Crystallizing compact semisimple Lie groups

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(joint work with Marco Matassa)

IECL Metz

Sofia, 28 May 2024

Executive summary

1. **Ultimate goal:** Set-theoretic version of representation theory for semisimple Lie groups.
2. This is well-known for compact s.s. Lie groups K: **Crystal bases**.
3. This talk is about crystalization for the AN group (or for $\exp(\mathfrak{p})$).

Crystal bases

Irred. representations of a compact semisimple Lie group

K — compact semisimple Lie group (connected, simply connected)

$\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ — complexification of its Lie algebra
$\Rightarrow$ irred. unitary rep'ns of K $\equiv$ irred. rep'ns of $\mathfrak{g}$.

Theorem

Irred. rep'ns of $\mathfrak{g}$ are classified by highest weight $\lambda \in \mathbf{P}^+$.

Explicit structure. . . ?

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- **Weyl character formula (1925):** Action of Cartan subalgebra $\mathfrak{h}$.
- **Gelfand-Tsetlin (1950):** Explicit formulas for action of simple root vectors E_i, F_i, but for $\mathfrak{gl}_n(\mathbb{C})$ only.
- **Pand-Hecht, Wong (1967), & many others:** Then same for $\mathfrak{o}_n$, then $\mathfrak{sp}_n$, then all classical $\mathfrak{g}$.
- **Kashiwara, Lustzig (1990):** Crystal bases (asymptotic formulas + much more)

Example: $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$

To understand an irred. rep. of $\mathfrak{sl}_3(\mathbb{C})$, need to know the action of:

- the **Cartan subalgebra**: $\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \middle| \; a_1 + a_2 + a_3 = 0 \right\}$.

Any irred. rep. of $\mathfrak{g}$ has a basis of common eigenvectors for all $H \in \mathfrak{h}$
—"weight spaces"

- the simple root vectors: $F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Each F_i maps weight spaces to weight spaces.

Example: $V(2\varpi_1 + 2\varpi_2)$ for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$

REPN OF $\mathcal{U}_q(\mathfrak{sl}_3)$ WITH HIGHEST WEIGHT $(2,0,-2)$

28 September 2011

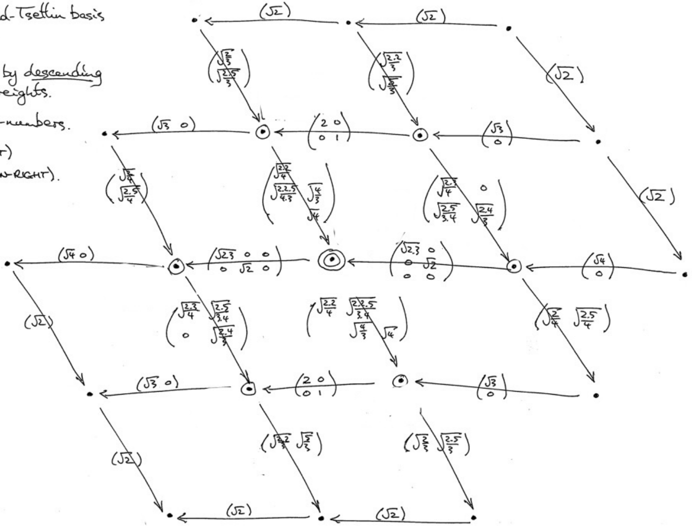
Given w.r.t. orthonormal Gelfand-Tsetlin basis

Notes

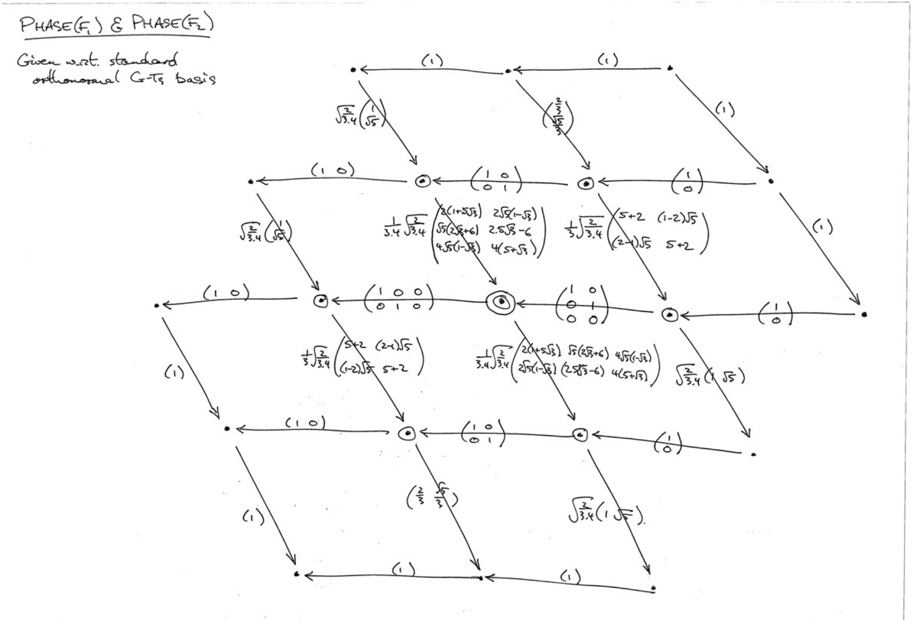
- All bases ordered by descending highest $\mathcal{U}_q(\mathfrak{sl}_2)$-weights.
- All numbers are q-numbers.

$\longleftarrow = F_1$ (LEFT)

$\longleftarrow = F_2$ (DOWN-RIGHT).



Example: $V(2\varpi_1 + 2\varpi_2)$ for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ — operator phase



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Definition

$[a]_q := \frac{q^a - q^{-a}}{q - q^{-1}}$

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Also, $[n]_q! := [n]_q \, [n-1]_q \ldots [2]_q \, [1]_q$, etc.

Quantized envelopping algebras (Drinfeld-Jimbo)

Ex. $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$,

$U(\mathfrak{g})$ is generated by elements E_i, F_i, H_i,

$$E_i = \begin{pmatrix} 0 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \; F_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & 1 & \ddots & \\ & & & 0 \end{pmatrix}, \; H_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

with the Chevalley-Serre relations:

$$[H_j, E_i] = \alpha_i(H_j)E_i$$

$$[H_j, F_i] = -\alpha_i(H_j)F_i$$

$$[E_i, F_j] = \delta_{ij}H_i$$

$$E_i^2E_{i\pm1} - 2E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 = 0$$

$$F_i^2F_{i\pm1} - 2F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 = 0$$

$$\alpha_i : \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n+1} \end{pmatrix} \mapsto a_i - a_{i+1}$$

Quantized envelopping algebras (Drinfeld-Jimbo)

Ex. $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$, $q \in \mathbb{R}_+^\times,\ q \neq 1$

$U_q(\mathfrak{g})$ is generated by elements E_i, F_i, H_i,

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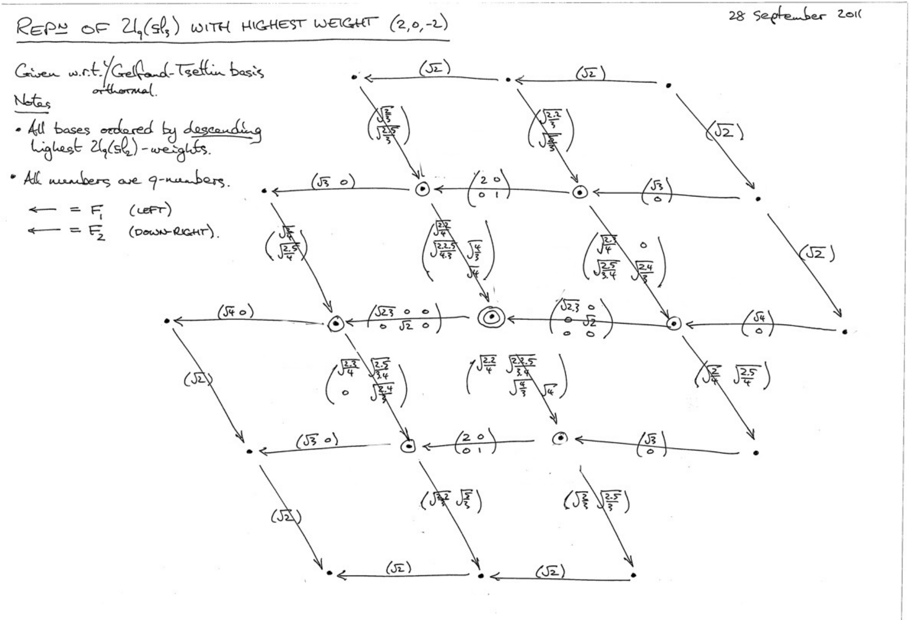
$$F_i^2 F_{i\pm1} - [2]_q F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0$$

where: $[2]_q = q + q^{-1}$, $[H]_q = \frac{q^H - q^{-H}}{q - q^{-1}}$.

Rmk. Actually, one uses $K_i = q^{H_i}$ instead of H_i as generators.

Example: $V(2\varpi_1 + 2\varpi_2)$



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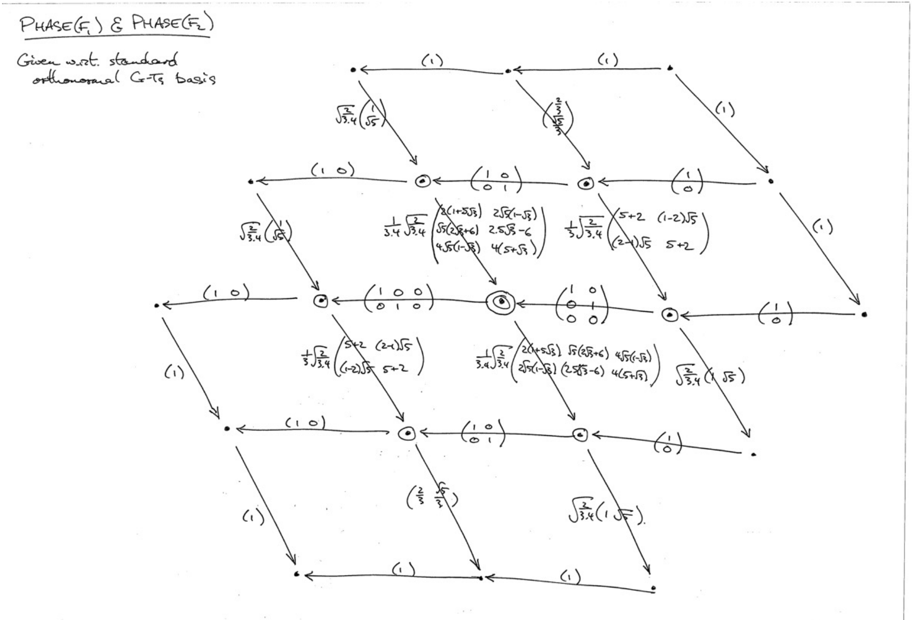

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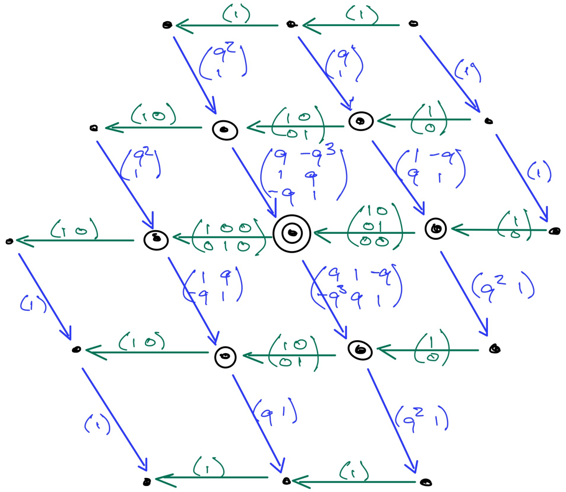
DOMINANT TERMS
IN PH(F_1) & PH(F_2)
AS $q \to 0$



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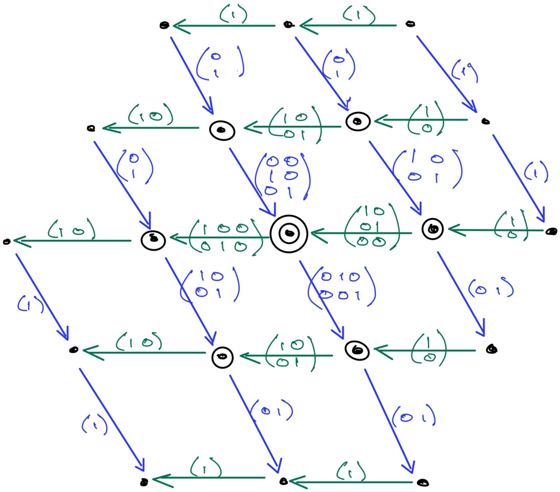
LIMIT AS $q \to 0$



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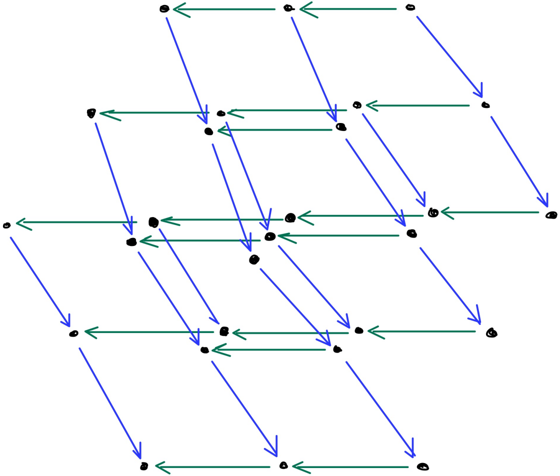
LIMIT AS $q \to 0$

(CRYSTAL GRAPH)



Tensor product of crystals

The crystal limit ($q \to 0$) simplifies not just the action of the generators, but also the Clebsch-Gordan coefficients, branching rules, . . .

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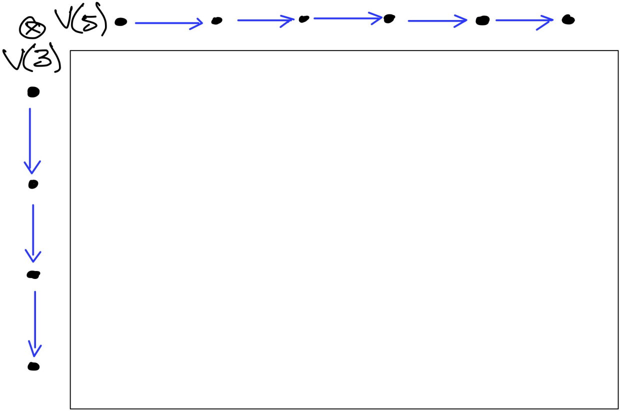
Theorem (Tensor product rule)

If $(\mathcal{L}, \mathcal{B})$, $(\mathcal{L}', \mathcal{B}')$ are crystal bases for V, V', then $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \times \mathcal{B}')$ is a crystal basis for $V \otimes V'$, with action

$$\tilde{f}_i : b \otimes c \mapsto \begin{cases} (\tilde{f}_i b) \otimes c, & \text{if } \varphi_i(b) > \varepsilon_i(c) \\ b \otimes (\tilde{f}_i c), & \text{if } \varphi_i(b) \leqslant \varepsilon_i(c) \end{cases}$$

where $\varepsilon_i(b) = \max\{n \mid \tilde{e}_i^n b \neq 0\}$ *and* $\varphi_i(b) = \max\{n \mid \tilde{f}_i^n b \neq 0\}$.

Example: Tensor product of $\mathfrak{sl}_2$ representations



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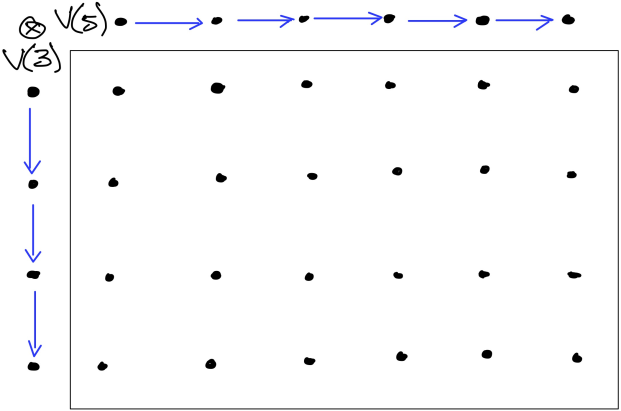

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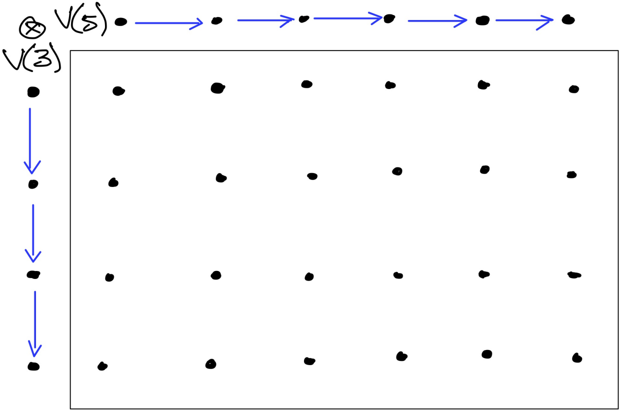

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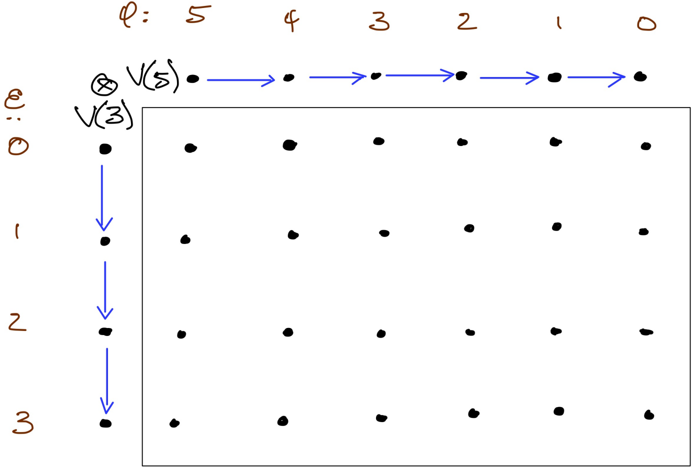

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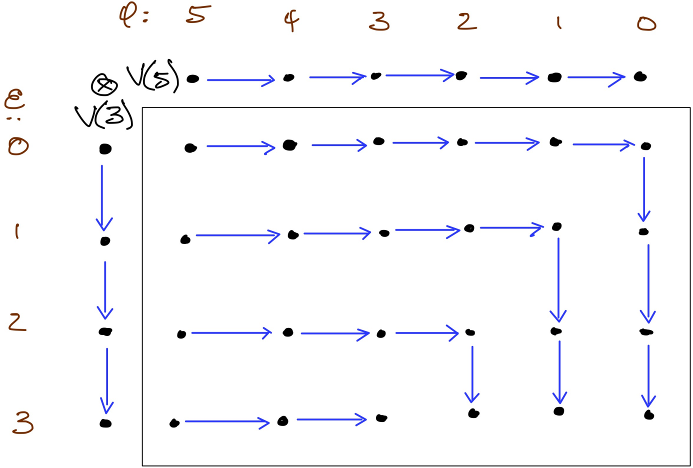

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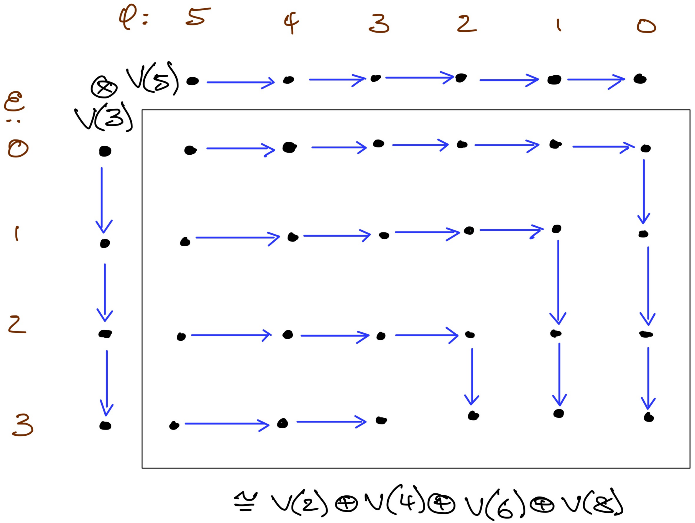
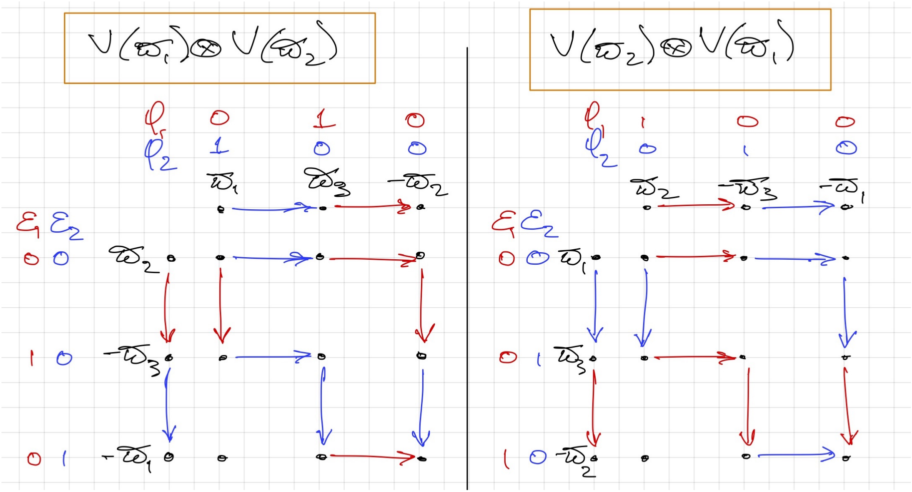
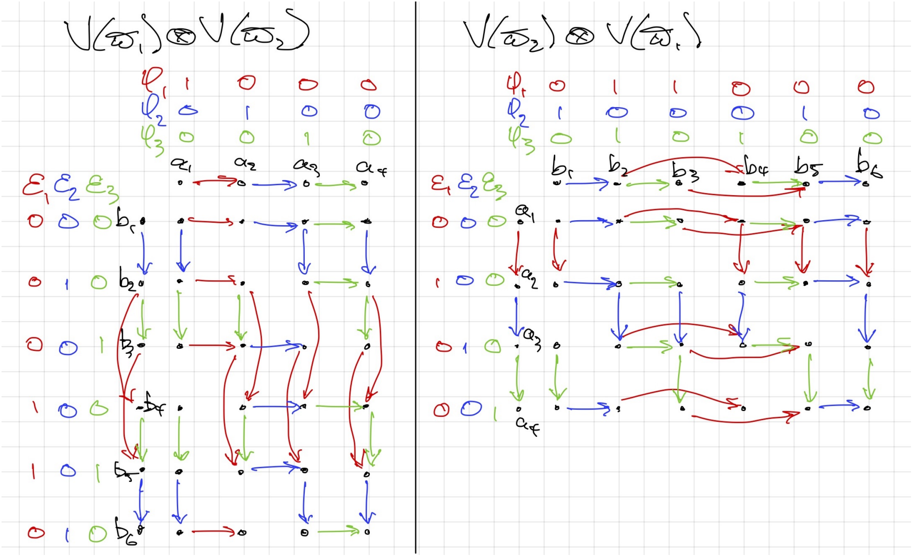


Example: Tensor product of $\mathfrak{sl}_3$ representations



Example: Tensor product of $\mathfrak{sl}_4$ representations



“Crystallization” in analysis:
Quantized algebras of functions

Quantized algebras of functions

$$\begin{aligned}\mathcal{O}(K) &= \{\text{polynomial functions on } K\} \\ &= \{\langle\xi|\cdot|\eta\rangle \mid \xi,\eta \in V \text{ (irred. integrable } \mathfrak{g}\text{-modules)}\}\\ C(K) &= \overline{\mathcal{O}(K)}^{\|\cdot\|} \qquad \text{— } C^*\text{-closure}\end{aligned}$$

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$$\begin{aligned}\mathcal{O}(K_q) &= \{\text{polynomial functions on } K_q\} \\ &= \{\langle\xi|\cdot|\eta\rangle \mid \xi,\eta \in V \text{ (integrable } \mathcal{U}_q(\mathfrak{g})\text{-modules)}\}\end{aligned}$$

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More generally, we can define $\mathcal{O}(X_q)$ and $C(X_q)$ for any $X = K/H$ with H a Poisson subgroup.

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$C(K_q) = \overline{\mathcal{O}(K_q)}^{\|\cdot\|}$ — C^*-closure

More generally, we can define $\mathcal{O}(X_q)$ and $C(X_q)$ for any $X = K/H$ with H a Poisson subgroup.

Remarks.

- **..., Neshveyev-Tuset (2012):** The algebras $C(K_q)$ form a continuous field of C^*-algebras for $0 < q < \infty$.
- **..., Giselsson (2020):** $C(K_q)$ are all isomorphic for $q \in (0, \infty) \setminus \{1\}$.
- The $\mathcal{O}(K_q)$ are not, though.

Quantized algebras of functions: $q = 0$ limit

Theorem (Woronowicz '87, Hong-Szymanski '02, Giselsson ('24))

For $X_q = \mathrm{SU}_q(2)$, $\mathbb{C}P^n_q$, $\mathrm{SU}_q(3)$, the continuous field $(C(X_q))$ extends to $q = 0, \infty$.
All fibres for $q \neq 1$ are isomorphic, and they are **graph C^*-algebras**.

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Remarks.

- Again, the $\mathcal{O}(X_q)$ are not all isomorphic.
- $\mathcal{O}(X_0)$ is a **Leavitt path algebra**
  (= algebraic analog of a graph C^*-algebra).
- More precisely:
  - $C(\mathbb{C}P^n_q)$ is the AF-core of a graph C^*-algebra,
  - $C(\mathrm{SU}_q(3))$ is a higher-rank graph C^*-algebra.
- Recent work by Brzeziski, Krähmer, Ò Buachalla & Strung shows all quantum flag varieties X_q give graph C^*-algebras.

Graph algebras

(Λ^0, Λ^1) — directed graph.

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Definition

The **Leavitt path algebra** $\mathrm{KP}^*(\Lambda)$ is the universal $*$-algebra generated by projections p_v ($v \in \Lambda^0$) and partial isometries s_e ($e \in \Lambda^1$) satisfying:

1. p_v are mutually orthogonal projections
2. $s_e^* s_{e'} = \delta_{ee'} p_{s(e)}$
3. $p_v = \sum_{r(e)=v} s_e s_e^*$

The **graph** C^***-algebra** $C^*(\Lambda)$ is the enveloping C^*-algebra.

Examples

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Classic examples

- $C^*(\,\bullet\!\circlearrowright\,) \cong C(\mathbb{T})$
- $C^*(\,\bullet \longrightarrow \bullet\!\circlearrowright\,) \cong \mathcal{T}$ (Toeplitz alg.)

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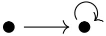
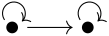
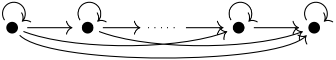
- $C^*($  $) \cong \mathcal{T}$ (Toeplitz alg.)

Quantized function algebras $(q \neq 1)$

- $C^*($  $) \cong C(\mathrm{SU}_q(2))$ —Woronowicz
- $C^*($  $) \cong C(Y_q)$ — Hong-Szymanski

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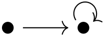
- $C^*($  $) \cong \mathcal{T}$ (Toeplitz alg.)

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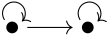
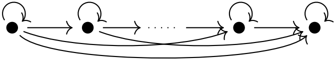
- $C^*($  $) \cong C(\mathrm{SU}_q(2))$ —Woronowicz
- $C^*($  $) \cong C(Y_q)$ — Hong-Szymanski

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Question: What are these graphs?
What about higher rank?

Definition (k-graph)

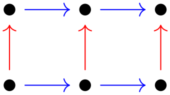
A k-**graph** is a category Λ (of paths) with a morphism $\mathbf{d} : \Lambda \to \mathbb{N}^k$ (*degree* or *coloured length*) satisfying the *factorization property*:

$$\mathbf{d}(e) = m + n \quad \Rightarrow \quad e = e_1 e_2 \text{ uniquely with } \mathbf{d}(e_1) = m,\ \mathbf{d}(e_2) = n.$$

Notation:

- $\Lambda^n := \mathbf{d}^{-1}(n)$ — paths of coloured length $n \in \mathbb{N}^k$.
- $\Lambda^{(0,\dots,0)}$ is the set of **vertices**.
- $\Lambda^{(0,\dots,1,\dots,0)}$ is the set of **edges of colour** i.

eg,



Higher rank graph algebras

Definition

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1. p_v are mutually orthogonal projections.
2. $s_e^* s_{e'} = \delta_{ee'} p_{s(e)}$ when $\mathbf{d}(e) = \mathbf{d}(e')$.
3. $p_v = \sum_{\substack{r(e)=v \\ \mathbf{d}(e)=n}} s_e s_e^*$ for any $n \neq 0$ fixed.

The **higher rank graph C^*-algebra** $C^*(\Lambda)$ is the enveloping C^*-algebra.

Crystallized algebras of functions

Theorem (Matassa-Y.)

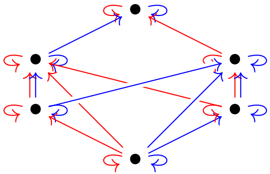
K — connected compact semisimple Lie group.
The continuous field of C^*-algebras $(C(K_q))_{q\in(0,\infty)}$ extends to $q = 0, \infty$ with $C(K_0) \cong C(K_\infty) \cong C^*(\Lambda)$ for some explicit **higher rank graph** *Λ of rank $\mathrm{rk}(K)$.*

Remarks

- For $K = \mathrm{SU}(3)$, this is a result of Giselsson.
  He also shows $C(\mathrm{SU}_q(3)) \cong C^*(\Lambda)$ for every $q \in [0, \infty]\setminus\{1\}$.
  Conjecture: This is true in generality.
- Can replace K by the canonical torus bundle Y over a flag manifold. The algebra of functions on the flag manifold itself is the gauge-invariant subalgebra. At $q = 0, \infty$, this is the AF-core.

Crystallized algebras of functions

Examples

- $C(SU_0(3)) \cong C^*(\Lambda)$, where $\Lambda =$ 

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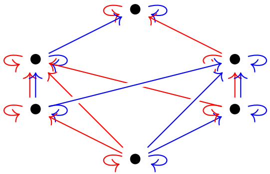
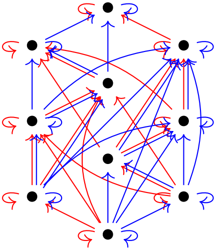
- $C(SU_0(3)) \cong C^*(\Lambda)$, where $\Lambda =$ 

- $C(Sp_q(2)) \cong C^*(\Lambda)$ at $q = 0$, where $\Lambda =$ 

The r-graph of a crystallized function algebra

Q: How to compute these k-graphs?

A (roughly): The union of all irreducible crystals defines an r-graph with

- length given by highest weight:

$$b \in \mathcal{B}(\lambda) \quad \Rightarrow \quad \mathbf{d}(b) = \lambda \in \mathbf{P}^+ \cong \mathbb{N}^r.$$

- composition given by tensor product projected to highest weight component:

$$b \in \mathcal{B}(\lambda), b' \in \mathcal{B}(\mu) \quad \Rightarrow$$
$$bb' = \begin{cases} b \otimes b' & \text{if } b \otimes b' \in \mathcal{B}(\lambda+\mu) \subset \mathcal{B}(\mu) \otimes \mathcal{B}(\lambda), \\ \varnothing & \text{else.} \end{cases}$$

Thank you.