Local operators and massless flows

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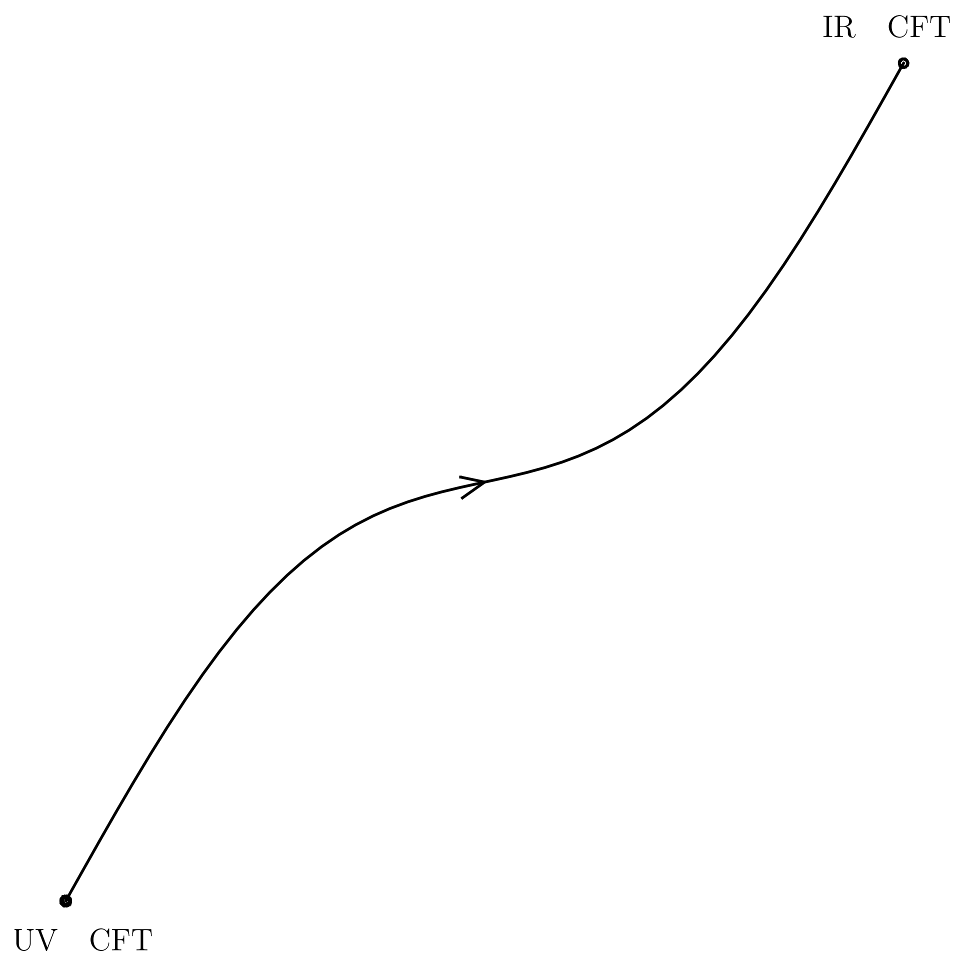
Join project with Stefano Negro

1. Massless flows

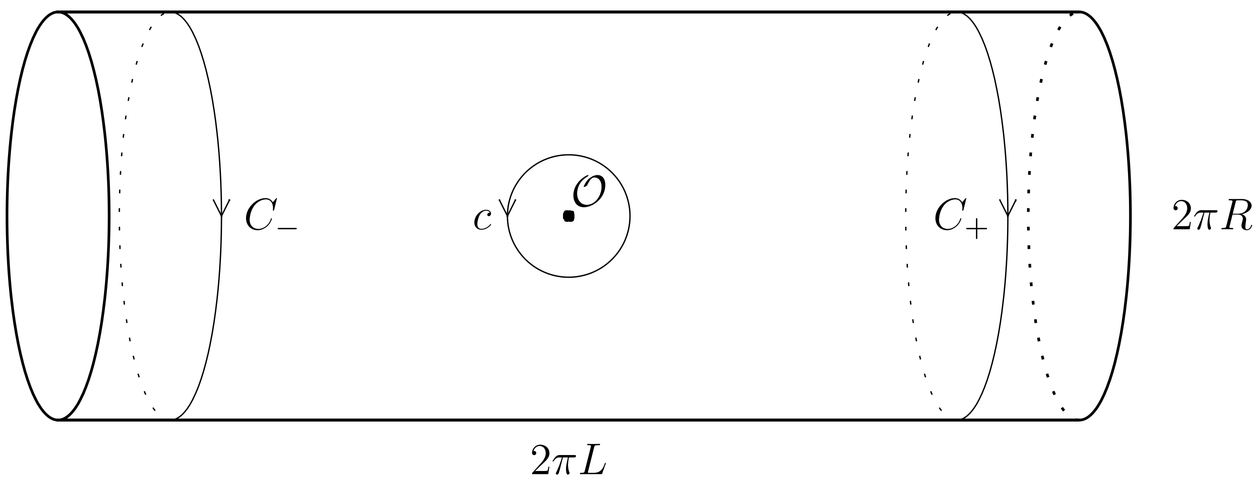


Famous example of integrable massless flow is that proposed by Zamolodchikov: $M_p \to M_{p-1}$ as $\phi_{1,3}$-perturbation with negative coupling constant.

The action

$$\mathcal{A} = \int \Big\{ \Big(\frac{1}{4\pi} \partial_z \varphi(z,\bar{z}) \partial_{\bar{z}} \varphi(z,\bar{z}) + e^{-i\beta\varphi(z,\bar{z})} \Big) \pm \mu^2 e^{i\beta\boldsymbol{\varphi}(z,\bar{z})} \Big\} d^2 z \, .$$

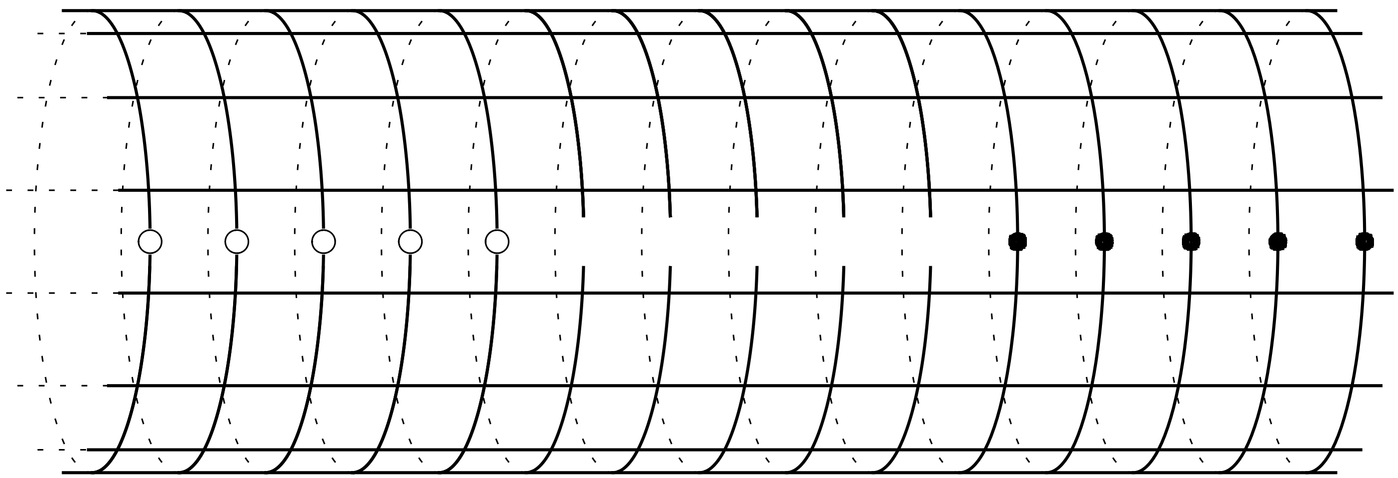
As world sheet we consider an infinite cylinder



We shall use the parameter p:

$$p = \frac{\beta^2}{1-\beta^2} \, .$$

The situation is not very usual, so, to be on the safe ground we begin with lattice model.



We consider the inhomogeneous in both directions six-vertex model on a cylinder. Two cases:

1. sine-Gordon model: staggering $\zeta_0^{\pm 1}$.

2. Massless flow: staggering: $\zeta_0 e^{\frac{\pi i}{4}}$ at even sites and $\zeta_0^{-1} e^{-\frac{\pi i}{4}}$ at odd sites.

In addition we shall later consider the “primary field" $q^{\alpha \sum_{j=-\infty}^{0} \sigma^3}$ put the twist κ in Matrubara direction, i.e. insert $q^{(\kappa+\alpha)\sigma^3}$ to the left and $q^{\kappa\sigma^3}$ to the right.

From this construction we Matsubara ground state is described using the nonlinear equations (which were conjectured by Al. Zamolodchikov from massless factorised scattering).

Non-linear equations, massive

$$\frac{1}{2i}F(\theta) = \pi M R \sinh\theta - \pi\,\frac{p+1}{p}P$$
$$- \operatorname{Im}\int\limits_{-\infty}^{\infty} \varphi(\theta-\theta')\log\bigl(1+e^{F(\theta'+i0)}\bigr)d\theta'\,,$$

The kernel is

$$\varphi(\theta) = \int_{-\infty}^{\infty} \frac{\sinh(\frac{1}{2}(\pi k(1-p))}{4\pi\sinh(\frac{1}{2}\pi k p)\cosh\left(\frac{1}{2}\pi k\right)} e^{ik\theta}dk\,.$$

Non-linear equations, massless

$$f_L(\theta) = -iLd(\theta) - \pi i \frac{p}{p-1} P$$

$$+ \int_{\mathbb{R}+i0} d\theta' \varphi(\theta-\theta') \log\left(1+e^{f_L(\theta')}\right) - \int_{\mathbb{R}-i0} d\theta' \varphi(\theta-\theta') \log\left(1+e^{-f_L(\theta')}\right)$$

$$+ \int_{\mathbb{R}+i0} d\theta' \chi(\theta-\theta') \log\left(1+e^{-f_R(\theta')}\right) - \int_{\mathbb{R}-i0} d\theta' \chi(\theta-\theta') \log\left(1+e^{f_R(\theta')}\right) ,$$

$$f_R(\theta) = -iLd(-\theta) + \pi i \frac{p}{p-1} P$$

$$+ \int_{\mathbb{R}-i0} d\theta' \varphi(\theta-\theta') \log\left(1+e^{f_R(\theta')}\right) - \int_{\mathbb{R}+i0} d\theta' \varphi(\theta-\theta') \log\left(1+e^{-f_R(\theta')}\right)$$

$$+ \int_{\mathbb{R}-i0} d\theta' \chi(\theta-\theta') \log\left(1+e^{-f_L(\theta')}\right) - \int_{\mathbb{R}+i0} d\theta' \chi(\theta-\theta') \log\left(1+e^{f_L(\theta')}\right) ,$$

Kernels

$$\varphi(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\theta} \frac{\sinh\left(\frac{p-2}{2}\pi k\right)}{2\cosh\left(\frac{\pi}{2}k\right)\sinh\left(\frac{p-1}{2}\pi k\right)} \,,$$

$$\chi(\theta) = -\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\theta} \frac{\sinh\left(\frac{\pi}{2}k\right)}{2\cosh\left(\frac{\pi}{2}k\right)\sinh\left(\frac{p-1}{2}\pi k\right)} \,,$$

the zero mode P is related to the twist as

$$P = \frac{\kappa}{p} \,.$$

The scaling limit consists in sending L and ζ_0 to infinity in a consistent way, obtaining

$$\lim Ld(\pm\theta) = 2\pi MRe^{\pm\theta} \,,$$

with some scale M. We shall use dimensionless parameter

$$\pi MR = e^{x} \,.$$

2. Two CFT limits

We work on the cylinder of radius R. Local and global Virasoro algebras:

$$T(z) = \sum_{n=-\infty}^{\infty} \mathbf{l}_n z^{-n-2}\,, \quad T(z) = \sum_{n=-\infty}^{\infty} L_n e^{\frac{1}{R}nz} - \frac{c}{24}\,.$$

Require

$$\mathrm{w}\!-\!\lim_{z\to\pm\infty} T(z) = \Delta - \frac{c}{24}\,.$$

We shall consider two cases

- In UV limit $\langle O(0)\rangle_{\mathrm{UV},p,P}$ corresponding to

$$c = c(p)\,, \quad \Delta = \frac{(pP)^2-1}{p(p+1)}\,.$$

- In IR limit $\langle O(0)\rangle_{\mathrm{IR},p,P}$ corresponding to

$$c = c(p-1)\,, \quad \Delta = \frac{(pP)^2-1}{p(p-1)}\,.$$

where $c(p) = 1 - \frac{6}{p(p+1)}$.

3. Integrals of motion

Eigenvalues of the local integrals of motion (k is odd):

$$\mathbf{I}_k(x) = e^{kx}\frac{1}{\pi}\mathrm{Im}\int\limits_{-\infty}^{\infty} e^{k\theta}\log\left(1+e^{-f_L(\theta+i0)}\right)d\theta\,.$$

Eigenvalues of local integrals of motion in CFT.

$$\mathcal{I}_1(k) = k^2 - \frac{1}{24}\,,\quad \mathcal{I}_3(k,c) = \mathcal{I}_1(k)^2 - \frac{1}{6}\mathcal{I}_1(k) + \frac{c}{1440}\,.$$

Introduce UV and IR $I_k(P)$ and $\widetilde{I}_k(P)$ obtained from $\mathcal{I}_1(k)$ by substitutions

$$\begin{cases} \mathrm{UV}\ : \quad k = \sqrt{\frac{p}{p+1}}\cdot P, \quad c = c(p)\,, \\ \mathrm{IR}\ : \quad k = \sqrt{\frac{p}{p-1}}\cdot P, \quad c = c(p-1)\,. \end{cases}$$

In the UV ($R \to 0$) and IR ($R \to \infty$) the massless flow should be compared with these CFT, namely,

$$\mathbf{I}_k(x) \to \begin{cases} C_k(p) m(p)^k I_k(P), & x \to -\infty \\ \\ C_k(p-1) m(p-1)^k \widetilde{I}_k(P), & x \to \infty \end{cases}$$

where for fine tuning we introduce

$$m(p) = 2\sqrt{\pi}\frac{\Gamma(p/2)}{\Gamma((p+1)/2)}$$

(Al. Zamolodchikov) ,

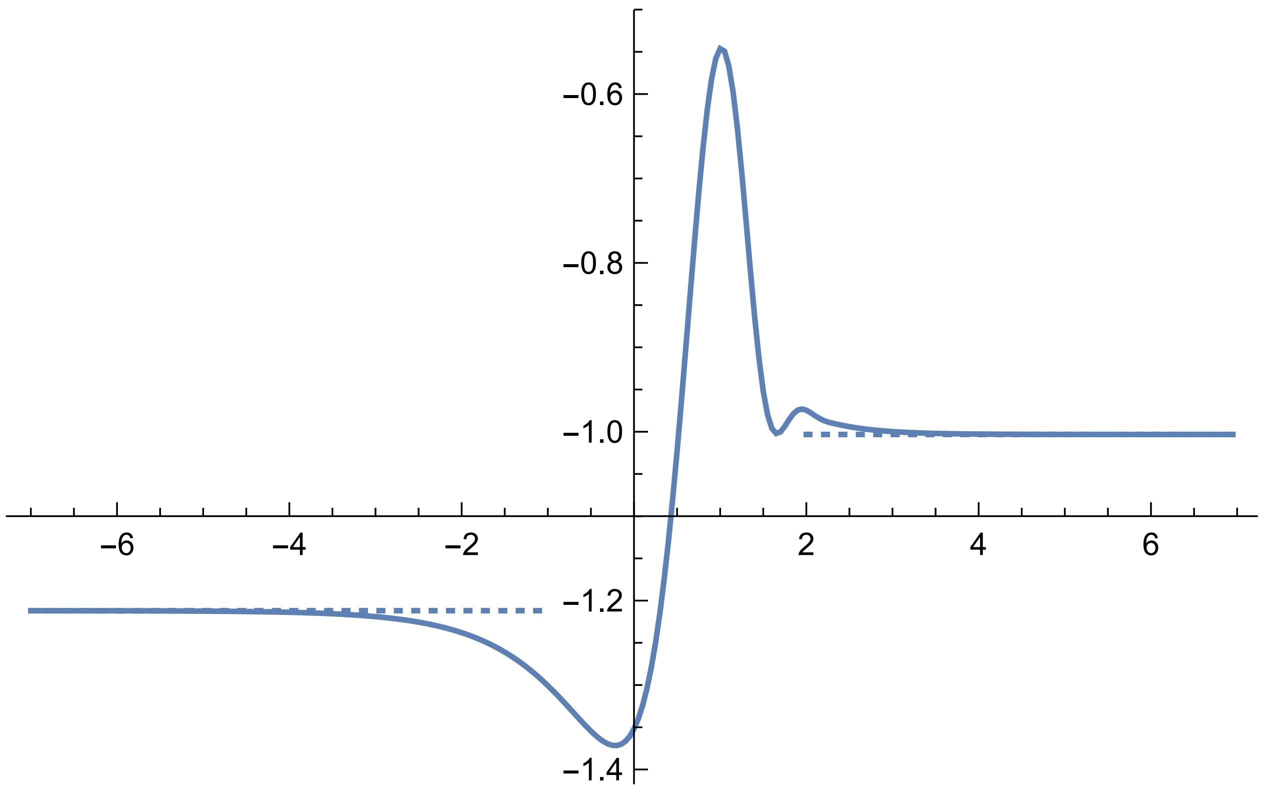
and

$$C_k(p) = -\frac{\sqrt{\pi}p}{(\frac{k+1}{2})!}\Big(\frac{p}{p+1}\Big)^{\frac{k-1}{2}}\frac{\Gamma(\frac{k(p+1)}{2})}{\Gamma(1+\frac{kp}{2})}$$

(BLZ) .

Example $k = 3$, $p = 2$, $P = .02$, The horizontal lines are respectively

$$C_3(p)m(p)^3 I_3(P), \quad C_3(p-1)m(p-1)^3 \widetilde{I}_3(P)$$



4.Matrix ω, massive case.

$$G_n(\theta,\alpha)=e^{n\theta}-\mathrm{Re}\Big(\int\limits_{\mathbb{R}+i0}\varphi(\theta-\theta',\alpha)\frac{1}{1+e^{F(\theta'+i0)}}G_n(\theta',\alpha)d\theta'\Big)\,.$$

$$\varphi(\theta,\alpha)=\int\frac{dk}{2\pi}e^{i\theta k}\frac{\sinh(\frac{\pi}{2}((1-p)k-i\alpha)}{\sinh\frac{\pi}{2}(pk+i\alpha)\cosh(\frac{\pi k}{2})}\,.$$

$$\omega_{m,n}=\mathrm{Re}\Big(\int\limits_{\mathbb{R}+i0}e^{m\theta}\frac{1}{1+e^{F(\theta+i0)}}G_n(\theta,\alpha)\Big)\,.$$

5. Matrix omega, massless

Equation:

$$
\begin{aligned}
G_{L,n}(\theta) = \Theta(n)e^{n\theta} + \Theta(-n)e^{-n\theta} \\
+ \int\limits_{\mathbb{R}-i0} \varphi(\theta-\theta',\alpha)G_{L,n}(\theta')\frac{d\theta'}{1+e^{f_L(\theta')}} \\
+ \int\limits_{\mathbb{R}+i0} \varphi(\theta-\theta',\alpha)G_{L,n}(\theta')\frac{d\theta'}{1+e^{-f_L(\theta')}} \\
- e^{\frac{\pi i\alpha}{2}} \int\limits_{\mathbb{R}-i0} \chi(\theta-\theta',\alpha)G_{R,n}(\theta')\frac{d\theta'}{1+e^{-f_R(\theta')}} \\
- e^{\frac{\pi i\alpha}{2}} \int\limits_{\mathbb{R}+i0} \chi(\theta-\theta',\alpha)G_{R,n}(\theta')\frac{d\theta'}{1+e^{f_R(\theta')}} ,
\end{aligned}
$$

In order to define $G_{R,n},\ G_{L,n}$ we need deformed kernels

$$\varphi(\theta,\alpha)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik\theta}\frac{\sinh\left(\frac{\pi}{2}((p-2)k+i\alpha\right)}{2\cosh\left(\frac{\pi}{2}k\right)\sinh\left(\frac{\pi}{2}((p-1)k+i\alpha)\right)}\,,$$

$$\chi(\theta,\alpha)=-\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik\theta}\frac{\sinh\left(\frac{\pi}{2}k\right)}{2\cosh\left(\frac{\pi}{2}k\right)\sinh\left(\frac{\pi}{2}((p-1)k+i\alpha\right)}\,.$$

Similarly another equation with
$G_{R,n}\to G_{L,n},\ G_{L,n}\to G_{R,n},\ f_L\to -f_R,\ f_R\to f_L.$
Symmetry $\alpha\to 2-\alpha, \quad \theta\to -\theta$, switches two chiralities.

For odd integers m,n we define the matrix ω:

$$\omega_{m,n}=\begin{cases}\int\limits_{\mathbb{R}-i0}e^{m\theta}G_{L,n}(\theta)\frac{d\theta}{1+e^{f_L(\theta)}}+\int\limits_{\mathbb{R}+i0}e^{m\theta}G_{L,n}(\theta)\frac{d\theta}{1+e^{-f_L(\theta)}}\,, & m>0\,,\\ \int\limits_{\mathbb{R}-i0}e^{m\theta}G_{R,n}(\theta)\frac{d\theta}{1+e^{-f_R(\theta)}}+\int\limits_{\mathbb{R}+i0}e^{m\theta}G_{R,n}(\theta)\frac{d\theta}{1+e^{f_R(\theta')}}\,, & m<0\,.\end{cases}$$

6. Fermiomic basis

The building block for computing the one-point function is the infinite matrix ω above.

Define

$$\omega_{M,N} = \det \omega_{m_p,n_q}\Big|_{p,q=1,\cdots,\#(M)} ,$$

then

$$\frac{\langle \beta^*_{I^+} \bar{\beta}^*_{\bar{I}^+} \bar{\gamma}^*_{\bar{I}^-} \gamma^*_{I^-} \Phi_\alpha \rangle}{\langle \Phi_\alpha \rangle} = \omega_{I^+ \sqcup (-\bar{I}^+),(-\bar{I}^-)\sqcup I^-} .$$

The scaling dimension for UV CFT are

$$\Delta(\alpha, p) = \frac{\alpha(\alpha-2)}{4p(p+1)} .$$

The following several formulae are well known.

Set $\bar{I}^+ = \bar{I}^- = \emptyset$. Then in UV CFT limit

$$\beta^*_{I^+}\gamma^*_{I^-}\Phi_\alpha(0) = \prod_{m\in I^+, n\in I^-} D_m(\alpha,p)D_n(2-\alpha,p)P_{I^+,I^-}(\alpha,p)\Phi_\alpha(0)\,,$$

where the multiplies are

$$D_m(\alpha,p) = \frac{\sqrt{\pi}}{\left(\frac{m-1}{2}\right)!}\left(\frac{p}{p+1}\right)^{\frac{m}{2}}\frac{\Gamma\left(\frac{1}{2}(\alpha+m(p+1))\right)}{\Gamma\left(\frac{1}{2}(\alpha+mp)\right)}\,,$$

and $P_{I^+,I^-}(\alpha,p)$ is a polynomial in even generators of the Virasoro algebra, it consists of even and odd parts:

$$P_{I^+,I^-}(\alpha,p) = P^{\text{even}}_{I^+,I^-}(\alpha,p) + d(\alpha,p)P^{\text{odd}}_{I^+,I^-}(\alpha,p)\,,$$

with

$$d(\alpha,p) = \frac{(2p+1)(\alpha-1)}{p(p+1)}\,.$$

the coefficients of the even and odd parts depend on α, p through $\Delta(\alpha,p)$, $c(p)$ only.

7. Ratio of one-point functions

CFT three-point functions in LZ normalizations, UV

$$t(\alpha,p) = -\pi\tan(\alpha+p)\,.$$

$$\mathcal{G}(\alpha,p,P) = t(\alpha,p)\frac{F(\alpha,p)}{F(\alpha+2p,p)}\frac{C(\alpha+2p,P)}{C(\alpha,P)}$$

$$\frac{F(\alpha,p)}{F(\alpha+2p,p)} = \frac{2\gamma\left(\frac{1+\alpha+p}{2}\right)\gamma\left(\frac{\alpha+p}{1+p}\right)}{(p+1)\gamma\left(\frac{\alpha+p}{2}\right)}\,,$$

$$\frac{C(\alpha+2p,P)}{C(\alpha,P)} = \frac{\gamma\left(\frac{\alpha+2p}{2+2p}\right)^2}{\gamma\left(\frac{\alpha+p}{1+p}\right)\gamma\left(\frac{\alpha+2p}{1+p}\right)} \times\gamma\left(\frac{\alpha+2p-2pP}{2+2p}\right)\gamma\left(\frac{\alpha+2p+2pP}{2+2p}\right)\,,$$

where $C(\alpha,P)$ is the three-point function.

Real coupling

Here is an example of the result of numerical computation of $\omega_{1,-1}(\alpha, R)$ for $p = 9/2, \alpha = 1/4, P = .1$:

$$r^{2\frac{\alpha+p-1}{p+1}} \frac{\langle \Phi_{\alpha+2p}(0)\rangle_R}{\langle \Phi_{\alpha}(0)\rangle_R} \tag{1}$$

$$= 0.0388085 - 0.72352 \cdot r^{\frac{4}{p+1}} - 7.99058 \cdot r^{\frac{8}{p+1}} + 0.404402 \cdot r^{\frac{12}{p+1}} \, .$$

The agreement of 0.0388085 with $\mathcal{G}(\alpha, p, P)$ is perfect.

Imaginary coupling For the same values as before ($p = 9/2, \alpha = 1/4, P = .1$) numerical calculation gives

$$r^{2\frac{\alpha+p-1}{p+1}} \frac{\langle \Phi_{\alpha+2p}(0)\rangle_R}{\langle \Phi_{\alpha}(0)\rangle_R}$$

$$= 0.0388085 + 0.72352 \cdot r^{\frac{4}{p+1}} - 7.99058 \cdot r^{\frac{8}{p+1}} - 0.404582 \cdot r^{\frac{12}{p+1}} \, ,$$

CFT three-point functions in LZ normalizations, IR.

$$\tilde{t}(\alpha, p) = -\pi \sec \tfrac{\pi}{2}(\alpha + p)\,.$$

$$\tilde{\mathcal{G}}(\alpha, p, P) = \frac{F(\alpha, p-1)}{F(\alpha+2p, p-1)} \frac{\widetilde{C}(\alpha+2p, P)}{\widetilde{C}(\alpha, P)}$$

$$\frac{F(\alpha, p-1)}{F(\alpha+2p, p-1)} = \left(\tfrac{p-1}{p}\right)^{2\frac{\alpha+p}{p-1}} \frac{2\gamma\left(\frac{1-\alpha-p}{2}\right)\gamma\left(\frac{2+\alpha+p}{2}\right)}{(p-1)\gamma\left(\frac{1+\alpha}{1-p}\right)}\,,$$

$$\frac{\widetilde{C}(\alpha+2p, P)}{\widetilde{C}(\alpha, P)} = \left(\tfrac{p-1}{p}\right)^{-\frac{2}{p-1}} \frac{\gamma\left(\frac{\alpha}{1-p}\right)\gamma\left(\frac{\alpha+p}{1-p}\right)}{\gamma\left(\frac{\alpha}{2-2p}\right)^2}$$

$$\times\,\gamma\left(1 - \frac{\alpha - 2pP}{2-2p}\right)\gamma\left(1 - \frac{\alpha + 2pP}{2-2p}\right).$$

IR behaviour

The main idea is that since the fermionic basis solves the reflection relations the formula for $\widehat{\Phi}_{\alpha+2p}$ in this basis must have as coefficients "CDD-multipliers". Here is an example which I found from numerics:

$$\widehat{\Phi}_{\alpha+2p} = e^{\frac{\pi i \alpha}{2}}\Big(\pi \sec \tfrac{\pi}{2}(\alpha+p) + \beta_1^*\bar{\gamma}_1^* + \tfrac{1}{\pi}\sec\tfrac{\pi}{2}(p-\alpha)\bar{\beta}_1^*\beta_1^*\gamma_1^*\bar{\gamma}_1^* + \cdots\Big)\Phi_\alpha \,.$$

The one-point function of $\beta_1^*\bar{\gamma}_1^*\Phi_\alpha$ is

$$\frac{\langle \beta_1^*\bar{\gamma}_1^*\Phi_\alpha\rangle_R}{\langle\Phi_\alpha\rangle_R} = \tilde{\omega}_{1,-1}(\alpha,R) + \tilde{t}(\alpha,p)$$

The one-point function of $\bar{\beta}_1^*\beta_1^*\gamma_1^*\bar{\gamma}_1^*\Phi_\alpha$ is given by the determinant formula (similar to $T\bar{T}$):

$$\frac{\langle \bar{\beta}_1^*\beta_1^*\gamma_1^*\bar{\gamma}_1^*\Phi_\alpha\rangle_R}{\langle\Phi_\alpha\rangle_R} = \begin{vmatrix} \tilde{\omega}_{1,1}(\alpha,R) & \tilde{\omega}_{1,-1}(\alpha,R)+\tilde{t}(\alpha,p) \\ \tilde{\omega}_{-1,1}(\alpha,R)+\tilde{t}(-\alpha,p) & \tilde{\omega}_{-1,-1}(\alpha,R) \end{vmatrix}$$

8. Numerical checks. It is convenient to use

$$\pi MR = e^{x}\,.$$

Introduce

$$\overline{\omega}_{I^+,I^-}(x,\alpha,p,P) = e^{-x(|I^+|+|I^-|)}\omega_{I^+,I^-}(x,\alpha,p,P)\,.$$

Then we are supposed to have

$$\overline{\omega}_{I^+,I^-}(x,\alpha,p,P) \to \begin{cases} \Omega_{I^+,I^-}(\alpha,p,P), & x\to-\infty \\ \Omega_{I^+,I^-}(\alpha,p-1,P), & x\to\infty \end{cases}$$

where

$$\Omega_{I^+,I^-}(\alpha,p,P) = \prod_{m\in I^+,n\in I^-} D_m(\alpha,p-1)D_n(2-\alpha,p-1)\mathcal{P}_{I^+,I^-}(\alpha,p)\,.$$

In our examples

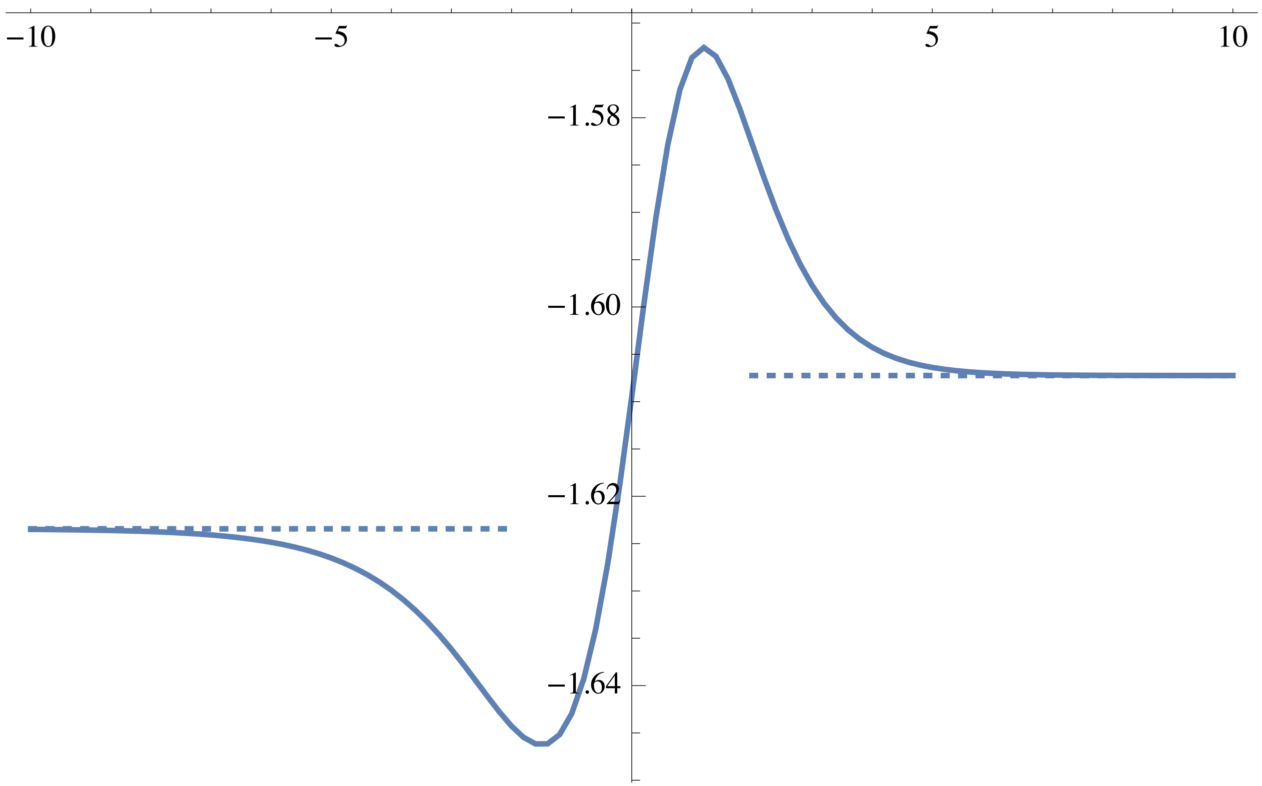
$$\mathcal{P}_{\{1\},\{1\}}(\alpha,p) = I_1(P) - \frac{\Delta(\alpha,p)}{12}\,,$$

$$\mathcal{P}_{\substack{\{1\},\{3\}\\ \{3\},\{1\}}}(\alpha,p)$$

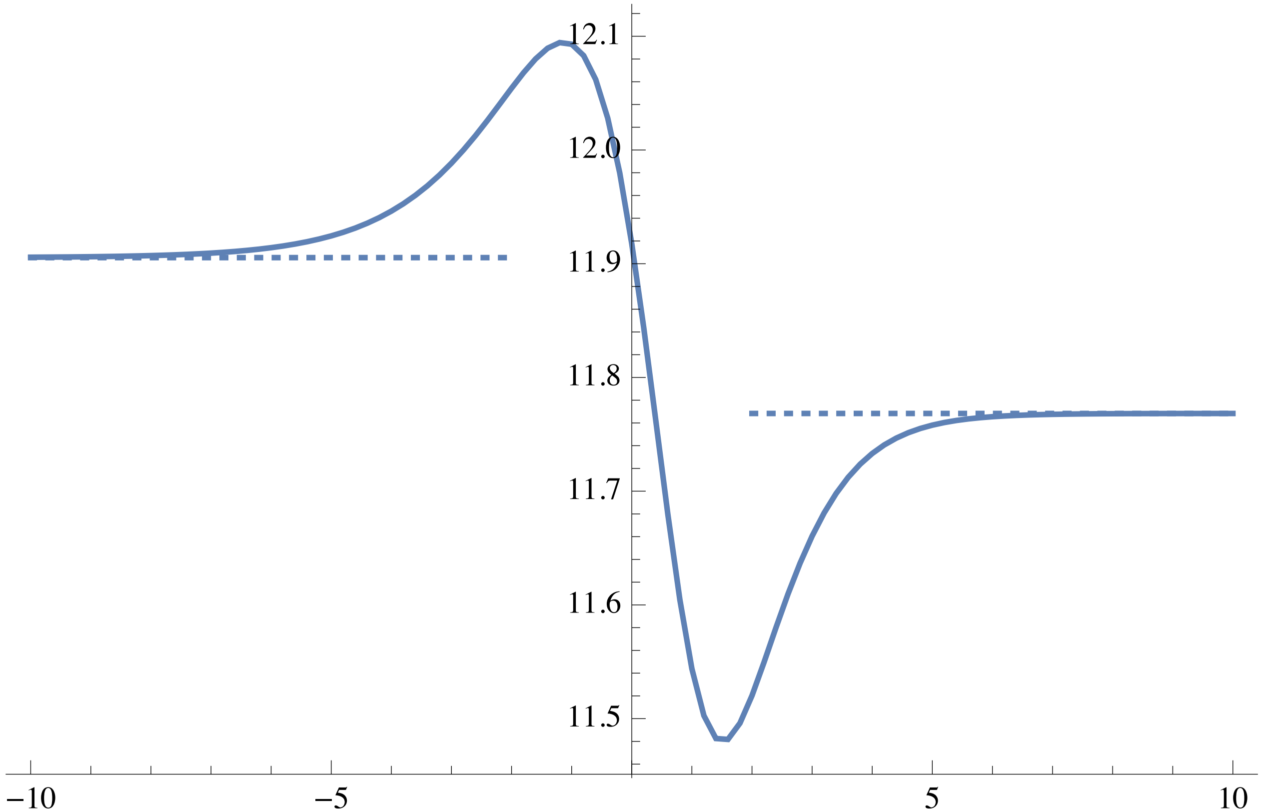
$$= I_3(P) - \frac{\Delta(\alpha,p)}{6}I_1(P) + \frac{\Delta(\alpha,p)}{144} + \frac{c(p)+5}{1080}\Delta(\alpha,p) \mp d(\alpha,p)\frac{\Delta(\alpha,p)}{360}\,.$$

Here $p = 4$, $\alpha = 3/4$, $P = .05$, $x = -10$ to $x = 10$ with step $1/2$.

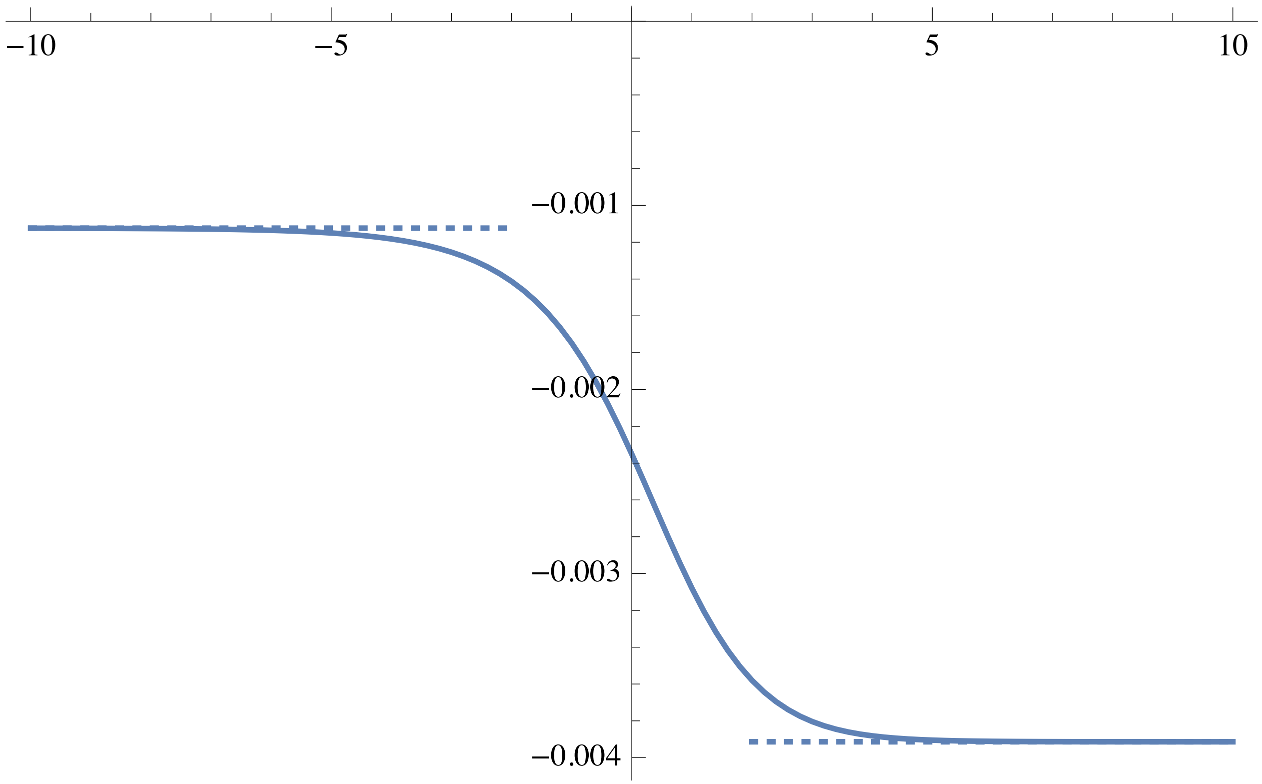
For $m, n = 1, 1$:



For $m, n = 1, 3$:



For $m, n = 3, 1$ is very close to $m, n = 1, 3$. Here is the difference



I computed up to level 8 when the first determinant appears.

Perturbative correction

In the UV limit we have a relevant perturbation by the field with the scaling dimension

$$\Delta_{\mathrm{UV}} = \frac{p-1}{p+1}\,.$$

In generic situation we arrive at the IR limit via the irrelevant operator of dimension

$$\Delta_{\mathrm{IR}} = \frac{p+1}{p-1}\,,$$

then there are further irrelevant contributions. Let us check the perturbative corrections. The following expressions must go to constants for respectively $x \to -\infty$, $x \to \infty$:

$$d_{\mathrm{UV}}(x) = e^{-2(1-\Delta_{\mathrm{UV}})x}(\overline{\omega}_{I^+,I^-}(x) - \Omega_{I^+,I^-})\,,$$

$$d_{\mathrm{IR}}(x) = e^{-2(1-\Delta_{\mathrm{IR}})x}(\overline{\omega}_{I^+,I^-}(x) - \widetilde{\Omega}_{I^+,I^-})\,.$$

In our case $2(1-\Delta_{\mathrm{UV}}) = 4/5$, $2(1-\Delta_{\mathrm{IR}}) = -4/3$.

Here are the numerical data for $m, n = 1, 1$

x	-46/5,	-47/5,	-48/5,	-49/5	-10
$d_{\rm UV}(x)$	-0.175483	-0.175522	-0.175556	-0.175584	-0.175608

x	46/5,	47/5,	48/5,	49/5	10
$d_{\rm IR}(x)$	0.725784	0.72623	0.72662	0.726961	0.727259

Here are the numerical data for $m, n = 1, 3$

x	-46/5,	-47/5,	-48/5,	-49/5	-10
$d_{\rm UV}(x)$	1.07584	1.07602	1.07617	1.07629	1.0764

x	46/5,	47/5,	48/5,	49/5	10
$d_{\rm IR}(x)$	-8.85654	-8.86307	-8.86878	-8.87378	-8.87815

We see that we are doing quite good.