Line Operators in Chern-Simons-Matter Theories and Bosonization in Three Dimensions

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New Mathematical Methods in Solvable Models and Gauge/String Dualities

2022/08/16 Varna

ArXiv: 2204.05262 with Barak Gabai and Amit Sever

Chern-Simons Duality

❖ **Chern-Simons Theory** $G = SU(N)$

$$S_{CS} = \frac{\mathrm{i}k}{4\pi} \int \mathrm{d}^3x \, \epsilon^{\mu\nu\rho} \, \mathrm{tr}\Big(A_\mu \partial_\nu A_\rho - \frac{2\mathrm{i}}{3} A_\mu A_\nu A_\rho\Big) \qquad k \in \Pi_3(SU(N)) = \mathbb{Z}$$

❖ **Topological:** Correlators of Wilson loops inv under small deformation

❖ **Level-rank Duality: non-perturbative**

- Rank N of gauge group SU(N)
- Level k of the action
- Correlation functions: $\langle \cdots \rangle_{N,k} = \langle \cdots \rangle_{k,N}$

Chern-Simons Matter Duality

❖ **Couple to fundamental matter**

- No longer topological, but slightly broken ($1/N$) higher spin symmetry

- Expected to extend to a duality between **bosons** and **fermions**

extensive evidence, especially at large N

- correlation function of local operators
- spectrum of monopole/baryon operators
- thermal free-energies
- S-matrices
- relating non-susy dualities to well-established susy ones

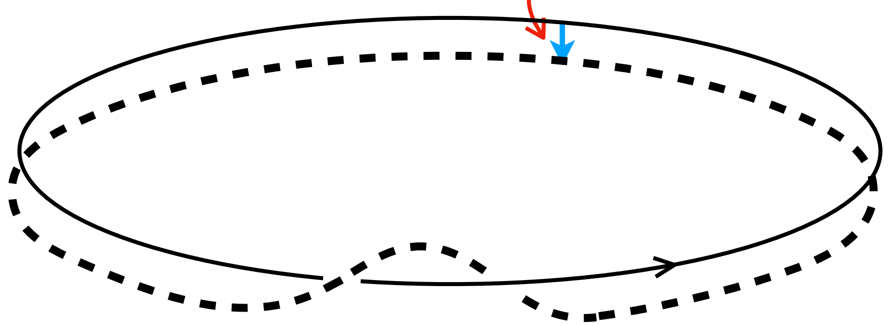
[Witten, Minwalla, Prakash, Trivedi, Wadia, Yin, Aharony, Gur-Ari, Yacoby, Maldacena, Zhiboedov, Giombi, Gaiotto, Kapustin, Hsin, Seiberg, Naculich, Schnitzer, Mlawer, Naculich, Riggs, Schnitzer, Camperi, Levstein, Zemba, Bedhotiya, Prakash, Gurucharan, Kirilin, Prakash, Skvortsov, Radivcevic, Jain, Yokoyama, Sharma, Takimi, Mandlik, Inbasekar, Mazumdar, Giveon, Kutasov, Benini, Closset, Cremonesi, …]

Chern-Simons Matter Duality

- **Why such duality?**

  - Pure Chern-Simons: framing regularization (normal point-splitting)

framing vector: normal to the loop

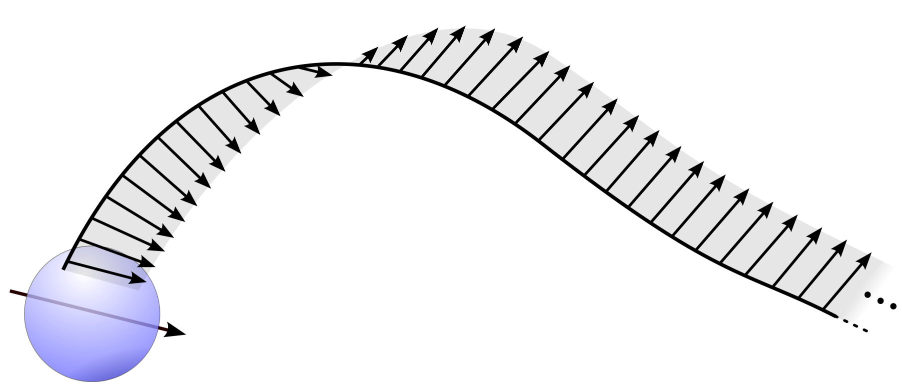


$$\left\langle W^{\mathfrak{f}}_{\text{unknot}} \right\rangle = e^{i\pi\lambda\mathfrak{f}} \times k\frac{\sin(\pi\lambda)}{\pi}$$

self linking number

  - Planar Limit: $N \to \infty$ with $\lambda \equiv \dfrac{N}{k} \in [-1, 1]$ fixed

  - Couple to fund source: dependence on framing $\leftrightarrow$ fractional statistics



$$2\pi \text{ rotation} : e^{2\pi \mathrm{i} s} \to e^{2\pi \mathrm{i} s + \mathrm{i}\pi\lambda}$$

Chern-Simons Matter Duality

❖ **Statement of the Duality:**

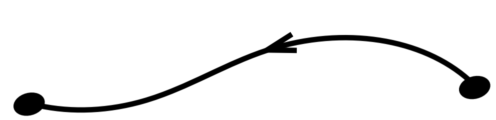
- CS couple to fundamental boson or fermion

$$S_E^{\text{bos}} = S_{CS} + \int d^3x\, (D_\mu\phi)^\dagger \cdot D^\mu\phi + \frac{\lambda_6}{N^2}(\phi\phi^\dagger)^3$$
$$S_E^{\text{fer}} = S_{CS} + \int d^3x\, \bar\psi \cdot \gamma^\mu D_\mu \psi$$

- Both theory has conformal fixed points: UV and IR, related by RG

$$\lambda \int d^3x\, (J^{(0)})^2 \qquad\qquad J^{(0)}_{\text{bos}} = \phi^\dagger \cdot \phi, \quad J^{(0)}_{\text{fer}} = \bar\psi\cdot\psi$$

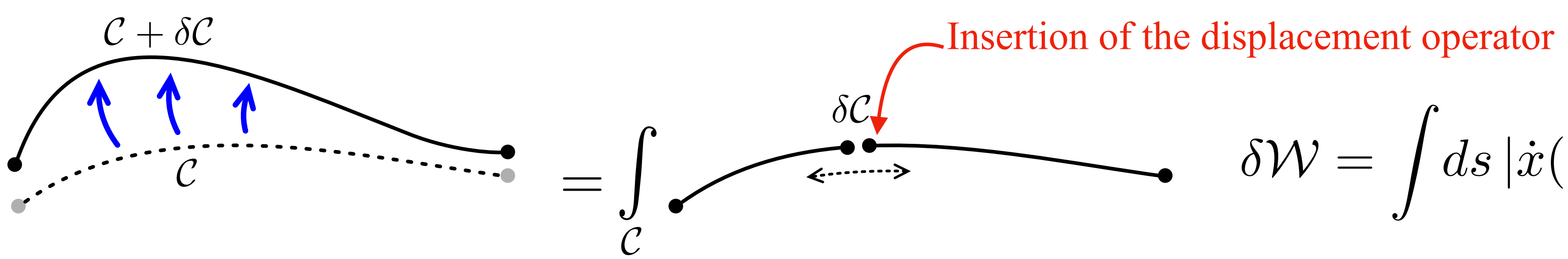
- The most fundamental observable: the mesonic line operator



e.g. $$\underbrace{\phi^\dagger}_{\text{anti-fund}}(x_1)\mathcal{P}e^{\mathrm{i}\int_C A\cdot dx}\underbrace{\phi}_{\text{fund}}(x_0)$$

Results

- Classify the conformal line operators: straight-line
- Classify the operators on the lines and the ones the lines can end on @ finite coupling: exact spectrum at finite coupling
- Evolution equation for deformation of the line: 1st order



$$\delta\mathcal{W} = \int ds\, |\dot{x}(s)|\, v^{\mu}(s)\, \mathcal{P}\Big[\mathbb{D}_{\mu}(s)\mathcal{W}\Big]$$

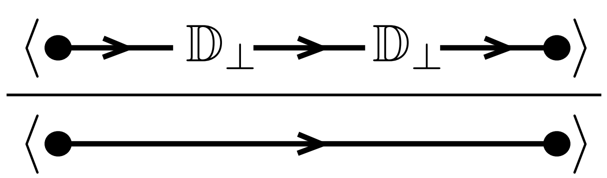
- Line bootstrap: **spectrum + evo eqn** uniquely determine the expectation values of the mesonic line operators
- Duality: same evo eqn, same spectrum in both theories

$$(k,\lambda) \;\leftrightarrow\; (-k,\lambda-\mathrm{sign}(k)) \;\;\Rightarrow\;\; \lambda_f = \lambda_b - \mathrm{sign}(k_b)$$

Results

❖ Duality: two point function of the displacement operator

displacement operator



$$= \frac{\langle\langle \mathcal{O}_L | \mathbb{D}_i(x_s)\mathbb{D}^i(x_t) | \mathcal{O}_R \rangle\rangle}{\langle\langle \mathcal{O}_L | \mathcal{O}_R \rangle\rangle} = \frac{\Lambda(\Delta)}{x_{st}^4}\left(\frac{x_{10}x_{st}}{x_{1s}x_{t0}}\right)^{2\Delta}$$

Straight-line expectation value

boundary operators of minimal dimension

Δ and opposite transverse spin

$$\Lambda(\Delta) = -\frac{(2\Delta-1)(2\Delta-2)(2\Delta-3)\sin(2\pi\Delta)}{2\pi}$$

Relation to λ depends on theory

❖ We will assume $k_b > 0 \;\; \Rightarrow \;\; \lambda_b \in [0,1] \;\; \lambda_f \in [-1,0]$

(The other sign is related to this by parity)

Part I: Duality

Bosonic Line Operators

- At conformal fixed points, the line can still have RG flow
- The naive Wilson line is not conformal! e.g @2-loops
- At conformal fixed points: bi-scalar adjoint

$$\alpha = \pm 1$$

$$\mathcal{W}^{\alpha}[\mathcal{C}, n] \equiv \left[\mathcal{P} e^{\, i \int_{\mathcal{C}} \left(A \cdot dx + i\alpha \frac{2\pi\lambda}{N} \phi\phi^{\dagger} |dx| \right)}\right]_n$$

- RG: $\alpha = 1$ is stable, while $\alpha = -1$ is unstable

Mesonic Line Operators (Bosonic)

- Straight line with two boundary operators:

$$M = \mathcal{O}_L \overset{\mathcal{W} = \mathcal{W}_{\alpha=1}}{\bullet\!\!\longrightarrow\!\!\bullet} \mathcal{O}_R$$

anti-fund fund

- Adjoint operators on the line: $\mathcal{O}_{\text{inner}} = \mathcal{O}_R \times \mathcal{O}_L$

- Classification of boundary operators
  - Infinite straight-line breaks conformal symmetry to $SL(2,\mathbb{R}) \times U(1)$
  - Quantum number: scaling dimension Δ and transverse spin $\mathfrak{s}$
  - Complete set @tree level: light-cone coordinate $ds^2 = 2dx^+dx^- + (dx^3)^2$

$$\mathcal{O}^{(n,s)}_{R,\,\text{tree}} = \frac{1}{\sqrt{N}} \times \begin{cases} \partial^n_{x^3_R}\partial^s_{x^+_R}\phi & s \geq 1 \\ \partial^n_{x^3_R}\partial^{-s}_{x^-_R}\phi & s \leq 0 \end{cases} \qquad \Delta^{(n,s)}_{\text{tree}} = 1/2 + n + |s|$$

Classical spin. No mixed derivative $\partial_{x^+_R}\partial_{x^-_R} \propto \partial^2_{x^3_R}$

Mesonic Line Operators (Bosonic)

- Boundary operators @ finite coupling
  - Complete set of operators: total number cannot jump!
  - Ordinary derivative becomes covariant derivative
  - Operators related by path derivatives: same anomalous dimension

$$\mathcal{O}_L^{(0,s+1)}\mathcal{W}\ldots = \delta_{x_L^+}\mathcal{O}_L^{(0,s)}\mathcal{W}\ldots\,, \qquad s \geq 0$$

$$\mathcal{O}_L^{(0,-s-1)}\mathcal{W}\ldots = \delta_{x_L^-}\mathcal{O}^{(0,-s)}\mathcal{W}\ldots\,, \quad s \geq 1$$

$$\mathcal{O}_L^{(n+1,s)}\mathcal{W}\ldots = \delta_{x_L^3}\mathcal{O}_L^{(n,s)}\mathcal{W}\ldots$$

  - Bottom of the tower: four operators

$$\{\mathcal{O}_L^{(0,0)}, \mathcal{O}_L^{(0,-1)}\} \qquad \{\mathcal{O}_R^{(0,0)}, \mathcal{O}_R^{(0,1)}\}$$

  - No relation between different towers!

Mesonic Line Operators (Bosonic)

❖ Methods & Results

- Resum pert theory: exact anomalous dimensions $\pm\lambda/2$
- Lifting the line to 1/2-BPS line in $\mathcal{N}=2$ CS matter theory
- Four bottom boundary operators fits in 1/2-BPS multiplets of the **line** SUSY
- Line 1/2-BPS condition relates spin to dimension
- Spectrum: $\Delta^{(n,s)} = 1/2 + n + |\mathfrak{s}|$
- Exact transverse spin: e.g. $\lambda = 1$, boson $\to$ fermion

$$\mathfrak{s}_L = s_L + \lambda/2, \quad \mathfrak{s}_R = s_R - \lambda/2$$

Classical spin

- Stable under RG: adjoint operator on the line with minimal dimension

$$\mathcal{O}_R^{(0,0)} \times \mathcal{O}_L^{(0,0)} \qquad \Delta_{\min} = \Delta_L^{(0,0)} + \Delta_R^{(0,0)} = 1 + \lambda > 1$$

Mesonic Line Operators (Bosonic)

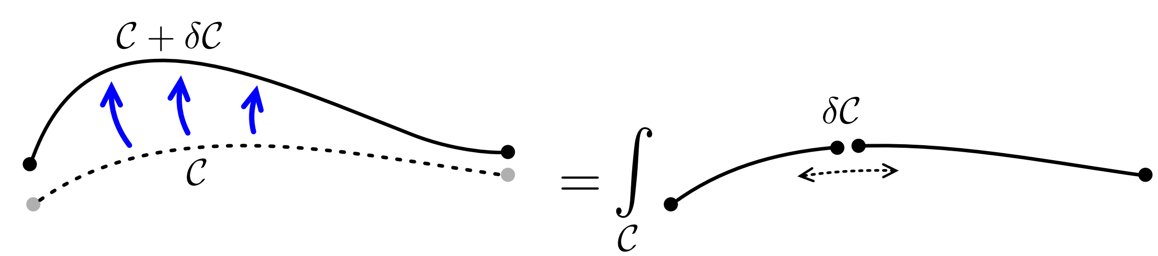
- General non-straight line: evolution equation
  - Conformal symmetry dictates the form of the mesonic operator:

$$\left\langle \mathcal{O}_L^{(0,0)} \mathcal{W} \mathcal{O}_R^{(0,0)} \right\rangle = \left(\frac{n_L^+ \, n_R^-}{n_L^- \, n_R^+} \right)^{\frac{\lambda}{4}} \times \frac{F^{(0,0)}[x(\cdot)]}{|x_L - x_R|^{1+\lambda}}$$

Conf inv functional of the path

  - Expanding the smooth deformation: $x(\cdot) \mapsto x(\cdot) + v(\cdot)$ around straight-line

$$\delta \mathcal{W} = \int ds \, |\dot{x}(s)| \, v^\mu(s) \, \mathcal{P} \left[\mathbb{D}_\mu(s) \mathcal{W} \right]$$



1st order equation!

  - Chiral displacement operator: **exact dimension two and spin one!**

$$\mathbb{D}_- = -4\pi\lambda \, \mathcal{O}_R^{(0,0)} \, \mathcal{O}_L^{(0,-1)} \qquad \mathbb{D}_+ = -4\pi\lambda \, \mathcal{O}_R^{(0,1)} \, \mathcal{O}_L^{(0,0)}$$

Anomalous dim/spin cancelled between RL operators!

$$\Rightarrow \qquad \Delta(\mathbb{D}_\pm) = 2 \,, \quad \mathfrak{s}(\mathbb{D}_\pm) = \pm 1$$

Mesonic Line Operators (Bosonic)

- Final missing ingredient: the boundary equation
  - @Tree level: $\partial_{x_R^+}\partial_{x_R^-} \propto \partial^2_{x_R^3}$
  - @ Finite coupling: $SL(2,\mathbb{R})$ primary/descendent related by

$$\delta_{x_L^-}\mathcal{O}_L^{(0,s+1)}\mathcal{W}\ldots = \bar{\beta}\,\mathcal{O}_L^{(2,s)}\mathcal{W}\ldots\,, \quad s\geq 0$$

Unique operator! same spin, twist reduced by 2

$$\delta_{x_L^+}\mathcal{O}_L^{(0,-s-1)}\mathcal{W}\ldots = \beta\,\mathcal{O}_L^{(2,-s)}\mathcal{W}\ldots\,, \quad s\geq 1$$

  - Different tower cannot mix! (Opposite anomalous spin/dimension)
  - Bootstrap: $\beta = \bar{\beta} = \dfrac{1}{2}$

Mesonic Line Operators (Fermionic)

❖ What is the dual fermonic line operator?

$$M = \mathcal{O}_L \overset{\mathcal{W} = \mathcal{W}_{\alpha=1}}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet} \mathcal{O}_R$$

anti-fund

fund

- To leading order in $1/N$, boson adjoint decouple, naive Wilson line is good

❖ Classification of boundary operators

- Complete set @tree level:

$$\mathcal{O}_R^{(n,s)} = \frac{1}{\sqrt{N}} \times \begin{cases} D_3^n D_+^{|s|-\frac{1}{2}} \psi_+(x_R) & s \geq +\frac{1}{2} \\ D_3^n D_-^{|s|-\frac{1}{2}} \psi_-(x_R) & s \leq -\frac{1}{2} \end{cases}$$

Mesonic Line Operators (Fermionic)

- Straight line with two boundary operators:

$$M = \mathcal{O}_L \overset{\mathcal{W}=\mathcal{W}_{\alpha=1}}{\bullet\!\!\longrightarrow\!\!\bullet} \mathcal{O}_R$$

anti-fund fund

- Matching: same **spectrum** & same **evolution equation**!

Fermionic	Tree	Bosonic	Tree	$\mathfrak{s}$	Δ
$\mathcal{O}_R^{(0,-\frac{1}{2})}$	ψ_-	$\mathcal{O}_R^{(0,0)}$	ϕ	$-\frac{\lambda_b}{2}$	$\frac{1+\lambda_b}{2}$
$\mathcal{O}_R^{(0,\frac{1}{2})}$	ψ_+	$\mathcal{O}_R^{(0,1)}$	$\partial_+\phi$	$\frac{2-\lambda_b}{2}$	$\frac{3-\lambda_b}{2}$
$\mathcal{O}_L^{(0,\frac{1}{2})}$	$\bar{\psi}_+$	$\mathcal{O}_L^{(0,0)}$	$\phi^\dagger$	$\frac{\lambda_b}{2}$	$\frac{1+\lambda_b}{2}$
$\mathcal{O}_L^{(0,-\frac{1}{2})}$	$\bar{\psi}_-$	$\mathcal{O}_L^{(0,-1)}$	$\partial_-\phi^\dagger$	$\frac{\lambda_b-2}{2}$	$\frac{3-\lambda_b}{2}$

Other Mesonic Line Operators

❖ $\alpha = -1$ mesonic operator and its dual

- Same anomalous spin $\mathfrak{s}_L = s_L + \lambda/2, \quad \mathfrak{s}_R = s_R - \lambda/2$
- Different spectrum of boundary operators
- Chiral displacement operator: **exact dimension two and spin one!**

$$\widetilde{\mathbb{D}}_+ = +4\pi\lambda\, \widetilde{\mathcal{O}}_R^{(0,0)}\, \widetilde{\mathcal{O}}_L^{(0,1)}, \quad \widetilde{\mathbb{D}}_- = +4\pi\lambda\, \widetilde{\mathcal{O}}_R^{(0,-1)}\, \widetilde{\mathcal{O}}_L^{(0,0)}$$

Chirality flipped!

- Dual? @ $\lambda_b = 1, \lambda_f = 0$, $\widetilde{\mathcal{O}}_L^{(0,0)}$ and $\widetilde{\mathcal{O}}_R^{(0,0)}$ have dim 0 and spin $\pm 1/2$

Wilson line $\mathcal{P}e^{\int A\cdot dx}$

Answer: $\widetilde{M}^{(\frac{1}{2},-\frac{1}{2})} = \sum_n \cdots \underset{1}{\bar{\psi}_+ \text{———} \psi_-} \cdots \underset{2}{\bar{\psi}_+ \text{—} \psi_-} \cdots \ \cdots \ \cdots \underset{n}{\bar{\psi}_+ \text{—} \psi_-} \cdots$

empty line with spin transport $\mathcal{P}e^{\int \Gamma\cdot dx}$

non-trivial topological spin connection

Other Mesonic Line Operators

❖ $\alpha = -1$ mesonic operator and its dual

- Matching: same **spectrum** & **evolution equation**!

Fermionic	Tree	Bosonic	Tree	$\mathfrak{s}$	Δ
$\widetilde{\mathcal{O}}_R^{(0,-\frac{1}{2})}$	$\mathbf{1}$	$\widetilde{\mathcal{O}}_R^{(0,0)}$	ϕ	$-\frac{\lambda_b}{2}$	$\frac{1-\lambda_b}{2}$
$\widetilde{\mathcal{O}}_R^{(0,-\frac{3}{2})}$	$\partial_-\psi_-$	$\widetilde{\mathcal{O}}_R^{(0,-1)}$	$\partial_-\phi$	$-\frac{2+\lambda_b}{2}$	$\frac{3+\lambda_b}{2}$
$\widetilde{\mathcal{O}}_L^{(0,\frac{1}{2})}$	$\mathbf{1}$	$\widetilde{\mathcal{O}}_L^{(0,0)}$	$\phi^\dagger$	$\frac{\lambda_b}{2}$	$\frac{1-\lambda_b}{2}$
$\widetilde{\mathcal{O}}_L^{(0,\frac{3}{2})}$	$\partial_+\bar{\psi}_+$	$\widetilde{\mathcal{O}}_L^{(0,1)}$	$\partial_+\phi^\dagger$	$\frac{2+\lambda_b}{2}$	$\frac{3+\lambda_b}{2}$

- Unstable under RG: minimal dim adjoint operator on the line

$$\widetilde{\mathcal{O}}_R^{(0,0)} \times \widetilde{\mathcal{O}}_L^{(0,0)} \qquad \Delta_{\min} = \widetilde{\Delta}_L^{(0,0)} + \widetilde{\Delta}_R^{(0,0)} = 1 - \lambda < 1$$

Part II: Bootstrap

Bootstrapping Mesonic Line Operators

- ❖ Evolution equation is powerful!
  - **Spectrum** + **evolution equation** uniquely determine the expectation values of the mesonic line operators
  - Deformation of the line = conformal pert theory
    - ✤ At given order of the deformation, list all relevant/marginal operators

      $$e.g. \quad \left(\gamma_0 v_0^+ v_0^- \delta_{x_0^3}^2 + \gamma_1^+ (v_0^+ dv_0^- + v_0^- dv_0^+) \delta_{x_0^3}\right) M_{10}^{(0,0)}$$

    - ✤ Fix coefficients by demanding conformal symmetry
    - ✤ Any appropriate scheme is allowed: differ by counter term coefficients

Bootstrapping Mesonic Line Operators

- ❖ Evolution equation dictates the expectation value!

  - Result:

$$\delta_{x_s}\delta_{x_t} M_{10}^{(0,0)} = M_{1s}^{(0,0)} \boxed{M_{st}^{(-1,1)} M_{t0}^{(0,0)}}$$

Normalization-independent!

$$\Downarrow$$

$$\frac{\langle \bullet \longrightarrow \mathbb{D}_\perp \longrightarrow \mathbb{D}_\perp \longrightarrow \bullet \rangle}{\langle \bullet \longrightarrow \bullet \rangle} = \frac{\langle\!\langle \mathcal{O}_L | \mathbb{D}_i(x_s) \mathbb{D}^i(x_t) | \mathcal{O}_R \rangle\!\rangle}{\langle\!\langle \mathcal{O}_L | \mathcal{O}_R \rangle\!\rangle} = \frac{\Lambda(\Delta)}{x_{st}^4} \left(\frac{x_{10} x_{st}}{x_{1s} x_{t0}} \right)^{2\Delta}$$

$$\Lambda(\Delta) = -\frac{(2\Delta - 1)(2\Delta - 2)(2\Delta - 3)\sin(2\pi\Delta)}{2\pi}$$

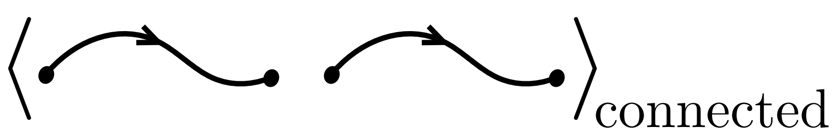
  - Δ is the dimension of any of the four bottom operators. For instance,

$$\alpha = 1 \qquad \Delta = (1 + \lambda_b)/2 \quad \text{or} \quad \Delta = (3 - \lambda_b)/2$$

  - $\Lambda(\Delta) = \Lambda(2 - \Delta)$: 2pt function of displacement ops on circular loop

Outlooks

- **Bootstrap the connected correlation functions**



Now, $J^{(0)}$ plays a role!

Spectrum of single trace local operators

+ Spectrum of boundary operators

+ Evolution equation

- **Solve the path dependence explicitly**
- **Derive the holographic dual:** Vasiliev's higher-spin theory.
- **Better regularization scheme?**

Thank you!