Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

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- Dual superconformal symmetry: NMHV superamplitudes
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On-shell scattering amplitudes in $\mathcal{N} = 4$ SYM

✓ Scattering amplitudes in $\mathcal{N} = 4$ SYM

× Quantum numbers of on-shell states $|i\rangle = |p_i, h_i, a_i\rangle$:
  momentum ($p_i^2 = 0$), helicity ($h_i$), color ($a_i$)

× IR divergences $\mapsto$ dimensional regularization
  $A_n = \text{Div}(p_i, 1/\epsilon, \mu) \times \text{Fin}(p_i) \rightarrow$ subject of this talk

× Perturbative expansion in 't Hooft coupling
  $a = g^2 N/8\pi^2$:
  $$A_n(p_i, h_i) = A_{n;0} + a \sum_H A^H_{n;1} M^H_{n;1}(p_i) + O(a^2)$$

✓ Simplest example: Maximally Helicity Violating (MHV) amplitudes, e.g. for gluons:
  $(- - + \ldots +), (- + - + \ldots +)$, etc.
  Unique helicity structure (tree):
  $$A_n^{\text{MHV}}(p_1^-, p_2^-, p_3^+, \ldots, p_n^+) = A_{n;0} M_n^{\text{MHV}}(p_i), \quad M_n^{\text{MHV}} = 1 + a M_n^{(1)} + O(a^2)$$

✓ $\mathcal{N} = 4$ SYM is a (super)conformal theory $\Rightarrow$ conformal symmetry of $A_n(p_i)$? Two problems:
  (i) Conformal boosts realized on momenta are 2nd-order differential operators [Witten'03]
  (ii) IR divergences break conformal symmetry
Hidden symmetry of $A_n$ of dynamical origin

Linear action on the particle momenta in dual space

$\times \; p_i = x_i - x_{i+1} \equiv x_i \; i+1 \; \Leftrightarrow \; \sum_i p_i = 0 \; \text{if} \; x_{n+1} \equiv x_1$

$\times \; p_i^2 = 0 \; \Leftrightarrow \; x_{i+1}^{2} = 0$

Simple change of variables, not a Fourier transform!

Conformal group $SO(4,2)$ acting on the dual coordinates $x_i$

$\Rightarrow$ dual conformal symmetry.

Example: MHV amplitudes

$$\ln M_n^{\text{MHV}} = \ln Z_n(x_i, \epsilon, \mu) + \ln F_n(x_i) + O(\epsilon), \quad \ln Z_n = \sum_{l \geq 1} a^l \sum_{i=1}^n (-x_{i,i+2}^2 \mu^2)^l \epsilon \left( \frac{\Gamma(l)}{(l \epsilon)^2} + \frac{\Gamma(l)}{l \epsilon} \right)$$

Duality MHV amplitude/Wilson loop

$$\ln F_n = \ln \langle 0 \mid \text{Tr} \exp \left( ig \oint_{C_n} dx^\mu A_\mu (x) \right) \mid 0 \rangle + \text{const} + O(\epsilon)$$

WL has conformal invariance in dual space $\Rightarrow$ Anomalous CWI:

$$K^\mu \ln F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^n \ln \frac{x_{i,i+2}}{x_{i,i-1,i+1}} \frac{x_{i,i+1}'}{x_{i,i+1}'} \Rightarrow \text{Fixes } \ln F_n \text{ for } n = 4, 5 \text{ but not for } n \geq 6$$
Dual conformal symmetry II

✔ Can we generalize dual conformal symmetry to non-MHV amplitudes?
Need to study the helicity structures and the loop corrections

✔ Spinor helicity formalism: commuting spinors \( \lambda^\alpha \) (helicity -1/2), \( \tilde{\lambda}^{\dot{\alpha}} \) (helicity 1/2) [Xu,Zhang,Chang'87]

\[
p_i^2 = 0 \iff p_i^{\alpha\dot{\alpha}} \equiv p_i^{\alpha} (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}}
\]

✔ Simplest case: MHV tree level [Parke, Taylor’86]

\[
A_{n;0}^{\text{MHV}} (\ldots i^- \ldots j^- \ldots) = \delta^{(4)} \left( \sum_{k=1}^{n} p_k \right) \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle}, \quad \langle i j \rangle = -\langle j i \rangle = \epsilon^{\alpha\beta \lambda \dot{\lambda}} \lambda_i^{\alpha} \lambda_j^{\beta}
\]

Is it dual conformal?

✔ Dual conformal transformations of spinors [Drummond,Henn,Korchemsky, ES’08]

× Conformal group = Poincaré + inversion:

\[
I[x^{\mu}] = \frac{x^{\mu}}{x^2} \equiv x^{-1}, \quad I[x_i - x_j] = x_i^{-1} (x_i - x_j) x_j^{-1}
\]

× Transforming spinors: \( p_i^{\alpha\dot{\alpha}} = (x_i - x_{i+1})^{\alpha\dot{\alpha}} = \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \Rightarrow
\]

\[
I[\lambda_i^{\alpha}] = \frac{\lambda_i^{\alpha} (x_i)^{\alpha\dot{\alpha}}}{x_i^2} \equiv \lambda_i x_i^{-1} = \lambda_i^{\alpha} \frac{(x_{i+1})^{\alpha\dot{\alpha}}}{x_{i+1}^2} \Rightarrow I[\langle i i+1 \rangle] = \langle i \rangle \frac{x_{i+1}}{x_i} \frac{x_i^{-1} |i + 1\rangle}{x_{i+1}^2} = \langle i i+1 \rangle \frac{x_i^2}{x_{i+1}}
\]
Dual conformal symmetry III

✔ What about the momentum conservation delta function \( \delta^{(4)}(\sum_{i=1}^{n} p_i) \)?

✗ Cyclic symmetry: \( \sum_{i=1}^{n} p_i = 0 \iff \sum_{i=1}^{n} (x_i - x_{i+1}) = 0 \iff x_{n+1} \equiv x_1 \)

✗ Relax cyclicity, \( x_1 \neq x_{n+1} \), and then impose it by

\[
\delta^{(4)}(x_1 - x_{n+1}) \to \text{manifestly dual conformal}
\]

✔ Split-helicity tree amplitudes: all negative-helicity gluons appear contiguously

✗ Known explicitly from recursion relations [Britto, Cachazo, Feng, Roiban, Spradlin, Volovich, Witten]

✗ Example: split-helicity MHV tree amplitude – manifestly dual conformal!

\[
\mathcal{A}_{\text{MHV}}^{\text{MHV}}(\ldots-n+\ldots+) = \delta^{(4)}(x_1 - x_{n+1}) \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle}
\]

✗ All split-helicity tree amplitudes are dual conformal

✗ Non-split-helicity amplitudes are dual conformal not on their own, but as parts of superamplitudes

✗ Superamplitudes in dual superspace exhibit dual superconformal symmetry.
Superamplitudes in on-shell superspace

\( \mathcal{N} = 4 \) gluon supermultiplet \( \rightarrow \) PCT self-conjugate \( \rightarrow \) holomorphic (chiral) description

\[
\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \eta^A \eta^B S_{AB}(p) + \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p) + \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p)
\]

\( \eta^A (SU(4) \) index \( A = 1 \ldots 4 \), helicity \( 1/2 \)) are Grassmann variables of on-shell superspace

Superamplitudes \( \mathcal{A}_n(\Phi(1) \ldots \Phi(n)) \) = expansion in powers of \( \eta_i^A \)

Example: Nair’s description of tree MHV amplitudes

\[
\mathcal{A}^\text{MHV}_n = \frac{\delta^{(4)} \left( \sum_{i=1}^n p_i \right) \delta^{(8)} \left( \sum_{j=1}^n \lambda_j \alpha \eta_j^A \right)}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} = \frac{\delta^{(4)} \left( \sum_{i=1}^n p_i \right)}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \left( \langle 12 \rangle^4 \eta_1^4 \eta_2^4 \eta_3^0 \ldots \eta_n^0 + \ldots \right)
\]

On-shell \( \mathcal{N} = 4 \) supersymmetry

Clifford algebra for massless Poincaré states:

\[
q^A = \eta^A , \quad \bar{q}_A = \partial / \partial \eta^A , \quad \{ q^A , \bar{q}_B \} = \delta^A_B
\]

Covariant description with the help of \( \lambda_\alpha \):

\[
q^{\alpha}_A = \lambda_\alpha \eta^A , \quad \bar{q}_A \dot{\alpha} = \bar{\lambda}_{\dot{\alpha}} \partial / \partial \eta^A
\]

On-shell \( \mathcal{N} = 4 \) supersymmetry algebra (with \( p^2 = 0 \))

\[
\{ q^A_\alpha , \bar{q}_B \dot{\alpha} \} = \delta^A_B \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} = \delta^A_B p_{\alpha\dot{\alpha}}
\]
Invariance of the superamplitude: \( p_{\alpha \dot{\alpha}} A_n = q^A_{\alpha} A_n = 0 \Rightarrow \)

\[ A_n(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) \delta^{(8)}\left(\sum_{j=1}^{n} \lambda_j \eta_j\right) \left[ A_n^{(0)} + A_n^{(4)} + \ldots + A_n^{(4n-16)}\right] \]

- \( A_n^{(4k)}(\eta) \) – homogeneous polynomials in \( \eta \) of degree \( 4k \):
  - \( k = 0 \rightarrow \text{MHV}, \ k = 1 \rightarrow \text{Next-to-MHV}, \ldots, \ k = n - 4 \rightarrow \overline{\text{MHV}} \)

- Simplest case – All-order MHV superamplitude:
  \[ A_n^{\text{MHV}}(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) \delta^{(8)}\left(\sum_{j=1}^{n} \lambda_j \eta_j\right) \left[ \frac{M_n(p)}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle}\right] \]

- Define ‘ratio’ \( R = \text{general/MHV superamplitude} : \)
  \[ A_n = A_n^{\text{MHV}} \times \left[ R_n(\lambda, \tilde{\lambda}, \eta) + O(\epsilon)\right] = A_n^{\text{MHV}} \left[ 1 + R_n^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon)\right] \]

- \( R_n^{(4k)} \) : finite homogeneous polynomials in \( \eta \rightarrow \) helicity structures and loop corrections for all \( n \)–particle amplitudes.

- Conjecture: all \( R_n^{(4k)} \) are exactly dual conformal. The conformal anomaly is confined to the IR divergent MHV prefactor.
Dual $\mathcal{N} = 4$ superconformal symmetry

✔ Chiral dual superspace $(x_{\alpha \dot{\alpha}}, \theta^A_{\alpha}, \lambda_\alpha)$:

$\times$ $p = \sum_{i=1}^n p_i = 0 \rightarrow p_i = x_i - x_{i+1}, \quad x_{n+1} = x_1$

$\times$ $q = \sum_{i=1}^n \lambda_i \eta_i = 0 \rightarrow \lambda_i \eta_i^A = (\theta_i - \theta_{i+1})^A_{\alpha}, \quad \theta_{n+1} = \theta_1$

Defining constraints: $\lambda_i^\alpha (x_i - x_{i+1})_{\alpha \dot{\alpha}} = 0 \rightarrow$ derive $\tilde{\lambda}_i^{\dot{\alpha}}$

$\lambda_i^\alpha (\theta_i - \theta_{i+1})_{\alpha}^A = 0 \rightarrow$ derive $\eta_i^A$

✔ Dual $\mathcal{N} = 4$ superconformal symmetry in dual superspace

$\times$ $\mathcal{N} = 4$ super-Poincaré algebra

$Q_A^\alpha = \sum_{i=1}^n \partial/\partial \theta_i^A_{\alpha}, \quad \bar{Q}_I^{\dot{\alpha}} = \sum_{i=1}^n \theta_i^A_{\alpha} \partial/\partial x_i^\dot{\alpha}, \quad P_{\alpha \dot{\alpha}} = \sum_{i=1}^n \partial/\partial x_i^\dot{\alpha}; \quad \{Q_A^\alpha, \bar{Q}_I^{\dot{\alpha}}\} = \delta^B_A \quad P_{\alpha \dot{\alpha}}$

$\times$ Conformal inversion: $I[x_i] = x_i^{-1}, \quad I[\theta_i] = \theta_i x_i^{-1}, \quad I[\lambda_i] = \lambda_i x_i^{-1}$

$\times$ From Poincaré to conformal supersymmetry:
- Conformal boosts: $K = IPI$
- Special conformal supersymmetry: $S = I\bar{Q}I, \bar{S} = IQI \equiv \bar{q}$
Dual superconformal symmetry: MHV superamplitudes

✔ Impose cyclicity, \(x_{n+1} = x_1, \theta_{n+1} = \theta_1\), through delta functions. Then, only in \(\mathcal{N} = 4\),

\[
I[\delta^{(4)}(x_1 - x_{n+1})] = x_1^8 \delta^{(4)}(x_1 - x_{n+1}) \quad I[\delta^{(8)}(\theta_1 - \theta_{n+1})] = x_1^{-8} \delta^{(8)}(\theta_1 - \theta_{n+1})
\]

✔ MHV superamplitude in dual superspace

\[
\mathcal{A}_{\text{MHV}}^n(x, \theta, \lambda) = \frac{\delta^{(4)}(x_1 - x_{n+1}) \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle} M_n(x_{ij})
\]

✗ Tree – manifestly dual superconformal covariant.
✗ Loops – IR divergent factor \(M_n(x_{ij})\) satisfies anomalous dual conformal Ward identity

✔ Part of the superconformal algebra \((Q, \bar{S}, P)\) is a symmetry of the whole amplitude, and \((\bar{Q}, S, K, D)\) only of the tree and the helicity structures (due to anomalies):

\[
\bar{Q}A = \bar{S}A = 0 \Rightarrow \{\bar{Q}, \bar{S}\}A = (D - C)A = 0 \Rightarrow D = C
\]

\(D = \) conformal weight \(\leftrightarrow\) anomalous; \(C = \) helicity \(\leftrightarrow\) protected ?
Needs better understanding!
Dual superconformal symmetry: NMHV superamplitudes I

✔ General superamplitude: \( A_n = A_n^{\text{MHV}}(a, 1/\epsilon) \left[ 1 + R_n^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon) \right] \)

✗ \( A_n^{\text{MHV}} \) is IR divergent and satisfies an anomalous dual CWI ⇔ Wilson loop

✗ Conjecture: \( R_n^{(4)} \) are finite dual (super)conformal invariants

[Drummond, Henn, Korchemsky, ES'08]

✔ Evidence: One-loop NMHV superamplitudes

✗ \( n \)-gluon NMHV known

✗ New result: One-loop NMHV superamplitude ⇔ dual (super)conformal invariant

\[
R_n^{(4)} = \sum_{p, q, r=1}^n \delta^{(4)}(\Xi_{pqr}) c_{pqr} M_{pqr}(x_{ij})
\]

■ dual superconformal covariant

\[
\Xi_{pqr} = \langle p | x_{pq} x_{qr} | \theta_{rp} \rangle + \langle p | x_{pr} x_{rq} | \theta_{qp} \rangle = \langle p | x_{pq} x_{qr} \sum_{i=p}^{r-1} | i \rangle \eta_i + \langle p | x_{pr} x_{rq} \sum_{i=p}^{q-1} | i \rangle \eta_i
\]

■ dual conformal covariant with matching conformal and helicity weights

\[
c_{pqr} = \frac{\langle q-1 | q \rangle \langle r-1 | r \rangle}{x_{qr}^2 \langle p | x_{pr} x_{rq} | q-1 \rangle \langle p | x_{pr} x_{rq} | q \rangle \langle p | x_{pq} x_{qr} | r-1 \rangle \langle p | x_{pq} x_{qr} | r \rangle}
\]

■ dual conformal invariant \( M_{pqr}(x_{ij}) = 1 + aM_{pqr}^{(\text{one-loop})} + O(a^2) \), made of finite combinations of one-loop scalar box integrals
The superstructure

\[ \delta^{(8)} \left( \sum_{i=1}^{n} \lambda_i \alpha \eta_i^A \right) \delta^{(4)}(\Xi_{pqr}) \ c_{pqr} = \mathcal{H}_{m_1 m_2 m_3} \eta_{m_1}^4 \eta_{m_2}^4 \eta_{m_2}^4 + \ldots \]

encodes all helicity structures for gluons, gluinos, scalars.

\[ \mathcal{H}_{m_1 m_2 m_3} \Leftrightarrow \text{3-mass-box coefficients} \quad \text{[Bern,Dixon, Kosower'04]} \]

Expanding in \( \eta_i \) breaks manifest dual conformal symmetry, except for split-helicity terms. The non-split-helicity ones transform into each other.

NMHV tree-level superamplitudes: new, manifestly Lorentz covariant form of the NMHV tree superamplitude obtained by setting all \( M_{pqr}(x_{ij}) = 1 \)

\[ A_{n;0}^{\text{NMHV}} = \delta^{(4)} \left( \sum_{i=1}^{n} \lambda_i \bar{\lambda}_i \right) \delta^{(8)} \left( \sum_{j=1}^{n} \lambda_j \eta_j \right) \sum_{p,q,r=1}^{n} c_{pqr} \delta^{(4)}(\Xi_{pqr}) \]

No need for reference spinor! \quad \text{[Cachazo, Svrcek, Witten'04],[Georgio, Glover, Khoze'04]}

“Twistor coplanarity” \quad \text{[Witten'03], [Cachazo, Svrcek, Witten'04]}

is a direct corollary of dual \( \bar{Q} \) supersymmetry and of the obvious property

\[ \left( \frac{\partial^2}{\partial \eta_i^A \eta_j^B} + A \leftrightarrow B \right) R_n^{(4)} = 0 \]
Generalized unitarity in superspace

- Generalized unitarity – efficient method for computing one-loop corrections [Britto, Cachazo, Feng’04]
- Supersymmetrization – replace the sum over exchange particles by a Grassmann integral $\int d^4 \eta$
- Allows to compute the complete one-loop NMHV superamplitude [Drummond, Henn, Korchemsky, ES’08]
- Example: 3-mass-box coefficients

- 3-particle MHV tree superamplitude $\leftrightarrow$ complexify the momenta, $\tilde{\lambda} \neq \bar{\lambda}$
- Grassmann Fourier transform of the 3-particle $\text{MHV}$ tree superamplitude (needed to get the right degree in $\eta$)

$$\mathcal{A}_{3;0}^{\text{MHV}} = i(2\pi)^4 \delta(4) \left( \sum_{i=1}^{3} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \right) \frac{\delta(4)(\eta_1[23] + \eta_2[31] + \eta_3[12])}{[12][23][31]}$$

The result is exactly the 3-point dual superconformal invariant $c_{pqr} \delta(4)(\Xi_{pqr})$

- 4-mass box coefficients contribute to NNNMHV amplitudes. New type of superconformal invariant.
Conclusions and outlook

✔ Dual superconformal symmetry is a universal feature of $\mathcal{N} = 4$ scattering amplitudes

✔ Its field-theory origin is unknown (dynamical). Recent explanation from string theory.
[Berkovits, Maldacena’08], [Beisert, Ricci, Tseytlin, Wolf’08]

✔ Probably the “tip of an iceberg” of an (infinite?) set of (non-local?) symmetries $\rightarrow$ integrability?

✔ non-MHV amplitudes $\rightarrow$ finite exactly dual conformal functions. Can we find differential equations for them? $\rightarrow$ integrability?

✔ The MHV/Wilson loop duality does not see the helicity structure. Need to generalize the WL (supersymmetry? vertex operators?) and test if it is dual to non-MHV superamplitudes.