XI. International Workshop
LIE THEORY AND ITS APPLICATIONS IN PHYSICS

# The Heun functions and their applications in astrophysics.

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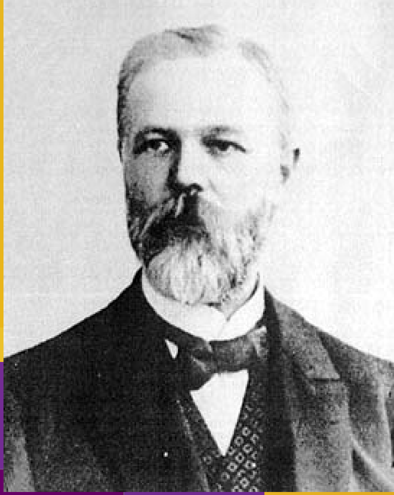
# The Heun functions



- Heun 1889:

$$\frac{d^2}{dz^2}H+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right]\frac{dH}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}H=0$$

$\varepsilon=\alpha+\beta-\gamma-\delta+1$, 4 regular singularities, $z=0,1,a,\infty$

- Confluence: CHE, BHE, DHE, THE
- Generalize: the hypergeometric function, the Lame function, Mathieu function, the spheroidal wave functions
- Numerous applications: in the Schrodinger equation with anharmonic potential, in water molecule, in the Stark effect, in gravitational physics of scalar, spinor, electromagnetic and gravitational waves, in crystalline materials, in 3d waves in atmosphere, in Bethe anzatz systems, in Collogero-Moser-Sutherland systems
- Group of symmetries of order 192

## General Heun Function (GHE)

Singularities: regular={0,1,a,∞}

$$GHE := \frac{d^2}{dz^2} y(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\left(\frac{d}{dz} y(z)\right) + \frac{(\alpha\beta z - q)\, y(z)}{z(z-1)(z-a)} = 0$$

## Confluent Heun Function (CHE)

Singularities: regular={0,1},irregular={∞}

$$CHE := \frac{d^2}{dz^2} y(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} - \varepsilon\right)\left(\frac{d}{dz} y(z)\right) + \left(\frac{q-\alpha\beta}{z-1} - \frac{q}{z}\right) y(z) = 0$$

Converging series solution only for |z|<1

## Biconfluent Heun Function (DHE)

Singularities: regular={0}, irregular={∞}

$$BHE := \frac{d^2}{dz^2} y(z) + \left(-2z - \beta + \frac{1+\alpha}{z}\right)\left(\frac{d}{dz} y(z)\right) + \left(\gamma - \alpha - 2 - \frac{1((1+\alpha)\beta + \delta)}{2z}\right) y(z) = 0$$

## Double confluent Heun Function (DHE)

Singularities: regular={}, irregular={-1,1}

$$DHE := \frac{d^2}{dz^2} y(z) - \frac{(\alpha + 2z + z^2\alpha - 2z^3)\left(\frac{d}{dz} y(z)\right)}{(z+1)^2 (z-1)^2} + \frac{(\delta + (2\alpha+\gamma) z + \beta z^2)\, y(z)}{(z-1)^3 (z+1)^3}$$

## Triconfluent Heun Function (THE)

Singularities: regular={}, irregular={∞}

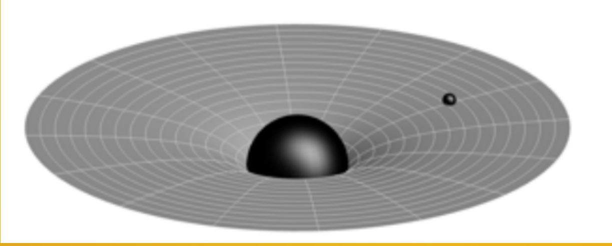
$$THE := \frac{d^2}{dz^2} y(z) + (-\gamma - 3z^2)\left(\frac{d}{dz} y(z)\right) + (\alpha + z\beta - 3z)\, y(z) = 0$$

**Hypergeometric DE:**
Regular singularities at z=0,1,∞

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0.$$

*Regularity condition: if for P(x)y'' + Q(x)y' + R(x)y = 0, the limits:

$\lim_{x \to x0} \frac{Q(x)}{P(x)} \cdot (x - x_0)$ and $\lim_{x \to x0} \frac{R(x)}{P(x)} \cdot (x - x_0)^2$ are finite.



# Applications in astrophysics: Perturbations of a black hole

$$\Psi = e^{i(\omega t + m\phi)} S(\theta) R(r)$$

## The Teukolsky Angular Equation:

$$\left[\left(1-u^2\right) S_{lm,u}\right]_{,u} + \left[(a\omega u)^2 + 2a\omega s u + {}_sE_{lm} - s^2 - \frac{(m+su)^2}{1-u^2}\right] S_{lm} = 0,$$

- where u=cos(θ) **Singularities: ±π,∞**

**BH parameters: a, M**
**Unknowns: $\omega_{nlm}$ and $E_{nlm}$!**

## The Teukolsky Radial Equation:

$$\frac{d^2 R_{\omega,E,m}}{dr^2} + (1+s)\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)\frac{dR_{\omega,E,m}}{dr} +$$

$$+\left(\frac{K^2}{(r-r_+)(r-r_-)} - is\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)K - \lambda - 4is\omega r\right)\frac{R_{\omega,E,m}}{(r-r_+)(r-r_-)} = 0$$

$$\Delta = r^2 - Mr + a^2 = (r-r_-)(r-r_+),\ K = \omega(r^2+a^2) - ma,\ \lambda = E - s(s+1) - a^2\omega^2 - 2am\omega$$

- **Singularities: $r_-$, $r_+$, ∞**

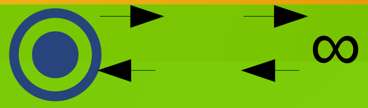
Both ODEs of the CHE type!!!

$$\frac{d^2}{dz^2}H(z) + \left(\alpha + \frac{\beta+1}{z} + \frac{\gamma+1}{z-1}\right)\frac{d}{dz}H(z) + \left(\frac{\mu}{z} + \frac{\nu}{z-1}\right)H(z) = 0.$$

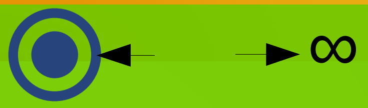
The case a=0, s=-2: P. Fiziev, D.S. ,PRD.84, 127502 (2011)

# The EM QNMs of the KBH (s=-1)

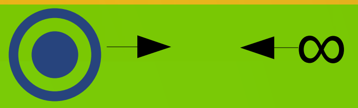
BC:



TTM/left or right/



QNM



QBM

- The spectral system after imposing the boundary conditions:
- TRE (BHBC): $$C_{\leftarrow} = r^{2+i\omega+\frac{2i\,m\,a+i\,\omega}{r_+-r_-}}\,\mathrm{HeunC}(\alpha,-\beta,\gamma,\delta,\eta,z)=0,$$
- TAE (regularity on the sphere):

$$W[S_1,S_2] = \frac{\mathrm{HeunC}'(\alpha_1,\beta_1,\gamma_1,\delta_1,\eta_1,(\cos(\pi/6))^2)}{\mathrm{HeunC}(\alpha_1,\beta_1,\gamma_1,\delta_1,\eta_1,(\cos(\pi/6))^2)} + \frac{\mathrm{HeunC}'(\alpha_2,\beta_2,\gamma_2,\delta_2,\eta_2,(\sin(\pi/6))^2)}{\mathrm{HeunC}(\alpha_2,\beta_2,\gamma_2,\delta_2,\eta_2,(\sin(\pi/6))^2)} + p = 0$$

Fiziev P., D. S., arXiv:1112.0310v1 [astro-ph.HE],
American Journal of Computational Mathematics Vol. 02 : 02, pp.95 (2012)

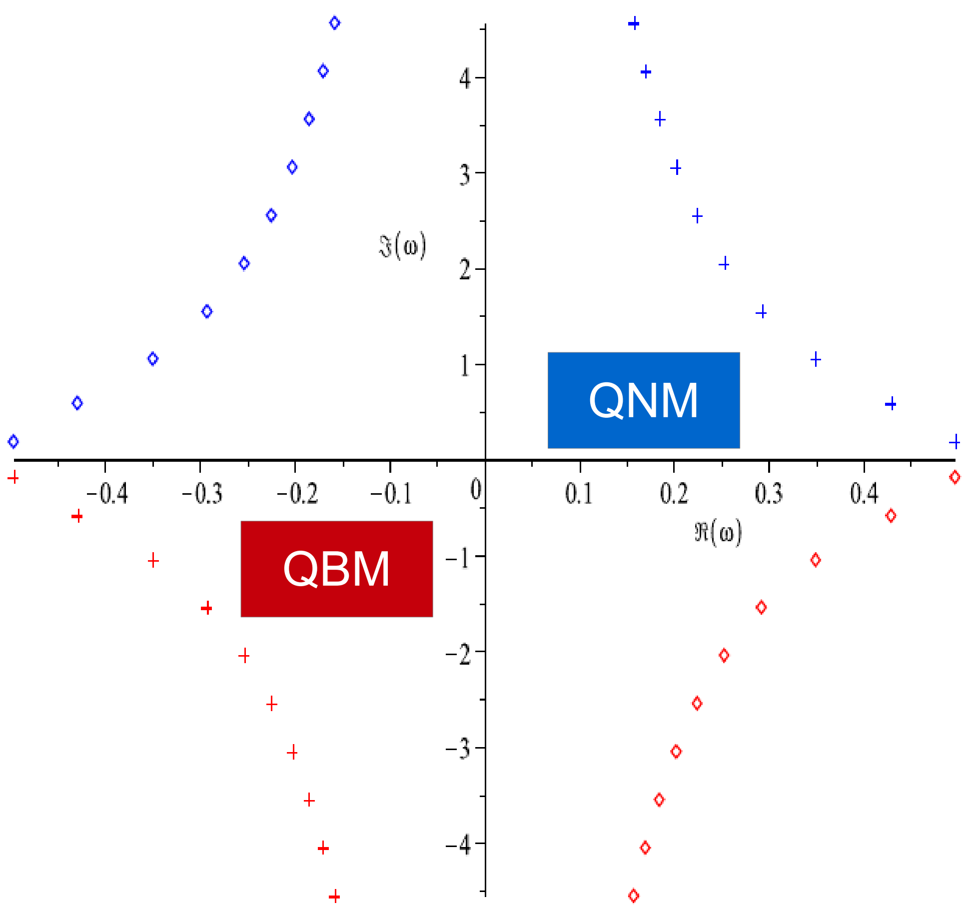
D.S., P. Fiziev, Astrophysics and Space Science, 358:10, 2015

# The results, a=0:

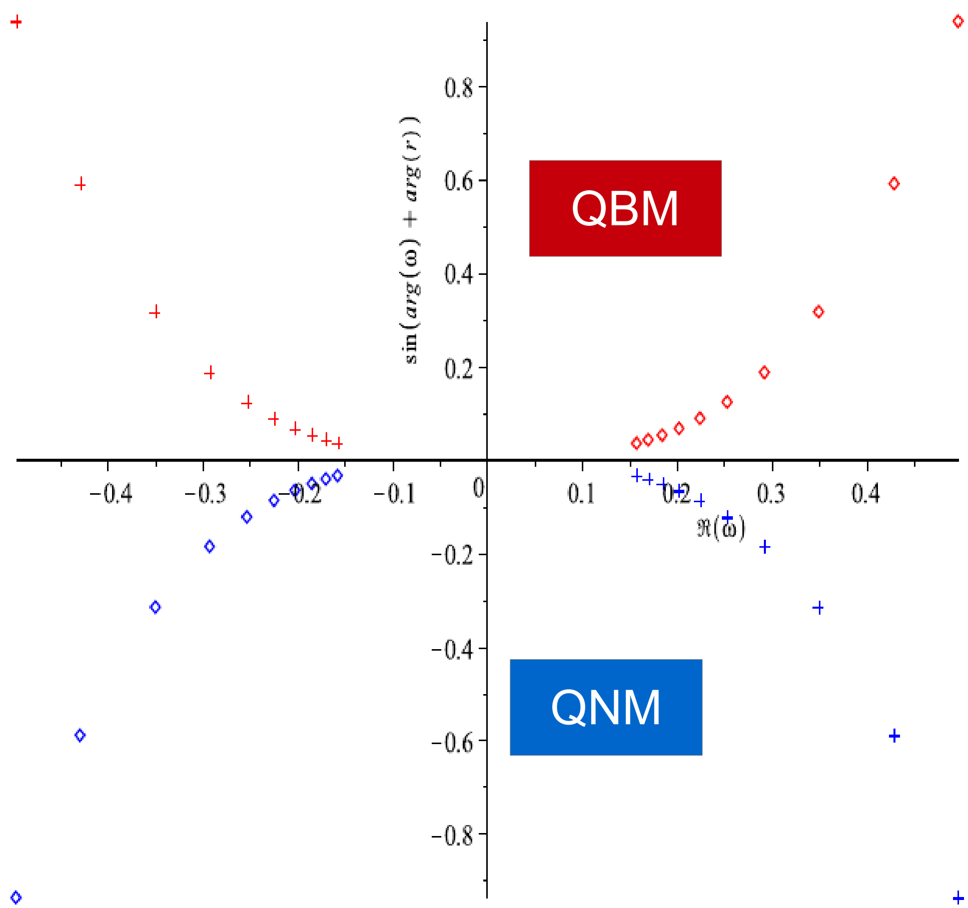
- From the boundary conditions:

QNM $\sin(\arg(\omega) + arg(r)) < 0, r = |r|e^{i\frac{3}{2}\pi}$

QBM $\sin(\arg(\omega) + arg(r)) > 0, r = |r|e^{i\frac{1}{2}\pi}$

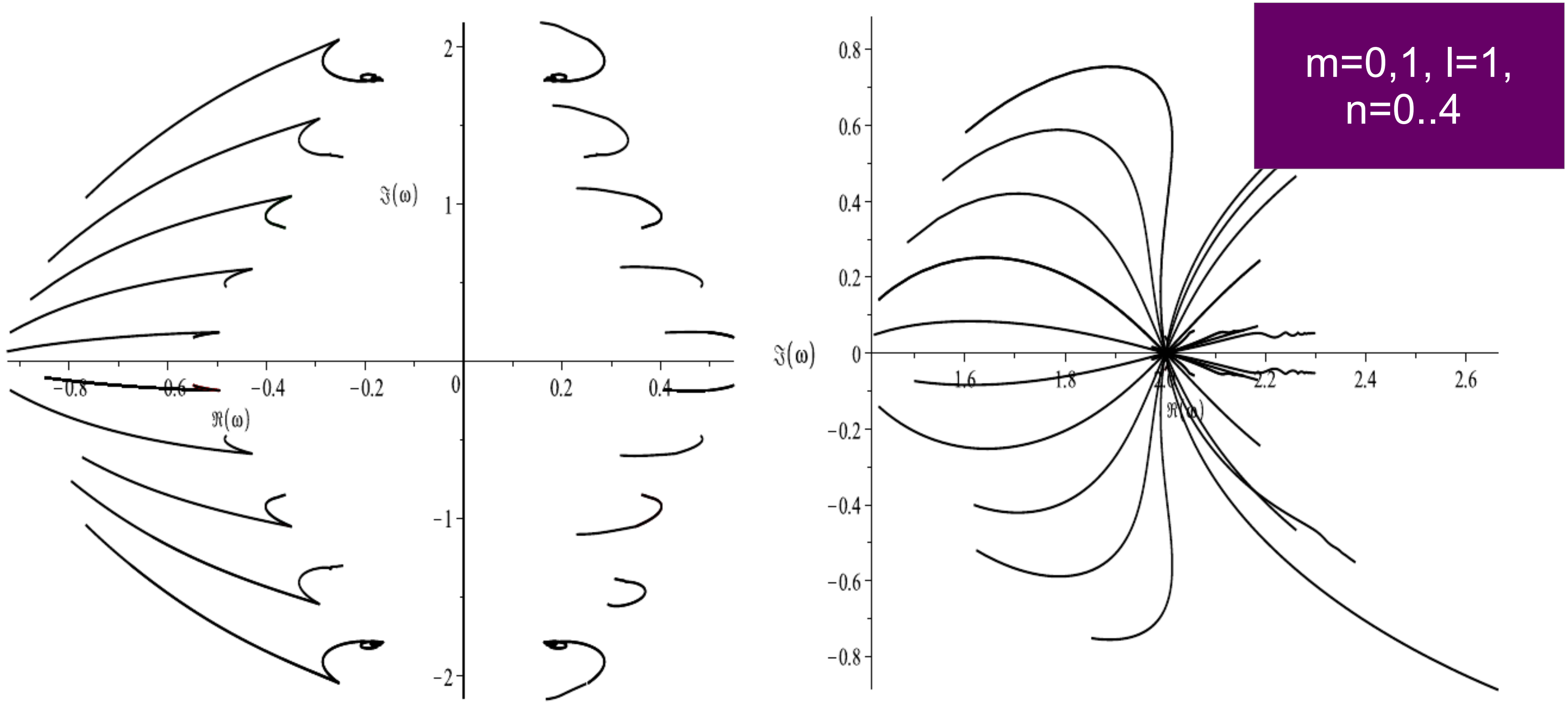



(a) m=0, l=1, $\omega$




(b) m=0, l=1, $\sin(\arg(\omega) + \arg(r))$

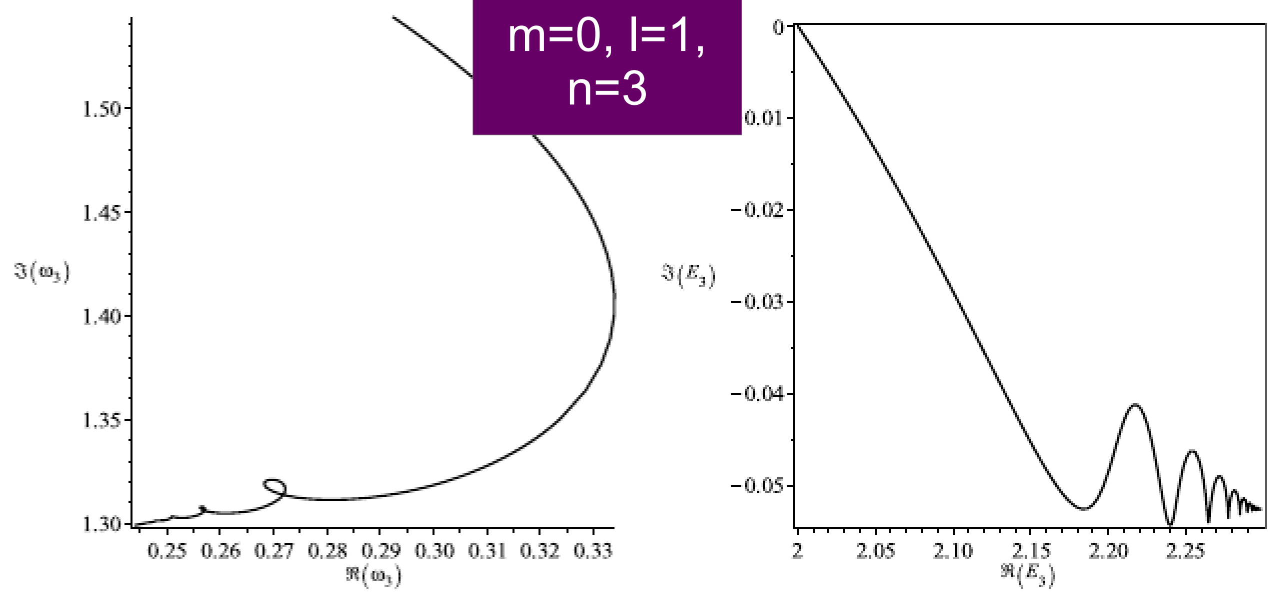
a=0..M




m=0,1, l=1, n=0..4

When we add rotation:

m=0, l=1, n=3

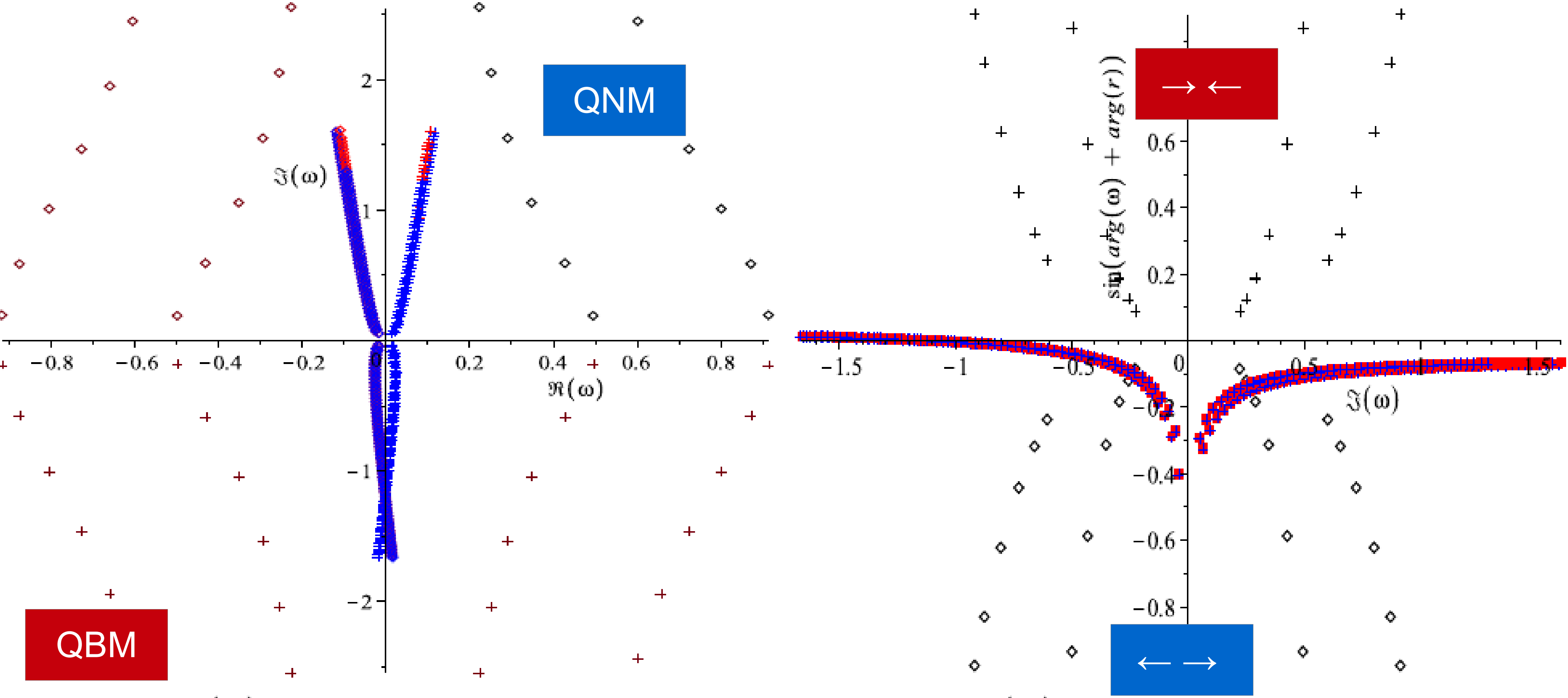



(a) $\omega_{0,3}(a)$ (b) $E_{0,3}(a)$

In the extreme regime (a→M), QNMs can be fit with analytical formula by Hod
*Phys.Rev. D 78:084035,(2008)*

$$\omega^1 = m\Omega - i2\pi T_{BH}(n+1/2)$$

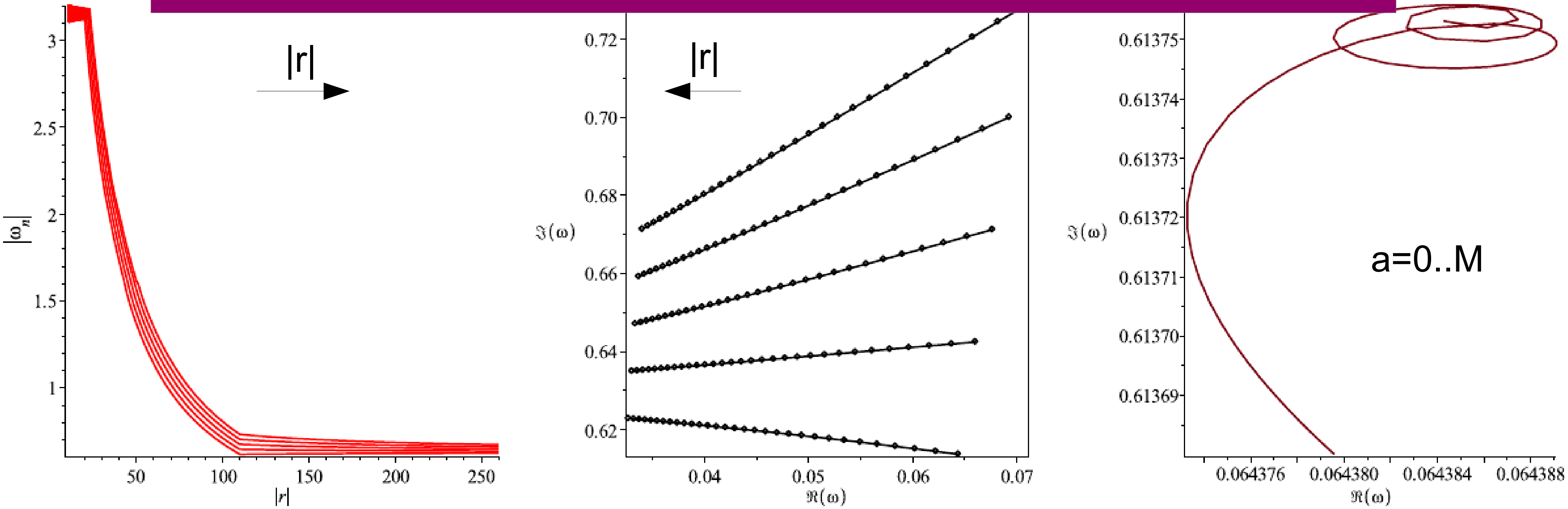
# The spurious spectrum




(a) m=0, l=1, 2 (b) m=0, l=1, 2

Even though the spurious spectrum follows the QNM/QBM, it doesn't correspond to physical BC

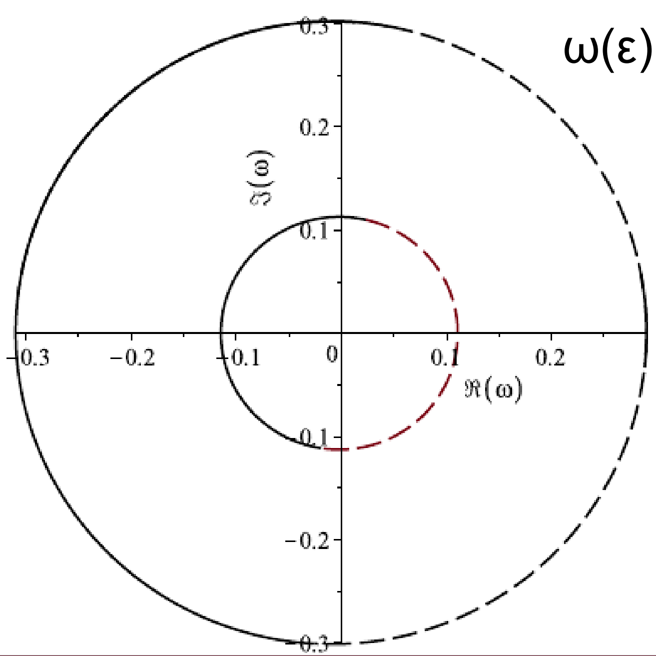
# Stability of the modes with respect to the radial variable r




(a) (b) (c)

$$r = |r| e^{i \frac{3+\epsilon}{2} \pi}$$

$$\epsilon = 0.. \pm 1$$




D.S, P. Fiziev, in prep.

The spurious spectrum fails the r-test – they change with changes of |r| and arg(r)

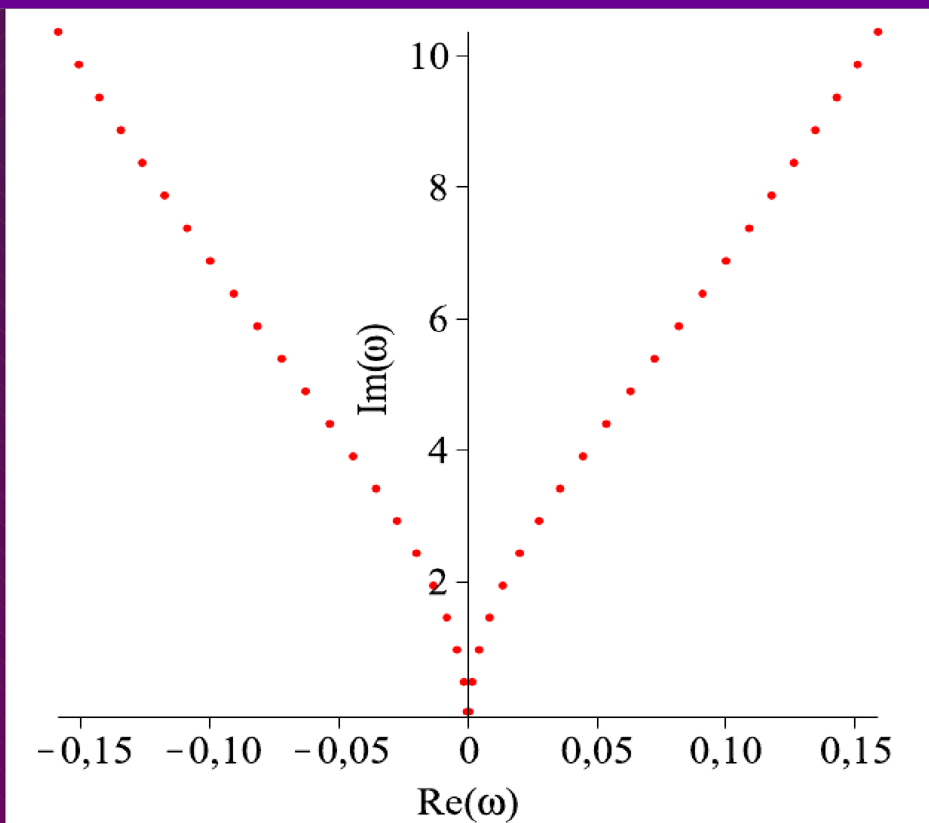
# Comparison with jet BC (s=-1, M=1/2):

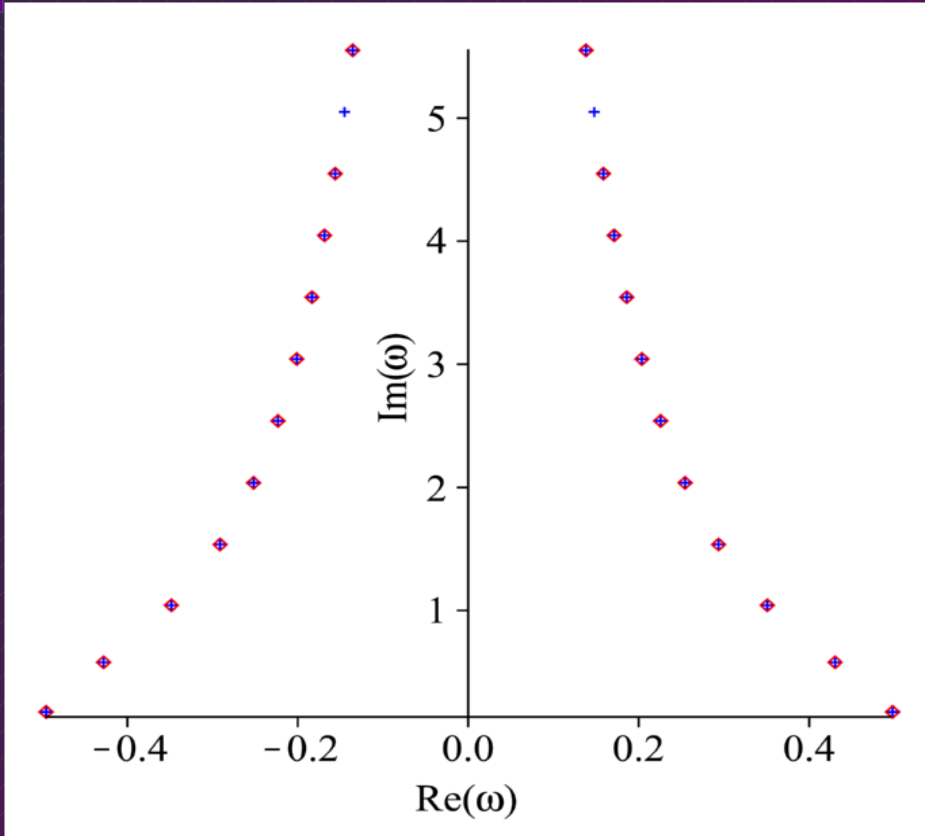
**Jets:** D.S., Fiziev P.
Astrophys Space Sci (2011) 332: 385-401

**QNM:** Fiziev P., D.S. (2010)
arXiv:1005.5375 [cs.NA]

$$E_{s=-1,m}(\omega) = -(a\omega)^2 - 2a\omega m \pm 2\sqrt{(a\omega)^2 + a\omega m}$$

$$\omega_{n=0,1,m} = \left(-m + i N\sqrt{b^2-1}\right)\Omega_{+},\ N=0,1$$




We observe **qualitative difference** due to the different boundary conditions!

# Thank you!

# Values of the parameters for TAE:

For $m = 0$: $\alpha_1 = 4\,a\omega, \beta_1 = 1, \gamma_1 = -1, \delta_1 = 4\,a\omega, \eta_1 = 1/2 - E - 2\,a\omega - a^2\omega^2$ and

$\alpha_2 = -4\,a\omega, \beta_2 = 1, \gamma_2 = 1, \delta_2 = -4\,a\omega, \eta_2 =, 1/2 - E + 2\,a\omega - a^2\omega^2$, $p = \frac{1}{(\sin(\pi/6))^2}$

For $m = 1$: $\alpha_1 = -4\,a\omega, \beta_1 = 2, \gamma_1 = 0, \delta_1 = 4\,a\omega, \eta_1 = 1 - E - 2\,a\omega - a^2\omega^2$ and

$\alpha_2 = -4\,a\omega, \beta_2 = 0, \gamma_2 = 2, \delta_2 = -4\,a\omega, , \eta_2 = 1 - E + 2\,a\omega - a^2\omega^2$ and $p = -4\,a\omega$

For $m = 2$: $\alpha_1 = -4\,a\omega, \beta_1 = 3, \gamma_1 = -1, \delta_1 = 4\,a\omega, \eta_1 = 5/2 - E - 2\,a\omega - a^2\omega^2$ and

$\alpha_2 = -4\,a\omega, \beta_2 = 1, \gamma_2 = -3, \delta_2 = -4\,a\omega, \eta_2 = 5/2 - E + 2\,a\omega - a^2\omega^2$ and $p = 8 - 4a\omega$.

# Values of the parameters for TRE:

where $z = -\frac{r - r_+}{r_+ - r_-}$ and the parameters are:

$$\alpha = -2\,i\,(r_+ - r_-)\,\omega,\ \beta = -\frac{2\,i(\omega\,(a^2 + r_+{}^2) + am)}{r_+ - r_-} - 1,$$

$$\gamma = \frac{2\,i(\omega\,(a^2 + r_-{}^2) + am)}{r_+ - r_-} - 1,$$

$$\delta = -2i(r_+ - r_-)\omega(1 - i\,(r_- + r_+)\,\omega),$$

$$\eta = \frac{1}{2}\frac{1}{(r_+ - r_-)^2} \times$$

$$\Big[4\omega^2 r_+{}^4 + 4\,(i\omega - 2\omega^2 r_-)\,r_+{}^3 + (1 - 4a\omega\,m - 2\omega^2 a^2 - 2E) \times$$

$$(r_+{}^2 + r_-{}^2) +$$

$$4\,(i\omega\,r_- - 2i\omega\,r_+ + E - \omega^2 a^2 - \tfrac{1}{2})\,r_-\,r_+ - 4a^2\,(m + \omega\,a)^2\Big].$$

# The polynomial condition:

P. Fiziev, Class. Quant. Grav.27:135001, 2010 arXiv:0908.4234

P. Fiziev, J.Phys. A: Math. Theor. 43 (2010). 035203

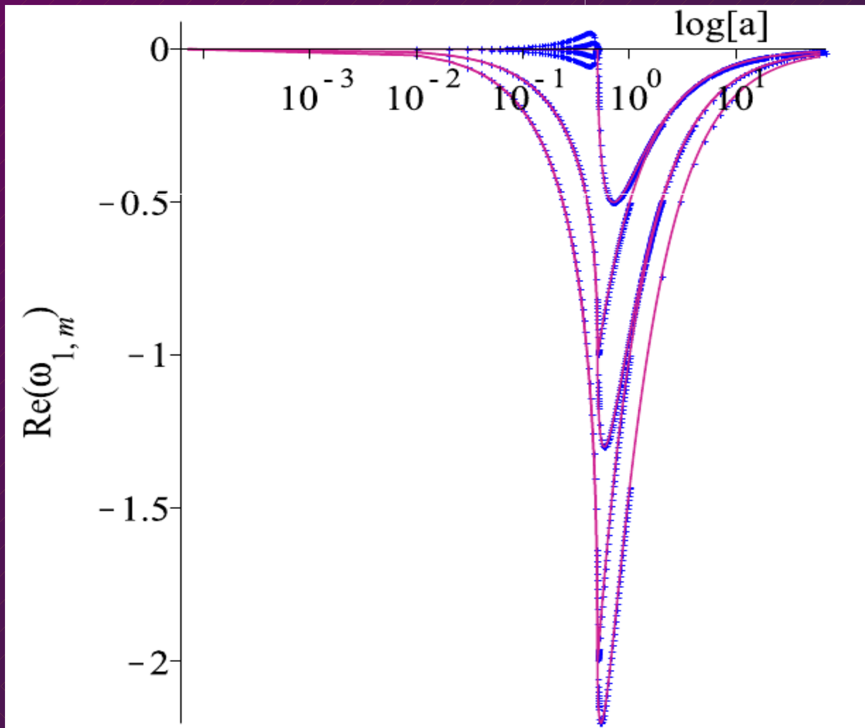
$$\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + N + 1 = 0,$$
$$\Delta_{N+1}(\mu) = 0.$$

We represent the three-diagonal determinant $\Delta_{N+1}(\mu)$ in the following specific explicit form:

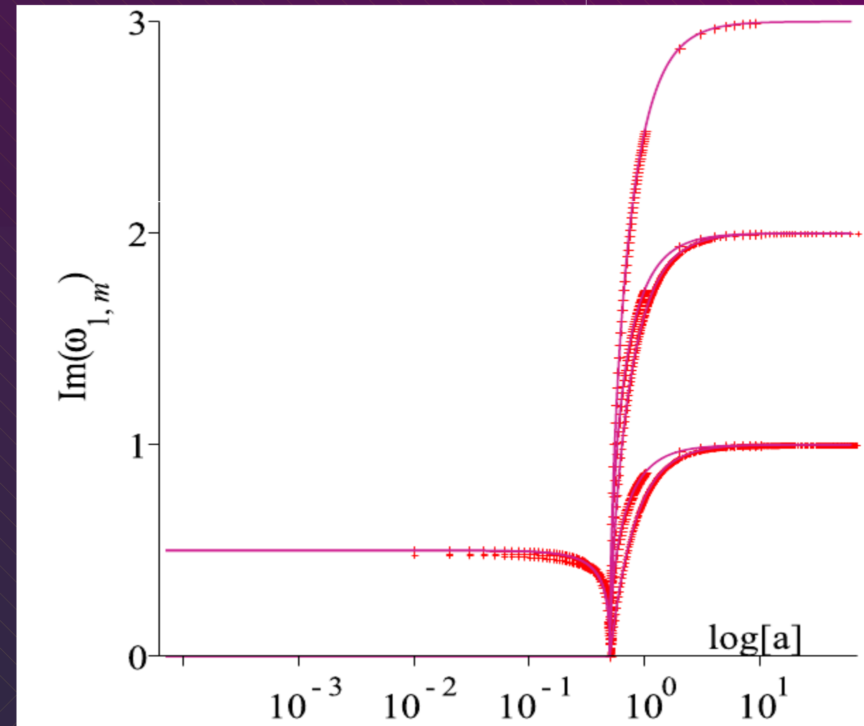
$$\begin{vmatrix} \mu-q_1 & 1(1+\beta) & 0 & \dots & 0 & 0 & 0 \\ N\alpha & \mu-q_2+1\alpha & 2(2+\beta) & \dots & 0 & 0 & 0 \\ 0 & (N-1)\alpha & \mu-q_3+2\alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu-q_{N-1}+(N-2)\alpha & (N-1)(N-1+\beta) & 0 \\ 0 & 0 & 0 & \dots & 2\alpha & \mu-q_N+(N-1)\alpha & N(N+\beta) \\ 0 & 0 & 0 & \dots & 0 & 1\alpha & \mu-q_{N+1}+N\alpha \end{vmatrix},$$

which turns to be useful for calculations. Here $q_n = (n-1)(n+\beta+\gamma)$.

a<M a=M a>M




a<M a=M a>M




Our numerical results are best fit by the formula:

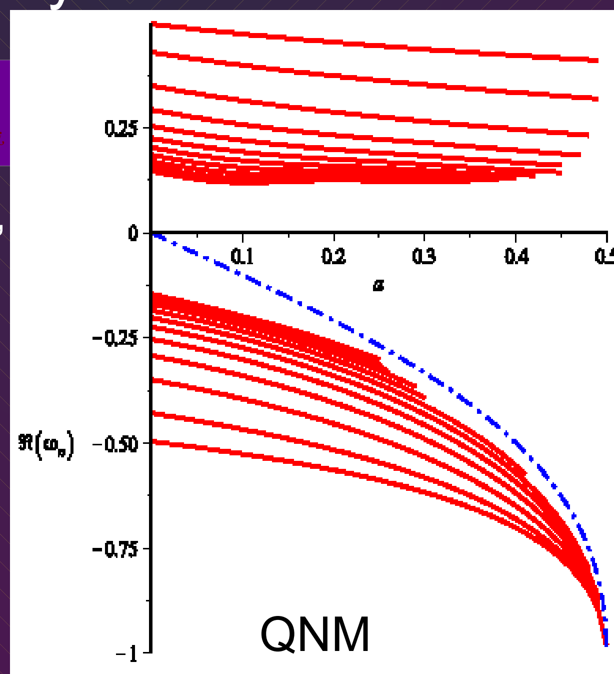
$$\omega_{n=0,1,m} = \left(-m + i\,N\sqrt{b^2-1}\right)\Omega_+$$

Fiziev P. P., ,Class. Quant. Grav.27:135001, 2010

$b = M/a$ – bifurcation parameter

$\Omega_+ = a/2Mr_+$ - angular velocity of the horizon

N = 0, 1




m=0,1, l=1