$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie (super)algebras and generalized quantum statistics

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Why $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras?

- Quantum physics: commutators $[x, y]$ and anticommutators $\{x, y\}$ between operators x and y
- Starting from an associative algebra, bracket $[x, y] = xy - yx$ leads to a Lie algebra
- Starting from a $\mathbb{Z}_2$-graded associative algebra, bracket $[\![x, y]\!] = xy - (-1)^{\xi\eta} yx$ leads to a Lie superalgebra
- Shall we go beyond and why?

Why $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras?

- For two elements x, y in an associative algebra, the trivial product identity can be rewritten as

$$[x, y] + [y, x] = 0, \qquad \text{or} \qquad \{x, y\} - \{y, x\} = 0$$

For three elements x, y, z in an associative algebra, the trivial product identity can be rewritten in (essentially) four ways:

$$
\begin{aligned}
&(1) \qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \\
&(2) \qquad [x, \{y, z\}] + [y, \{z, x\}] + [z, \{x, y\}] = 0, \\
&(3) \qquad [x, \{y, z\}] + \{y, [z, x]\} - \{z, [x, y]\} = 0, \\
&(4) \qquad [x, [y, z]] + \{y, \{z, x\}\} - \{z, \{x, y\}\} = 0.
\end{aligned}
$$

(1) Jacobi identity for Lie algebras (LA); (1)–(3) Jacobi identity for Lie superalgebras (LSA); (4) can appear only as Jacobi identity for $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras or $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras

Bosons and Fermions

- Bose operators $B_i^{\pm}$:

$$[B_i^-, B_j^+] = \delta_{ij} \qquad \text{all other } [\cdot, \cdot] \text{ zero}$$

  Bose-Einstein statistics

- Fermi operators $F_i^{\pm}$:

$$\{F_i^-, F_j^+\} = \delta_{ij} \qquad \text{all other } \{\cdot, \cdot\} \text{ zero}$$

  Fermi-Dirac statistics

- many open problems; quantum theory allows for the existence of infinitely many families of paraparticles, obeying mixed-symmetry statistics.

Parabosons and parafermions

- parabosons $b_j^\pm$ [Green 1953]:

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \eta)\delta_{kl} b_j^\xi + (\epsilon - \xi)\delta_{jl} b_k^\eta$$

- Fock space $V(p)$ characterized by $(b_j^\pm)^\dagger = b_j^\mp$ and $b_j^- |0\rangle = 0$ and [Greenberg & Messiah 1965]

$$\{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle$$

- parafermions $f_j^\pm$ [Green 1953]:

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = |\epsilon - \eta|\delta_{kl} f_j^\xi - |\epsilon - \xi|\delta_{jl} f_k^\eta$$

- Fock space $W(p)$ characterized by $(f_j^\pm)^\dagger = f_j^\mp$ and $f_j^- |0\rangle = 0$ and [Greenberg & Messiah 1965]

$$[f_j^-, f_k^+]|0\rangle = p\,\delta_{jk}\,|0\rangle$$

Paraboson and parafermion algebra

Theorem (LA by generators and relations) [Kamefuchi & Takahishi 1962; Ryan & Sudarshan 1963]

The Lie algebra (LA) generated by $2m$ elements $f_j^{\pm}$ subject to the parafermion triple relations is $\mathfrak{so}(2m+1)$. The Fock space $W(p)$ is the unitary irreducible representation of $\mathfrak{so}(2m+1)$ with lowest weight $(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$.

$p = 1$

Theorem (LSA by generators and relations) [Ganchev & Palev 1980]

The Lie superalgebra (LSA) generated by $2n$ odd elements $b_j^{\pm}$ subject to the paraboson triple relations is $\mathfrak{osp}(1|2n)$. The Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$.

$p = 1$

Parastatistics, parastatistics algebra

Simultaneous system: can be combined in 2 non-trivial ways [Greenberg, Messiah]. The first of these are the so-called: *relative parafermion relations:*

$$[[f_j^\xi, f_k^\eta], b_l^\epsilon] = 0, \qquad [\{b_j^\xi, b_k^\eta\}, f_l^\epsilon] = 0,$$
$$[[f_j^\xi, b_k^\eta], f_l^\epsilon] = -|\epsilon - \xi|\delta_{jl} b_k^\eta, \qquad \{[f_j^\xi, b_k^\eta], b_l^\epsilon\} = (\epsilon - \eta)\delta_{kl} f_j^\xi.$$

Theorem [Palev 1982]

The Lie superalgebra (LSA) generated by $2m$ even elements $f_j^\pm$ and $2n$ odd elements $b_j^\pm$ subject to the above relations is $\mathfrak{osp}(2m+1|2n)$. The Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(2m+1|2n)$ with lowest weight $[-\frac{p}{2}, \ldots, -\frac{p}{2} | \frac{p}{2}, \ldots, \frac{p}{2}]$.

Parastatistics, parastatistics algebra

Simultaneous system: the **second** non-trivial relative commutation relations (the so-called paraboson relations) between parafermions and parabosons are defined by:

$$[[\bar{f}_j^{\,\xi}, \bar{f}_k^{\,\eta}], \bar{b}_l^{\epsilon}] = 0, \qquad [\{\bar{b}_j^{\xi}, \bar{b}_k^{\eta}\}, \bar{f}_l^{\,\epsilon}] = 0,$$

$$\{\{\bar{f}_j^{\,\xi}, \bar{b}_k^{\eta}\}, \bar{f}_l^{\,\epsilon}\} = |\epsilon - \xi|\delta_{jl}\bar{b}_k^{\eta}, \qquad [\{\bar{f}_j^{\,\xi}, \bar{b}_k^{\eta}\}, \bar{b}_l^{\epsilon}] = (\epsilon - \eta)\delta_{kl}\bar{f}_j^{\,\xi}.$$

The second case leads to an algebra which is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-**graded Lie superalgebra.**

Theorem [Tolstoy 2014]

The algebra generated by $2m$ parafermions $f_j^{\pm}$ and $2n$ parabosons $b_j^{\pm}$ subject to the above relations is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra denoted by $\mathfrak{osp}(1, 2m|2n, 0) \equiv \mathfrak{pso}(2m+1|2n)$. The Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{pso}(2m+1|2n)$ with lowest weight $[-\frac{p}{2}, \dots, -\frac{p}{2}|\frac{p}{2}, \dots, \frac{p}{2}]$.

Renewed interest

- symmetries of Lévy–Leblond equations [Aizawa et al 2016, 2017]
- graded (quantum) mechanics and quantization [Bruce 2020; Aizawa, Kuznetsova, Toppan 2020, 2021; Quesne 2021]
- $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded two-dimensional models [Bruce 2021, Toppan 2021]
- parastatistics [Tolstoy 2014, Stoilova and Van der Jeugt 2018]
- alternative descriptions of parabosons and parafermions [Toppan 2021]
- algebraic structute and representation theory [Aizawa 2018-2021, Issac 2019, 2024, Rui Lu 2023]

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras

V. Rittenberg and D. Wyler (1978)

- $\mathfrak{g} = \bigoplus_{\boldsymbol{a}} \mathfrak{g}_{\boldsymbol{a}} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}$
  with $\boldsymbol{a} = (a_1, a_2)$ an element of $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- homogeneous elements of $\mathfrak{g}_{\boldsymbol{a}}$: $x_{\boldsymbol{a}}$ with degree $deg\ x_{\boldsymbol{a}}$
- $\mathfrak{g}$ with bracket $[\![.,.]\!]$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra, resp. $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra:

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] \in \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}},$$
$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = -(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, x_{\boldsymbol{a}}]\!],$$
$$[\![x_{\boldsymbol{a}}, [\![y_{\boldsymbol{b}}, z_{\boldsymbol{c}}]\!]]\!] = [\![[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!], z_{\boldsymbol{c}}]\!] + (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, [\![x_{\boldsymbol{a}}, z_{\boldsymbol{c}}]\!]]\!],$$

where
$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$,
$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_2 - a_2 b_1$ - $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra
$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$ - $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra

General remarks

- Note: in general, a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra is NOT a Lie algebra, nor a Lie superalgebra.
- (Similarly: a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra is NOT a Lie superalgebra.)
- $\mathfrak{g}_{(0,0)}$ is a Lie subalgebra;
  $\mathfrak{g}_{(0,1)}$, $\mathfrak{g}_{(1,0)}$ and $\mathfrak{g}_{(1,1)}$ are $\mathfrak{g}_{(0,0)}$-modules.
- $[\mathfrak{g}_{(0,0)}, \mathfrak{g}_{\boldsymbol{a}}] \subset \mathfrak{g}_{\boldsymbol{a}}, \quad [\![\mathfrak{g}_{\boldsymbol{a}}, \mathfrak{g}_{\boldsymbol{a}}]\!] \subset \mathfrak{g}_{(0,0)}, \qquad \boldsymbol{a} \in \mathbb{Z}_2 \times \mathbb{Z}_2$
- Let $\mathfrak{g}$ be an associative $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra, with a product denoted by $x \cdot y$:

$$\mathfrak{g}_{\boldsymbol{a}} \cdot \mathfrak{g}_{\boldsymbol{b}} \subset \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}$$

  then $(\mathfrak{g}, [\![\cdot,\cdot]\!])$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra, resp. a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra, by defining

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = x_{\boldsymbol{a}} \cdot y_{\boldsymbol{b}} - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} y_{\boldsymbol{b}} \cdot x_{\boldsymbol{a}},$$

  with $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_2 - a_2 b_1$, resp. with $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$.

General remarks: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras

- Now consider: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras
- Assume at least two nontrivial subspaces in $\mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}$
- $\{\mathfrak{g}_{\boldsymbol{a}}, \mathfrak{g}_{\boldsymbol{b}}\} \subset \mathfrak{g}_{\boldsymbol{c}}$ if $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are mutually distinct elements of $\{(1,0),(0,1),(1,1)\}$.
- If $\mathfrak{g} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra, any permutation of the last three subspaces maps $\mathfrak{g}$ into another $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra. ("trivial permutation transformations")
- Moreover: natural to assume that $\mathfrak{g}$ is generated by $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$.
- Then one can deduce

$$\mathfrak{g}_{(0,0)} = [\![\mathfrak{g}_{(1,0)}, \mathfrak{g}_{(1,0)}]\!] + [\![\mathfrak{g}_{(0,1)}, \mathfrak{g}_{(0,1)}]\!]$$
$$\mathfrak{g}_{(1,1)} = [\![\mathfrak{g}_{(1,0)}, \mathfrak{g}_{(0,1)}]\!].$$

Construction of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras

Let V be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded linear space of dimension n:
$V = V_{(0,0)} \oplus V_{(0,1)} \oplus V_{(1,0)} \oplus V_{(1,1)}$, subspaces of dimension $p+q+r+s=n$.
$\mathrm{End}(V)$ is then a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded associative algebra, and turned into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra by the bracket $[\![\cdot,\cdot]\!]$. Denoted by $\mathfrak{gl}_{p,q,r,s}(n)$. In matrix form:

$$\begin{array}{cc} \begin{array}{cccc} p & q & r & s \end{array} & \\ \left(\begin{array}{cccc} a_{(0,0)} & a_{(0,1)} & a_{(1,0)} & a_{(1,1)} \\ b_{(0,1)} & b_{(0,0)} & b_{(1,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(1,1)} & c_{(0,0)} & c_{(0,1)} \\ d_{(1,1)} & d_{(1,0)} & d_{(0,1)} & d_{(0,0)} \end{array}\right) & \begin{array}{c} p \\ q \\ r \\ s \end{array} \end{array}$$

The indices of the matrix blocks refer to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.
One can check: $\mathrm{Tr}[\![A,B]\!]=0$, hence $\mathfrak{g}=\mathfrak{sl}_{p,q,r,s}(n)$ is subalgebra of traceless elements.

$$\begin{array}{ll} \mathfrak{g}_{(0,0)} & p^2+q^2+r^2+s^2-1 \\ \mathfrak{g}_{(0,1)} & 2pq+2rs \\ \mathfrak{g}_{(1,0)} & 2pr+2qs \\ \mathfrak{g}_{(1,1)} & 2qr+2ps \end{array}$$

Graded transpose

If $A \in \mathfrak{sl}_{p,q,r,s}(n) \subset \mathrm{End}(V)$, then $A^* \in \mathrm{End}(V^*)$ by requirement:

$$\langle A^* y_{\boldsymbol{b}}, x\rangle = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} \langle y_{\boldsymbol{b}}, Ax\rangle$$

where $\langle \cdot, \cdot \rangle$ is natural pairing of V and V^*.
In matrix form, this leads to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded transpose A^T of A:

$$A = \begin{pmatrix} a_{(0,0)} & a_{(0,1)} & a_{(1,0)} & a_{(1,1)} \\ b_{(0,1)} & b_{(0,0)} & b_{(1,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(1,1)} & c_{(0,0)} & c_{(0,1)} \\ d_{(1,1)} & d_{(1,0)} & d_{(0,1)} & d_{(0,0)} \end{pmatrix}, A^T = \begin{pmatrix} a^t_{(0,0)} & b^t_{(0,1)} & c^t_{(1,0)} & d^t_{(1,1)} \\ a^t_{(0,1)} & b^t_{(0,0)} & -c^t_{(1,1)} & -d^t_{(1,0)} \\ a^t_{(1,0)} & -b^t_{(1,1)} & c^t_{(0,0)} & -d^t_{(0,1)} \\ a^t_{(1,1)} & -b^t_{(1,0)} & -c^t_{(0,1)} & d^t_{(0,0)} \end{pmatrix}$$

Property:

$$\boxed{(AB)^T = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} B^T A^T}$$

Subalgebra $\mathfrak{g} = \mathfrak{so}_{p,q,r,s}(n) \subset \mathfrak{sl}_{p,q,r,s}(n)$

$$\mathfrak{g} = \mathfrak{so}_{p,q,r,s}(n) = \{A \in \mathfrak{sl}_{p,q,r,s}(n) \mid A^T + A = 0\}$$

If $A, B \in \mathfrak{g}$, then

$$\begin{aligned}
[\![A, B]\!]^T &= (AB - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} BA)^T \\
&= (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} B^T A^T - A^T B^T = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} BA - AB = -[\![A, B]\!]
\end{aligned}$$

Matrices of the form:

$$\begin{array}{c c}
\begin{array}{cccc} p & q & r & s \end{array} & \\
\begin{pmatrix}
a_{(0,0)} & a_{(0,1)} & a_{(1,0)} & a_{(1,1)} \\
-a^t_{(0,1)} & b_{(0,0)} & b_{(1,1)} & b_{(1,0)} \\
-a^t_{(1,0)} & b^t_{(1,1)} & c_{(0,0)} & c_{(0,1)} \\
-a^t_{(1,1)} & b^t_{(1,0)} & c^t_{(0,1)} & d_{(0,0)}
\end{pmatrix} & \begin{array}{c} p \\ q \\ r \\ s \end{array}
\end{array}$$

where $a_{(0,0)}$, $b_{(0,0)}$, $c_{(0,0)}$ and $d_{(0,0)}$ are antisymmetric matrices.

Disadvantages: Cartan subalgebra? (classical choice not abelian)

Different approach

Analogues of classical Lie algebras of type B, C, D?

$G = \mathfrak{so}(2n+1)$
$(\dim G = 2n^2 + n)$
$$\begin{array}{cc} \begin{array}{ccc} n & n & 1 \end{array} & \\ \begin{pmatrix} a & b & c \\ d & -a^t & e \\ -e^t & -c^t & 0 \end{pmatrix} & \begin{array}{c} n \\ n \\ 1 \end{array} \end{array}$$
b and d antisymmetric;

$G = \mathfrak{sp}(2n)$
$(\dim G = 2n^2 + n)$
$$\begin{array}{cc} \begin{array}{cc} n & n \end{array} & \\ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} & \begin{array}{c} n \\ n \end{array} \end{array}$$
b and c symmetric;

$G = \mathfrak{so}(2n)$
$(\dim G = 2n^2 - n)$
$$\begin{array}{cc} \begin{array}{cc} n & n \end{array} & \\ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} & \begin{array}{c} n \\ n \end{array} \end{array}$$
b and c antisymmetric,

Different approach

- start from a set of generators of the classical Lie algebra (in the defining matrix form)
- associate a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on these generators
- compute new elements with these generators using the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded bracket, and see which matrix structures and algebras arise in this way.

How to do this systematically?

- Let generating subspace S of the classical Lie algebra G correspond to the subspace $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$ of the associated $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{g}$, and generate $\mathfrak{g}$.
- Thus we are looking for generating subspaces S of a classical Lie algebra G such that $G = S + [S, S]$ (as vector space).
- Use all so-called 5-gradings $G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$ of G such that G is generated by $S = G_{-1} \oplus G_1$.

Different approach

Classification of those 5-gradings: [Stoilova and Van der Jeugt 2005]

Procedure:

- For each of the 5-gradings of G, let $S = G_{-1} \oplus G_1$ (as a subspace of the vector space of G).
- Partition S in all possible ways in two subspaces $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$.
- Construct from here the matrix elements of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{g}$ using the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded bracket.

This construction process is straightforward but very elaborate.

For $\mathfrak{sl}(n)$: same graded algebras $\mathfrak{sl}_{p,q,r,s}(n)$.
Results on following slides.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type C

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\boxed{\mathfrak{g} = \mathfrak{sp}_p(2n)}$ consists of all matrices of the following block form:

$$\begin{array}{cc} \begin{array}{cccc} p & n-p & p & n-p \end{array} & \\ \left(\begin{array}{cc|cc} a_{(0,0)} & a_{(1,0)} & b_{(1,1)} & b_{(0,1)} \\ \tilde{a}_{(1,0)} & \tilde{a}_{(0,0)} & -b_{(0,1)}^t & \tilde{b}_{(1,1)} \\ \hline c_{(1,1)} & c_{(0,1)} & -a_{(0,0)}^t & -\tilde{a}_{(1,0)}^t \\ -c_{(0,1)}^t & \tilde{c}_{(1,1)} & -a_{(1,0)}^t & -\tilde{a}_{(0,0)}^t \end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array} \end{array}$$

where $b_{(1,1)}$, $\tilde{b}_{(1,1)}$, $c_{(1,1)}$ and $\tilde{c}_{(1,1)}$ are symmetric matrices.

$$\dim \mathfrak{g}_{(0,0)} = p^2 + (n-p)^2$$
$$\dim \mathfrak{g}_{(0,1)} = 2p(n-p), \quad \dim \mathfrak{g}_{(1,0)} = 2p(n-p)$$
$$\dim \mathfrak{g}_{(1,1)} = p(p+1) + (n-p)(n-p+1).$$

Note: $\dim \mathfrak{sp}_p(2n) = \dim \mathfrak{sp}(2n)$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type C

Having this form, one can verify that $\mathfrak{sp}_p(2n)$ consists of all matrices A of $\mathfrak{sl}_{p,n-p,p,n-p}(2n)$ that satisfy

$$\boxed{A^T J + JA = 0} \qquad\qquad (*)$$

where

$$J = \begin{array}{cc} \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array} \end{array}$$

Note: $J^T = -J$, $J^{-1} = J^t$.

Easy to show that $[\![A, B]\!]$ satisfies (*) when A and B satisfy (*).

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type D

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\boxed{\mathfrak{g} = \mathfrak{so}_p(2n)}$ consists of all matrices of the following block form:

$$
\begin{array}{cc}
\begin{array}{cccc} p & n-p & p & n-p \end{array} & \\
\left(\begin{array}{cc|cc}
a_{(0,0)} & a_{(1,0)} & b_{(1,1)} & b_{(0,1)} \\
\tilde{a}_{(1,0)} & \tilde{a}_{(0,0)} & b^t_{(0,1)} & \tilde{b}_{(1,1)} \\
\hline
c_{(1,1)} & c_{(0,1)} & -a^t_{(0,0)} & -\tilde{a}^t_{(1,0)} \\
c^t_{(0,1)} & \tilde{c}_{(1,1)} & -a^t_{(1,0)} & -\tilde{a}^t_{(0,0)}
\end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array}
\end{array}
$$

where $b_{(1,1)}$, $\tilde{b}_{(1,1)}$, $c_{(1,1)}$ and $\tilde{c}_{(1,1)}$ are antisymmetric matrices.

$$\dim \mathfrak{g}_{(0,0)} = p^2 + (n-p)^2$$
$$\dim \mathfrak{g}_{(0,1)} = 2p(n-p), \quad \dim \mathfrak{g}_{(1,0)} = 2p(n-p)$$
$$\dim \mathfrak{g}_{(1,1)} = p(p-1) + (n-p)(n-p-1).$$

Note: $\dim \mathfrak{so}_p(2n) = \dim \mathfrak{so}(2n)$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type D

One can verify that $\mathfrak{so}_p(2n)$ consists of all matrices A of $\mathfrak{sl}_{p,n-p,p,n-p}(2n)$ that satisfy

$$\boxed{A^T K + KA = 0}$$

where

$$K = \begin{array}{c} \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array}$$

Note: $K^T = K$, $K^{-1} = K^t$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type B

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{so}_p(2n+1)$ consists of all matrices of the following block form:

$$\begin{array}{c c}
\begin{array}{ccccc} p & n-p & p & n-p & 1 \end{array} & \\
\left(\begin{array}{cc:cc:c}
a_{(0,0)} & a_{(1,1)} & b_{(0,0)} & b_{(1,1)} & c_{(0,1)} \\
\tilde{a}_{(1,1)} & \tilde{a}_{(0,0)} & b^t_{(1,1)} & \tilde{b}_{(0,0)} & c_{(1,0)} \\
\hdashline
d_{(0,0)} & d_{(1,1)} & -a^t_{(0,0)} & \tilde{a}^t_{(1,1)} & e_{(0,1)} \\
d^t_{(1,1)} & \tilde{d}_{(0,0)} & a^t_{(1,1)} & -\tilde{a}^t_{(0,0)} & e_{(1,0)} \\
\hdashline
-e^t_{(0,1)} & -e^t_{(1,0)} & -c^t_{(0,1)} & -c^t_{(1,0)} & 0
\end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \\ 1 \end{array}
\end{array}$$

where $b_{(0,0)}$, $\tilde{b}_{(0,0)}$, $d_{(0,0)}$ and $\tilde{d}_{(0,0)}$ are antisymmetric matrices.

$$\dim \mathfrak{g}_{(0,0)} = 2n^2 - n - 4p(n-p)^2$$
$$\dim \mathfrak{g}_{(0,1)} = 2p, \quad \dim \mathfrak{g}_{(1,0)} = 2(n-p)$$
$$\dim \mathfrak{g}_{(1,1)} = 4p(n-p).$$

Note: $\dim \mathfrak{so}_p(2n+1) = \dim \mathfrak{so}(2n+1)$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type B

One can verify that $\mathfrak{g} = \mathfrak{so}_p(2n+1)$ consists of all matrices A of $\mathfrak{sl}_{2p,0,2n-2p,1}(2n)$ that satisfy

$$\boxed{A^T K' + K' A = 0}$$

where

$$K' = \begin{array}{c} \left(\begin{array}{cc|cc|c} 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} p \\ n-p \\ p \\ n-p \\ 1 \end{array}$$

Note: $K'^T = K'$, $K'^{-1} = K'^t$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras

- Now consider: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras
- Let $\mathfrak{g}$ be an associative $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra, with a product denoted by $x \cdot y$:

$$\mathfrak{g}_{\boldsymbol{a}} \cdot \mathfrak{g}_{\boldsymbol{b}} \subset \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}$$

then $(\mathfrak{g}, [\![\cdot, \cdot]\!])$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra by defining

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = x_{\boldsymbol{a}} \cdot y_{\boldsymbol{b}} - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} y_{\boldsymbol{b}} \cdot x_{\boldsymbol{a}} \ ,$$

with $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded general linear Lie superalgebra

Let V be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded linear space, $V = V_{(0,0)} \oplus V_{(1,1)} \oplus V_{(1,0)} \oplus V_{(0,1)}$, with subspaces of dimension m_1, m_2, n_1 and n_2 respectively. $\mathrm{End}(V)$ is then a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded associative algebra, and by the previous property it is turned into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra. This algebra is usually denoted by $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. In matrix form, the elements are written as:

$$A = \begin{array}{cl} \begin{array}{cccc} m_1 & m_2 & n_1 & n_2 \end{array} & \\ \begin{pmatrix} a_{(0,0)} & a_{(1,1)} & a_{(1,0)} & a_{(0,1)} \\ b_{(1,1)} & b_{(0,0)} & b_{(0,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(0,1)} & c_{(0,0)} & c_{(1,1)} \\ d_{(0,1)} & d_{(1,0)} & d_{(1,1)} & d_{(0,0)} \end{pmatrix} & \begin{array}{l} m_1 \\ m_2 \\ n_1 \\ n_2 \end{array} \end{array}$$

The indices of the matrix blocks refer to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading, and the size of the blocks is indicated in the lines above and to the right of the matrix.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra

$$A = \begin{array}{cc} & \begin{array}{cccc} m_1 & m_2 & n_1 & n_2 \end{array} \\ & \left(\begin{array}{cccc} a_{(0,0)} & a_{(1,1)} & a_{(1,0)} & a_{(0,1)} \\ b_{(1,1)} & b_{(0,0)} & b_{(0,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(0,1)} & c_{(0,0)} & c_{(1,1)} \\ d_{(0,1)} & d_{(1,0)} & d_{(1,1)} & d_{(0,0)} \end{array} \right) \begin{array}{c} m_1 \\ m_2 \\ n_1 \\ n_2 \end{array} \end{array}$$

The matrices of the Lie algebra $\mathfrak{gl}(m_1 + m_2 + n_1 + n_2)$, of the Lie superalgebra $\mathfrak{gl}(m_1 + m_2|n_1 + n_2)$ and of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ are all the same, but of course the bracket is different in all of these cases.

One can check that $\mathrm{Str}[\![A, B]\!] = 0$, where $\mathrm{Str}(A) = \mathrm{tr}(a_{(0,0)}) + \mathrm{tr}(b_{(0,0)}) - \mathrm{tr}(c_{(0,0)}) - \mathrm{tr}(d_{(0,0)})$ is the graded supertrace in terms of the ordinary trace tr. Hence $\mathfrak{sl}(m_1, m_2|n_1, n_2)$ is defined as the subalgebra of elements of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ with graded supertrace equal to 0.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranspose A^T of A

Let $A \in \mathfrak{sl}(m_1, m_2|n_1, n_2) \subset \mathrm{End}(V)$ of degree $\boldsymbol{a} \in \mathbb{Z}_2 \times \mathbb{Z}_2$; V^* - dual to V, inheriting the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading from V; $\langle \cdot, \cdot \rangle$ - the natural pairing of V and V^*. Then $A^* \in \mathrm{End}(V^*)$ is determined by:

$$\langle A^* y_{\boldsymbol{b}}, x \rangle = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} \langle y_{\boldsymbol{b}}, Ax \rangle, \qquad \forall y_{\boldsymbol{b}} \in V^*_{\boldsymbol{b}}, \forall x \in V. \tag{1}$$

This is extended by linearity to all elements of $\mathfrak{sl}(m_1, m_2|n_1, n_2)$. In matrix form, this yields the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranspose A^T of A:

$$A^T = \begin{pmatrix} a^t_{(0,0)} & b^t_{(1,1)} & -c^t_{(1,0)} & -d^t_{(0,1)} \\ a^t_{(1,1)} & b^t_{(0,0)} & c^t_{(0,1)} & d^t_{(1,0)} \\ a^t_{(1,0)} & -b^t_{(0,1)} & c^t_{(0,0)} & -d^t_{(1,1)} \\ a^t_{(0,1)} & -b^t_{(1,0)} & -c^t_{(1,1)} & d^t_{(0,0)} \end{pmatrix}, \tag{2}$$

a^t - ordinary matrix transpose. One can check (case by case, according to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading) that the graded supertranspose of matrices satisfies

$$(AB)^T = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} B^T A^T$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n_1, 2n_2)$ consists of the set of matrices of the following block form:

$$
A = \begin{array}{cc}
\begin{array}{ccccccccc} m_1 & m_2 & m_1 & m_2 & 1 & n_1 & n_2 & n_1 & n_2 \end{array} & \\
\begin{pmatrix}
a^{[1,1]}_{(0,0)} & a^{[1,2]}_{(1,1)} & a^{[1,3]}_{(0,0)} & a^{[1,4]}_{(1,1)} & a^{[1,5]}_{(0,0)} & b^{[1,1]}_{(1,0)} & b^{[1,2]}_{(0,1)} & b^{[1,3]}_{(1,0)} & b^{[1,4]}_{(0,1)} \\
a^{[2,1]}_{(1,1)} & a^{[2,2]}_{(0,0)} & a^{[2,3]}_{(1,1)} & a^{[2,4]}_{(0,0)} & a^{[2,5]}_{(1,1)} & b^{[2,1]}_{(0,1)} & b^{[2,2]}_{(1,0)} & b^{[2,3]}_{(0,1)} & b^{[2,4]}_{(1,0)} \\
a^{[3,1]}_{(0,0)} & a^{[3,2]}_{(1,1)} & a^{[3,3]}_{(0,0)} & a^{[3,4]}_{(1,1)} & a^{[3,5]}_{(0,0)} & b^{[3,1]}_{(1,0)} & b^{[3,2]}_{(0,1)} & b^{[3,3]}_{(1,0)} & b^{[3,4]}_{(0,1)} \\
a^{[4,1]}_{(1,1)} & a^{[4,2]}_{(0,0)} & a^{[4,3]}_{(1,1)} & a^{[4,4]}_{(0,0)} & a^{[4,5]}_{(1,1)} & b^{[4,1]}_{(0,1)} & b^{[4,2]}_{(1,0)} & b^{[4,3]}_{(0,1)} & b^{[4,4]}_{(1,0)} \\
a^{[5,1]}_{(0,0)} & a^{[5,2]}_{(1,1)} & a^{[5,3]}_{(0,0)} & a^{[5,4]}_{(1,1)} & a^{[5,5]}_{(0,0)} & b^{[5,1]}_{(1,0)} & b^{[5,2]}_{(0,1)} & b^{[5,3]}_{(1,0)} & b^{[5,4]}_{(0,1)} \\
c^{[1,1]}_{(1,0)} & c^{[1,2]}_{(0,1)} & c^{[1,3]}_{(1,0)} & c^{[1,4]}_{(0,1)} & c^{[1,5]}_{(1,0)} & d^{[1,1]}_{(0,0)} & d^{[1,2]}_{(1,1)} & d^{[1,3]}_{(0,0)} & d^{[1,4]}_{(1,1)} \\
c^{[2,1]}_{(0,1)} & c^{[2,2]}_{(1,0)} & c^{[2,3]}_{(0,1)} & c^{[2,4]}_{(1,0)} & c^{[2,5]}_{(0,1)} & d^{[2,1]}_{(1,1)} & d^{[2,2]}_{(0,0)} & d^{[2,3]}_{(1,1)} & d^{[2,4]}_{(0,0)} \\
c^{[3,1]}_{(1,0)} & c^{[3,2]}_{(0,1)} & c^{[3,3]}_{(1,0)} & c^{[3,4]}_{(0,1)} & c^{[3,5]}_{(1,0)} & d^{[3,1]}_{(0,0)} & d^{[3,2]}_{(1,1)} & d^{[3,3]}_{(0,0)} & d^{[3,4]}_{(1,1)} \\
c^{[4,1]}_{(0,1)} & c^{[4,2]}_{(1,0)} & c^{[4,3]}_{(0,1)} & c^{[4,4]}_{(1,0)} & c^{[4,5]}_{(0,1)} & d^{[4,1]}_{(1,1)} & d^{[4,2]}_{(0,0)} & d^{[4,3]}_{(1,1)} & d^{[4,4]}_{(0,0)}
\end{pmatrix} &
\begin{array}{c} m_1 \\ m_2 \\ m_1 \\ m_2 \\ 1 \\ n_1 \\ n_2 \\ n_1 \\ n_2 \end{array}
\end{array}
$$

Orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n_1, 2n_2)$

such that

$$A^T J + JA = 0$$

where

$$J = \begin{pmatrix} 0 & I_{m_1+m_2} & 0 & 0 & 0 \\ I_{m_1+m_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_1+n_2} \\ 0 & 0 & 0 & -I_{n_1+n_2} & 0 \end{pmatrix}.$$

Note: the row and column indices of the matrix as an element of $\mathfrak{sl}(2m_1+1, 2m_2|2n_1, 2n_2)$ have been appropriately permuted. This was done in order to preserve an analogy with the matrices of $\mathfrak{osp}(2m+1|2n)$ $(m_1+m_2=m,\ n_1+n_2=n)$, and in order to have a proper relation with parafermions and parabosons.

$\mathfrak{osp}(2m_1+1, 2m_2|2n_1, 2n_2)$

Concretely, $\mathfrak{osp}(2m_1+1, 2m_2|2n_1, 2n_2)$ consists of matrices which satisfy

$$a^{[3,3]}_{(0,0)} = -a^{[1,1]\,t}_{(0,0)},\ a^{[3,4]}_{(1,1)} = -a^{[2,1]\,t}_{(1,1)},\ a^{[4,3]}_{(1,1)} = -a^{[1,2]\,t}_{(1,1)},\ a^{[4,4]}_{(0,0)} = -a^{[2,2]\,t}_{(0,0)},$$

$$a^{[2,3]}_{1,1)} = -a^{[1,4]\,t}_{(1,1)},\ a^{[4,1]}_{(1,1)} = -a^{[3,2]\,t}_{(1,1)};\ a^{[1,3]}_{(0,0)}, a^{[2,4]}_{(0,0)}, a^{[3,1]}_{(0,0)} \text{ and } a^{[4,2]}_{(0,0)} \text{ skew symmetric,}$$

$$a^{[5,1]}_{(0,0)} = -a^{[3,5]\,t}_{(0,0)},\ a^{[5,2]}_{(1,1)} = -a^{[4,5]\,t}_{(1,1)},\ a^{[5,3]}_{(0,0)} = -a^{[1,5]\,t}_{(0,0)},\ a^{[5,4]}_{(1,1)} = -a^{[2,5]\,t}_{(1,1)},\ a^{[5,5]}_{(0,0)} = 0$$

$$d^{[3,3]}_{(0,0)} = -d^{[1,1]\,t}_{(0,0)},\ d^{[3,4]}_{(1,1)} = d^{[2,1]\,t}_{(1,1)},\ d^{[4,3]}_{(1,1)} = d^{[1,2]\,t}_{(1,1)},\ d^{[4,4]}_{(0,0)} = -d^{[2,2]\,t}_{(0,0)},$$

$$d^{[2,3]}_{1,1)} = -d^{[1,4]\,t}_{(1,1)},\ d^{[4,1]}_{(1,1)} = -d^{[3,2]\,t}_{(1,1)};\ d^{[1,3]}_{(0,0)}, d^{[2,4]}_{(0,0)}, d^{[3,1]}_{(0,0)} \text{ and } d^{[4,2]}_{(0,0)} \text{ symmetric,}$$

$$c^{[1,1]}_{(1,0)} = b^{[3,3]\,t}_{(1,0)},\ c^{[1,2]}_{(0,1)} = -b^{[4,3]\,t}_{(0,1)},\ c^{[1,3]}_{(1,0)} = b^{[1,3]\,t}_{(1,0)},\ c^{[1,4]}_{(0,1)} = -b^{[2,3]\,t}_{(0,1)},\ c^{[1,5]}_{(1,0)} = b^{[5,3]}_{(1,0)}$$

$$c^{[2,1]}_{(0,1)} = b^{[3,4]\,t}_{(0,1)},\ c^{[2,2]}_{(1,0)} = -b^{[4,4]\,t}_{(1,0)},\ c^{[2,3]}_{(0,1)} = b^{[1,4]\,t}_{(0,1)},\ c^{[2,4]}_{(1,0)} = -b^{[2,4]\,t}_{(1,0)},\ c^{[2,5]}_{(0,1)} = b^{[5,4]}_{(0,1)}$$

$$c^{[3,1]}_{(1,0)} = -b^{[3,1]\,t}_{(1,0)}, c^{[3,2]}_{(0,1)} = b^{[4,1]\,t}_{(0,1)}, c^{[3,3]}_{(1,0)} = -b^{[1,1]\,t}_{(1,0)}, c^{[3,4]}_{(0,1)} = b^{[2,1]\,t}_{(0,1)}, c^{[3,5]}_{(1,0)} = -b^{[5,1]\,t}_{(1,0)}$$

$$c^{[4,1]}_{(0,1)} = -b^{[3,2]\,t}_{(0,1)}, c^{[4,2]}_{(1,0)} = b^{[4,2]\,t}_{(1,0)}, c^{[4,3]}_{(0,1)} = -b^{[1,2]\,t}_{(0,1)}, c^{[4,4]}_{(1,0)} = b^{[2,2]\,t}_{(1,0)}, c^{[4,5]}_{(0,1)} = -b^{[5,2]\,t}_{(0,1)}$$

Matrix conditions look complicated at first sight; they are not difficult to work with. Special cases:

- When $m_2 = n_2 = 0$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m_1+1, 0|2n_1, 0)$ just coincides with the ordinary Lie superalgebra $\mathfrak{osp}(2m_1+1|2n_1)$ (with appropriate $\mathbb{Z}_2$ grading).
- When $m_1 = n_2 = 0$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1, 2m_2|2n_1, 0)$ coincides with the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra denoted by $\mathfrak{pso}(2m_2+1|2n_1)$, or (up to a rearrangement of row and column indices) by $\mathfrak{osp}(1, 2m_2|2n_1, 0)$ (Tolstoy).
- When $n_1 = n_2 = 0$, $\mathfrak{osp}(2m_1+1, 2m_2|0, 0)$ reduces to the Lie algebra $\mathfrak{so}(2m_1+2m_2+1)$.

- When $m_1 = m_2 = n_2 = 0$, $\mathfrak{osp}(1, 0|2n_1, 0)$ reduces to the Lie superalgebra $\mathfrak{osp}(1|2n_1)$. Similarly, when $m_1 = m_2 = n_2 = 0$, $\mathfrak{osp}(1, 0|0, 2n_2)$ reduces to the Lie superalgebra $\mathfrak{osp}(1|2n_2)$. Note however that for $m_1 = m_2 = 0$, $\mathfrak{osp}(1, 0|2n_1, 2n_2)$ does not reduce to a Lie algebra or a Lie superalgebra, but remains a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra. This last case is interesting in parastatistics.

By deleting row $2m_1 + 2m_2 + 1$ and column $2m_1 + 2m_2 + 1$ in:

$$A = \begin{array}{c} \begin{array}{ccccccccc} m_1 & m_2 & m_1 & m_2 & 1 & n_1 & n_2 & n_1 & n_2 \end{array} \\ \left(\begin{array}{ccccccccc}
a^{[1,1]}_{(0,0)} & a^{[1,2]}_{(1,1)} & a^{[1,3]}_{(0,0)} & a^{[1,4]}_{(1,1)} & a^{[1,5]}_{(0,0)} & b^{[1,1]}_{(1,0)} & b^{[1,2]}_{(0,1)} & b^{[1,3]}_{(1,0)} & b^{[1,4]}_{(0,1)} \\
a^{[2,1]}_{(1,1)} & a^{[2,2]}_{(0,0)} & a^{[2,3]}_{(1,1)} & a^{[2,4]}_{(0,0)} & a^{[2,5]}_{(1,1)} & b^{[2,1]}_{(0,1)} & b^{[2,2]}_{(1,0)} & b^{[2,3]}_{(0,1)} & b^{[2,4]}_{(1,0)} \\
a^{[3,1]}_{(0,0)} & a^{[3,2]}_{(1,1)} & a^{[3,3]}_{(0,0)} & a^{[3,4]}_{(1,1)} & a^{[3,5]}_{(0,0)} & b^{[3,1]}_{(1,0)} & b^{[3,2]}_{(0,1)} & b^{[3,3]}_{(1,0)} & b^{[3,4]}_{(0,1)} \\
a^{[4,1]}_{(1,1)} & a^{[4,2]}_{(0,0)} & a^{[4,3]}_{(1,1)} & a^{[4,4]}_{(0,0)} & a^{[4,5]}_{(1,1)} & b^{[4,1]}_{(0,1)} & b^{[4,2]}_{(1,0)} & b^{[4,3]}_{(0,1)} & b^{[4,4]}_{(1,0)} \\
a^{[5,1]}_{(0,0)} & a^{[5,2]}_{(1,1)} & a^{[5,3]}_{(0,0)} & a^{[5,4]}_{(1,1)} & a^{[5,5]}_{(0,0)} & b^{[5,1]}_{(1,0)} & b^{[5,2]}_{(0,1)} & b^{[5,3]}_{(1,0)} & b^{[5,4]}_{(0,1)} \\
c^{[1,1]}_{(1,0)} & c^{[1,2]}_{(0,1)} & c^{[1,3]}_{(1,0)} & c^{[1,4]}_{(0,1)} & c^{[1,5]}_{(1,0)} & d^{[1,1]}_{(0,0)} & d^{[1,2]}_{(1,1)} & d^{[1,3]}_{(0,0)} & d^{[1,4]}_{(1,1)} \\
c^{[2,1]}_{(0,1)} & c^{[2,2]}_{(1,0)} & c^{[2,3]}_{(0,1)} & c^{[2,4]}_{(1,0)} & c^{[2,5]}_{(0,1)} & d^{[2,1]}_{(1,1)} & d^{[2,2]}_{(0,0)} & d^{[2,3]}_{(1,1)} & d^{[2,4]}_{(0,0)} \\
c^{[3,1]}_{(1,0)} & c^{[3,2]}_{(0,1)} & c^{[3,3]}_{(1,0)} & c^{[3,4]}_{(0,1)} & c^{[3,5]}_{(1,0)} & d^{[3,1]}_{(0,0)} & d^{[3,2]}_{(1,1)} & d^{[3,3]}_{(0,0)} & d^{[3,4]}_{(1,1)} \\
c^{[4,1]}_{(0,1)} & c^{[4,2]}_{(1,0)} & c^{[4,3]}_{(0,1)} & c^{[4,4]}_{(1,0)} & c^{[4,5]}_{(0,1)} & d^{[4,1]}_{(1,1)} & d^{[4,2]}_{(0,0)} & d^{[4,3]}_{(1,1)} & d^{[4,4]}_{(0,0)}
\end{array}\right) \end{array} \begin{array}{c} \\ m_1 \\ m_2 \\ m_1 \\ m_2 \\ 1 \\ n_1 \\ n_2 \\ n_1 \\ n_2 \end{array}$$

and the corresponding conditions, one obtains the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras $\mathfrak{osp}(2m_1, 2m_2|2n_1, 2n_2)$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras corresponding to the Lie superalgebras of type C and D.

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions

Generators from $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{so}_q(2n+1)$:

$$f_j^- = \sqrt{2}(e_{j,2n+1} - e_{2n+1,n+j}), \quad f_j^+ = \sqrt{2}(e_{2n+1,j} - e_{n+j,2n+1}), \quad j = 1, \ldots, n.$$

Subspaces:

$$\mathfrak{g}_{(0,1)} = \mathrm{span}\{f_k^\pm,\ k = 1, \ldots, q\}$$
$$\mathfrak{g}_{(1,0)} = \mathrm{span}\{f_k^\pm,\ k = q+1, \ldots, n\}$$
$$\mathfrak{g}_{(0,0)} = \mathrm{span}\{[f_k^\xi, f_l^\eta],\ \xi, \eta = \pm,\ k, l = 1, \ldots, q \text{ and } k, l = q+1, \ldots, n\}$$
$$\mathfrak{g}_{(1,1)} = \mathrm{span}\{\{f_k^\xi, f_l^\eta\},\ \xi, \eta = \pm,\ k = 1, \ldots, q,\ l = q+1, \ldots n\}.$$

Parafermion relations for $j, k, l = 1, \ldots, q$ or $j, k, l = q+1, \ldots, n$:

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f_j^\xi - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f_k^\eta, \quad \xi, \eta, \epsilon = \pm \text{ or } \pm 1.$$

But the "relative commutation relations" between the two sorts:

$$\{\{f_j^\xi, f_k^\eta\}, f_l^\epsilon\} = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f_j^\xi + \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f_k^\eta, \quad \xi, \eta, \epsilon = \pm \text{ or } \pm 1.$$

$(j = 1, \ldots, q,\ k = q+1, \ldots, n,\ l = 1, \ldots, n$ or
$j = q+1, \ldots, n,\ k = 1, \ldots, q,\ l = 1, \ldots, n)$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded A-statistics

Generators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{sl}_{1,q,n-q,0}(n+1)$:

$$a_j^- = e_{1,j+1}, \quad a_j^+ = e_{j+1,1}, \qquad j = 1,\ldots,n$$

$$\begin{aligned}
\mathfrak{g}_{(0,1)} &= \mathrm{span}\{a_j^- = e_{1,j+1},\ a_j^+ = e_{j+1,1},\ j = 1,\ldots,q\},\\
\mathfrak{g}_{(1,0)} &= \mathrm{span}\{a_j^- = e_{1,j+1},\ a_j^+ = e_{j+1,1},\ j = q+1,\ldots,n\},\\
\mathfrak{g}_{(0,0)} &= \mathrm{span}\{[a_j^+, a_k^-],\ j,k = 1,\ldots,q \text{ and } j,k = q+1,\ldots,n\},\\
\mathfrak{g}_{(1,1)} &= \mathrm{span}\{\{a_j^-, a_k^+\},\ \{a_j^+, a_k^-\},\ j = 1,\ldots,q \text{ and } k = q+1,\ldots,n\}.
\end{aligned}$$

Ordinary A-statistics for each sort separately

$$\begin{aligned}
[a_j^+, a_k^+] &= [a_j^-, a_k^-] = 0,\\
[[a_j^+, a_k^-], a_l^+] &= \delta_{jk} a_l^+ + \delta_{kl} a_j^+,\\
[[a_j^+, a_k^-], a_l^-] &= -\delta_{jk} a_l^- - \delta_{jl} a_k^-,
\end{aligned}$$

$(j,k,l = 1,\ldots,q \text{ and } j,k,l = q+1,\ldots,n)$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded A-statistics

The relative relations between the two sorts of operators are purely in terms of nested anticommutators:

$$\{a_j^+, a_k^+\} = \{a_j^-, a_k^-\} = 0,$$
$$\{\{a_j^+, a_k^-\}, a_l^+\} = \delta_{kl} a_j^+,$$
$$\{\{a_j^+, a_k^-\}, a_l^-\} = \delta_{jl} a_k^-.$$

$(j = 1, \ldots, q,\ k = q+1, \ldots, n,\ l = 1, \ldots, n$ and
$j = q+1, \ldots, n,\ k = 1, \ldots, q,\ l = 1, \ldots, n)$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons

- $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1,0|2n_1,2n_2)$
- set of generators for $\mathfrak{osp}(1,0|2n_1,2n_2)$:

$$b_i^- = \sqrt{2}(e_{1,i+1} - e_{n_1+n_2+i+1,1}),\ b_i^+ = \sqrt{2}(e_{1,n_1+n_2+i+1} + e_{i+1,1}),\ \ i = 1,\dots,n_1+n_2$$

- Note: $b_i^\pm \in \mathfrak{g}_{(1,0)}$, $i = 1,\dots,n_1$, and $b_i^\pm \in \mathfrak{g}_{(0,1)}$, $i = n_1+1,\dots,n_1+n_2$
- the two sets of elements satisfy the common relations of parabosons:

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi.$$

$\eta, \epsilon, \xi \in \{+,-\}$, either $j,k,l \in \{1,2,\dots,n_1\}$ or else $j,k,l \in \{n_1+1,\dots,n_1+n_2\}$.

- the mixed triple relations between the two families of parabosons:

$$\{[b_j^\xi, b_k^\eta], b_l^\epsilon\} = -(\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi,$$

where $j = 1,\dots,n_1$, $k = n_1+1,\dots,n_1+n_2$, $l = 1,\dots,n_1+n_2$ or else $j = n_1+1,\dots,n_1+n_2$, $k = 1,\dots,n_1$, $l = 1,\dots,n_1+n_2$.

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $A-$superstatistics

- consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{sl}(1,0|n_1,n_2)$
- define: $a_i^+ = e_{i+1,1}, \ a_i^- = e_{1,i+1}, i = 1,2,\ldots,n_1+n_2$
- $a_i^\pm \in \mathfrak{g}_{(1,0)}, \ i = 1,\ldots,n_1; \ a_i^\pm \in \mathfrak{g}_{(0,1)}, \ i = n_1+1,\ldots,n_1+n_2$
- if $i,j,k \in \{1,2,\ldots,n_1\}$ or $i,j,k \in \{n_1+1,n_1+2,\ldots,n_1+n_2\}$

$$\{a_i^+,a_j^+\} = \{a_i^-,a_j^-\} = 0,$$
$$[\{a_i^+,a_j^-\},a_k^+] = \delta_{jk}a_i^+ - \delta_{ij}a_k^+,$$
$$[\{a_i^+,a_j^-\},a_k^-] = -\delta_{ik}a_j^- + \delta_{ij}a_k^-$$

- mixed relations between the two families are as follows:

$$[a_i^+,a_j^+] = [a_i^-,a_j^-] = 0,$$
$$\{[a_i^+,a_j^-],a_k^+\} = \delta_{jk}a_i^+,$$
$$\{[a_i^+,a_j^-],a_k^-\} = \delta_{ik}a_j^-.$$

$i \in \{1,2,\ldots,n_1\}$, $j \in \{n_1+1,\ldots,n_1+n_2\}$, $k \in \{1,\ldots,n_1+n_2\}$,
or else $i \in \{n_1+1,\ldots,n_1+n_2\}$, $j \in \{1,2,\ldots,n_1\}$,
$k \in \{1,\ldots,n_1+n_2\}$.

Conclusions / Summary

- natural structure to consider, renewed interest
- interesting definition, both of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras
- reasonable definition of $\mathfrak{sl}_{p,q,r,s}(n)$ and $\mathfrak{so}_{p,q,r,s}(n)$, $\mathfrak{sl}(m_1, m_2|n_1, n_2)$ but we need more for better structure (roots, root space decomposition,...)
- our main result: classical analogues of Lie algebras and Lie superalgebras of type B, C and D as $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras
- applications in quantum statistics