Introduction

to AdS/CFT integrability

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New Mathematical Methods in Solvable Models and Gauge/String Dualities,

Varna, 15.08.22

AdS/CFT correspondence

Maldacena '97
Gubser, Klebanov, Polyakov '98
Witten '98

$\boxed{\begin{array}{c} \mathcal{N}=4 \quad D=4 \\ \text{super-Yang-Mills} \end{array}}$ $\Longleftrightarrow$ $\boxed{\begin{array}{c} \text{Strings} \\ \text{on } AdS_5 \times S^5 \end{array}}$

't Hooft coupling:

$$\lambda = g_{YM}^2 N$$

Number of colors:

$$N$$

String tension:

$$T = \frac{\sqrt{\lambda}}{2\pi}$$

String coupling:

$$g_s = \frac{\lambda}{4\pi N}$$

Large-N limit $\longleftrightarrow$ Free strings

$\mathcal{N}=4$ Super-Yang-Mills

- dimensional reduction of $\mathcal{N}=1$ $D=10$ SYM:

Gauge field: $A_M = (A_\mu, \Phi_i)$ $\quad i = 1 \dots 6$

Fermions: Ψ_α (10D Majorana-Weyl spinor) $\quad \Gamma^M = (\Gamma^\mu, \Gamma^i)$

$$\mathcal{L} = \frac{1}{g_{YM}^2} \operatorname{tr} \Big\{ \underbrace{-\frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_i)^2 + \frac{1}{2} [\Phi_i, \Phi_j]^2}_{-\frac{1}{2} F_{MN}^2} + \underbrace{i \bar{\Psi} \Gamma^\mu D_\mu \Psi + \bar{\Psi} \Gamma^i [\Phi_i, \Psi]}_{i \bar{\Psi} \Gamma^M D_M \Psi} \Big\}$$

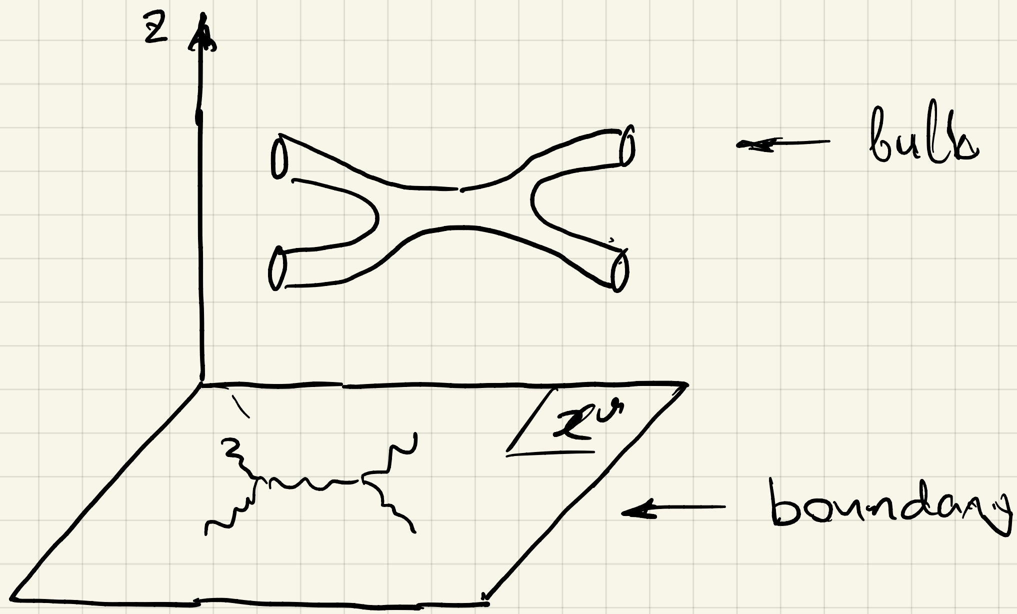
Symmetries:

$$\underbrace{SO(4,2)}_{\text{conformal}} \times \underbrace{SO(6)}_{\text{R-symmetry}}$$

+ 16 supercharges: Q_α

Anti-de-Sitter space

$$\boxed{ds^2 = \frac{dx_\mu^2 + dz^2}{z^2}}$$



- homogeneous space of the conformal group:

$$AdS_5 = SO(4,2)/SO(4,1): \qquad g = e^{iP_\mu x^\mu} z^{-D}$$

- $S^5 = SO(6)/SO(5)$ $\rightsquigarrow$ all symmetries realized geometrically

Local operators

$$\mathcal{O} = \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} \quad \text{or} \quad \mathrm{tr}\, \Phi_i \Phi_i \quad \text{or} \quad \mathrm{tr}\, \bar{\Psi} \Gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_3} \Psi \qquad \ldots$$

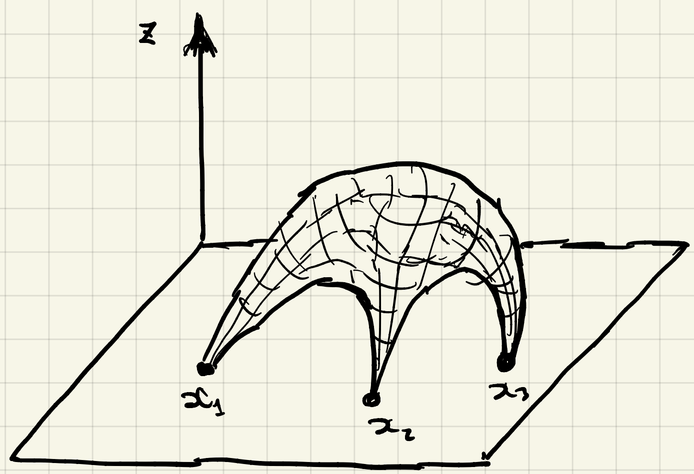
- require renormalization and therefore mix

$$\underbrace{[D}_{\text{dilatation operator}}, \mathcal{O}(x)] = \underbrace{\Delta}_{\text{scaling dimension}} \mathcal{O}(x)$$

$$\langle \mathcal{O}^\dagger(x)\, \mathcal{O}(0) \rangle = \frac{1}{|x|^{2\Delta}}$$

structure constant

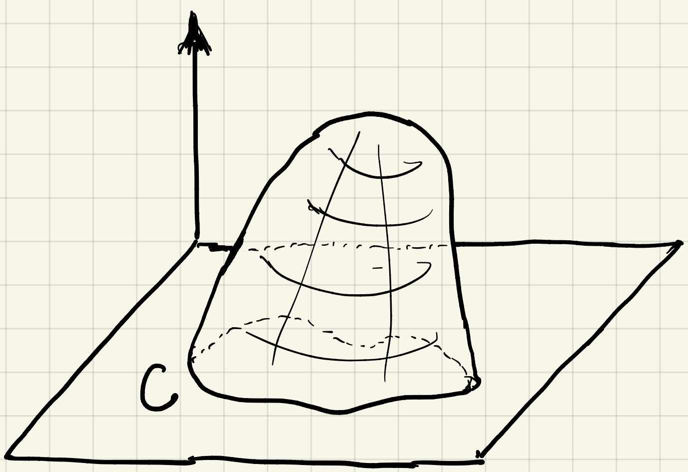
$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\, |x_{13}|^{\Delta_1+\Delta_3-\Delta_2}\, |x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$



Wilson Loops

$$W(C,\vec{n}) = \left\langle \frac{1}{N} \operatorname{tr} P \exp \oint_C ds \left(i \dot{x}^\mu A_\mu + |\dot{x}| n^i \Phi_i \right) \right\rangle$$

- UV finite and does not require renormalization



$$W(C) \overset{\lambda\to\infty}{\simeq} e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}(C)}$$

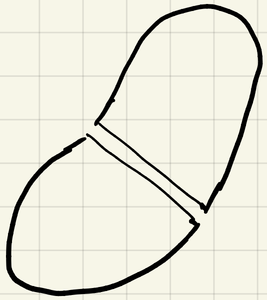
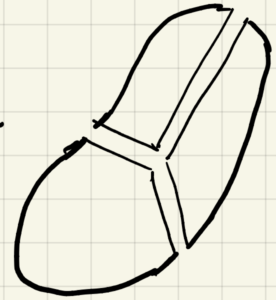
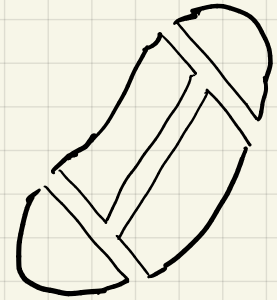
$A_{min}(C) < 0 \quad \forall C$

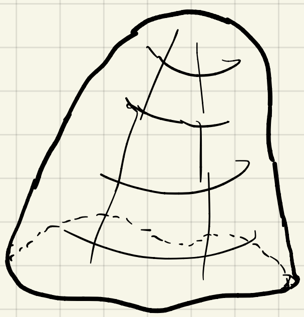
Compatible with conformal invariance:

$$W(L\times T) \overset{T\to\infty}{=} e^{-T V(L)} \qquad V(L) = -\frac{\alpha(\lambda)}{L}$$

$$\alpha(\lambda) = \begin{cases} \frac{1}{4\pi}\lambda & (\lambda\to 0) \\ \frac{4\pi^2}{\Gamma^4(1/4)}\sqrt{\lambda} & (\lambda\to\infty) \end{cases}$$

Weak – strong coupling interpolation

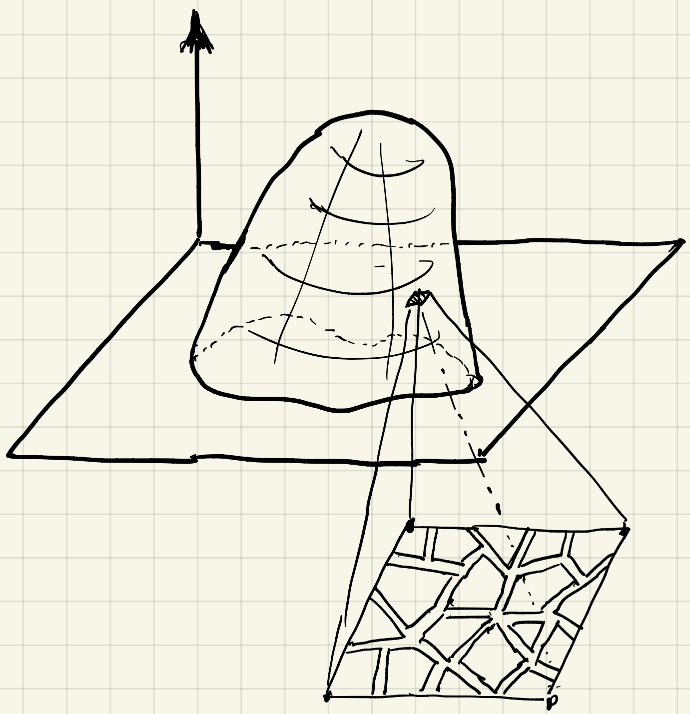
$\lambda \ll 1$: $\quad 1 +$  $+$  $+$  $+ \dots$

$\lambda \gg 1$:  diagrams condense into a string worldsheet

$$1 + c_1 \lambda + c_2 \lambda^2 + \dots \simeq e^{-\frac{\sqrt{\lambda}}{2\pi} A}\left(1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{\lambda} + \dots\right)$$

Two expansions of the same function

How dense are planar diagrams?



$$W(C) = \sum_{l=0}^{\infty} w_l \lambda^l$$

Average number of loops:

$$\bar{l} \equiv \frac{\sum_l l\, w_l \lambda^l}{\sum_l w_l \lambda^l} = \lambda \frac{\partial}{\partial \lambda} \ln W(C)$$

$$\ln W(C) \overset{\lambda\to\infty}{\simeq} \frac{\sqrt{\lambda}}{2\pi} A \quad \Longrightarrow \quad \bar{l} = \frac{\sqrt{\lambda}}{4\pi} A = \frac{A}{2\hbar}$$

- one gluon loop per two Planck units of area

Exact interpolating function: an example

Circular Wilson loop:

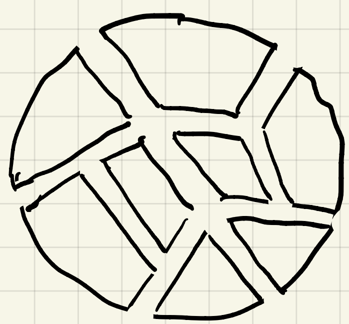
$$W(C) = \frac{2}{\sqrt{\lambda}}\, I_1\left(\sqrt{\lambda}\right)$$

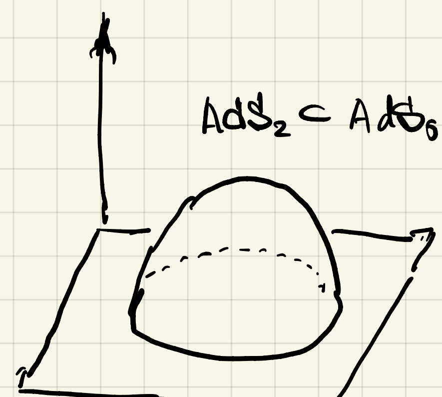
$\lambda \to 0$

$$1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \dots$$

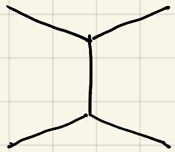
$\lambda \to \infty$

$$\sqrt{\frac{2}{\pi}}\, \lambda^{-3/4}\, e^{\sqrt{\lambda}} \left(1 - \frac{3}{8\sqrt{\lambda}} - \frac{15}{128\lambda} + \dots\right)$$

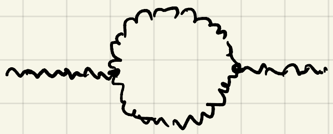




Computational complexity of QFT



$$\Longrightarrow \quad |A|^2 = \lambda^2\,\frac{s^2+u^2}{t^2}$$

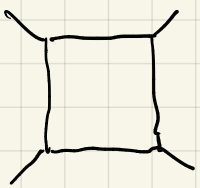


$\Big|_{div.}$

$$\beta = \frac{11}{3}\cdot\frac{\lambda^2}{16\pi^2}$$

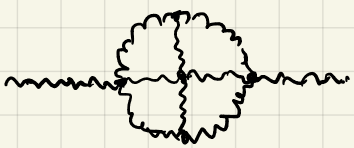
...

circular Wilson loop



$$\mathrm{Li}_n\left(\frac{s}{t}\right)$$

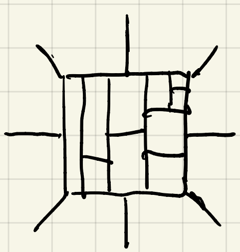
$$\Longrightarrow \quad \zeta(2n+1)$$



$\Big|_{div.}$

...

planar $\mathcal{N}=4$ SYM



$\Longrightarrow$

generic QFT

non-planar $\mathcal{N}=4$ SYM (?)

Transcendentality

- Harmonic sums:

$$S_{\pm a}(M) = \sum_{j=1}^{M} \frac{(\pm 1)^j}{j^a} \longrightarrow S_{\pm a_1 \ldots a_n}(M) = \sum_{j=1}^{M} \frac{(\pm 1)^j}{j^{a_1}} S_{a_2 \ldots a_n}(j)$$

$$\text{transcedentality} = \sum_i a_i$$

- ζ - values:

$$\zeta_{a_1 \ldots a_n} = S_{a_1 \ldots a_n}(\infty)$$

- Polylogarithms:

$$\mathrm{Li}_1(z) = -\log(1-z) \longrightarrow \mathrm{Li}_n(z) = \int_0^z \frac{dt}{t} \mathrm{Li}_{n-1}(z)$$

$$\text{transcedentality} = n$$

L-loop quantities in planar $\mathcal{N}=4$ SYM have uniform transcendentality $\sim 2L$.

Ex(1) Anomalous dimension of twist-2 operators

$$\mathcal{O}_M = \mathrm{tr}\, Z D_+^M Z$$

$$\Delta = M + 2 + \frac{\lambda}{2\pi^2} S_1 - \frac{\lambda^2}{16\pi^4}\left(S_3 + S_{-3} - 2 S_{-2,1} + 2 S_1 S_2 + 2 S_1 S_{-2} \right) + \ldots$$

Ex(2) Cusp anomalous dimension:

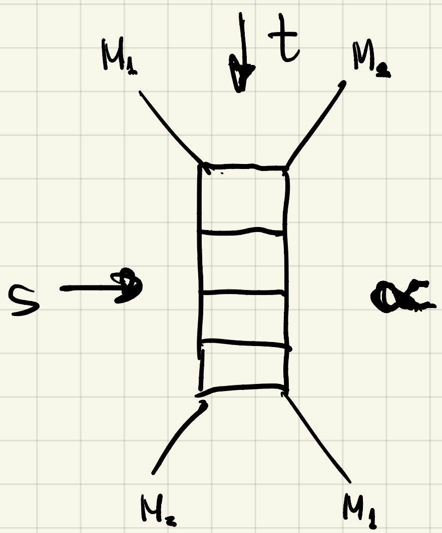
$$W(\measuredangle\varphi) \simeq \left(\frac{L}{\varepsilon}\right)^{\Gamma_{cusp}(\varphi)} \qquad \Gamma_{cusp}(i\theta) \overset{\theta\to\infty}{\simeq} -4 f(\lambda)\,\theta \qquad \text{Rem: } \Delta \overset{M\to\infty}{\simeq} f(\lambda) \ln M$$

$$f(\lambda) = \frac{\lambda}{2\pi^2} - \frac{\lambda^2}{32\pi^4} \times \frac{\pi^2}{3} + \frac{\lambda^3}{512\pi^6} \cdot \frac{11\pi^4}{45} - \frac{\lambda^4}{4024\pi^8}\left(\frac{73\pi^6}{630} + 4\zeta(3)^2 \right) + \ldots$$

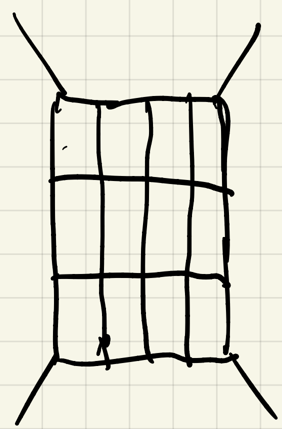
transcendentality $(\pi) = 1$!

Ex (3) Basso - Dixon and ladder integrals.

- Emerge in the fishnet limit of $\mathcal{N}=4$ SYM



$\propto \dfrac{L_n(z,\bar z)}{t^n s}$



$\propto \dfrac{\left[\frac{(1-z)(1-\bar z)}{z-\bar z}\right]^m}{t^n s^m} \det_{1\le i,j\le m} L_{i+j+n-m-1}(z,\bar z)$

transcendentality = $2n$

transcendentality = $2nm$

$$L_n(z,\bar z) = \sum_{j=n}^{2n} \frac{j!\,(n-1)!}{(j-n)!\,(2n-j)!} \left(-\ln z\bar z\right)^{2n-j} \left(\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar z)\right)$$

$$\frac{z\bar z}{(1-z)(1-\bar z)} = \frac{M_1^4}{st}$$

$$\frac{1}{(1-z)(1-\bar z)} = \frac{M_2^4}{st}$$

Operator renormalization

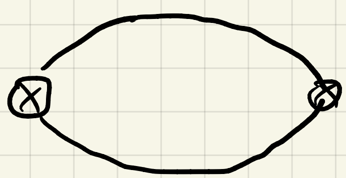
Konishi operator:

$$K = \mathrm{tr}\,\Phi_i \Phi_i$$

$$\Delta = 2$$

symmetry factor

$(\bullet\!-\!\!-\!\!-\!\bullet)^2$



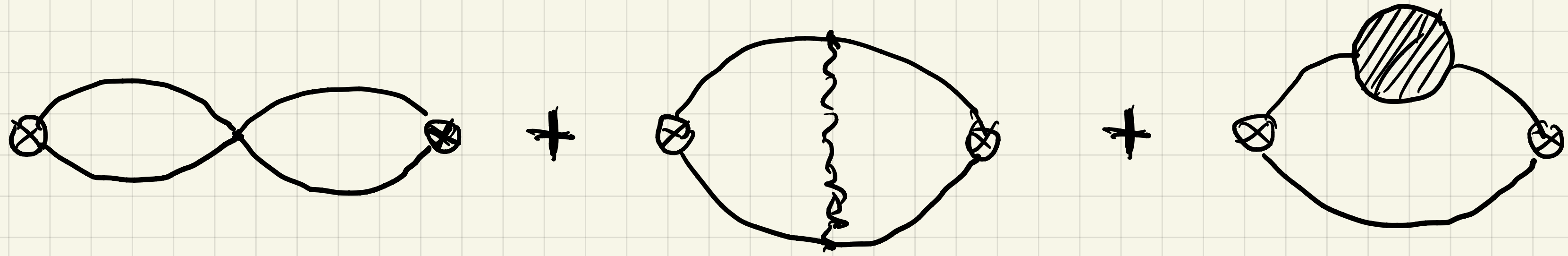
$$= 6 \times 2 \times \lambda^2 \times \left(\frac{1}{8\pi^2 x^2}\right)^2 = \frac{3\lambda^2}{16\pi^4 x^4}$$

2Δ

$\#\,\Phi_i$'s

loop counting

Next order in λ:



- these diagrams log-diverge:

$$\langle K(x)\, K(0)\rangle = \frac{3\lambda^2}{16\pi^4 x^4}\left(1 - \frac{3\lambda}{4\pi^2}\ln x^2\Lambda^2\right)$$

– need multiplicative renormalization:

$$K_R = Z K$$

$$Z = 1 + \frac{3\lambda}{4\pi^2}\ln\Lambda$$

– log-dependence on x contradicts conformal invariance, hence $\frac{1}{x^4}\left(1 - \gamma \ln x^2\Lambda^2\right)$ has to be regarded as $\frac{\Lambda^{-2\gamma}}{x^{2(2+\gamma)}}$ expanded in $\gamma = \frac{3\lambda}{4\pi^2}$

both are manifestations of quantum corrections to scaling

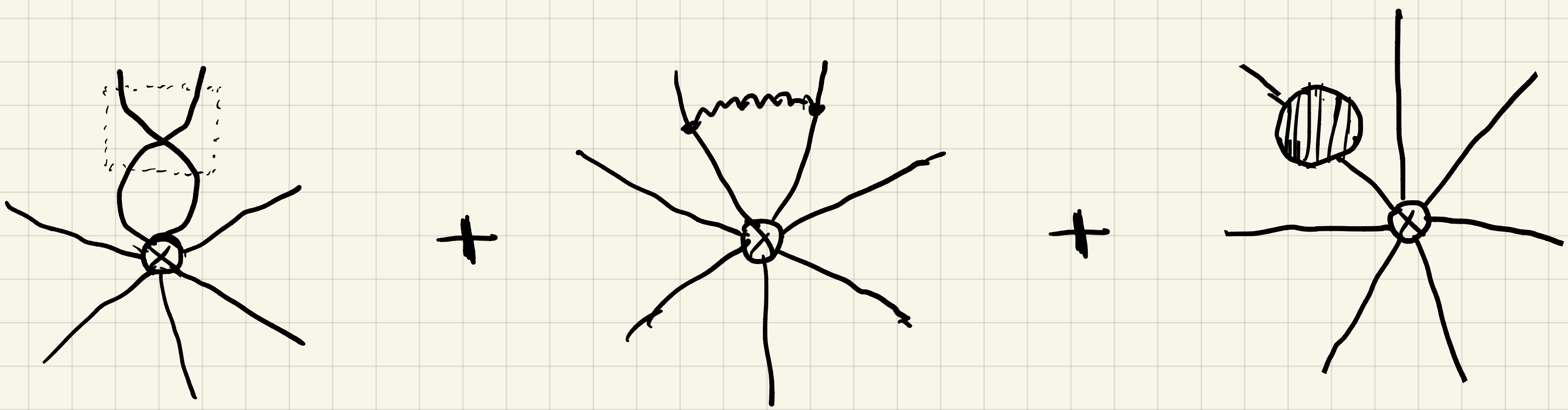
$$\langle K_R(x)\, K_R(0)\rangle = \frac{1}{x^{4+\frac{3\lambda}{2\pi^2}+\dots}} \Longrightarrow \boxed{\Delta = 2 + \frac{3\lambda}{4\pi^2} + O(\lambda^2)}$$

Operator mixing

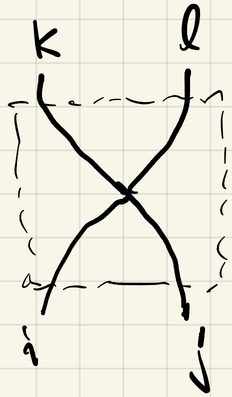
Generic scalar operator:

$$\mathcal{O} = \Psi^{i_1 \dots i_L}\, \mathrm{tr}\, \Phi_{i_1} \dots \Phi_{i_L}$$

Ψ: cyclically symmetric SO(6) tensor of rank-L

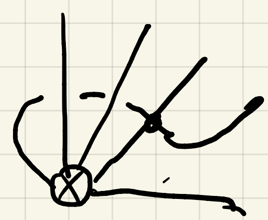


- Operator mixing:



$$= \delta_i^k \delta_j^l - 2\, \delta_i^l \delta_j^k + \delta_{ij}\, \delta^{kl}$$

- nearest-neighbour "interactions":



is $1/N$-suppressed

Spin chain

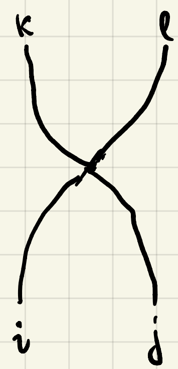
$$\mathcal{O}_R^\alpha = Z^\alpha{}_\beta\, \mathcal{O}^\beta$$

Dilatation operator (mixing matrix): $\quad D = Z^{-1}\dfrac{dZ}{d\ln\Lambda} + L$

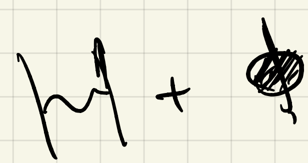
$$D\cdot\Psi_n = \Delta_n \Psi_n$$

$$P_{ij}^{kl} = \delta_i^l\, \delta_j^k$$

$$K_{ij}^{kl} = \delta_{ij}\, \delta^{kl}$$



$$= (1 - 2P + K)_{ij}^{kl}$$



$$\boxed{D = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(2 - 2P_{l,l+1} + K_{l,l+1}\right)}$$

$$\mathcal{O} = \Psi^{i_1 \dots i_L} \, \mathrm{tr}\, \Phi_{i_1} \dots \Phi_{i_L}$$

$$\boxed{D = L + \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(2 - 2 P_{l,l+1} + K_{l,l+1} \right)}$$

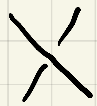
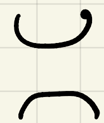
- D is the Hamiltonian of an $SO(6)$ spin chain
- $\Psi^{i_1 \dots i_L}$ is the spin-chain wavefunction

$$\Psi \in \mathbb{R}^6 / \mathbb{Z}_L$$

translational inv. $\longleftrightarrow$ trace cyclicity

SO(6) spin chain: an example

$$H = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(2 - 2P_{l,l+1} + K_{l,l+1}\right)$$

$P =$  $K =$ 

Konishi operator: $K = \delta^{ij}\,\mathrm{tr}\,\Phi_i\Phi_j$

Spin chain of length 2:

The eigenvalue:

$$\gamma = \frac{\lambda}{16\pi^2}\cdot 2\cdot\left(2-\underset{\uparrow \atop P}{2}+\underset{\uparrow \atop K}{6}\right) = \frac{3\lambda}{4\pi^2}$$

Anomalous dim. of Konishi op.

Protected operators

$$H = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(2 - 2P_{l,l+1} + K_{l,l+1}\right)$$

Ground state:

$$H|0\rangle = 0$$

$$(1-P)\,|0\rangle = 0 \qquad \& \qquad K|0\rangle = 0$$

$$\Downarrow$$

Chiral Primary Operators:

$$CPO = C^{i_1 \dots i_L} \, \mathrm{tr}\, \Phi_{i_1} \dots \Phi_{i_L}$$

symmetric traceless

<u>Ex</u>: $20' = \mathrm{tr}\, \Phi_{(i} \Phi_{j)} - \frac{1}{6} \delta_{ij} \Phi_k^2$

Explicit parameterization:

$$\mathrm{tr}\,(\vec{y} \cdot \vec{\Phi})^L$$

$\vec{y}$ - complex 6-vector with $\vec{y}^2 = 0$

SU(2) subsector and Heisenberg model

Consider restriction to operators built of

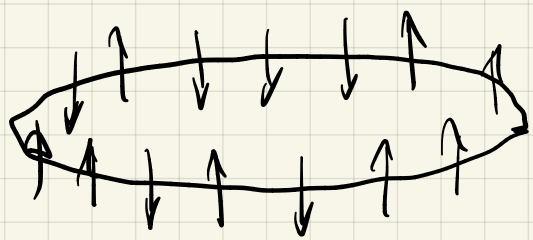
$$Z = \Phi_1 + i\,\Phi_2$$

$$W = \Phi_3 + i\,\Phi_4$$

$$\mathcal{O} = \operatorname{tr} Z^{L-M} W^{M} + \text{permutations.}$$

$$Z \longleftrightarrow \uparrow$$

$$W \longleftrightarrow \downarrow$$



Mixing matrix:

$$D = L + \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1 - P_{l,l+1}\right) = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1 - \vec{\sigma}_l\cdot\vec{\sigma}_{l+1}\right)$$

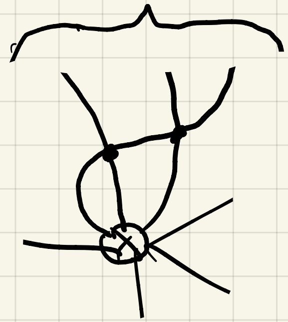
Heisenberg Hamiltonian!

- How to include other operators?

$$\mathcal{L} = \mathrm{tr}\, F_{\mu\nu}^2 \,, \quad \mathcal{O} = \mathrm{tr}\, \bar{\Phi}\Psi \,, \quad \mathcal{O} = \mathrm{tr}\, D_\mu \Phi D_\nu \Phi \,, \quad \ldots$$

- At higher loops range of interaction grows:

$H_{l,l+1,l+2}$



Also, operators of different length start to mix:

$$\mathrm{tr}\Big(\underset{L=2}{a_G F_{\mu\nu}^2} + \underset{L=4}{a_{s_1} \Phi_i \Phi_i \Phi_j \Phi_j} + \underset{L=4}{a_{s_2} \Phi_i \Phi_j \Phi_i \Phi_j} + \underset{L=3}{a_F \bar{\Psi} \Gamma^i \Phi_i \Psi}\Big)$$

- Mixing matrix by itself is not an invariant object.
  We are interested in the <u>eigenvalues</u>
  (and in principle eigenfunctions).

Magnons

$$H=\sum_{l=1}^{L}\left(1-P_{l,l+1}\right)$$

The ground state: $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ $\sim$ $\mathrm{tr}\, Z^L$ $\qquad Z=\Phi_1+i\Phi_2$

1-magnon state:

$$\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$$
$$\xrightarrow{\;e^{ipl}\;}$$

$$H|p\rangle=\varepsilon(p)|p\rangle \qquad \varepsilon(p)=4\sin^2\frac{p}{2}$$

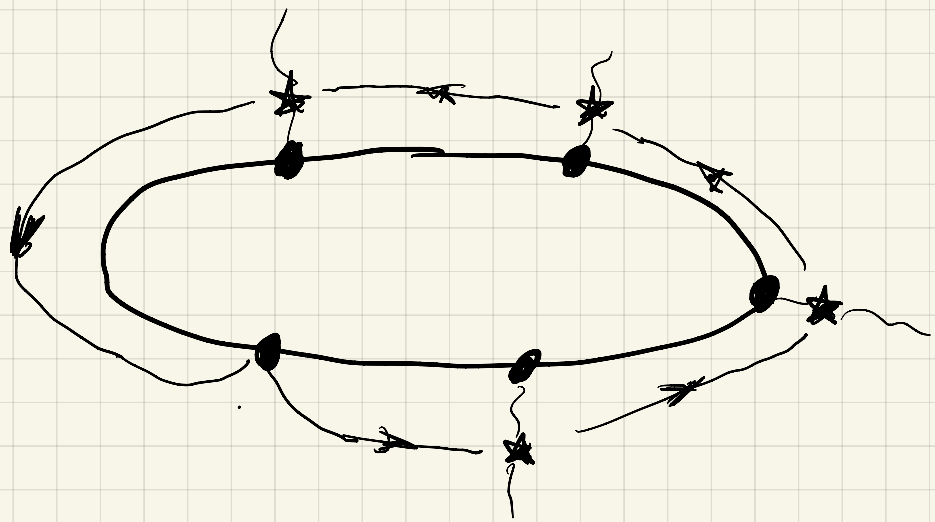
2-magnon state:

$$\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$$
$$\xrightarrow{\;p_1\;} \qquad \xleftarrow{\;p_2\;}$$

$$|p_1 p_2\rangle=\sum_{x_1<x_2}\left(e^{ip_1x_1+ip_2x_2}+S(p_1,p_2)\,e^{ip_1x_2+ip_2x_1}\right)|x_1,x_2\rangle$$

$$S(p_1p_2)=\frac{u_1-u_2-i}{u_1-u_2+i} \qquad u(p)=\frac{1}{2}\operatorname{ctg}\frac{p}{2}$$

Bethe Ansatz



Bethe equations:

$$e^{ip_j L} \prod_{k \neq j} S(p_j, p_k) = 1 \qquad \text{(periodicity of wavefunction)}$$

$$E = \sum_j \varepsilon(p_j)$$

- Only works in integrable models

Exact solution of Heisenberg model

$$H = \sum_{l=1}^{L} \left(1 - P_{l,l+1}\right)$$

Bethe Ansatz equations:

$$\boxed{\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}}$$

$$E = \sum_j \frac{1}{u_j^2 + \frac{1}{4}} \qquad \Delta = L + \frac{\lambda}{8\pi^2} E$$

$$e^{iP} = \prod_j \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \qquad P = 0 \text{ (trace cyclicity)}$$

Example: Konishi anomalous dimension

$\{u, -u\}$ $\rightsquigarrow$ total momentum is zero

and $L = 4$

$$\left(\frac{u+\frac{i}{2}}{u-\frac{i}{2}}\right)^4 = \frac{u-(-u)+i}{u-(-u)-i} \implies \left(u+\frac{i}{2}\right)^3 = \left(u-\frac{i}{2}\right)^3 \implies u^2 = \frac{1}{12}$$

$$\gamma = \frac{\lambda}{8\pi^2} \cdot \frac{2}{\frac{1}{12}+\frac{1}{4}} = \frac{3\lambda}{4\pi^2}$$

agrees w. explicit calculation for $K = \mathrm{tr}\, \Phi_i \Phi_i$

But we are dealing with $\mathcal{O}_4 = \mathrm{tr}\,\{Z, W\}^2$

Supersymmetry:

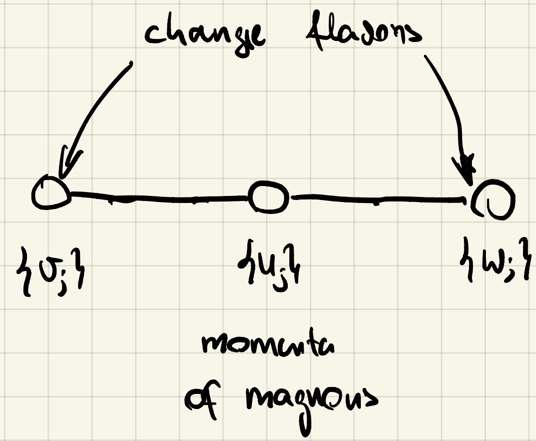
$$\mathcal{O}_4 = (Q_\alpha Q^\alpha)^2\, K$$

Nested Bethe Ansatz

SO(6) spin chain:

$$H = \sum_{l=1}^{L} \left(1 - P_{l,l+1} + \frac{1}{2} K_{l,l+1} \right)$$

Three sets of rapidities:



$$1 = \prod_k \frac{v_j - u_k - \frac{i}{2}}{v_j - u_k + \frac{i}{2}}$$

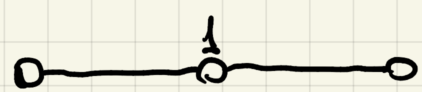
$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_k \frac{u_j - v_k - \frac{i}{2}}{u_j - v_k + \frac{i}{2}} \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_k \frac{u_j - w_k - \frac{i}{2}}{u_j - w_k + \frac{i}{2}}$$

$$1 = \prod_k \frac{w_j - u_k - \frac{i}{2}}{w_j - u_k + \frac{i}{2}}$$

$$\boxed{\left(\frac{u_{aj}+\frac{iq_a}{2}}{u_{aj}-\frac{iq_a}{2}}\right)^L = \prod_{bk\neq aj}\frac{u_{aj}-u_{bk}+\frac{i\mathcal{M}_{ab}}{2}}{u_{aj}-u_{bk}-\frac{i\mathcal{M}_{ab}}{2}}}$$

$$\mathcal{M} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ Cartan matrix of $SO(6)$

$$q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ Dynkin labels of the spin representation

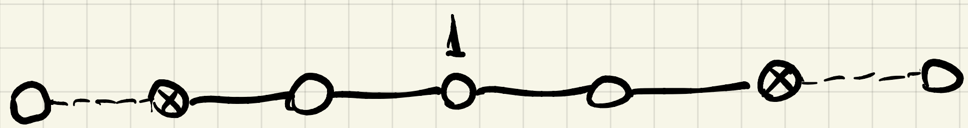


Including all operators

Global symmetry of $\mathcal{N}=4$ SYM : $\boxed{psu(2,2|4)}$ ($\mathcal{N}=4$ superconformal algebra)

$$\Downarrow$$

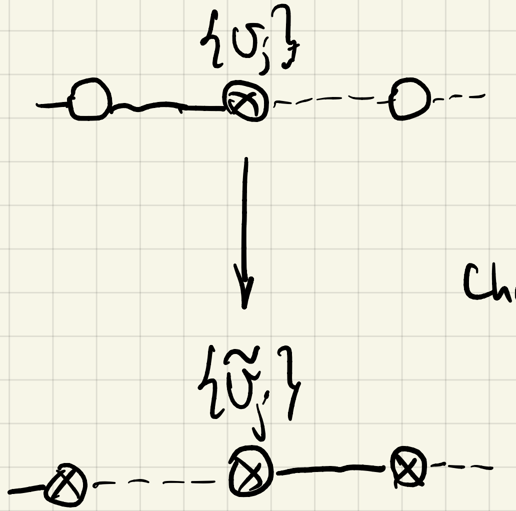
$$su(2,2) \oplus su(4) = \underbrace{so(4,2)}_{\text{conf.}} \oplus \underbrace{so(6)}_{\text{R-symm.}}$$



$$M = \begin{bmatrix} -2 & 1 & & & & & \\ 1 & 0 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & & 0 & 1 \\ & & & & & 1 & -2 \end{bmatrix}$$

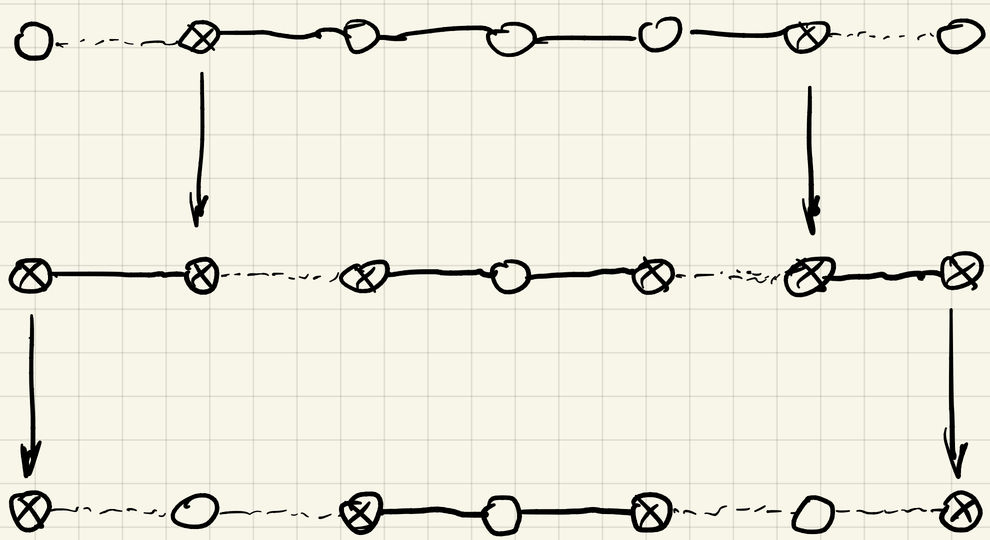
$$\boxed{\left(\frac{u_{aj} + \frac{i q_a}{2}}{u_{aj} - \frac{i q_a}{2}}\right)^L = \prod_{bk \neq aj} \frac{u_{aj} - u_{bk} + \frac{i M_{ab}}{2}}{u_{aj} - u_{bk} - \frac{i M_{ab}}{2}}}$$

Fermionic duality



Change of variables in the Bethe equations

Correspond to different gradings of $psu(2,2|4)$:



distinguished grading

(can be generalized to higher loops)

$\boxed{\text{Symmetries}}$

Kinematics:

$\varepsilon = \varepsilon(p)$

S-matrix:

e.g. $SO(6)$

$\Downarrow$

$$S_{ij}^{kl} = \sigma_0 \left(\mathbb{1} + a\, P + b\, K \right)_{ij}^{kl}$$

dressing phase

(fixed by unitarity and crossing)

fixed by Yang-Baxter equation

Integrability Bootstrap in AdS/CFT

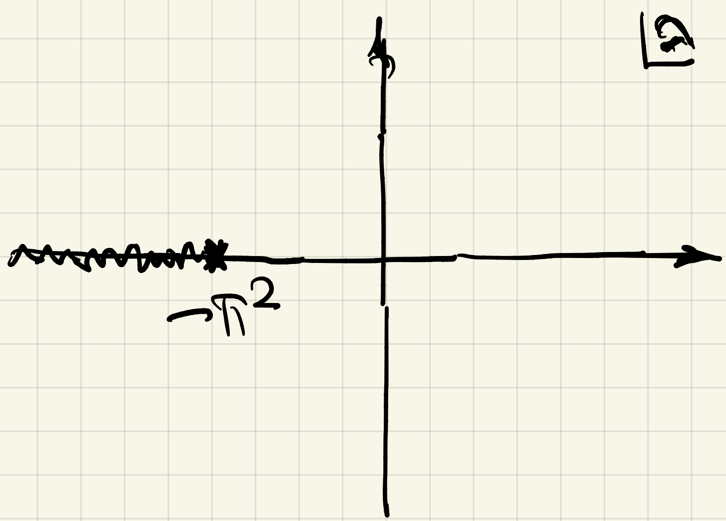
Magnon dispersion relation:

$$\varepsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$

$\lambda \to 0$ $\qquad\qquad\qquad$ $p = \frac{2\pi}{\sqrt{\lambda}}\, p_{str}\ ,\ \lambda \to \infty$

$$1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} + \ldots \qquad\qquad \sqrt{1+p^2}$$

(Heisenberg model) $\qquad$ (relativistic mode on the string worldsheet)



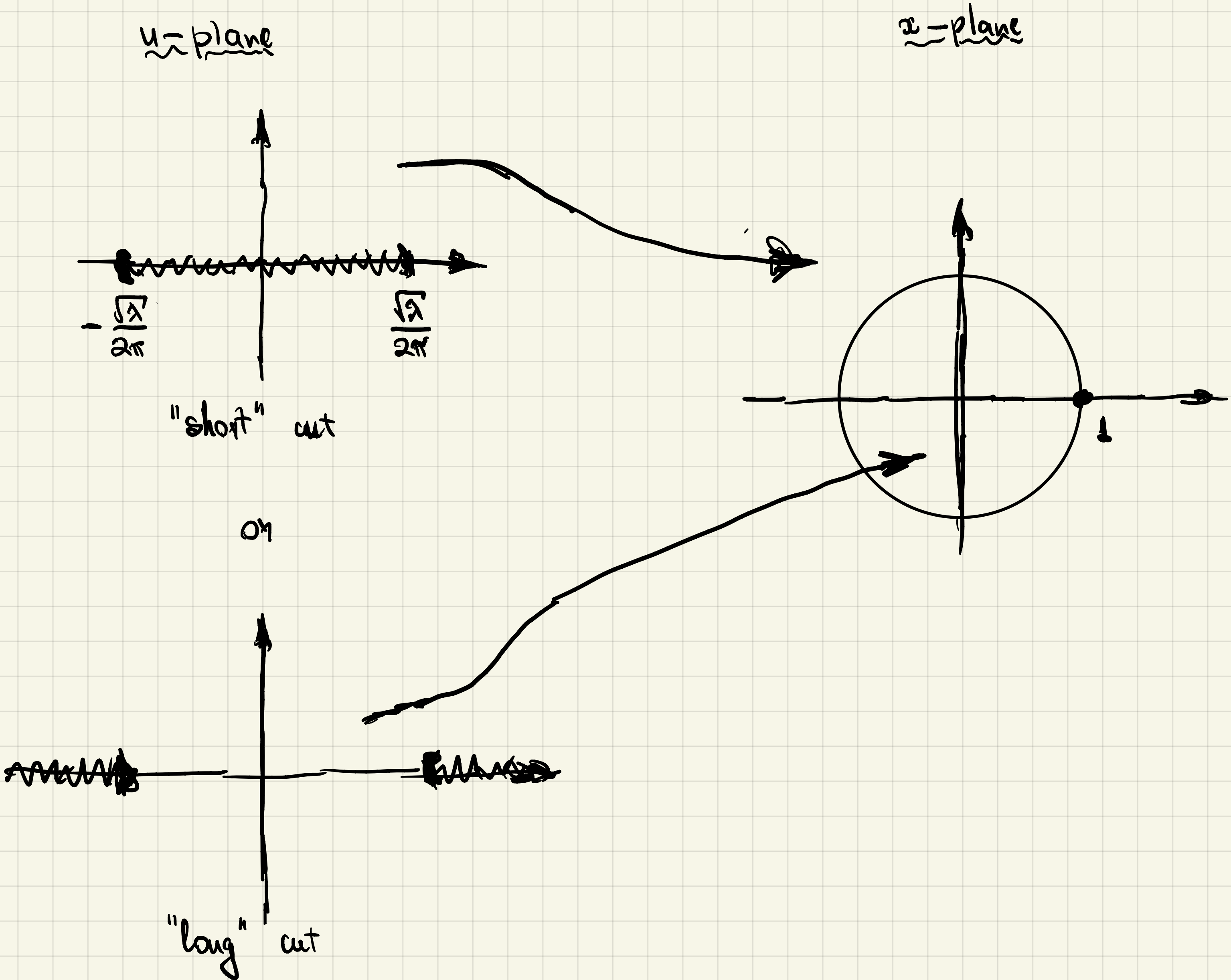
- planar perturbation theory has finite radius of convergence $\lambda_c = \pi^2$

- $\lambda \to -\pi^2$ scaling limit?
  AdS $\to$ dS /Polyakov/

Kinematics: Zhukowski variable

$x(u)$:

$$\boxed{x + \frac{1}{x} = \frac{4\pi u}{\sqrt{\lambda}}}$$



Dispersion relation from Zhukowski variable

$$f^{\pm}(u) \equiv f\left(u \pm \tfrac{i}{2}\right)$$

$$\boxed{\begin{aligned} e^{ip} &= \frac{x^+}{x^-} \\ \varepsilon(p) &= \frac{i\sqrt{\lambda}}{2\pi}\left(\frac{1}{x^+} - \frac{1}{x^-}\right) \end{aligned}}$$

Weak coupling:

$$x \simeq \frac{4\pi}{\sqrt{\lambda}} u + \dots \quad \text{(quantum deformation of rapidity)}$$

Strong coupling:

$$x + \frac{1}{x} = u_{str.}$$

$$f^{\pm}(u_{str}) = f(u_{str} \pm i\hbar) \qquad \hbar = \frac{2\pi}{\sqrt{\lambda}}$$

Asymptotic Bethe equations

su(2) sector: deformation of Heisenberg model

$$\boxed{\left(\frac{x_j^+}{x_j^-}\right)^L = \prod_{k\neq j} \sigma_{BES}(x_j, x_k)\, \frac{u_j - u_k + i}{u_j - u_k - i}}$$

dressing (Beisert - Eden - Staudacher) phase:

$$\sigma_{BES}(x,y) = e^{i\chi(x^+,y^-) - i\chi(x^-,y^-) - i\chi(x^+,y^+) + i\chi(x^-,y^+)}$$

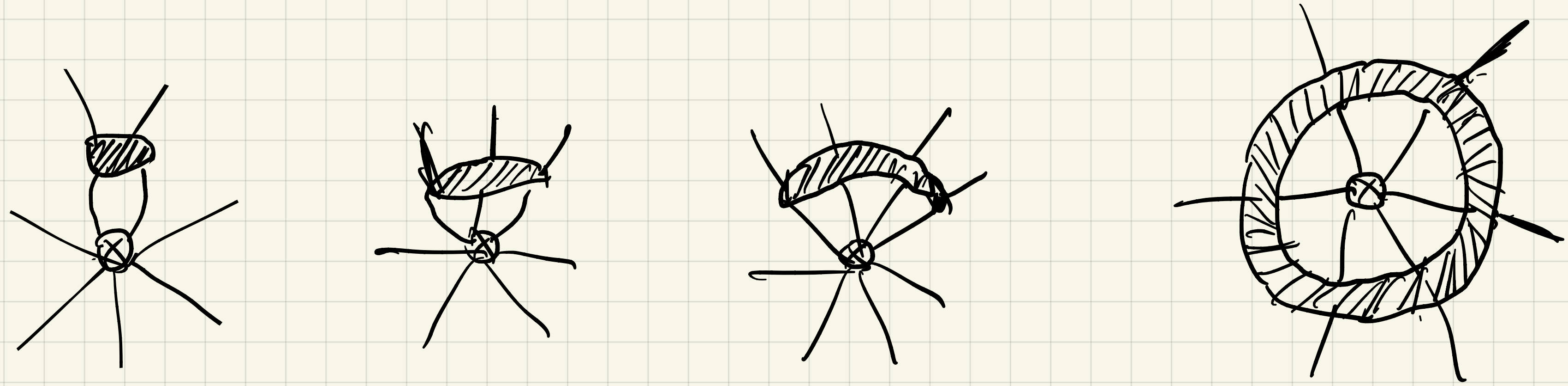
$$\chi(x,y) = -i \oint_{|z|=1} \frac{dz}{2\pi i} \oint_{|w|=1} \frac{dw}{2\pi i} \frac{1}{x-z} \cdot \frac{1}{y-w} \ln \frac{\Gamma\left(1 + \frac{i\sqrt{\lambda}}{4\pi}\left(z + \frac{1}{z} - w - \frac{1}{w}\right)\right)}{\Gamma\left(1 - \frac{i\sqrt{\lambda}}{4\pi}\left(z + \frac{1}{z} - w - \frac{1}{w}\right)\right)}$$

- minimal solution of the crossing equation

Wrapping corrections

Operator of length L:

$$\operatorname{tr} \Phi_{i_1} \dots \Phi_{i_L}$$



1 - loop   2 - loops   3 - loops $(L-1)$ - loops

- At $O(\lambda^L)$ — the <u>wrapping</u> order — the range of interactions becomes bigger than the length of the spin chain

$\Downarrow$

Bethe Ansatz breaks down.

<u>Q</u>: How to take wrapping corrections into account exactly?

QQ – equation : Heisenberg model

Bethe equations:

$$\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}$$

Baxter polynomial: and its dual:

$$Q(u) = \prod_{j=1}^{M} (u - u_j) \qquad \tilde{Q}(u) = \prod_{j=1}^{L-M-1} (u - \tilde{u}_j)$$

QQ – equation:

$$\boxed{Q^+ \tilde{Q}^- - Q^- \tilde{Q}^+ = i(2M - L - 1)\, u^L}$$

$L-1$ eqs. for $L-1$ unknowns

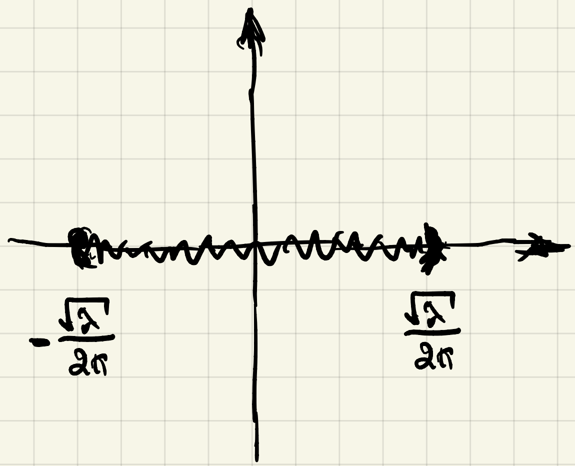
- determines both Q and $\tilde{Q}$

BAEs $\Longleftrightarrow$ QQ-eqs.

Quantum spectral curve

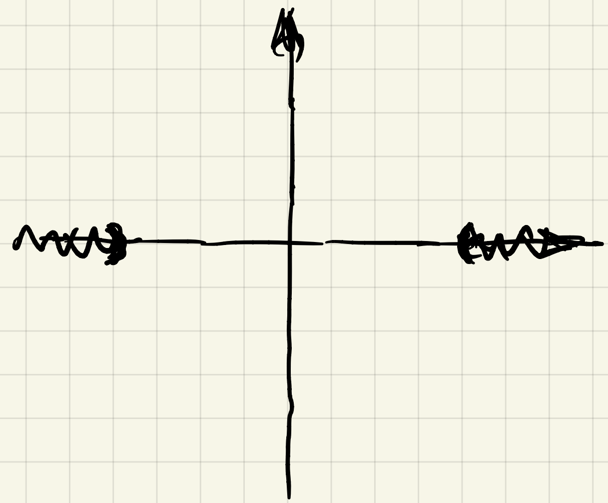
S^5:

$P_a \,,\ P^a \qquad a=1\ldots4$



AdS_5:

$Q_j \,,\ Q^j \qquad j=1\ldots4$



- QQ - relations
- conditions on analytic continuation

$\Big\}$ describe full non-perturbative spectrum of AdS/CFT

Generic integrable system	AdS/CFT
Systematically calculable	
- spectrum of states S-matrix	Beisert's S-matrix Asymptotic Bethe Ansatz
- thermodynamics: TBA	$\sim$ QSC
- 1pt functions: $\langle 0 \| \mathcal{O}(0) \| 0 \rangle$	
- More generally: formfactors $\langle p_1 \dots p_n \| \mathcal{O}(0) \| 0 \rangle$	Hexagon form-factors
- Overlaps with integrable boundary states: $\langle p_1 \dots p_n \| B \rangle$	Integrable D-branes

Correlation functions

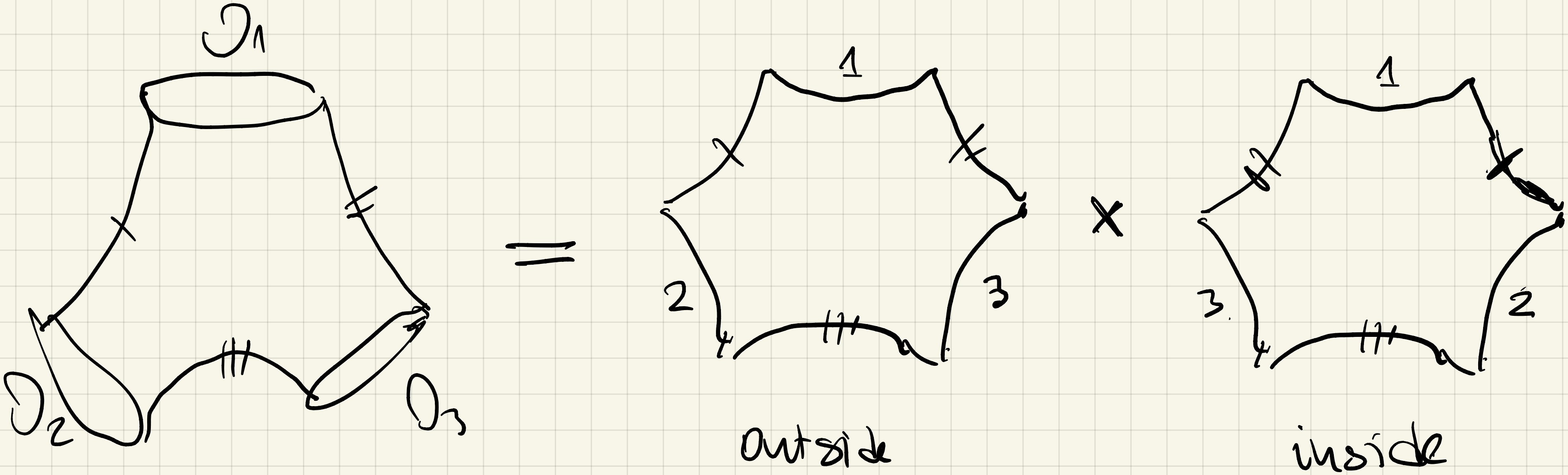
Form-factor expansion:

$$\langle 0|\mathcal{O}(x)\,\mathcal{O}(0)|0\rangle = \sum_N \int \prod_{j=1}^{N} \frac{dp_i}{2\pi\sqrt{E(p_i)}}\, e^{i(p_1+\dots+p_N)x}\, \left|\langle 0|\mathcal{O}|p_1\dots p_N\rangle\right|^2$$

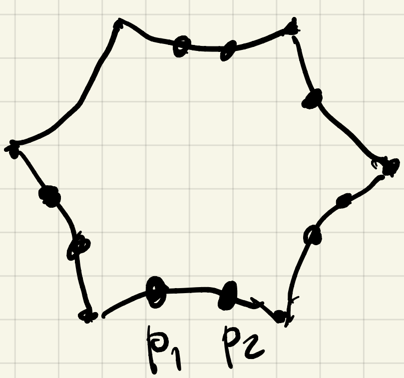
$$\Updownarrow$$

Hexagon expansion in AdS/CFT

Three-point functions and hexagons

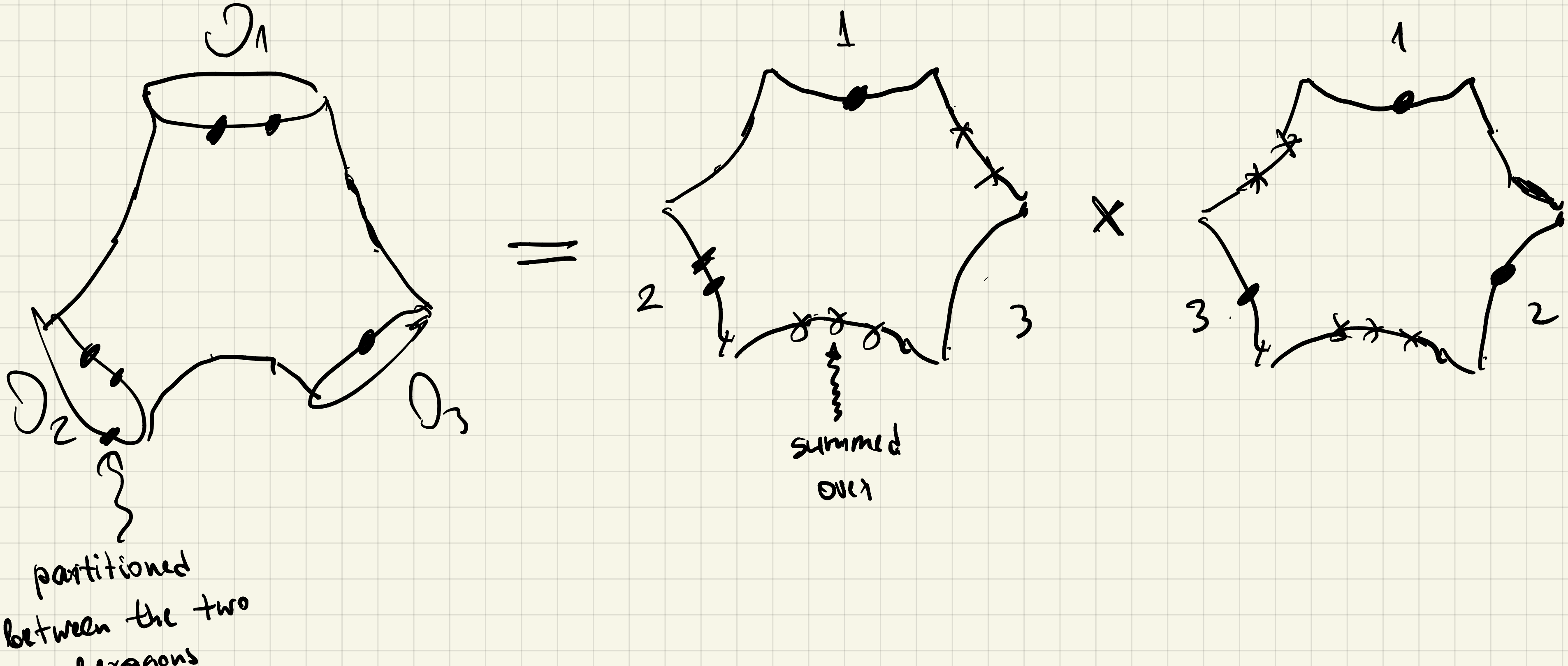


Hexagon form-factor:



$$= \langle p_1 \, p_2 \, \dots \, p_n \,|\, h \rangle$$

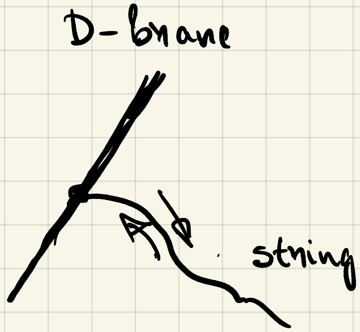
Three-point functions

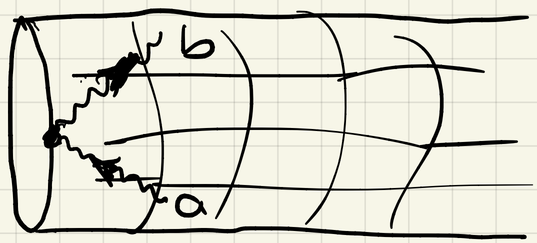


$$\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3 \rangle = \sum_{\text{partitions}} \sum_{n_1,n_2,n_3=0}^{\infty} \int \prod_{i=1}^{n_i} du_i\, \mu(u_i)\, \langle \{v_j\}, \{u_i\} | h \rangle \langle h | \{\tilde{v}_j\}, \{u_i\} \rangle$$

$\{v_j\} \cup \{\tilde{v}_j\}$ = all phys. rapidities

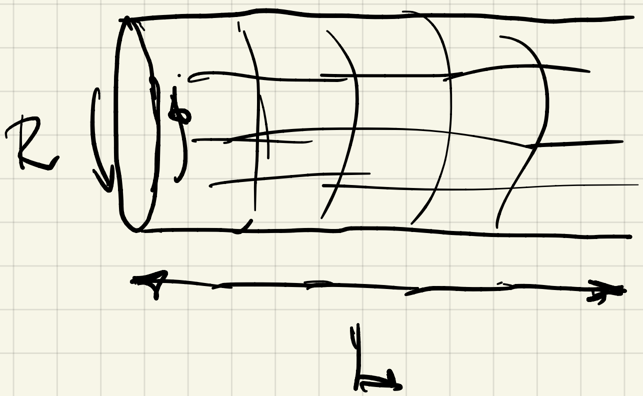
Integrable D-branes





Reflection amplitude: $K_{ab}(p)$

Elastic reflection $\Longleftrightarrow$ Integrability

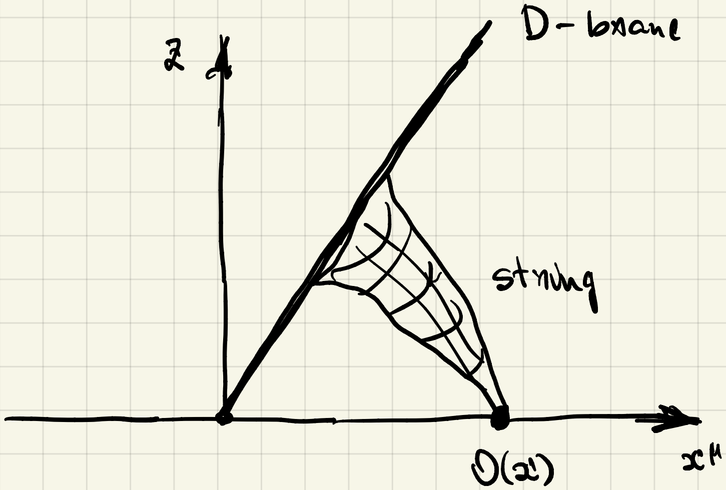


$$Z(R,L) \overset{L\to\infty}{\simeq} g(R)\, e^{-E_0 L}$$

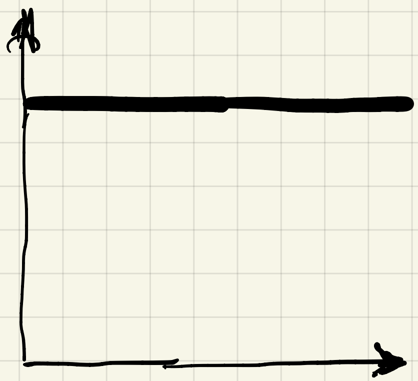
g - function

- calculable by TBA!

Affleck, Ludwig '91
Dorey, Fioravanti, Rim, Tateo '04
Pozsgay '10

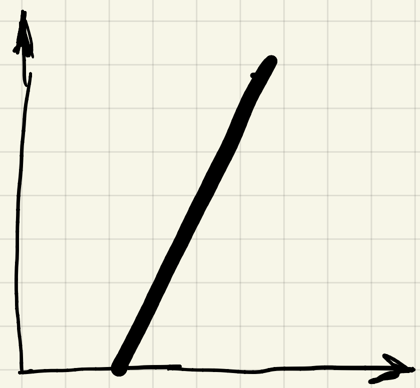


g-function $= \langle \mathcal{O}(x) \rangle$

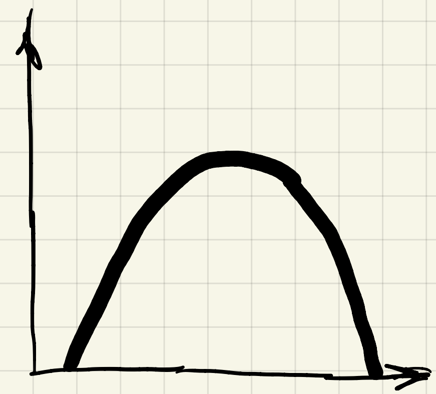


Coulomb branch:

$\langle \Phi^i \rangle = n^i v$



defect CFT



Heavy operator insertion:

$\langle H H \mathcal{O} \rangle$

Coulomb branch

Scalar potential in $\mathcal{N}=4$ SYM:

$$V = -\frac{1}{2}\,\mathrm{tr}\,[\Phi_i , \Phi_j]^2 \geq 0$$

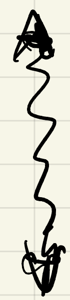
- reaches zero on

$$[\Phi_i , \Phi_j] = 0$$

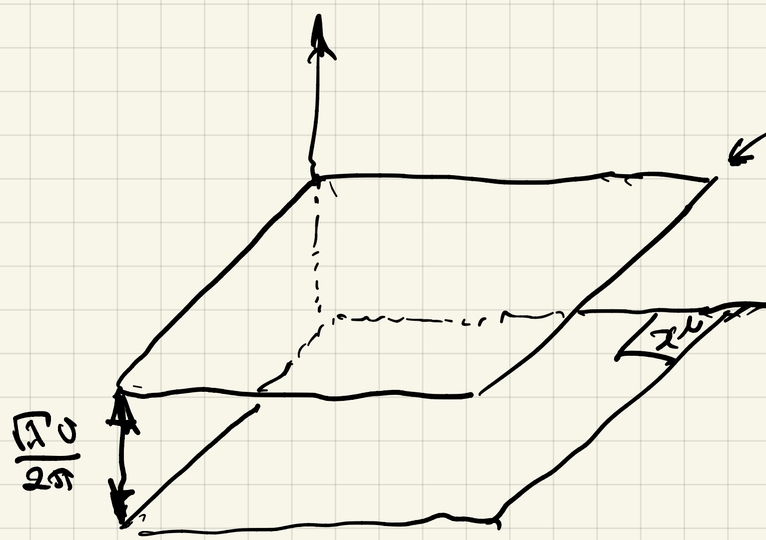
$$\Phi_i = \begin{bmatrix} \varphi_i^1 & & \\ & \ddots & \\ & & \varphi_i^N \end{bmatrix}$$ – "Coulomb branch" of vacua

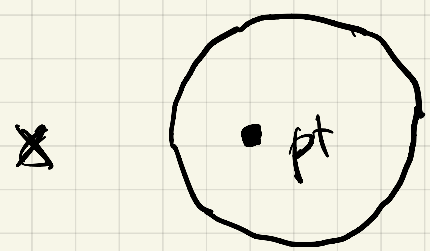
$SU(N) \longrightarrow SU(N-1) \times U(1)$:

$$\langle \Phi_i \rangle = \delta_{i6}\, v \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}$$



D3-brane in the bulk:





One-point functions

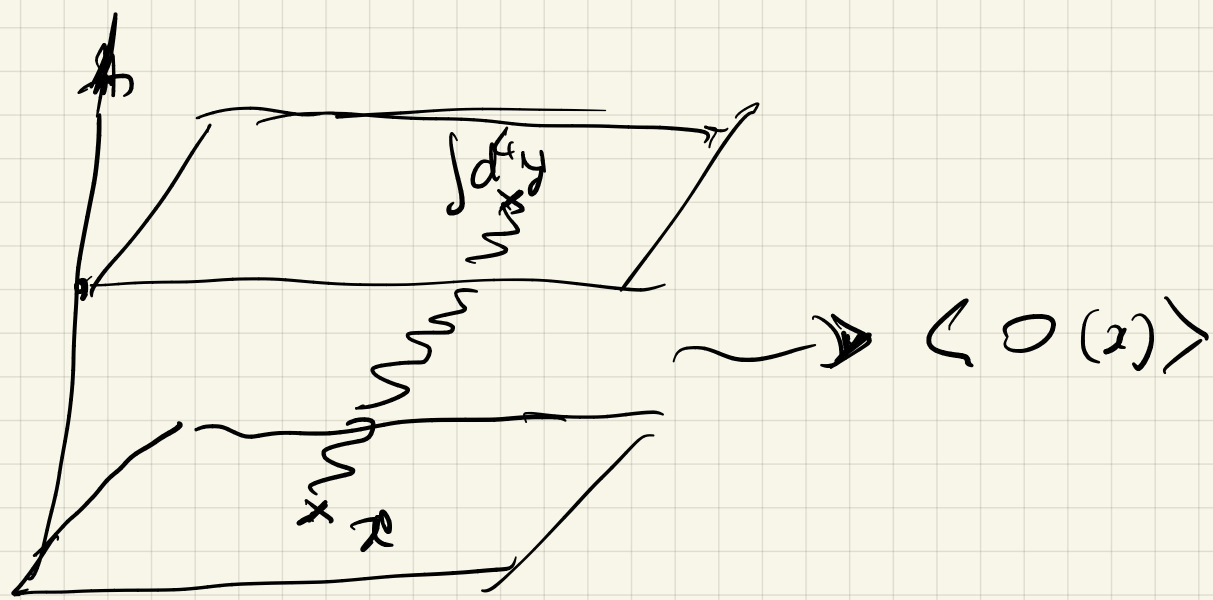
Weak coupling:

$$\mathcal{O} = \Psi^{i_1 \dots i_L} \, \mathrm{tr}\, \Phi_{i_1} \dots \Phi_{i_L}$$

interpolation using integrability?

$$\langle \mathcal{O} \rangle = v^L \, \Psi^{6 \dots 6}$$ need to know the wavefunction!

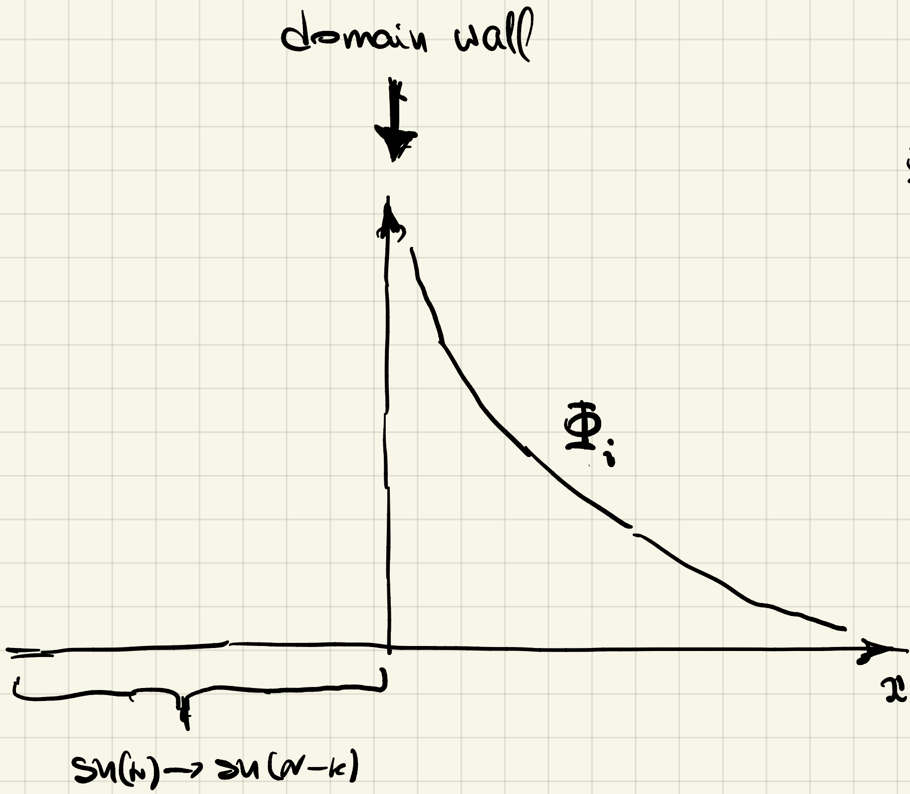
Strong coupling:



Defect CFT

$\Phi_i = \dfrac{t_i}{x} \qquad i = 1,2,3 \qquad\qquad \Phi_i = 0 \qquad i = 4,5,6$

- t_i form k-dim rep. of $su(2)$: $[t_i, t_j] = i\,\varepsilon_{ijk}\, t_k$



$$\Phi_i : \quad \begin{array}{cc} & \begin{array}{cc} k & \end{array} \\ k & \left[\begin{array}{c|c} & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$$

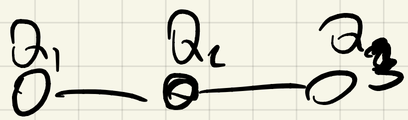
One-point functions

$$\mathcal{O} = \Psi^{i_1 \dots i_L} \operatorname{tr} \Phi_{i_1} \dots \Phi_{i_L}$$

$$\Phi_i \to \Phi_i^{cl} = \frac{t_i}{x}$$

$$\langle \mathcal{O}(x) \rangle = \frac{1}{x_1^L} \Psi^{i_1 \dots i_L} \operatorname{tr} t_{i_1} \dots t_{i_L} \equiv \frac{\langle \Psi | B \rangle}{x_1^L}$$

$|B\rangle_{i_1 \dots i_L} = \operatorname{tr} t_{i_1} \dots t_{i_L}$ - matrix product state.



$$\boxed{\langle \Psi | B \rangle = \sqrt{\frac{Q_1(0)\, Q_2(0)\, Q_3(0)}{Q_1(i/2)\, Q_2(i/2)\, Q_3(i/2)} \operatorname{Sdet} G}}$$

Q - functions

Gaudin superdeterminant.

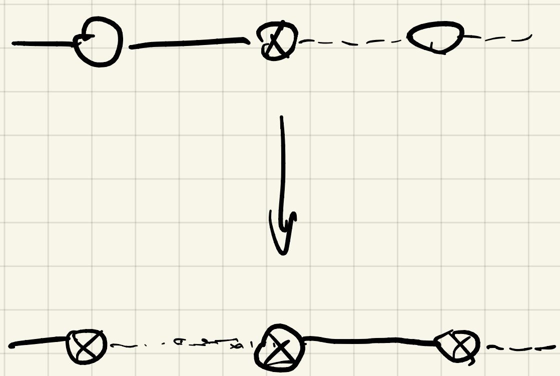
Gaudin superdeterminant:

$$\mathrm{Sdet}\, G = \frac{\det G^+}{\det G^-}$$

$$G^{\pm}_{jk} = \left(\frac{L q_a}{u_{aj}^2 + \frac{q_a^2}{4}} - \sum_{cl} K^+_{aj,cl} \right) \delta_{ab}\, \delta_{jk} + K^{\pm}_{aj,bk}$$

$$K^{\pm}_{aj,bk} = \frac{M_{ab}}{(u_{aj} - u_{bk})^2 + \frac{M_{ab}^2}{4}} \pm \frac{M_{ab}}{(u_{aj} + u_{bk})^2 + \frac{M_{ab}^2}{4}}$$

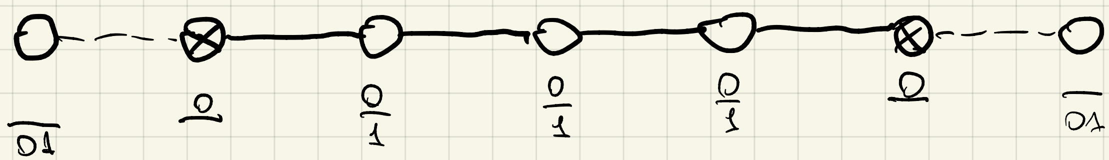
- transforms nicely under fermionic duality:

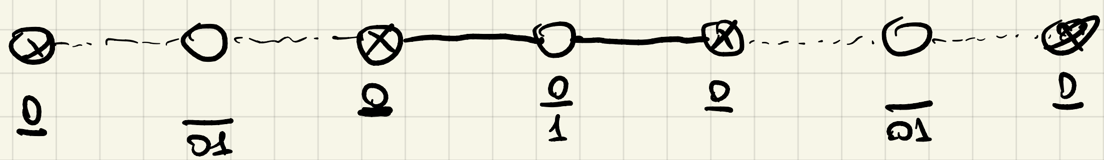


$$\mathrm{Sdet}\, \tilde{G} = \frac{Q_a(0)\, \tilde{Q}_a(0)}{Q_{a-1}(\frac{i}{2})\, Q_{a+1}(\frac{i}{2})}\, \mathrm{Sdet}\, G$$

$$\underset{\frac{\alpha\ldots}{\beta\ldots}}{-\!\bigcirc\!-} \equiv \sqrt{\frac{Q\left(\frac{i\alpha}{2}\right)\ldots}{Q\left(\frac{i\beta}{2}\right)\ldots}}$$

Overlap formula in different gradings:





• can be extended to all loops!