A few simple observations about $Y^{p,q}$ holoraphic background

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New Mathematical Methods in Solvable Models and Gauge/String Dualities

Varna, August, 2022

Why Sasaki-Einstein and holography

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• Recently: metric g_M on M emerges from the canonical ensemble (of N "point particles")in the large N-limit $\Longrightarrow$ emergent Saski-Einstein

Sasaki-Einstein $Y^{p,q}$

The metric tensor of $Y^{p,q}$ parameterized by two positive integers p, q $(p > q)$

$$ds^2 = \frac{1-y}{6}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\psi - \cos\theta d\phi)^2$$
$$+ w(y)\left[d\alpha + f(y)(d\psi - \cos\theta d\phi)\right]^2 \equiv ds^2(B) + w(y)(d\alpha + A)^2 .$$

The functions are

$$\begin{aligned} w(y) &= \frac{2(b-y^2)}{1-y} \ , \ q(y) = \frac{b - 3y^2 + 2y^3}{b - y^2} \ , \ f(y) = \frac{b - 2y + y^2}{6(b-y^2)} \ , \\ b &= \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2} \ . \end{aligned} \qquad (1)$$

The coordinates $\{y, \theta, \phi, \psi, \alpha\}$ have the following ranges $(0 < b < 1)$:

$$y_1 \leq y \leq y_2 \ , \ 0 \leq \theta \leq \pi \ , \ 0 \leq \phi \leq 2\pi \ , \ 0 \leq \psi \leq 2\pi \ , \ 0 \leq \alpha \leq 2\pi l \ . \qquad (2)$$

Schrödinger equation $\Box\Phi = -E\Phi$ with

$$\Box = \frac{1}{1-y}\frac{\partial}{\partial y}(1-y)w(y)q(y)\frac{\partial}{\partial y}$$
$$+\left(\frac{3}{2}\hat{Q}_R\right)^2 + \frac{1}{w(y)q(y)}\left(\frac{\partial}{\partial \alpha}+3y\hat{Q}_R\right)^2 + \frac{6}{1-y}\left[\hat{K}-\left(\frac{\partial}{\partial\psi}\right)^2\right]. \quad (3)$$

The R-symmetry operator is $\hat{Q}_R = 2\partial_\psi - 1/3\partial_\alpha$ and $\hat{K}$ is the second Casimir of $SU(2)$ - a part of the isometry $SU(2)\times U(1)^2$,

$$\hat{K} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi}+\cos\theta\frac{\partial}{\partial\psi}\right)^2 + \left(\frac{\partial}{\partial\psi}\right)^2 \quad (4)$$

Due to the isometry, the eigenfunction takes the form

$$\Phi(y,\theta,\phi,\psi,\alpha) \quad = \quad \exp\left[i\left(P_\phi\phi + P_\psi\psi + \frac{P_\alpha}{l}\alpha\right)\right]Y(y)\Theta(\theta) \quad (5)$$

with $P_\phi, P_\psi, P_\alpha \in \mathbb{Z}$, $\hat{K}$ acting on $SU(2)$ part.

The regular solutions of the equation below are given by Jacobi polynomials.

$$\hat{K}\underbrace{e^{i\left(P_\phi\phi+P_\psi\psi\right)}\Theta(\theta)}_{SU(2)\text{ part}} = -J(J+1)\,e^{i\left(P_\phi\phi+P_\psi\psi\right)}\Theta(\theta)\ , \tag{6}$$

The rest

$$\frac{1}{1-y}\frac{d}{dy}\left[(1-y)w(y)q(y)\frac{d}{dy}Y(y)\right]-\left[\left(\frac{3}{2}Q_R\right)^2+\right.$$
$$\left.\frac{1}{w(y)q(y)}\left(\frac{P_\alpha}{l}+3yQ_R\right)^2+\frac{6}{1-y}\left(J(J+1)-P_\psi^2\right)-E\right]Y(y)=0\ .$$

converts into Fuchsian-type with four regular singularities at $y=y_1,y_2,y_3$ and ∞, i.e. Heun's equation;

$$\frac{d^2}{dy^2}Y(y)+\left(\sum_{i=1}^{3}\frac{1}{y-y_i}\right)\frac{d}{dy}Y(y)+o(y)Y(y)=0\ , \tag{7}$$

The functions and parameters

$$
\begin{aligned}
o(y) &= \frac{1}{P(y)}\left[\mu - \frac{y}{4}E - \sum_{i=1}^{3}\frac{\alpha_i^2 P'(y_i)}{y-y_i}\right]\ ,\ P(y) = \prod_{i=1}^{3}(y-y_i)\ , \\
\mu &= \frac{E}{4} - \frac{3}{2}J(J+1) + \frac{3}{2}\left(\frac{2}{3}\frac{P_\alpha}{l} - Q_R\right)^2 \qquad (8)
\end{aligned}
$$

where $l = \frac{q}{3q^2-2p^2+p\sqrt{4p^2-3q^2}}$ and

$$
\begin{aligned}
\alpha_1 &= \pm\frac{1}{4}\left[P_\alpha\left(p+q-\frac{1}{3l}\right) - Q_R\right]\ , \qquad (9) \\
\alpha_2 &= \pm\frac{1}{4}\left[P_\alpha\left(p-q+\frac{1}{3l}\right) + Q_R\right]\ , \qquad (10) \\
\alpha_3 &= \pm\frac{1}{4}\left[P_\alpha\left(\frac{-2p^2+q^2+p\sqrt{4p^2-3q^2}}{q} - \frac{1}{3l}\right) - Q_R\right]\ . \qquad (11) \\
y_{1,2} &= \frac{1}{4p}\left(2p \mp 3q - \sqrt{4p^2-3q^2}\right)\ , y_3 = \frac{1}{2} + \frac{\sqrt{4p^2-3q^2}}{2p}\ . \qquad (12)
\end{aligned}
$$

It is convenient to transform the singularities from $\{y_1, y_2, y_3, \infty\}$ to $\{0, 1, t = \frac{y_1 - y_3}{y_1 - y_2}, \infty\}$. This is achieved by the transformation

$$\boxed{x = \frac{y - y_1}{y_2 - y_1}} \tag{13}$$

together with the rescaling

$$\boxed{Y = x^{\alpha_1}(1-x)^{\alpha_2}(t-x)^{\alpha_3}q(x)\ ,} \tag{14}$$

which transforms (7) to the standard form of Heun's equation

$$\boxed{\frac{d^2}{dx^2}q(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-t}\right)\frac{d}{dx}q(x) + \frac{\alpha\beta x - k}{x(x-1)(x-t)}q(x) = 0}$$

Bunch of Heun's parameters

$$\alpha = -\lambda + \sum_{i=1}^{3} |\alpha_i| \ , \ \beta = 2 + \lambda + \sum_{i=1}^{3} |\alpha_i| \ ,$$
$$\gamma = 1 + 2\alpha_1 \ , \ \delta = 1 + 2\alpha_2 \ , \ \epsilon = 1 + 2\alpha_3 \ , \tag{15}$$

The parameter k, the "accessory" parameter, is

$$\begin{aligned} k &= (|\alpha_1| + |\alpha_3|)(|\alpha_1| + |\alpha_3| + 1) - |\alpha_2|^2 \\ &\quad + t\left\{(|\alpha_1| + |\alpha_2|)(|\alpha_1| + |\alpha_2| + 1) - |\alpha_3|^2\right\} - \tilde{\mu} \end{aligned} \tag{16}$$

with

$$\begin{aligned} \tilde{\mu} &= -\frac{1}{y_1 - y_2}(\mu - y_1\lambda(\lambda+2)) \\ &= \frac{p}{q}\left[\frac{2}{3}(1-y_1)\lambda(\lambda+2) - J(J+1) + \frac{1}{16}\left(\frac{2}{3}\frac{N_\alpha}{l} - Q_R\right)^2\right] , \end{aligned} \tag{17}$$

$$t = \frac{1}{2}\left(1 + \frac{\sqrt{4p^2 - 3q^2}}{q}\right) \ . \tag{18}$$

Note that the parameter t satisfies the inequality $t > 1$ reflecting $p > q$.

A little holography of point-like string

- **Point-like strings**

$$S = \frac{\sqrt{\lambda}}{2}\int d\tau\left(-\dot{t}^2 + g_{ab}\dot{x}^a\dot{x}^b\right). \tag{19}$$

The standard equations of motion are supplemented also with the Virasoro constraint

$$-\dot{t}^2 + g_{ab}\dot{x}^a\dot{x}^b = 0. \tag{20}$$

For the metric at hand the action is reduces to

$$\begin{aligned} S = \frac{\sqrt{\lambda}}{2}\int d\tau \Big[& \frac{1-y}{6}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{\omega(y)q(y)}\dot{y}^2 + \frac{q(y)}{9}(\dot{\psi}^2 - \cos\theta\dot{\phi}^2) \\ & + w(y)\left[\dot{\alpha} + f(y)(\dot{\psi} - \cos\theta\dot{\phi})\right]^2\Big]. \end{aligned} \tag{21}$$

The Hamiltonian for the point-like string is

$$H = \frac{1}{2}g^{\mu\nu}P_\mu P_\nu. \tag{22}$$

The conjugate momenta to the coordinates $(\theta, \phi, y, \alpha, \psi)$ are:

$$
\begin{aligned}
\frac{1}{\sqrt{\lambda}} P_\theta &= \frac{1-y}{6}\dot{\theta}\,, \\
\frac{1}{\sqrt{\lambda}} P_y &= \frac{1}{6p(y)}\dot{y}\,, \\
\frac{1}{\sqrt{\lambda}} P_\alpha &= w(y)\left(\dot{\alpha} + f(y)\left(\dot{\psi} - \cos\theta\dot{\phi}\right)\right)\,, \\
\frac{1}{\sqrt{\lambda}} P_\psi &= w(y)f(y)\dot{\alpha} + \left[\frac{q(y)}{9} + w(y)f^2(y)\right]\left(\dot{\psi} - \cos\theta\dot{\phi}\right)\,, \\
\frac{1}{\sqrt{\lambda}} P_\phi &= \frac{1-y}{6}\sin^2\theta\dot{\phi} - \cos\theta P_\psi \\
&= \frac{1-y}{6}\sin^2\theta\dot{\phi} - \cos\theta w(y)f(y)\dot{\alpha} - \cos\theta\left[\frac{q(y)}{9} + w(y)f^2(y)\right]\dot{\psi} \\
&\quad + \cos^2\theta\left[\frac{q(y)}{9} + w(y)f^2(y)\right]\dot{\phi}\,,
\end{aligned}
\tag{23}
$$

where $p(y) = w(y)q(y)/6 = (b - 3y^2 + 2y^3)/[3(1-y)]$ and dot means proper time derivative.

• The momentum P_t conjugate to t is the energy of the string $\Longrightarrow$ equal to the conformal dimension Δ of the dual operator:

$$\Delta = P_t \equiv H = \sqrt{\lambda}\kappa \tag{24}$$

• The R-charge:

$$Q_R = 2P_\psi - \frac{1}{3}P_\alpha \tag{25}$$

• The energy/dispersion relations

$$\Delta^2 = \left(\frac{3}{2}Q_R\right)^2 + \frac{(P_\alpha + 3yQ_R)^2}{6p(y)} + 6p(y)P_y^2 + \frac{6(J^2 - P_\psi^2)}{1-y} \tag{26}$$

• Minimizing $H \Longrightarrow P_y = 0$; $y_0 = -\frac{P_\alpha}{3Q_R} \Longrightarrow \Delta = \frac{3}{2}Q_R \Longrightarrow$ BPS

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Summary:

a) The full set of point-like strings moving only in the transverse SE manifold is completely described by eq. (26);

b) for all BPS geodesics motion we obtain:

$$P_\alpha = -3y_0Q_R, \;\; Q_R = (2J - \frac{1}{3}P_\alpha) \;\; \Leftrightarrow \;\; \Delta = \frac{3}{2}Q_R, \;\; Q_R = 2P_\psi - \frac{1}{3}P_\alpha$$

Schlesinger and Heun

Statement: Painlevé VI associated with Heun equation (15) describes the isomonodromic flow of the Fuchsian system

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- Let us have a closer look at the Fuchsian equation

$$\frac{d\Psi}{dz} = \left[\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_2}{z-t}\right]\Psi, \tag{27}$$

where, without loss of generality, the coefficient matrices $A_\nu, \nu = 0, 1, 2,$ are traceless, and the system is diagonal at $z = \infty$, i.e.,

$$\mathrm{Tr}\, A_\nu = \theta, \quad \nu = 0, 1, 2; \qquad A_\infty = -A_0 - A_1 - A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2. \end{pmatrix} \tag{28}$$

Let us denote the eigenvalues of A_ν by

$$\pm\alpha;\, \pm\beta,\, \pm\gamma, \qquad 2\alpha,\, 2\beta,\, 2\gamma \notin \mathbb{Z}.$$

In a compact form Schlesinger equations reads

$$\frac{\partial A_i}{\partial a_j} = (1 - \delta_{ij})\frac{[A_i, A_j]}{a_i - a_j} - \delta_{ij}\sum_{k\neq i}\frac{[A_i, A_k]}{a_i - a_k};$$

- The second order ODE for the first component of $\Psi = (\psi_1\,,\,\psi_2)^T$:

$$\partial_z^2\psi_1 - (\mathrm{Tr}\,A(z) + \partial_z \log A_{12}(z))\partial_z\psi_1 + \left(\det A(z) + A_{11}(z)\partial_z \log \frac{A_{12}(z)}{A_{11}(z)}\right)\psi_1 = 0. \quad (29)$$

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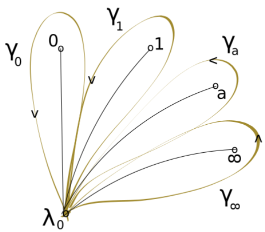

The monodromy group $\mathfrak{M}$; the base point λ_0; the branch cuts $[\lambda_0, 0]$; $[\lambda_0, 1]$; $[\lambda_0, a]$; $[\lambda_0, \infty]$ and the corresponding loops $\gamma_0, \gamma_1, \gamma_a, \gamma_\infty$. The complete monodromy data - in M_ν, $\nu = 0, 1, a, \infty$ realizing representation of $SL(2, \mathbb{Z})$ of the loops γ_ν. Conditions on monodromy matrices are:

$$\det M_\nu = 1, \qquad \nu = 0, 1, a, \infty \quad M_\infty M_t M_1 M_0 = 1, \qquad \text{(cyclic condition)}$$

$$M_\infty = \begin{pmatrix} e^{2\pi i\delta} & 0 \\ 0 & e^{-2\pi i\delta} \end{pmatrix} \quad (30)$$

- Monodromy data ($M_t \equiv M_2$, $M_\infty \equiv M_3$) w/ inv. coordinates on it

$$\begin{aligned} a_\nu &= \mathrm{Tr}\, M_\nu = 2\cos 2\pi\alpha_\nu, \qquad \nu = 0,1,2,3 \\ t_{\mu\nu} &= \mathrm{Tr}\, M_\mu M_\nu = 2\cos\sigma_{\mu\nu}, \qquad \mu,\nu = 0,1,2. \end{aligned} \tag{31}$$

- For Heun equation - take $\mathrm{tr}\, A_i = \theta_i$ and fix

$A_\infty = -\sum_{i=0,1,t} A_i = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$

+ Fricke-Jimbo relation (leaves two independent t_{ij}):

$$\begin{aligned} W(t_{0t},t_{1t},t_{01}) = t_{0t}t_{1t}t_{01} + t_{0t}^2 + t_{1t}^2 + t_{01}^2 - t_{0t}(a_1 a_\infty + a_0 a_t) - t_{1t}(a_0 a_\infty + a_1 a_t) \\ - t_{01}(a_t a_\infty + a_0 a_1) + a_0^2 + a_1^2 + a_t^2 + a_\infty^2 + a_0 a_1 a_t a_\infty = 4. \end{aligned}$$

Thus $2\theta_\infty = \kappa_1 - \kappa_2 - 1$ and $\kappa_1 + \kappa_2 = -2(\theta_0 + \theta_1 + \theta_t)$. These last conditions can be solved as

$$\kappa_1 = \theta_\infty + \frac{1}{2} - \sum_{i=0,1,t} \theta_i, \quad \kappa_2 = -\theta_\infty - \frac{1}{2} - \sum_{i=0,1,t} \theta_i. \tag{32}$$

$$\mu := \sum_{i=0,1,t} \frac{p_i + 2\theta_i}{\lambda - a_i}; \quad A_{12}(z) = k\frac{z-\lambda}{z(z-1)(z-t)}, \quad k \in \mathbb{C}, \tag{33}$$

Canonical form of deformed Heun equation

$$\partial_z^2\psi_1 + g_1(z)\partial_z\psi_1 + g_2(z)\psi_1 = 0, \tag{34a}$$

$$g_1(z) = \frac{1-2\theta_0}{z} + \frac{1-2\theta_1}{z-1} + \frac{1-2\theta_t}{z-t} - \frac{1}{z-\lambda}, \tag{34b}$$

$$g_2(z) = \frac{\kappa_1(\kappa_2+1)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{34c}$$

with the accessory parameter $K = K(\theta; x, \mu, t)$ given by

$$\begin{aligned} K(\theta;\lambda,\mu,t) = {}& \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \\ & \times\left[\mu^2 - \left(\frac{2\theta_0}{\lambda} + \frac{2\theta_1}{\lambda-1} + \frac{2\theta_t-1}{\lambda-t}\right)\mu + \frac{\kappa_1(\kappa_2+1)}{\lambda(\lambda-1)}\right]. \end{aligned} \tag{35}$$

• Define

$$A(z,t) = \left[\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right]\Psi(z,x); \quad B(z,t) = -\frac{A_t}{z-t}\Psi(z,t). \qquad (36)$$

Zero-curvature cond $\partial_z A - \partial_t B - [A,B] = 0$ is satisfied if A_i satisfy Schlesinger eqs.

$\rightarrow$ Write Schlesinger for deformed Heun and parmetrize A_i as

$$A_i = \begin{pmatrix} p_i + 2\theta_i & p_i q_i \\ -\frac{(p_i+2\theta_i)}{q_i} & -p_i \end{pmatrix}, \quad A_\infty = -\sum_{i=0,1,t} A_i = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

where p_i and q_i now are functions of (λ, t) and the fixed parameters.

• Compatibility condition for (36)

$$\frac{d\lambda}{dt} = \{K, \lambda\}, \qquad \frac{d\mu}{dt} = \{K, \mu\}, \qquad (\{,\} = \partial_\mu\partial_\lambda - \partial_\lambda\partial\mu)$$

- a change of the true singularity $t \Longrightarrow$ a change in the parameters.

- μ and λ are canonically conjugated coordinates in the phase space of isomonodromic deformations.

Explicitly

$$\dot{\lambda} = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[2\mu - \left(\frac{2\theta_0}{\lambda} + \frac{2\theta_1}{\lambda-1} + \frac{2\theta_t-1}{\lambda-t}\right)\right] \qquad (37)$$

$$\dot{\mu} = \Big\{\left[-3\lambda^2 2(1+t)\lambda - t\right]\mu^2 + [2(2\lambda-1-t)\theta_0 + 2(2\lambda-t)\theta_1 \\ +(2\lambda-1)(2\theta_t-1)]\,\mu - \kappa_1(\kappa_2).\Big\} \qquad (38)$$

Explicitly

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Equivalently, for λ only this is Painleve VI

$$\begin{aligned}\ddot{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\dot{\lambda}^2 - \left(\frac{1}{t} + \frac{1}{t-1}\frac{1}{\lambda-t}\right)\dot{\lambda} \\ + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha - \gamma\frac{t}{\lambda^2} + \beta\frac{t-1}{(\lambda-1)^2} + (\frac{1}{2}-\delta)\frac{t(t-1)}{(\lambda-t)^2}\right)\end{aligned} \tag{39}$$

where

$$\alpha = \frac{1}{2}(2\theta_\infty - 1)^2 \quad \gamma = 2\theta_0, \quad \beta = 2\theta_1^2, \quad \delta = 2\theta_t(\theta_t - 1) \tag{40}$$

- Painleve VI equation describes isomonodromy flow!

Reductions of Painleve VI

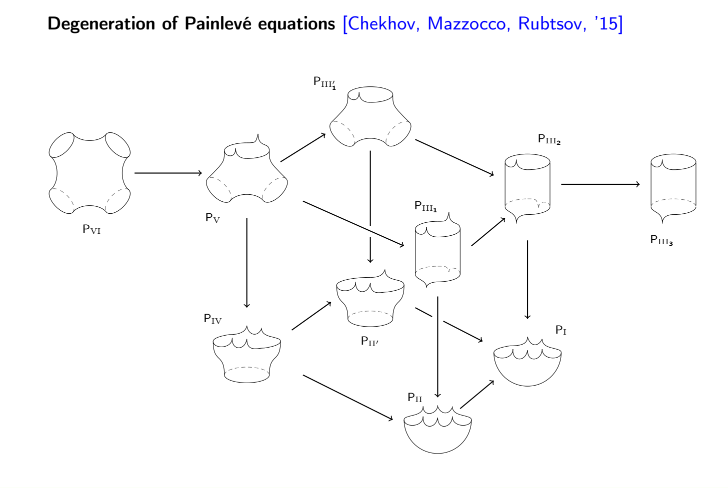
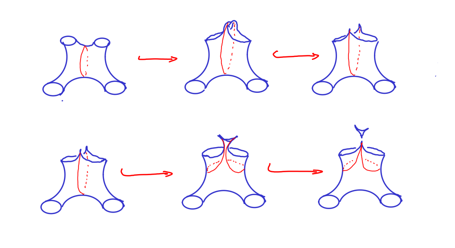


Figure: The table of confluences of Riemann surfaces from the Painlevé perspective.

• Degeneration of surfaces corresponding to reductions of Painleve equations (from [Chekhov, Mazzocco, Rubtsov 15'].)



PE w/ 4 singular points have reps in terms of Riemann surfaces. Geometric transition between different Painleve's - different types degeneration of the corresponding Riemann surfaces.

For instance, degeneration as in the first line of the figure gives

$$P_{VI} \to P_V : t \to 1 + \epsilon t_1, \quad \beta \to -\beta_1, \quad \gamma \to \delta_1 \epsilon^{-2} + \gamma_1 \epsilon^{-1}$$
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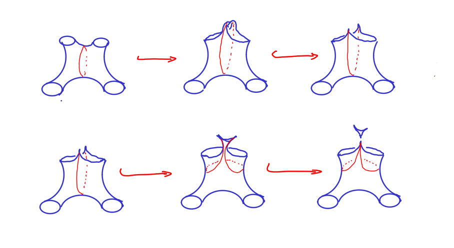
• Degeneration of surfaces corresponding to reductions of Painleve equations (from [Chekhov, Mazzocco, Rubtsov 15'].)



PE w/ 4 singular points have reps in terms of Riemann surfaces. Geometric transition between different Painleve's - different types degeneration of the corresponding Riemann surfaces.

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• Functions corresponding to some surfaces

Figure: Gauss hypergeometric (3 regular punctures), Whittaker (1 regular + 1 of Poincaré rank 1) and Bessel (1 regular + 1 of rank 1/2) [Gavrilenko, Lisovyy 16'].

A side remark on Schwarz-Christoffel map and ...

Schwarz-Christoffel accessory parameters. We start with the formula of Christoffel-Schwarz mapping

$$\frac{df(w)}{dw} = \gamma \prod_{i=1}^{n} (w - w_i)^{\theta_i - 1}, \tag{41}$$

where w_i are called pre-vertices (on the line), and z_i - the pre-images of the vertices (vertices of the polygon, $z_i = f(w_i)$).

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The Schwarzian differential equation

$$\boxed{\{f(w), w\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \sum_{i=1}^{n} \left[\frac{1-\theta_i^2}{2(w-w_i)^2} + \frac{2\beta_i}{w-w_i}\right]}, \tag{42}$$

where n is the number of vertices and $\pi\theta_i$ are the interior angles at each vertex z_i.

The solutions of the above equation is given by $z = f(w)$ which can be written as $\boxed{f(w) = \tilde{y}_1/\tilde{y}_2}$. Here $\tilde{y}_i$ are the two independent solutions of

$$\tilde{y}''(w) + \sum_{i=1}^{n} \left[\frac{1-\theta_i^2}{4(w-w_i)^2} + \frac{\beta_i}{w-w_i} \right] \tilde{y}(w) = 0. \tag{43}$$

Requiring that the solutions behave well at $w = \infty$ imposes algebraic constraints on the accessory parameters

$$\sum_i \beta_i = \sum_i (w_i \beta_i + 1 - \theta_i^2) = \sum_i (2 w_i \beta_i^2 + w_i (1 - \theta_i^2)) = 0. \tag{44}$$

By applying the transformation

$$\boxed{\tilde{y}(w) = w^{-\theta_0/2} (w-1)^{-\theta_1/2} (w-t)^{-\theta_t/2} y(w),} \tag{45}$$

we find the Heun equation in *canonical form*

$$y''(w) + \left(\frac{1-\theta_0}{w} + \frac{1-\theta_t}{w-t} + \frac{1-\theta_1}{w-1} \right) y'(w) + \left(\frac{\kappa_- \kappa_+}{w(w-1)} - \frac{t(t-1)K_0}{w(w-1)(w-t)} \right) y(w) = 0. \tag{46}$$

The constants and undefomed Hamiltonian K_0 are

$$\kappa_{\pm} = 1 - \frac{1}{2}(\theta_0 + \theta_t + \theta_1 \pm \theta_\infty) \qquad K_0 = -\beta_t + \sum_{i \neq t} \frac{(1-\theta_t)(1-\theta_i)}{2(w_i - t)}.$$

Examples of Schwarz-Christoffel maps

- The straight line passing through z_1 and z_2

$$\bar{z} = S(z) = \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} z + \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{z_1 - z_2}. \tag{47}$$

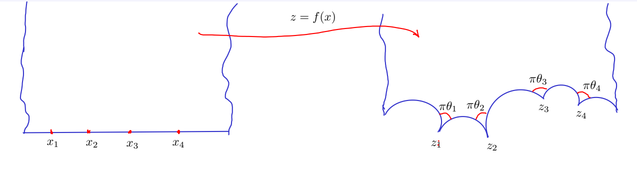
- The circle of radius r, center at z_0

$$\bar{z} = S(z) = \frac{r^2}{z - z_0} + \bar{z}_0. \tag{48}$$

- The ellipse $(z^2/a^2) + (y^2/b^2) = 1, \quad (a > b)$

$$\bar{z} = S(z) = \frac{a^2 + b^2}{a^2 - b^2} z + \frac{2ab}{a^2 - b^2} \sqrt{z^2 + b^2 - a^2}. \tag{49}$$

As a map from UHP to a polycircular-shaped domain



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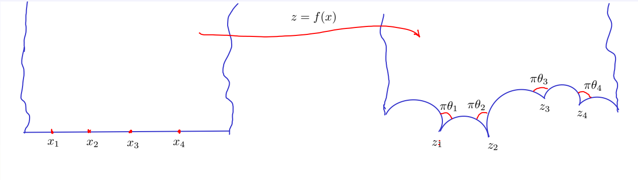
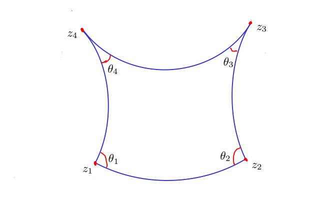


Schwarz-Christoffell graph



For $f(w) = y_1(w)/y_2(w)$

$$\bar{z} = S_i(z) = \frac{\bar{x}_i z + r_i^2 - |x|^2}{z - x_i}.$$

The centers of circle arcs C_i: x_i; radius: r_i; angles: $\pi\theta_i$.

In terms of the single monodromy parameters ($M_i = S_{i+1}\bar{S}_i$)

$$2\cos\theta_i = \frac{x_i\bar{x}_{i+1} + r_i^2 - |x_i|^2 + \bar{x}_i x_{i+1} + r_{i+1}^2 - |x_{i+1}|^2}{r_i r_{i+1}}.$$

$\Longrightarrow$ Schwarz-Christoffel graph is built out from the single monodromy parameters.

(Non)integrability issues

- For PVI non-integrability:
  Theorem 1. Let $\theta_\infty = \theta_1 + \theta_2 + \theta_t$ and at least one $\theta_j \in \mathbb{Z}$ and at least one $\theta_k \notin \mathbb{Q}$. Then the sixth Painleve equation is not integrable.
  Theorem 2. Let $\theta_\infty = \theta_1 + \theta_2 + \theta_t$ and at least two θ_j are integers. Then the sixth Painleve equation is not integrable.

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  Theorem 2. Let $\theta_\infty = \theta_1 + \theta_2 + \theta_t$ and at least two θ_j are integers. Then the sixth Painleve equation is not integrable.
- For non-integrability of strings in $Y^{p,q}$ background:
  - Basu & Pando Zayas 11' considered $Y^{p,q}$ with the simplest ansatz

$$\theta = \theta(\tau), \quad \mu = \mu(\tau), \quad y = y(\tau), \quad \phi = \alpha_1 \sigma, \quad \psi = \alpha_2 \sigma. \tag{50}$$

- $\dot{\theta}(\tau) = \theta(\tau) = 0$ solves string EoM.
- for remaining y-eq

$$\ddot{y} - \frac{p'}{p}\dot{y}^2 + \frac{p\,p'}{2}(\alpha_2 + c\,\alpha_1)^2 + \frac{2}{3}p(\alpha_2 + c\,\alpha_1)(y(\alpha_2 + c\,\alpha_1) - \alpha_1) = 0. \tag{51}$$

- the Normal Variational Equation takes the form

$$\ddot{\eta} - \frac{c\,\dot{y}_s}{1 - cy_s}\dot{\eta} + \alpha_1 \left(\alpha_1 - \frac{c\,p(y_s)}{1 - c\,y_s}(\alpha_2 + c\,\alpha_1) - \frac{2}{3}((\alpha_2 + c\,\alpha_1)y_s - \alpha_1) \right)\eta = 0$$

(Non)integrability issues

- wrining ormal Variational Equation in appropriate form and ppplying systematically Kovacic' algorithm fails to yield a solution pointing to the fact that the system is generically non-integrable.
- consider the simper geometry $T^{1,1}$

$$ds^2 = R^2 \Big(-\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega_3^2 + \frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) + \frac{1}{9}(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i)^2 \Big) . \quad (52)$$

with tha ansatz

$$\phi_1 = \alpha_1 \sigma, \quad \phi_2 = \alpha_2 \sigma, \quad t = t(\tau), \quad \psi = \psi(\tau), \quad \theta_i = \theta_i(\tau).$$

$\Longrightarrow$ Kovacic' algorithm fails again for generic values of constants.

• For these solutions, we found that the condition for firts theorem for non-integrability of Painleve VI is satisfied!

• **Conjecture**: There exist correspondence between string non-integrability in strings in $Y^{p,q}$ background and PVI non-integrability.

Other issues

- Different SE backgrounds $\rightarrow$ different Heun equation $\rightarrow$ Painleve equations $\rightarrow$ different sinularity

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• **Conjecture**: Confluent limits of Painleve VI encode the changes of background geometry.

Again: the confluent limit PVI $\rightarrow$ PV

$$P_{VI} \rightarrow P_V : t \rightarrow 1 + \epsilon t_1, \quad \beta \rightarrow -\beta_1, \quad \gamma \rightarrow \delta_1 \epsilon^{-2} + \gamma_1 \epsilon^{-1}$$
$$\delta \rightarrow -\delta_1 \epsilon^{-2}, \quad (\epsilon \rightarrow 0)$$

The corresponding confluent Heun equation is

$$y''(z) + \left[\frac{1-2\tilde{\theta}_0}{z} + \frac{1-2\tilde{\theta}_t}{z-t} - \frac{1}{z-\lambda}\right] y'(z)$$
$$+ \left[-\frac{1}{4} + \frac{2\tilde{\theta}_\infty - 1}{2z} - \frac{tc}{z(z-t)} + \frac{\lambda\mu}{z(z-\lambda)}\right] y(z) = 0.$$

Thus

$$t = \frac{1}{2}\left(1 + \frac{\sqrt{4p^2 - 3q^2}}{q}\right) \longrightarrow 1 \quad \Longrightarrow \quad Y^{p,q} \longrightarrow T^{p,p}\ (T^{1,1})$$

Summary

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Future directions:

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- (black hole) backgrounds w/ monodromies associaed to other Painleve's
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THANK YOU!