Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

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Based on work in collaboration with

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arXiv[hep-th]: 0707.0243, 0709.2368, 0712.1223, 0712.4138, 0803.1466, 0807.1095, 0808.0491

Outline

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- Dual conformal symmetry
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- Dual $\mathcal{N} = 4$ superconformal symmetry
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- Dual superconformal symmetry: NMHV superamplitudes
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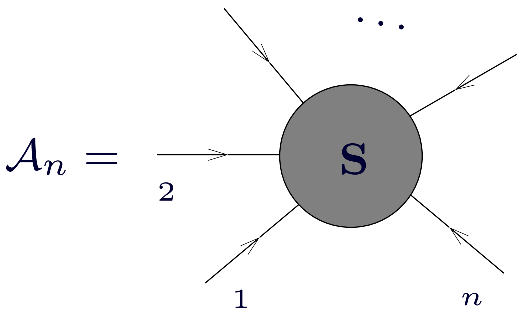
On-shell scattering amplitudes in $\mathcal{N}=4$ SYM

✔ Scattering amplitudes in $\mathcal{N}=4$ SYM



✗ Quantum numbers of on-shell states $|i\rangle = |p_i, h_i, a_i\rangle$:
momentum ($p_i^2 = 0$), helicity (h_i), color (a_i)

✗ IR divergences $\mapsto$ dimensional regularization
$\mathcal{A}_n = \mathrm{Div}(p_i, 1/\epsilon, \mu) \times \mathrm{Fin}(p_i)$ $\longrightarrow$ subject of this talk

✗ Perturbative expansion in 't Hooft coupling
$a = g^2 N/8\pi^2$:

$$\mathcal{A}_n(p_i, h_i) = \mathcal{A}_{n;0} + a \sum_H \mathcal{A}^H_{n;1}\, M^H_{n;1}(p_i) + O(a^2)$$

✔ Simplest example: Maximally Helicity Violating (MHV) amplitudes, e.g. for gluons:
$(- - + \dots +)$, $(- + - + \dots +)$, etc.
Unique helicity structure (tree):

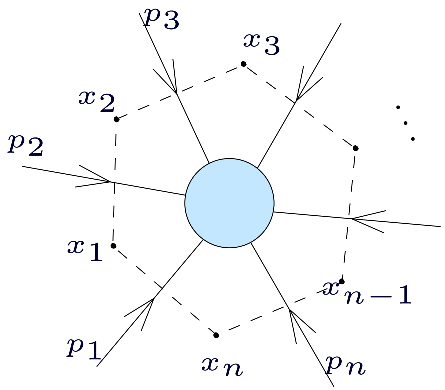
$$\mathcal{A}_n^{\mathrm{MHV}}(p_1^-, p_2^-, p_3^+, \dots, p_n^+) = \mathcal{A}^{\mathrm{MHV}}_{n;0}\, M_n^{\mathrm{MHV}}(p_i)\,, \qquad M_n^{\mathrm{MHV}} = 1 + aM_n^{(1)} + O(a^2)$$

✔ $\mathcal{N}=4$ SYM is a (super)conformal theory $\Rightarrow$ conformal symmetry of $\mathcal{A}_n(p_i)$? Two problems:
(i) Conformal boosts realized on momenta are 2nd-order differential operators [Witten'03]

(ii) IR divergences break conformal symmetry

Dual conformal symmetry I

- ✔ Hidden symmetry of $\mathcal{A}_n$ of dynamical origin
- ✔ Linear action on the particle momenta in dual space [Broadhurst'95],[Drummond,Henn,Smirnov,ES'06]



- ✗ $p_i = x_i - x_{i+1} \equiv x_{i\ i+1} \quad \Leftrightarrow \quad \sum_i p_i = 0 \text{ if } x_{n+1} \equiv x_1$
- ✗ $p_i^2 = 0 \quad \Leftrightarrow \quad x_{i\ i+1}^2 = 0$
- ✗ Simple change of variables, not a Fourier transform!
- ✗ Conformal group $SO(4,2)$ acting on the dual coordinates x_i $\Longrightarrow$ dual conformal symmetry.

- ✔ Example: MHV amplitudes

$$\ln M_n^{\text{MHV}} = \ln Z_n(x_i,\epsilon,\mu) + \ln F_n(x_i) + O(\epsilon)\,, \quad \ln Z_n = \sum_{l\geq 1} a^l \sum_{i=1}^n \left(-x_{i,i+2}^2\mu^2\right)^{l\epsilon}\left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon}\right)$$

- ✔ Duality MHV amplitude/Wilson loop [Alday,Maldacena'07], [Drummond,Korchemsky,ES'07], [Brandhuber,Heslop,Travaglini'07]

$$\ln F_n = \ln\langle 0|\text{Tr}\,\text{P}\exp\left(ig\oint_{C_n} dx^\mu A_\mu(x)\right)|0\rangle + \text{const} + O(\epsilon)$$

WL has conformal invariance in dual space $\Rightarrow$ Anomalous CWI : [Drummond,Henn,Korchemsky, ES'07]

$$K^\mu \ln F_n = \frac{1}{2}\Gamma_{\text{cusp}}(a)\sum_{i=1}^n \ln\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\, x_{i,i+1}^\nu \quad \Rightarrow \quad \text{Fixes } \ln F_n \text{ for } n=4,5 \text{ but not for } n\geq 6$$

Dual conformal symmetry II

- ✔ Can we generalize dual conformal symmetry to non-MHV amplitudes?
  Need to study the helicity structures and the loop corrections

- ✔ Spinor helicity formalism: commuting spinors λ^α (helicity -1/2), $\tilde\lambda^{\dot\alpha}$ (helicity 1/2) [Xu,Zhang,Chang'87]

$$p_i^2 = 0 \;\Leftrightarrow\; p_i^{\alpha\dot\alpha} \equiv p_i^\mu (\sigma_\mu)^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha}$$

- ✔ Simplest case: MHV tree level [Parke, Taylor'86]

$$\mathcal{A}_{n;0}^{\mathrm{MHV}}(\ldots i^- \ldots j^- \ldots) = \delta^{(4)}(\sum_{k=1}^{n} p_k)\, \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle}\,, \qquad \langle i\, j\rangle = -\langle j\, i\rangle = \epsilon^{\alpha\beta}\lambda_{i\,\alpha}\,\lambda_{j\,\beta}$$

  Is it dual conformal?

- ✔ Dual conformal transformations of spinors [Drummond,Henn,Korchemsky, ES'08]
  - ✗ Conformal group $=$ Poincaré $+$ inversion:

$$I[x^\mu] = \frac{x^\mu}{x^2} \equiv x^{-1}\,, \qquad I[x_i - x_j] = x_i^{-1}(x_i - x_j)x_j^{-1}$$

  - ✗ Transforming spinors: $p_i^{\alpha\dot\alpha} = (x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha} \qquad \Rightarrow$

$$I[\lambda_i^\alpha] = \frac{\lambda_i^\alpha (x_i)_{\alpha\dot\alpha}}{x_i^2} \equiv \lambda_i\, x_i^{-1} = \lambda_i^\alpha \frac{(x_{i+1})_{\alpha\dot\alpha}}{x_i^2} \;\Rightarrow\; I[\langle i\; i+1\rangle] = \langle i| \frac{x_{i+1}}{x_i^2} x_{i+1}^{-1} |i+1\rangle = \frac{\langle i\; i+1\rangle}{x_i^2}$$

Dual conformal symmetry III

- ✔ What about the momentum conservation delta function $\delta^{(4)}(\sum_{i=1}^n p_i)$?
  - ✗ Cyclic symmetry : $\sum_{i=1}^n p_i = 0 \Leftrightarrow \sum_{i=1}^n (x_i - x_{i+1}) = 0$ iff $x_{n+1} \equiv x_1$
  - ✗ Relax cyclicity, $x_1 \neq x_{n+1}$, and then impose it by

$$\delta^{(4)}(x_1 - x_{n+1}) \;\rightarrow\; \text{manifestly dual conformal}$$

- ✔ Split-helicity tree amplitudes: all negative-helicity gluons appear contiguously
  - ✗ Known explicitly from recursion relations [Britto, Cachazo, Feng, Roiban, Spradlin, Volovich, Witten]
  - ✗ Example: split-helicity MHV tree amplitude – manifestly dual conformal!

$$\mathcal{A}_n^{\mathrm{MHV}}(--+\ldots+) = \delta^{(4)}(x_1 - x_{n+1})\, \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}$$

  - ✗ All split-helicity tree amplitudes are dual conformal
  - ✗ Non-split-helicity amplitudes are dual conformal not on their own, but as parts of superamplitudes
  - ✗ Superamplitudes in dual superspace exhibit dual superconformal symmetry.

Superamplitudes in on-shell superspace I

- ✔ $\mathcal{N}=4$ gluon supermultiplet $\rightarrow$ PCT self-conjugate $\rightarrow$ holomorphic (chiral) description

$$\Phi(p,\eta)=G^+(p)+\eta^A\Gamma_A(p)+\eta^A\eta^B S_{AB}(p)+\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\Gamma}^D(p)+\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p)$$

  η^A ($SU(4)$ index $A=1\ldots4$, helicity $1/2$) are Grassmann variables of on-shell superspace

- ✔ Superamplitudes $\mathcal{A}_n(\Phi(1)\ldots\Phi(n))$ = expansion in powers of η_i^A

- ✔ Example: Nair's description of tree MHV amplitudes [Nair'88]

$$\mathcal{A}_n^{\mathrm{MHV}}=\frac{\delta^{(4)}(\sum_{i=1}^n p_i)\,\delta^{(8)}(\sum_{j=1}^n \lambda_{j\,\alpha}\,\eta_j^A)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}=\frac{\delta^{(4)}(\sum_{i=1}^n p_i)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\Big(\langle 12\rangle^4\,\eta_1^4\eta_2^4\eta_3^0\ldots\eta_n^0+\ldots\Big)$$

- ✔ On-shell $\mathcal{N}=4$ supersymmetry
  - ✗ Clifford algebra for massless Poincaré states:

$$q^A=\eta^A\,,\qquad \bar{q}_A=\partial/\partial\eta^A\,,\qquad \{q^A,\bar{q}_B\}=\delta_B^A$$

  - ✗ Covariant description with the help of λ_α:

$$q_\alpha^A=\lambda_\alpha\eta^A\,,\qquad \bar{q}_{A\,\dot\alpha}=\tilde{\lambda}_{\dot\alpha}\partial/\partial\eta^A$$

  - ✗ On-shell $\mathcal{N}=4$ supersymmetry algebra (with $p^2=0$)

$$\{q_\alpha^A,\bar{q}_{B\,\dot\alpha}\}=\delta_B^A\,\lambda_\alpha\tilde{\lambda}_{\dot\alpha}=\delta_B^A\,p_{\alpha\dot\alpha}$$

Superamplitudes in on-shell superspace II

- ✔ Invariance of the superamplitude: $p_{\alpha\dot\alpha}\mathcal{A}_n = q^A_\alpha \mathcal{A}_n = 0 \;\Rightarrow$

$$\mathcal{A}_n(\lambda,\tilde\lambda,\eta) = \delta^{(4)}(\sum_{i=1}^n p_i)\,\delta^{(8)}(\sum_{j=1}^n\,\lambda_j\,\eta_j)\Big[\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(4)} + \ldots + \mathcal{A}_n^{(4n-16)}\Big]$$

- ✔ $\mathcal{A}_n^{(4k)}(\eta)$ – homogeneous polynomials in η of degree $4k$:
  $k=0 \;\longrightarrow\;$ MHV, $k=1 \;\longrightarrow\;$ Next-to-MHV, ..., $k=n-4 \;\longrightarrow\; \overline{\mathrm{MHV}}$
- ✔ Simplest case – All-order MHV superamplitude:

$$\mathcal{A}_n^{\mathrm{MHV}}(\lambda,\tilde\lambda,\eta) = \delta^{(4)}(\sum_{i=1}^n\,p_i)\,\delta^{(8)}(\sum_{j=1}^n\,\lambda_{j\,\alpha}\,\eta_j^A)\Big[\frac{M_n(p)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\Big]$$

- ✔ Define 'ratio' R = general/MHV superamplitude:

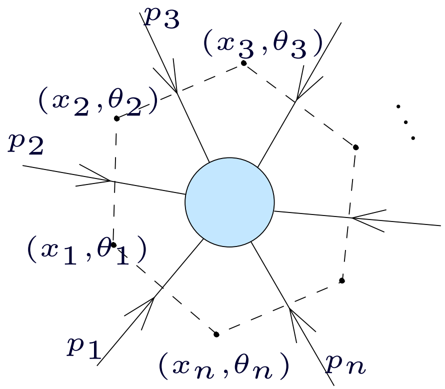
$$\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}} \times \Big[R_n(\lambda,\tilde\lambda,\eta) + O(\epsilon)\Big] = \mathcal{A}_n^{\mathrm{MHV}}\;\Big[1 + R_n^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon)\Big]$$

  $R_n^{(4k)}$: finite homogeneous polynomials in $\eta \longrightarrow$ helicity structures and loop corrections for all n–particle amplitudes.

- ✔ Conjecture: all $R_n^{(4k)}$ are exactly dual conformal. The conformal anomaly is confined to the IR divergent MHV prefactor.

Dual $\mathcal{N}=4$ superconformal symmetry

✔ Chiral dual superspace $(x_{\alpha\dot\alpha}\,,\,\theta^A_\alpha\,,\,\lambda_\alpha)$:



✗ $p=\sum_{i=1}^n p_i = 0 \;\rightarrow\; p_i = x_i - x_{i+1}\,, \quad x_{n+1}=x_1$

✗ $q=\sum_{i=1}^n \lambda_i\eta_i = 0 \;\rightarrow\; \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i-\theta_{i+1})^A_\alpha\,, \quad \theta_{n+1}=\theta_1$

✗

Defining constraints: $\lambda_i^\alpha\,(x_i - x_{i+1})_{\alpha\dot\alpha} = 0 \;\rightarrow$ derive $\tilde\lambda_i^{\dot\alpha}$

$\lambda_i^\alpha\,(\theta_i - \theta_{i+1})^A_\alpha = 0 \;\rightarrow$ derive η_i^A

✔ Dual $\mathcal{N}=4$ superconformal symmetry in dual superspace

✗ $\mathcal{N}=4$ super-Poincaré algebra

$$Q_{A\,\alpha} = \sum_{i=1}^n \partial/\partial\theta_i^{A\,\alpha}, \quad \bar{Q}^A_{\dot\alpha} = \sum_{i=1}^n \theta_i^{A\,\alpha}\partial/\partial x_i^{\dot\alpha\alpha},\; P_{\alpha\dot\alpha} = \sum_{i=1}^n \partial/\partial x_i^{\dot\alpha\alpha}\,; \quad \{Q_{A\,\alpha}, \bar{Q}^B_{\dot\alpha}\} = \delta^B_A\, P_{\alpha\dot\alpha}$$

✗ Conformal inversion: $I[x_i] = x_i^{-1}\,, \quad I[\theta_i] = \theta_i x_i^{-1}\,, \quad I[\lambda_i] = \lambda_i x_i^{-1}$

✗ From Poincaré to conformal supersymmetry:

- Conformal boosts: $K = IPI$
- Special conformal supersymmetry : $S = I\bar{Q}I$, $\bar{S} = IQI \equiv \bar{q}$

Dual superconformal symmetry: MHV superamplitudes

- ✔ Impose cyclicity, $x_{n+1} = x_1$, $\theta_{n+1} = \theta_1$, through delta functions. Then, only in $\mathcal{N} = 4$,

$$I[\delta^{(4)}(x_1 - x_{n+1})] = x_1^8\, \delta^{(4)}(x_1 - x_{n+1}) \qquad I[\delta^{(8)}(\theta_1 - \theta_{n+1})] = x_1^{-8}\, \delta^{(8)}(\theta_1 - \theta_{n+1})$$

- ✔ MHV superamplitude in dual superspace

$$\mathcal{A}_n^{\mathrm{MHV}}(x, \theta, \lambda) = \frac{\delta^{(4)}(x_1 - x_{n+1})\, \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle}\, M_n(x_{ij})$$

  - ✗ Tree – manifestly dual superconformal covariant.
  - ✗ Loops – IR divergent factor $M_n(x_{ij})$ satisfies anomalous dual conformal Ward identity

- ✔ Part of the superconformal algebra $(Q, \bar{S}, P)$ is a symmetry of the whole amplitude, and $(\bar{Q}, S, K, D)$ only of the tree and the helicity structures (due to anomalies):

$$\bar{Q}\mathcal{A} = \bar{S}\mathcal{A} = 0 \;\Rightarrow\; \{\bar{Q}, \bar{S}\}\mathcal{A} = (D - C)\mathcal{A} = 0 \;\Rightarrow\; D = C$$

  $D =$ conformal weight $\mapsto$ anomalous; $\qquad C =$ helicity $\mapsto$ protected ?
  Needs better understanding !

Dual superconformal symmetry: NMHV superamplitudes I

- ✔ General superamplitude: $\mathcal{A}_n = \mathcal{A}_n^{\rm MHV}(a, 1/\epsilon)\left[1 + R_n^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon)\right]$
  - ✗ $\mathcal{A}_n^{\rm MHV}$ is IR divergent and satisfies an anomalous dual CWI $\Leftrightarrow$ Wilson loop
  - ✗ Conjecture: $R_n^{(4)}$ are finite dual (super)conformal invariants [Drummond,Henn,Korchemsky,ES'08]
- ✔ Evidence: One-loop NMHV superamplitudes
  - ✗ n–gluon NMHV known [Bern, Dixon, Kosower'04]
  - ✗ New result: One-loop NMHV superamplitude $\Leftrightarrow$ dual (super)conformal invariant

$$R_n^{(4)} = \sum_{p,q,r=1}^{n} \delta^{(4)}(\Xi_{pqr})\, c_{pqr}\, M_{pqr}(x_{ij})$$

- dual superconformal covariant

$$\Xi_{pqr} = \langle p|x_{pq}x_{qr}|\theta_{rp}\rangle + \langle p|x_{pr}x_{rq}|\theta_{qp}\rangle = \langle p|x_{pq}x_{qr}\sum_{i=p}^{r-1}|i\rangle\eta_i + \langle p|x_{pr}x_{rq}\sum_{i=p}^{q-1}|i\rangle\eta_i$$

- dual conformal covariant with matching conformal and helicity weights

$$c_{pqr} = \frac{\langle q-1\, q\rangle\langle r-1\, r\rangle}{x_{qr}^2\langle p|x_{pr}x_{r\, q-1}|q-1\rangle\langle p|x_{pr}x_{r\, q}|q\rangle\langle p|x_{pq}x_{q\, r-1}|r-1\rangle\langle p|x_{pq}x_{q\, r}|r\rangle}$$

- dual conformal invariant $M_{pqr}(x_{ij}) = 1 + aM_{pqr}^{(\rm one-loop)} + ?\, O(a^2)$, made of finite combinations of one-loop scalar box integrals

Dual superconformal symmetry: NMHV superamplitudes II

✔ The superstructure

$$\delta^{(8)}(\sum_{i=1}^{n} \lambda_{i\,\alpha}\,\eta_i^A)\,\delta^{(4)}(\Xi_{pqr})\,c_{pqr} = \mathcal{H}_{m_1 m_2 m_3}\,\eta_{m_1}^4\,\eta_{m_2}^4\,\eta_{m_2}^4 + \dots$$

encodes all helicity structures for gluons, gluinos, scalars.

$\mathcal{H}_{m_1 m_2 m_3} \Leftrightarrow$ 3-mass-box coefficients [Bern,Dixon, Kosower'04]

✔ Expanding in η_i breaks manifest dual conformal symmetry, except for split-helicity terms. The non-split-helicity ones transform into each other

✔ NMHV tree-level superamplitudes: new, manifestly Lorentz covariant form of the NMHV tree superamplitude obtained by setting all $M_{pqr}(x_{ij}) = 1$

$$\mathcal{A}_{n;0}^{\mathrm{NMHV}} = \delta^{(4)}(\sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i)\,\delta^{(8)}(\sum_{j=1}^{n} \lambda_j\,\eta_j)\sum_{p,q,r=1}^{n} c_{pqr}\,\delta^{(4)}(\Xi_{pqr})$$

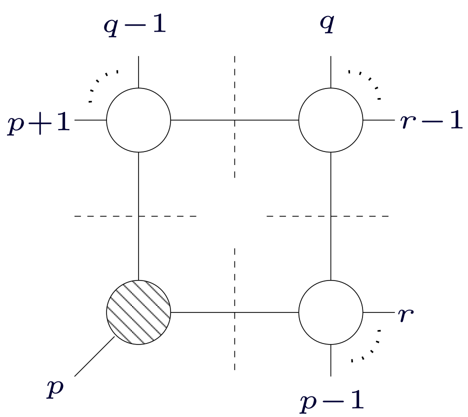
No need for reference spinor ! [Cachazo, Svrcek, Witten'04],[Georgio, Glover, Khoze'04]

✔ “Twistor coplanarity" [Witten'03], [Cachazo, Svrcek, Witten'04]
is a direct corollary of dual $\bar{Q}$ supersymmetry and of the obvious property

$$(\frac{\partial^2}{\partial\eta_i^A \eta_j^B} + A \leftrightarrow B) R_n^{(4)} = 0$$

Generalized unitarity in superspace

- ✔ Generalized unitarity – efficient method for computing one-loop corrections [Britto, Cachazo, Feng'04]
- ✔ Supersymmetrization – replace the sum over exchange particles by a Grassmann integral $\int d^4\eta$
- ✔ Allows to compute the complete one-loop NMHV superamplitude [Drummond,Henn,Korchemsky,ES'08]
- ✔ Example: 3-mass-box coefficients



- ✗ 3-particle MHV tree superamplitude $\leftrightarrow$ complexify the momenta, $\tilde\lambda \neq \bar\lambda$
- ✗ Grassmann Fourier transform of the 3-particle $\overline{\mathrm{MHV}}$ tree superamplitude (needed to get the right degree in η)

$$\mathcal{A}_{3;0}^{\overline{\mathrm{MHV}}} = i(2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{3} \lambda_i^{\alpha}\, \tilde\lambda_i^{\dot\alpha}\Big) \frac{\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12])}{[12][23][31]}$$

The result is exactly the 3-point dual superconformal invariant $c_{pqr}\ \delta^{(4)}(\Xi_{pqr})$

- ✔ 4-mass box coefficients contribute to NNMHV amplitudes. New type of superconformal invariant.

Conclusions and outlook

- ✔ Dual superconformal symmetry is a universal feature of $\mathcal{N} = 4$ scattering amplitudes
- ✔ Its field-theory origin is unknown (dynamical). Recent explanation from string theory.

  [Berkovits, Maldacena'08], [Beisert,Ricci,Tseytlin,Wolf'08]
- ✔ Probably the "tip of an iceberg" of an (infinite?) set of (non-local?) symmetries $\longrightarrow$ integrability?
- ✔ non-MHV amplitudes $\longrightarrow$ finite exactly dual conformal functions. Can we find differential equations for them? $\longrightarrow$ integrability?
- ✔ The MHV/Wislon loop duality does not see the helicity structure. Need to generalize the WL (supersymmetry? vertex operators?) and test if it is dual to non-MHV superamplitudes.