Subfactors and coset models[1]

Karl-Henning Rehren

[...]

The "fusion rules" which are read off the local solution are

$$[3][3] = [0] + [3]$$

in contrast to the standard fusion rules

$$[3][3] = [0] + [1] + [2] + [3] + [4].$$

This seems to contradict the message from the general theory of superselection sectors [2] that the fusion rules are *intrinsic* to a given local quantum field theory. Moreover, the isospin 3 sector is known to have non-trivial braid group statistics, so here is the surprising fact that one can associate it with local correlation functions.

[...]

[1] Talk at the Workshop on "Generalized Symmetries in Physics", Clausthal (FRG), July 1993, to appear in the proceedings, eds. H.-D. Doebner *et al.*

Conformal QFT in various dimensions:
some new results and ideas

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References

Joint work with **Nikolay Nikolov, Ivan Todorov [1,2,3]** and joint work in progress with diploma students: **Marcel Bischoff, Ingo Wagner, Daniel Meise (2009)**

1. Partial wave expansion and **Wightman positivity** in conformal field theory, Nucl. Phys. **B 722**, 266–296 (2005)
2. Harmonic **bilocal fields** generated by globally conformal invariant scalar fields, Commun. Math. Phys. **279**, 225-250 (2008)
3. **Pole structure** and biharmonic fields in conformal QFT in four dimensions, Bulg. J. Phys. **35 s1**, 113 (2008)

Introduction

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Conformal fields

Conformal quantum fields are classified according to **unitary PE representations of the conformal group** $SO(2, D)$ (Mack 1977) ($D =$ spacetime dimension)

Conformal fields

Conformal quantum fields are classified according to **unitary PE representations of the conformal group** $SO(2, D)$ (Mack 1977) (D = spacetime dimension)

Distinguished fields:

- conserved tensor fields (currents, SET, ...)
- have **"twist"** $D - 2$ (twist := scaling dimension $-$ spin)
- decompose into **local chiral** fields in $D = 2$
- are generated by **bi-harmonic bi-fields** $V(x_1, x_2)$ in $D = 4$, arise in OPE in globally conformal QFT.

Bi-harmonic bi-fields

Scaling dimension $(1,1)$, bi-harmonicity:

$$\Box_1 \mathbf{V}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0} = \Box_2 \mathbf{V}(\mathbf{x}_1, \mathbf{x}_2),$$

regular at $x_1 = x_2$, rational leading part (in $(12) = (x_1 - x_2)^2$) of correlation functions.

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These properties are highly restrictive. The leading part determines the full correlation ("harmonic completion"). $\Leftrightarrow$ It can exhibit at most **"cross double poles"**

$$\frac{\dots}{(1k)^p(1l)^q \cdot (2k)^r(2l)^s},$$

(with $(kl) = (x_k - x_l)^2$), but no triple poles:

$$\frac{\dots}{(1k)^p(1l)^q(1m)^r \dots}.$$

Example (6 points)

Leading singularity: $\langle V\phi\phi V\rangle = u_0 + \dots$

$$u_0 = \left[\frac{\frac{1}{2}(15)(26)(34) - (15)(23)(46) - (15)(24)(36)}{(13)(14)(23)(24)\cdot(34)\cdot(35)(36)(45)(46)}\right]_{[1,2],[5,6]}.$$

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$$u_0 = \left[\frac{\frac12(15)(26)(34) - (15)(23)(46) - (15)(24)(36)}{(13)(14)(23)(24)\cdot(34)\cdot(35)(36)(45)(46)}\right]_{[1,2],[5,6]}.$$

Harmonic completion $v = u_0 + O((12)) =$

$$u_0\cdot g(t,s)g(t',s') + \left[\frac{(13)(24)\cdot(35)(46)}{\dots\cdot(34)^2\cdot\dots}\right]_{[1,2],[5,6]}\cdot(1-g(t,s)g(t',s')),$$

where

$$g(t,s) \quad = \quad \frac{1}{s}\cdot\Big[Li_2(u)+Li_2(v)-Li_2(u+v-uv)\Big] + \text{perm's}$$

with $s = \frac{(12)(34)}{(13)(24)}$, $t = \frac{(14)(23)}{(13)(24)}$, s' and t' similar with $1,2 \to 5,6$, and $u(s,t)$ and $v(s,t)$ the "4D chiral" variables defined by $s = uv$ and $t = (1-u)(1-v)$.

Pole structure

Signal for non-triviality:

Cross double poles **cannot arise from free fields.**

If they occur: transcendental correlations, violation of Huygens locality, presumably local wrt interval (x_1, x_2).

Classification of admissible cross double pole structures (M. Bischoff):

- no 5-point CDP's, at least 6-point functions
- arise in multipletts of $sl(2, \mathbb{R})$

Open problem:

Admissible by Hilbert space positivity??? PWE for 6 points?

Dimension Hopping

DIMENSION HOPPING

Passing from D to D'

Reducing dimensions should simplify the analysis.

Options:

- Restriction to **hypersurfaces**
  - distinguished (twist $D-2$) fields do not pass to distinguished fields.
- Restriction to **subgroups**
  - There are subgroups $Conf(D')$ of $Conf(D)$ which do not come from hypersurfaces.

Timelike hypersurfaces

Prop. (Borchers 1964): Quantum fields can be restricted to timelike hypersurfaces (in the axiomatic sense).

- Drawback: time-slice property will be lost.
- Consolation: Non-free conformal fields do not fulfil the time-slice property anyway.

Lemma (Folklore): Restricted conformal fields remain conformal.

More precisely:

- Derivatives of conformal fields are **not** conformal.
- Transverse derivatives **are** conformal on the hypersurface.
- Representations split according "naive counting" of tensor components and transverse Taylor expansion.

Example:

$\phi =$ massless free field in $D = 4$.

- Restricts to dimension 1 scalar in $D = 2$:

$$\stackrel{y=z=0}{\longrightarrow} \lim_{N\to\infty} \Big(\frac{1}{\sqrt{N}} \sum^{N} \psi_L^{(n)}(t+x) \otimes \psi_R^{(n)}(t-x) \Big)$$

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- Surprise: this field restricts further to the time axis, $D = 1$:

$$\xrightarrow{x=0} j(t)$$

$=$ chiral current, $c = 1$.

(trivially checked by inspection of correlation functions)

Inducing $D \to D' > D$

Prop. (Bakalov, Nikolov): A low-dimensional CFT induces a higher-dimensional CFT on the same Hilbert space, provided the "missing part" of the higher-dimensional conformal group is present as an inner symmetry (suitable field multipletts).

- Are there solutions with finite c?

Restriction to null hypersurfaces

Restriction to null hypersurfaces ($x_- = x^0 - x^1 = 0$) is not covered by Borchers' result.

Yet:

In $D = 2$,

$$\langle \phi\phi \rangle = \frac{1}{x_+^{2d_L} x_-^{2d_R}}$$

A restriction to $x_- = 0$ is possible when $d_R = 0$. This is precisely the case when ϕ is a **chiral component** of one of the "distinguished fields" ($=$ conserved tensors).

... continued

In $D = 4$,

$$\langle\phi\phi\rangle = \frac{1}{(x_+x_- - y^2)^d}$$

does not admit a restriction to $x_- = 0$.

Yet, for $d = 1$ (free massless field)

$$\langle\partial_+\phi\partial_+\phi\rangle = \frac{x_-^2}{(x_+x_- - y^2)^3}$$

restricts (as $x_- \to 0$, taking care of $i\varepsilon$) to

$$\langle\partial_+\phi\partial_+\phi\rangle|_{x_-=0} = \frac{1}{x_+^2} \cdot \delta(y).$$

This is an **infinite-component ($c = \infty$) chiral field**. The transverse directions have turned into inner symmetries.

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- **Conjecture:** It works for all bi-harmonic bi-fields (related to the **"distinguished"** twist-2 fields) in $D = 4$.
  Their correlation functions are of the form

  $$\langle V(x_1, x_2)V(x_3, x_4)\rangle = \frac{1}{(13)(24)} \frac{f(u) - f(v)}{u - v},$$

  where u and v restrict in $D = 2$ to chiral cross ratios.

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  where u and v restrict in $D = 2$ to chiral cross ratios.

Recall that ∂_+ separates the L- and R-runner solutions of $\Box_{2D} = \partial_+\partial_- = 0$.

Understand how the factor $1/(u-v)$ "intertwines" this mechanism, upon restriction, for solutions of $\Box_{4D} = \partial_+\partial_- - \Delta_y = 0$!

Group Theory

GROUP THEORY

An exotic excursion

Embedding of conformal groups

$$\mathbf{D=2:}\quad so(2,2) \quad = \quad so(2,1)\oplus so(2,1) \quad \textbf{chiral}$$

$$\cap$$

$$\mathbf{D=4:}\quad so(2,4)$$

An exotic excursion

Embedding of conformal groups

$$\begin{array}{lccc} \mathbf{D=2:} & so(2,2) & = & so(2,1)\oplus so(2,1) \quad \textbf{chiral} \\ & \cap & & \\ \mathbf{D=4:} & so(2,4) & & \\ & \cup & & \\ & so(1,2)\oplus so(1,2) & & \textbf{“exotic”}\ 2D \end{array}$$

Is the exotic embedding potentially useful?

The exotic embedding

The exotic embedding has three-dimensional rather than two-dimensional orbits within $4D$ conformal space.

Yet, one could attempt to define $2D$ fields **algebraically** by selecting a suitable operator "$\phi(0)$" and "transporting it around" by the $2D$ conformal group.

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Check list:

- Locality?
- Positive energy?

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Check list:

- Locality?
- Positive energy?

Indeed, PE representations of $so(2,4)$ split into a continuum of non-PE rep's of $so(1,2) \oplus so(1,2)$ (D. Meise).

Double Pole Positivity ?

DOUBLE POLE POSITIVITY ?

The problem

Positivity of $\langle V\phi\phi V\rangle$ is equivalent to positivity of all partial wave coefficient matrices $B^{(\Lambda)}$ in

$$\langle V\,\Pi_{2,L}\,\phi\,\Pi_\Lambda\,\phi\,\Pi_{2,L'}\,V\rangle = B^{(\Lambda)}_{LL'}\cdot\beta^{(\Lambda)}_{LL'}(x_1,\dots,x_6).$$

Need to know the **partial waves** $\beta^{(\Lambda)}_{LL'}$ in order to expand a given correlation function and read off the coefficients.

Problem:

6-point partial waves in $4D$ are very difficult to obtain.

First idea

First idea:

$4D$ positivity implies positivity of the $2D$ restriction (necessary but not sufficient). $2D$ partial waves are easier to obtain.

$$\beta^{4D}_{k,L} - c_{k+L}\,\beta^{4D}_{k+1,L} - \sum_{\nu=1}^{[L/2]} d_{k,L,\nu}\,\beta^{4D}_{k+\nu+1,L-2\nu} = \sum_{\substack{m,n\geq 0\\ m+n=L}} \beta^{2D}_{k+m,k+n}$$

($2k$ = twist, L = spin,

$c_k = \frac{k^2}{4(4k^2-1)}$, $d_{k,L,\nu} = c_{k+L-\nu} - c_{k+\nu-1} \geq 0$).

- shows that $2D$ positivity is weaker than $4D$ positivity
- allows recursive computation of $4D$ partial waves
- reflects branching of repn's
- 6 points? (D. Meise)

Second idea

"Twist" is an algebraic function of the three Casimir operators $\Leftrightarrow$ no PDE to single out the twist in general. But twist-2 bi-fields are characterized by bi-harmonicity $\Box_y \langle V(y,z)A(x_1)\cdots C(x_n)\rangle = 0 = \Box_z \langle V(y,z)A(x_1)\cdots C(x_n)\rangle$.

Second idea:

Use conformal cross ratios as "collective variables" to turn PDE's wrt y, z into **PDE's wrt** $x_1, \dots, x_n$. These latter PDE's should then hold also for, eg,

$$\langle C(y_n)\cdots A(y_1)\Pi_{\text{twist 2}} A(x_1)\cdots C(x_n)\rangle.$$

Strategy works well for 4 points (quite non-trivial if $d_A \neq d_B$). For 6 points only partial results (I. Wagner).

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- Correlation structures can be classified.
- **Nontrivial structures arise at ≥ 6 points.**

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- Twist $D-2$ fields are "distinguished" due to conservation laws.
- In $D=4$: as a consequence, twist-2 bi-field correlations are highly constrained.
- Correlation structures can be classified.
- **Nontrivial structures arise at ≥ 6 points.**
- **Hilbert space positivity is a big challenge.**
- Several new ideas, but only partial results sofar.

Thank you

THANK YOU

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THANK YOU

AND ALL THE BEST, IVAN