On intransitive 3-nondegenerate CR manifolds in dimension 7

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Plan of the talk:

- Holomorphic nondegeneracy for CR mnfds $(\mathcal{M}, \mathcal{D}, \mathcal{J})$
- 3-nondegenerate CR mnfds in $\dim \mathcal{M} = 7$: the transitive case
- 3-nondegenerate CR mnfds in $\dim \mathcal{M} = 7$: the intransitive case

First Part

History of the problem

- In 1907 H. Poincaré proved that two generic real hypersurfaces $\mathcal{M}$ and $\mathcal{M}'$ in $\mathbb{C}^2$ are not biholomorphically equivalent
- É. Cartan realized that $\mathcal{M} \subset \mathbb{C}^2$ has non-trivial geometric structure given by maximal complex distribution $\mathcal{D} \subset T\mathcal{M}$ with $\mathcal{J} : \mathcal{D} \to \mathcal{D}$
- He solved equivalence problem for 3-dimensional *Levi nondegenerate* $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ in 1932 by associating bundle $\pi : P \to \mathcal{M}$ with absolute parallelism Φ s.t. $\mathrm{Aut}\,(\mathcal{M}, \mathcal{D}, \mathcal{J}) \cong \mathrm{Aut}\,(P, \Phi)$
- The construction was generalized by S.-S. Chern and J. Moser to *Levi nondegenerate* hypersurfaces $\mathcal{M} \subset \mathbb{C}^n$, $n \geqslant 2$, in 1974

Tanaka method (70's)

Let $\mathcal{M}$ be mnfd with distribution $\mathcal{D} \subset T\mathcal{M}$. Consider *filtration* of Lie algebra of vector fields defined by $\underline{\mathcal{D}}^{-1} = \underline{\mathcal{D}}$ and for any $k > 1$ by

$$\underline{\mathcal{D}}^{-k} = \underline{\mathcal{D}}^{-k+1} + [\underline{\mathcal{D}}, \underline{\mathcal{D}}^{-k+1}]$$

Evaluating at $x \in \mathcal{M}$, we get flag

$$\ldots = \mathcal{D}^{-\mu-1}(x) = \mathcal{D}^{-\mu}(x) \supset \mathcal{D}^{-\mu+1}(x) \supset \cdots \supset \mathcal{D}^{-2}(x) \supset \mathcal{D}^{-1}(x) = \mathcal{D}(x)$$

and assuming $T_x\mathcal{M} = \mathcal{D}^{-\mu}(x)$, the commutator of v.f. induces a structure of nilpotent graded Lie algebra on $\mathfrak{m}(x) = \mathrm{gr}(T_x\mathcal{M}) = \mathfrak{m}(x)_{-\mu} + \cdots + \mathfrak{m}(x)_{-1}$.

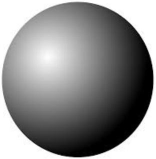
Def. $\mathcal{D}$ is *strongly regular* if we have flag $T\mathcal{M} = \mathcal{D}^{-\mu} \supset \cdots \supset \mathcal{D}^{-1} = \mathcal{D}$ of distributions and all $\mathfrak{m}(x)$ are isomorphic to $\mathfrak{m} = \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$

Strongly-regular Levi nondegenerate CR mnfds

Thm[N. Tanaka '70s] Let $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ be a strongly regular CR mnfd with CR symbol $(\mathfrak{m}, J)$ and set $\mathfrak{g}_0 = \mathrm{der}(\mathfrak{m}, J)$. Then:

1. $\dim \mathfrak{g}_\infty < \infty$ iff for any nonzero $v \in \mathfrak{m}_{-1}$, there is a $w \in \mathfrak{m}_{-1}$ s.t. $[v, w] \neq 0$ (i.e, the Levi form is nondegenerate); in this case
2. $\exists\ \pi : P \to \mathcal{M}$ and parallelism Φ s.t. $\dim \mathrm{Aut}\,(\mathcal{M}, \mathcal{D}, \mathcal{J}) \leqslant \dim(\mathfrak{g}_\infty)$

Strictly-pseudoconvex hypersurface case

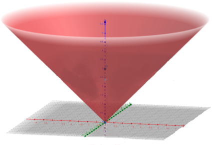


- $\Phi : TP \to \mathfrak{g} \cong \mathfrak{su}(n+1, 1)$ is *Cartan connection*
- $S^{2n+1} \cong G/H$ projectivization of null cone in $\mathbb{C}^{n+2}$
- $G = SU(n+1, 1)$, H = stabilizer of isotropic line
- Locally isomorphic to tube over hyperquadric

$$\mathcal{M} = \{z \in \mathbb{C}^{n+1} : \mathrm{Re}\, z_0 = (\mathrm{Re}\, z_1)^2 + \cdots + (\mathrm{Re}\, z_n)^2\}$$

Levi degenerate CR mnfds

$$\mathcal{M} = \{z \in \mathbb{C}^3 : (\mathrm{Re}\, z_1)^2 + (\mathrm{Re}\, z_2)^2 - (\mathrm{Re}\, z_3)^2 = 0\,,\ \mathrm{Re}\, z_3 > 0\} \subset \mathbb{C}^3$$



- It is Levi degenerate at all points (its completion in $\mathbb{C}P^4$ is homogeneous for $G = SO^o(3,2)$), foliated by complex leaves (the future light cone is union of real half-lines, so $\mathcal{M}$ is union of complex half-lines), yet it admits no CR straightening
- This can be seen by checking the necessary condition for local CR straightenings found by Freeman (1977)

Freeman sequence

The *Freeman sequence* is a sequence

$$\underline{\mathcal{F}}^{-1} \supset \underline{\mathcal{F}}^{0} \supset \underline{\mathcal{F}}^{1} \supset \cdots \supset \underline{\mathcal{F}}^{p-1} \supset \underline{\mathcal{F}}^{p} \supset \underline{\mathcal{F}}^{p+1} \supset \cdots$$

of complex vector fields $\underline{\mathcal{F}}^{p} = \underline{\mathcal{F}}^{p}_{10} \oplus \overline{\underline{\mathcal{F}}^{p}_{10}}$ given by $\underline{\mathcal{F}}^{-1} = \underline{\mathcal{D}}^{\mathbb{C}}$ and $\underline{\mathcal{F}}^{p}_{10}$ for $p \geqslant 0$ is the left kernel of the *higher order Levi form*

$$\begin{aligned} \mathcal{L}^{p+1} : \underline{\mathcal{F}}^{p-1}_{10} \otimes \underline{\mathcal{D}}_{01} &\longrightarrow \mathfrak{X}(\mathcal{M})^{\mathbb{C}} / (\underline{\mathcal{F}}^{p-1}_{10} \oplus \underline{\mathcal{D}}_{01}) \\ (X, Y) &\longrightarrow [X, Y] \mod \underline{\mathcal{F}}^{p-1}_{10} \oplus \underline{\mathcal{D}}_{01} \end{aligned}$$

Def.

1. $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ is *regular* if we have flags of distributions corresponding to the Tanaka and Freeman sequences.
2. $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ is *k-nondegenerate* if $\mathcal{F}^p \neq 0$ for all $-1 \leqslant p \leqslant k-2$ and $\mathcal{F}^{k-1} = 0$. Otherwise, we say it is *holomorphically degenerate*.

Classification of 2-nondegenerate homogeneous hypersurfaces of $\mathbb{C}^3$

The classification of all locally homogeneous 2-nondegenerate CR-mnfds in dimension 5 was achieved in 2008 by the celebrated work of Fels-Kaup. All such CR mnfds are tubes $\mathcal{M} = S + i\mathbb{R}^3$ over surfaces $S \subset \mathbb{R}^3$:

1. S the future light cone,
2. $S = \{r\,(\cos t, \sin t, e^{\omega t})\}$ (for any fixed $\omega > 0$)
3. $S = \{r\,(1, t, e^{t})\}$
4. $S = \{r\,(1, e^{t}, e^{\theta t})\}$ (for any fixed $\theta > 2$)
5. $S = \{c(t) + rc'(t)\}$, with $c(t) := (t, t^2, t^3)$ the *twisted cubic*

where $r \in \mathbb{R}^+$, $t \in \mathbb{R}$. For every $\mathcal{M} = S + i\mathbb{R}^3$ in $(2) - (5)$, the symmetry algebra is solvable and it has dimension 5.

Beloshapka's conjecture

- The lowest manifold dimension for which 3-nondegenerate CR-mnfds can exist is 7. Beloshapka '21 showed that the *upper bound* for the dimension of symmetry algebra of all 3-*nondegenerate* 7-dimensional CR-hypersurfaces is 20
- There is evidence supporting conjecture that CR hypersurfaces with maximal finite dimensional symmetry algebra are Levi nondegenerate:

  Conjecture (V. Beloshapka) For any real-analytic and connected holomorphically nondegenerate CR-hypersurface $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ with CR-dimension n one has $\dim \mathfrak{inf}(\mathcal{M}, \mathcal{D}, \mathcal{J}) \leqslant n^2 + 4n + 3$, with the maximal value attained only if $\mathcal{M}$ is everywhere spherical.

Second Part

Model $\mathcal{R}^7 \subset \mathbb{C}^4$ for 7-dimensional 3-nondegenerate CR-hypersurfaces

Together with B. Kruglikov, we have recently shown that the abstract model I derived in '15 is also a tube $\mathcal{R}^7 = S + i\mathbb{R}^4 \subset \mathbb{C}^4$. To realize it, consider the *tangent variety $\Sigma = TR$ to the cone R over twisted cubic*, parametrized as

$$x_0 = r^3,\ x_1 = r^2(s+t),\ x_2 = r\,s\,(s+2t),\ x_3 = s^2(s+3t),$$

with $r, t \neq 0$, and with global defining equations

$$\Sigma = \left\{x \in \mathbb{R}^4 \mid x_0^2 x_3^2 - 6x_0x_1x_2x_3 + 4x_0x_2^3 + 4x_1^3x_3 - 3x_1^2x_2^2 = 0\right\}.$$

These formulae give geometric interpretation to an example of Fels-Kaup '08. The $GL_2(\mathbb{R})$-orbits on $\Sigma \subset S^3\mathbb{R}^2 \cong \mathbb{R}^4$ are three: $S := \Sigma \backslash R$, $R\backslash\{0\}$, and $\{0\}$.

Thm. [Kruglikov, S. '23] $\mathcal{R}^7 = S + i\mathbb{R}^4$ is a 7-dimensional 3-nondegenerate globally homogeneous CR mnfd diffeomorphic to $S^1 \times \mathbb{R}^6$. Its automorphism group is $G \cong GL_2(\mathbb{R}) \ltimes S^3\mathbb{R}^2$ and stabilizer $\mathrm{Stab} = \{\mathrm{diag}(a^2, 1/a) \mid a \in \mathbb{R}^\times\}$.

Main classification result

Thm. [Kruglikov, S. '23]

1. *∃! locally homogeneous 3-nondegenerate CR mnfd in dimension 7* (hence any such structure is locally isomorphic to the model $\mathcal{R}^7$)
2. Every connected *globally homogenous* 3-nondegenerate CR mnfd in dimension 7 with the automorphism group of *maximal dimension* is finite or countable covering of $\mathcal{R}^7$
3. The *submaximal bound* $\dim \mathrm{Aut}\,(\mathcal{M}, \mathcal{D}, \mathcal{J}) \leqslant 7$ is achieved on the tube over (the nonsingular part of) the tangent variety to the cone over the *punctured twisted cubic*, for which

$$\mathrm{Aut}\,(\mathcal{M}, \mathcal{D}, \mathcal{J}) \cong B \ltimes S^3\mathbb{R}^2 \ ,$$

where B is the Borel subgroup in $GL_2(\mathbb{R})$.

Third Part

Main classification result – intransitive case

Thm. [Kruglikov, S. '23]

1. If symmetry algebra $\mathfrak{g}$ of 3-nondegenerate 7-dimensional real-analytic CR-hypersurface $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ acts *locally intransitively*, i.e., generic orbits have dimension < 7, then $\dim \mathfrak{g} < 7$ as well (in particular the symmetry dimension 7 is also non-realizable in the intransitive case)
2. The *submaximal symmetry dimension is 6*. There is a continuum of pairwise non-equivalent such CR-hypersurfaces with $\dim \mathfrak{g} = 6$

Rem.

1. $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ can be assumed regular, i.e., rank of involved bundles are constant. (In fact, they are constant on an open dense subset of $\mathcal{M}$ due to upper-semicontinuity, and one may localize to it by analyticity.)
2. It uses nondegenerate projective curves in $\mathbb{R}P^3$ with a 1-dimensional projective symmetry algebra (instead of the twisted cubic as before).

Ingredients of the proof of $\boxed{1}$

Variety of methods depending on the dimension of generic orbits:

- *Tanaka–Weisfeiler filtration* of $\mathfrak{g}$ at a *regular intransitive point*, retaining information along directions transverse to the orbits;
- The complex structure $\mathcal{J}$ does not project to the leaf space of Cauchy characteristic distribution, however $\mathfrak{g}$ *projects faithfully* (as Lie algebra of infinitesimal contact symmetries);
- Often $\mathfrak{g}$ is effectively represented on orbit $\mathcal{O}_p^G$ as infinitesimal symmetries of a smaller dimensional geometric structure $\rightsquigarrow$ *Tanaka–Weisfeiler filtration of* $\mathfrak{g} \cong \mathfrak{g}|_{\mathcal{O}_p^G}$ *at a transitive point*
- In the most involved cases, such filtrations are combined with a study in Cartan's spirit of *structure equations* of adapted frames.

An example: the case of orbits $\mathcal{O}^G_p$ of small dimension $d = 3$

First of all

$$\mathcal{M} \cong U \times V$$

near a regular point $p \in \mathcal{M}$, where U and V have the local coordinates $(u^i)_{i=1}^3$ and $(v^j)_{j=1}^4$, $TU = \langle \partial_{u^i} \rangle$ is the distribution $q \mapsto \mathfrak{g}|_q$ given by $\mathfrak{g}$. In particular, any *infinitesimal CR-symmetry* has the form

$$\xi = \sum_{i=1}^{3} \xi^i(u,v)\partial_{u^i} \ .$$

We also set

$$\begin{aligned}
TU_{\mathcal{D}} &= TU \cap \mathcal{D}\,, \quad TU^{\mathcal{J}}_{\mathcal{D}} = TU_{\mathcal{D}} \cap \mathcal{J}(TU_{\mathcal{D}})\,,\\
TU_{\mathcal{K}} &= TU \cap \mathcal{K}\,, \quad TU^{\mathcal{J}}_{\mathcal{K}} = TU_{\mathcal{K}} \cap \mathcal{J}(TU_{\mathcal{K}})\,,\\
TU_{\mathcal{L}} &= TU \cap \mathcal{L}\,, \quad TU^{\mathcal{J}}_{\mathcal{L}} = TU_{\mathcal{L}} \cap \mathcal{J}(TU_{\mathcal{L}})\,,
\end{aligned}$$

where $\mathcal{D} \supset \mathcal{K} \supset \mathcal{L}$ is the Freeman filtration.

An example: the case of orbits $\mathcal{O}_p^G$ of small dimension $d = 3$

- The subbundle $TU_{\mathcal{D}} \subset TU$ has codimension 1, hence it has rank 2. (If $TU_{\mathcal{D}} = TU$, then $\mathfrak{g}_{-2}(q) = 0$ for all $q \in \mathcal{M}$, hence a symmetry ξ would be tangent to $\mathcal{D}$ everywhere, and $\xi = 0$ by 3-nondegeneracy.)
- In particular $TU_{\mathcal{D}}^{\mathcal{J}}$ is either trivial or equal to $TU_{\mathcal{D}}$.
- If $TU_{\mathcal{D}}^{\mathcal{J}} = 0$, then $\mathfrak{g}_0(p) = 0 \implies \mathfrak{g}$ acts simply transitively on orbits and $\dim \mathfrak{g} = 3$. So we may consider the case where $TU_{\mathcal{D}}$ is $\mathcal{J}$-stable.
- In that case $TU_{\mathcal{K}}$ is $\mathcal{J}$-stable too. If $TU_{\mathcal{K}} = TU_{\mathcal{D}}$, then $TU_{\mathcal{K}}$ has codimension 1 in $TU \implies \dim \mathfrak{g} = 1$. (In fact $\mathfrak{g}$ projects faithfully to leaf space of Cauchy characteristic distribution $\mathcal{K}$ and two non-trivial analytic contact vector fields are locally proportional iff homothetic.)

An example: the case of orbits $\mathcal{O}_p^G$ of small dimension $d = 3$

Hence $TU_{\mathcal{K}} = 0$ and each orbit carries a natural structure of 3-dimensional contact CR mnfd

$$(\mathcal{O}_q^G, TU_{\mathcal{D}}, \mathcal{J}|_{TU_{\mathcal{D}}}) \ .$$

Since $T\mathcal{M} = TU \oplus \mathcal{K}$ as complementary integrable distributions, we may take rectifying coordinates $(u^i)_{i=1}^3$ and $(v^j)_{j=1}^4$ on $\mathcal{M} \cong U \times V$, so that $TU = \langle \partial_{u^i} \rangle$, $\mathcal{K} = \langle \partial_{v^j} \rangle$. Moreover

$$\xi = \sum_{i=1}^{3} \xi^i(u, v) \partial_{u^i}$$

preserves $\mathcal{K}$, whence $\xi^i = \xi^i(u) \Longrightarrow \mathfrak{g} \cong \mathfrak{g}|_{\mathcal{O}_q^G}$ *is effectively represented on each orbit* $\mathcal{O}_q^G$ *as symmetries of a* 3*-dimensional contact CR structure*.
We now use Tanaka–Weisfeiler filtration of $\mathfrak{g} \cong \mathfrak{g}|_{\mathcal{O}_q^G}$ at a transitive point, but still retaining global information.

An example: the case of orbits $\mathcal{O}_p^G$ of small dimension $d = 3$

- Any maximally symmetric $(\mathcal{O}_q^G, TU_{\mathcal{D}}, \mathcal{J}|_{TU_{\mathcal{D}}})$ is locally isomorphic to $S^3 \cong SU(1,2)/B$, and submaximal CR symmetry dimension is 3.
- Thus we may assume *every orbit locally spherical* and $\mathfrak{g} \subset \mathfrak{su}(1,2)$. Since the maximal dimension of a proper subalgebra of $\mathfrak{su}(1,2)$ is 5 (Borel subalgebras), we may assume $\mathfrak{g}|_{\mathcal{O}_q^G} \cong \mathfrak{su}(1,2)$ at all $q \in \mathcal{M}$.
- So far we have obtained that $\dim \mathfrak{g} = \dim \mathfrak{g}|_{\mathcal{O}_p^G} \leqslant 5$, unless

$$\mathfrak{g}|_{\mathcal{O}_q^G} \cong \mathfrak{su}(1,2)$$

at all $q \in \mathcal{M}$. We claim this possibility contradicts 3-nondegeneracy.

An example: the case of orbits $\mathcal{O}_p^G$ of small dimension $d = 3$

Tanaka-Weisfeiler filtration at transitive point

$$\begin{aligned}\mathfrak{g} &= \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_{+2} \\ &\cong \mathbb{R} \oplus \mathbb{C} \oplus \mathfrak{gl}_1(\mathbb{C}) \oplus \mathbb{C}^* \oplus \mathbb{R}^*\end{aligned}$$

where $\mathfrak{b} = \mathfrak{g}_{\geqslant 0} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Simple algebraic fact: *there is a unique (up to sign) contact CR structure on $\mathcal{O}_q^G$ that is preserved by $\mathfrak{g} \cong \mathfrak{g}|_{\mathcal{O}_q^G}$*. However, symmetries in $\mathfrak{g}$ have the v-independent local form

$$\xi = \sum_{i=1}^{3} \xi^i(u) \partial_{u^i} \,,$$

so the Lie algebra $\mathfrak{g}|_{\mathcal{O}_q^G}$ is not only abstractly isomorphic to $\mathfrak{g}|_{\mathcal{O}_p^G}$ for all $q \in \mathcal{M}$ but actually *equal*. The CR structure on $\mathcal{O}_q^G$ is then equal to that on $\mathcal{O}_p^G$ up to a sign, but since $\mathcal{J}$ depends smoothly on v, they are equal $\Longrightarrow$ *$\mathcal{J}$ is projectable to leaf space of $\mathcal{K}$*, which is not possible $\blacksquare$.

Beloshapka's conjecture

Thm. [Kruglikov, S. '23]

1. A 3-nondegenerate 7-dimensional real-analytic CR-hypersurface $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ has symmetry dimension $\dim \mathfrak{g} \leqslant 8$ and bound is sharp.
2. A holomorphically nondegenerate 7-dimensional real-analytic CR-hypersurface $(\mathcal{M}, \mathcal{D}, \mathcal{J})$ has symmetry dimension $\dim \mathfrak{g} \leqslant 24$. If it is not everywhere spherical, then $\dim \mathfrak{g} \leqslant 17$ (this bound is sharp and it can be attained only on inhomogeneous CR-manifolds).

Thanks!