mathcal{H}_{x|y} = \oplus_{i,\bar{i}\in\mathcal{I}}\, \widetilde{V}_{i\bar{i}^*;\,x}{}^{y}\, \mathcal{V}_i \otimes \overline{\mathcal{V}}_{\bar{i}}\ ,$$

$\widetilde{V}_{i\bar{i};\,x}{}^{y}$ – multiplicities, $\widetilde{V}_{i\bar{i}^*;\,1}{}^{1} = Z_{i\bar{i}}$.

$$Z_{x|y} = \mathrm{tr}\,_{\mathcal{H}_{x|y}}\, q^{L_0 - c/24}\, q^{\bar{L}_0 - c/24} = \sum_{i,\bar{i}\in\mathcal{I}} \widetilde{V}_{i\bar{i};\,x}{}^{y}\ \chi_i(q)\,\chi_{\bar{i}}(q)\,.$$

Identify the two expressions for $Z_{x|y}$ using the modular transformation of the characters $\Rightarrow$

$$\widetilde{V}_{i\bar{i};\,x}{}^{y} = \sum_{j,\bar{j},\alpha,\alpha'} \frac{S_{ij}}{S_{1j}}\,\frac{S_{\overline{i}\overline{j}}}{S_{1\bar{j}}}\ \Psi_x^{(j,\bar{j};\alpha,\alpha')}\ \Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}\,, \qquad \widetilde{V}_{11;\,x}{}^{y} = \delta_{xy}\,,$$

spectral decomposition, Ψ unitary, implies new set of **NIMreps**

$$\widetilde{V}_{ii'}\widetilde{V}_{jj'} = \sum_{k,k'} \mathcal{N}_{ij}{}^{k}\,\mathcal{N}_{i'j'}{}^{k'}\,\widetilde{V}_{kk'} \qquad (**)$$

In general $\quad Z_{y|z} = \sum_x\ \widetilde{N}_{yx}{}^{z}\ Z_{1|x}\,,$

$$\widetilde{N}_{yx}{}^{z} = \sum_{j,\bar{j};\alpha}\ \sum_{\beta,\gamma}\ \Psi_y^{(j,\bar{j};\,\alpha,\beta)}\,\frac{\Psi_x^{(j,\bar{j};\,\beta,\gamma)}}{\Psi_1^{(j,\bar{j})}}\,\Psi_z^{(j,\bar{j};\,\alpha,\gamma)\,*}$$

The matrices $\widetilde{N}_x = \{\widetilde{N}_{yx}{}^{z}\}$ span an associative algebra (non-commutative if some $Z_{j\bar{j}} > 1$), realised as well by the operators X_x

$$X_x\ X_y = \sum_z \widetilde{N}_{xy}{}^{z}\ X_z$$

"fusion algebra of defects" (Ocneanu graph algebra)

• Examples of solutions:

diagonal case $x \in \mathcal{I}$, $\widetilde{V}_{ij} = \mathcal{N}_i \mathcal{N}_j$, e.g., the diagonal case (A_2, A_3), of the minimal theories, i.e., Ising model, with Vir irreps labelled by the central charge $c = \frac{1}{2}$ and the dimensions $h_1 = 0\,, h_2 = \frac{1}{16}\,, h_3 = \frac{1}{2}$; $(x = 1, 2, 3)$

$$Z_{1|x} = \sum_j \frac{S_{xj}}{S_{1j}} \, |\chi_j(\tilde{q})|^2 = \sum_{i,j} N_{ij}^x \, \chi_i(q)\, \chi_j(q)^*$$

$$Z_{1|1} = |\chi_1(\tilde{q})|^2 + |\chi_3(\tilde{q})|^2 + |\chi_2(\tilde{q})|^2 = |\chi_1(q)|^2 + |\chi_3(q)|^2 + |\chi_2(q)|^2$$

$$Z_{1|3} = |\chi_1(\tilde{q})|^2 + |\chi_3(\tilde{q})|^2 - |\chi_2(\tilde{q})|^2 = \chi_1(q)\chi_3(q)^* + \text{c.c.} + |\chi_2(q)|^2$$

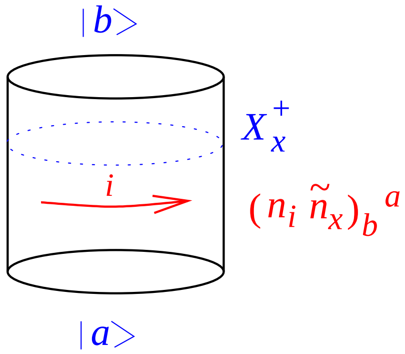
$$Z_{1|2} = \sqrt{2}(|\chi_1(\tilde{q})|^2 - |\chi_3(\tilde{q})|^2) = (\chi_1(q) + \chi_3(q))\chi_2(q)^* + \text{c.c.}$$

$Z_{1|1}$ - the mod invariant obtained as a system with periodic boundary conditions, or $V_{ij;1}^1 = N_{ij}^1 = \delta_{ij}$.

The second $Z_{1|3} = Z_{1|\epsilon(1)}$ reproduces the simplest of the torus partition functions with $\mathbf{Z}_2$ twisted boundary conditions, and is an example of a defect related to a group.

Describes the operator content - half-spin operators, appearing in the OPE of an order σ and disorder μ scalar operator, both of dimension $\frac{1}{16}$; this can be now interpreted as an OPE of one scalar operator in the presence of a defect $\sigma\, X_\sigma\, \sigma = \sigma\, \mu$.

• On a cylinder - both defects and boundaries



$$X_x|a>=\sum_c \tilde{n}_{ax}{}^c|c>$$

i.e., the defects map conformal boundary conditions into conformal boundary conditions, where

$$\tilde{n}_{ax}{}^c = \sum_{j,\alpha,\beta} \psi_a^{(j,\alpha)} \frac{\Psi_x^{(j,j;\alpha,\beta)}}{\sqrt{S_{1j}S_{1\bar{j}}}} \psi_c^{(j,\beta)*} \ .$$

Repeating the derivation of Cardy equation one finds a partition function a cylinder with one defect line X_x and boundary states a and b

$$Z_{b|x\,a} \sum_{i\in\mathcal{I}} (n_i \tilde{n}_x)_a{}^b \ \chi_{\bar{i}}(q)$$

The new set of multiplicities provide a representation of $\tilde{N}$

$$\sum_b \tilde{n}_{ax}^b\, \tilde{n}_{by}^c = \sum_z \tilde{N}_{xy}^z\, \tilde{n}_{az}^c\,,$$

the multiplicities $\tilde{n}_x\,,\widetilde{N}_x$ – dual analogs of $n_j\,,\mathcal{N}_j$, complete the combinatorial data in the construction of the Ocneanu quantum algebra.

• Defects on the cylinder - interpreted as source of boundary perturbations (deformations) of conformal models [Graham, Watts].

Up to now - defects treated in a formal purely algebraic way - explicit realisations?

• Examples of WZW defects

Diagonal defects, γ - h.w. of integrable representation of KM algebra

$$X_\gamma = \sum_\mu \frac{S_{\gamma\mu}}{S_{\mathbf{1}\mu}} P^{(\mu,\mu)}$$

Explicit realisation - by Wilson loop operators,

[Bachas, Gaberdiel (2004)]; earlier in condensed matter physics such operators encountered [Affleck et al] in the CFT interpretation of Kondo effect (screening of magnetic impurities in a metal).

$$\mathcal{O}(\lambda; R) = \mathrm{Tr}_R P\ \exp(i\lambda \oint_C dx^+ J^a t^a)\,,\ \lambda = \lambda^* = -\frac{1}{k}$$

$$J(x^+) = -ik\partial_+ g g^{-1}\,,\ \bar{J}(x^-) = ikg^{-1}\partial_- g\,,\ g(x^+, x^-) \in G$$

WZW currents, generate the symmetry of the WZW model

$$g \to u(x^+)^{-1}\, g\, \bar{u}(x^-)$$

and transform as (gauge fields)

$$J \to u^{-1} J u + ik\, u^{-1}\partial_+ u$$

Quantum operators - need regularisation - analysed perturbatively to some order in the powers of λ

$k \to k + h^\vee$

Eigenvalues in $\mathcal{H}_\mu \otimes \bar{\mathcal{H}}_\mu$

$$\mathcal{O}(\lambda^*; R_\gamma) = \frac{S_{\gamma\mu}}{S_{\mathbf{1}\mu}}$$

Non-perturbative quantisation - [Alekseev, Monier (2007)]

• Wilson loops - identified with central elements in (completion) of the universal enveloping algebra $U(\hat{g})$ of the affine KM algebra $\hat{g}$; - "generalised Casimir operators" constructed by [Kac (1984)].

• boundary perturbations - RG flows relating conformal boundary conditions.

• Correlators in the presence of defects

$$\langle 0|\Phi_{(J^*;\beta)}\Phi_{(I^*;\alpha)}X_x\,\Phi_{(I;\alpha')}\Phi_{(J;\beta')}\,X_x^\dagger|0\rangle$$

For $z_{12}\to 0$ we use the standard expansion, the defect contributes by its eigenvalue; alternatively in

$$\langle 0|\Phi_{(I^*;\alpha)}X_x\,\Phi_{(I;\alpha')}\Phi_{(J;\beta')}\,X_x^\dagger\,\Phi_{(J^*;\beta)}|0\rangle$$

we need to compute the OPE of $\qquad \Phi_{(J^*,\alpha)}X_x\Phi_{(J,\beta)}$

take the leading, identity field contribution and use that 2-point function

$$\langle 0|\Phi_{(J^*,\alpha)}X_x\Phi_{(J',\beta)}|0\rangle = \delta_{j,j'}\delta_{\bar{j},\bar{j}'}\frac{\Psi_x^{(J;\alpha,\beta)}}{\Psi_1^J}\langle 0|\Phi_{(J^*,\alpha)}\Phi_{(J,\beta)}|0\rangle$$

once again the two chiral blocks are the identity contribution is related by simple particular and known fusion coeffs $F_{k1}\bar{F}_{\bar{k}1}$ and we get from this cluster expansion

$$\frac{\Psi_x^{(I;\alpha,\alpha')}}{\Psi_x^{(1)}}\,\frac{\Psi_x^{(J;\beta,\beta')}}{\Psi_x^{(1)}} = \sum_{k,\bar{k},\gamma,\gamma'}\,\sum_{t,\bar{t}} d_{(I^*;\alpha)(J^*;\beta)}^{(K^*;\gamma;t,\bar{t})}\,d_{(I;\alpha')(J;\beta')}^{(K;\gamma';t,\bar{t})}\frac{\Psi_x^{(K;\gamma,\gamma')}}{\Psi_x^{(1)}}$$

these ratios - 1-dim reps of a **generalised Pasquier algebra** (dual to $\tilde{N}$) with structure constants

$$\widetilde{M}_{(I;\alpha,\alpha')\,(J;\beta,\beta')}{}^{(K;\gamma,\gamma')} = \sum_x \frac{\Psi_x^{(i,\bar{i};\alpha,\alpha')}}{\Psi_x^1}\,\Psi_x^{(j,\bar{j};\beta,\beta')}\,\Psi_x^{(k,\bar{k};\gamma,\gamma')\,*}$$

$\Rightarrow$ formula for the relative **spin - field** OPE coeffs

$$\widetilde{M}_{(i,\bar{i};\,\alpha,\alpha)\ \ (j,\bar{j};\,\beta,\beta)}{}^{(k,\bar{k};\,\gamma,\gamma)} = |d_{(i,\bar{i};\,\alpha)(j,\bar{j};\,\beta)}{}^{(k,\bar{k};\,\gamma)}|^2 \qquad (\star)$$

confirmed by the ADE cases - $\Psi_x^{(i,\bar{i};\alpha,\alpha')}$ computed.

• Full OPE coeffs in the presence of defects?

Expect some defect field analog of the boundary fields with multiplicity described by $\widetilde{V}_{ij;x}{}^{x}$; studied later extensively by [FFRS - Fröhlich, Fuchs, Runkel, Schweigert]

• Crossing relation in the presence of defects generalising the cluster relation?

Diagonal case: ansatz for - related to mod matrix of 2-point chiral correlator on the torus, $\sim S(y) \otimes I)F$ - the check is reduced to the use of the pentagon equation,

• Other important developments:

• "Duality defects" (in the sense of Kramers-Wannier) [FFRS]

(realise only some of the order-order disorder correlators in the minimal, $c < 1$ Vir models)

• Boundaries (and recently defects) - have been also generalised to Liouville theory -
the $c > 25$ Virasoro theory with continuous spectrum [FZZ -Fateev,Zamolodchikov^2, ZZ, Ponsot-Teschner]
and the main strc constants computed.

• applications in open and closed 2d non-critical string theories - combine generic level $c < 1$ and $c > 25$ Vir theories.

• defects - equivalently described as "permutation branes";

? other applications in string theory?