Applications of the Heun functions in astrophysics

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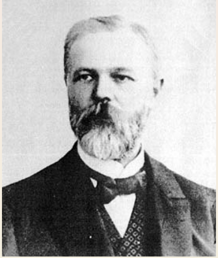
Contents

Why the name?

- First appear in the book: Heun, Karl (1889), **"Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten", Mathematische Annalen, 33: 161**
- Natural generalization of the hypergeometric function, the Lame function, Mathieu function, the spheroidal wave functions
- Numerous applications: Schrödinger equation with anharmoic potential, linear perturbations of black holes, transversable wormholes, quantum Rabi models, confinement of graphene electrons in different potentials, quantum critical systems etc

The Heun Project

http://theheunproject.org/



Very promising area of research in theory, numerics and applications!

Some bibliography

- Seminal theoretical works:
  A. Erdlyi, F. et al., "Higher Transcendental functions vol. 3" (McGraw Hill, NY, 1953).
  Ronveaux, A., ed. (1995), "Heun's differential equations", Oxford Science Publications
  S. Y. Slavyanov and W. Lay, "Special Functions, A Unified Theory Based on Singularities," Oxford Mathematical Monographs,2000.
- Applications (astrophysics):
  P. Fiziev : Class. Quant. Grav.27:135001, 2010 Phys. Rev. D 80: 124001 (2009) J. Phys. A: Math. Theor. 43 (2010) 035203 Class.Quant.Grav. 23 (2006): 2447-2468
- Numerics (MAPLE):
  Edgardo Cheb-Terrab: Journ. of Phys. A: Mathematical and General 37: 9923-9949 (2004)
  For more: https://theheunproject.org/bibliography.html

The hypergeometric function

The hypergeometric function is a solution of the following ODE:

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - abw(z) = 0$$

This equation has 3 regular singular points $z = 0, 1, \infty$ with solutions according to the Frobenius method:

Around $z = 0$	Around $z = 1$	Around $z = \infty$
${}_2F_1(a, b, c; z)$	${}_2F_1(a, b, 1+a+b-c; 1-z)$	$z^{-a}{}_2F_1(a, 1+a-c, 1+a-b; 1/z)$
$z^{1-c}{}_2F_1(1+a-c, 1+b-c, 2-c; z)$	$(1-z)^{c-a-b}{}_2F_1(c-a, c-b, 1+c-a-b; 1-z)$	$z^{-b}{}_2F_1(b, 1+b-c, 1+b-a; 1/z)$

where ${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{c_n} \frac{z^n}{n!}$ with $(q)_n = \begin{cases} 1, n = 0 \\ q(q+1)...(q+n-1), n > 0 \end{cases}$

Confluent Hypergeometric equation:

$$zw''(z) + (c-z)w'(z) - aw(z) = 0$$

With singularities: $z = 0$ – regular and $z = \infty$ – irregular and solutions: $C_1M(a, c, z) + C_2z^{1-b}M(a+1-c, 2-c, z)$,

the Kummer function:

$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} = \lim_{b\to\infty} {}_2F_1(a, b, c; z/b)$

The type of the singularities matters!

For an ODE of the form:

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0,$$

the point x_0 is singular if Q(x)/P(x) or R(x)/P(x) diverge at $x = x_0$.

- The point x_0 is **regular singularity** if the limits $\lim_{x\to x_0} \frac{Q(x)}{P(x)}(x - x_0)$ and $\lim_{x\to x_0} \frac{R(x)}{P(x)}(x - x_0)^2$ exist and are finite.
- Otherwise, it is **irregular or essential singularity**.
- The point $x_0 = \infty$ is treated the same way under the change $x = 1/z$.

Singularities are characterized by their s-rank.
Depending on the s-rank and number of singularities, one can consider classes of ODEs!

Hypergeometric class of equations

Table: The hypergeometric class of differential equations

Name Eq.	Equation	Singularities	s-rank
	$y''(z) = 0$	$z = \infty$	$\{1\}$
Euler equation	$(z - z_1)^2 y''(z) + A(z - z_1)y'(z) + By(z) = 0$	regular: $z = z_1, \infty$	$\{1, 1\}$ or $\{1/2, 1/2\}$
Confluent case	$y''(z) + Ey'(z) + Dy(z) = 0$	$z = \infty$	$\{2\}$
Generalized Legendre	$(1 - z^2)y''(z) + 2(s - (m + 1)z)y'(z) + \lambda y(z) = 0$	reg. $z = -1, 1, \infty$	$\{\}$
Gauss eq.	$z(1 - z)y''(z) + (c - (a + b + 1)z)y'(z) - aby(z)$	$z = 0, 1, \infty$	$\{1/2, 1/2, 1/2\}$ or $\{1/2, 1, 1\}$ or $\{1/2, 1/2, 1\}$
Confluent hyperg.	$zy''(z) + (c - z)y'(z) - ay(z) = 0$	$z = 0, \infty$	$\{1/2, 2\}$ or $\{1, 3/2\}$ or $\{1/2, 3/2\}$
Weber equation	$y''(z) + (\lambda - z^2)y(z) = 0$	$z = \infty$	$\{3\}$
Airy equation	$y''(z) - zy(z) = 0$	$z = \infty$	$\{5/2\}$

The general Heun equation

$$\frac{d^2}{dz^2}H(z) + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right]\frac{dH(z)}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}H(z) = 0 \quad (1)$$

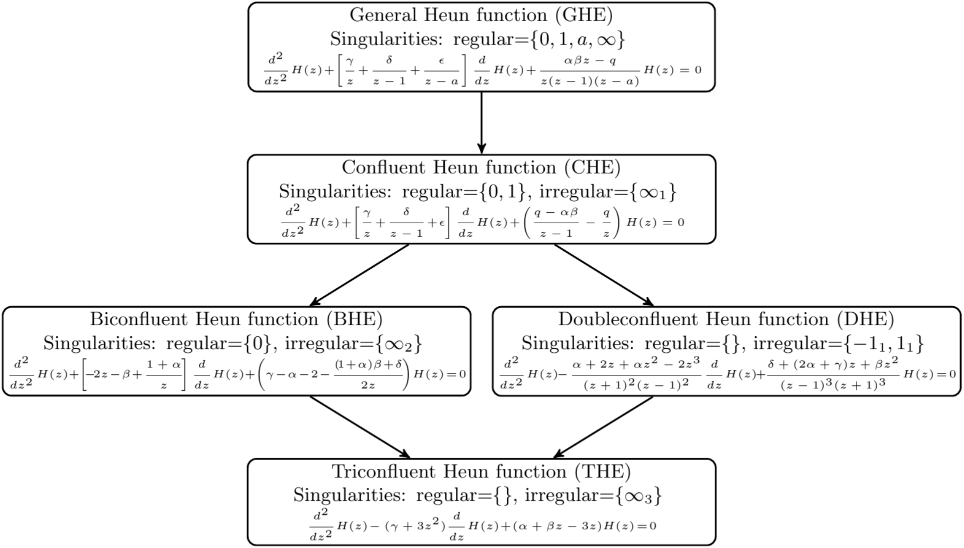
Here $\epsilon = \alpha + \beta - \gamma - \delta + 1$.
Regular singularities: $z = 0, 1, a, \infty$.
Solution of the type $y = \sum_{r=0}^{\infty} c_r z^n$ defines a 3-term recursion:

$$-qc_0 + a\gamma c_1 = 0, P_r c_{r-1} - (Q_r + q)c_r + R_r c_{r+1} = 0, c_0 = 1$$

$$P_r = (r-1+\alpha)(r-1+\beta), Q_r = r((r-1+\gamma)(1+a) + a\delta + \epsilon), R_r = (r+1)(r+\gamma)a$$

Its group of symmetries is of order of 192. (for the hypergeometric ODE, it is $n!2^{n-1} = 24$).

Under the process called confluence of singularities, one obtains 4 different types of confluent Heun functions with fewer singularities but of higher s-rank.


General Heun function (GHE)
Singularities: regular=$\{0, 1, a, \infty\}$
$\frac{d^2}{dz^2}H(z)+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right]\frac{d}{dz}H(z)+\frac{\alpha\beta z-q}{z(z-1)(z-a)}H(z)=0$
Confluent Heun function (CHE)
Singularities: regular=$\{0, 1\}$, irregular=$\{\infty_1\}$
$\frac{d^2}{dz^2}H(z)+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\epsilon\right]\frac{d}{dz}H(z)+\left(\frac{q-\alpha\beta}{z-1}-\frac{q}{z}\right)H(z)=0$
Biconfluent Heun function (BHE)
Singularities: regular=$\{0\}$, irregular=$\{\infty_2\}$
$\frac{d^2}{dz^2}H(z)+\left[-2z-\beta+\frac{1+\alpha}{z}\right]\frac{d}{dz}H(z)+\left(\gamma-\alpha-2-\frac{(1+\alpha)\beta+\delta}{2z}\right)H(z)=0$
Doubleconfluent Heun function (DHE)
Singularities: regular=$\{\}$, irregular=$\{-1_1, 1_1\}$
$\frac{d^2}{dz^2}H(z)-\frac{\alpha+2z+\alpha z^2-2z^3}{(z+1)^2(z-1)^2}\frac{d}{dz}H(z)+\frac{\delta+(2\alpha+\gamma)z+\beta z^2}{(z-1)^3(z+1)^3}H(z)=0$
Triconfluent Heun function (THE)
Singularities: regular=$\{\}$, irregular=$\{\infty_3\}$
$\frac{d^2}{dz^2}H(z)-(\gamma+3z^2)\frac{d}{dz}H(z)+(\alpha+\beta z-3z)H(z)=0$

The confluent Heun function

One starts with the general Heun equation:

$$\frac{d^2}{dz^2}H(z)+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right]\frac{dH(z)}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}H(z)=0, \quad (2)$$

redefines $\beta=\beta a, \epsilon=\epsilon a, q=qa$ and takes the limit $a\to\infty$ to obtain:

$$\frac{d^2}{dz^2}H(z)-\left(\epsilon-\frac{\delta}{z-1}-\frac{\gamma}{z}\right)\frac{\mathrm{d}}{\mathrm{d}z}H(z)-\left(\frac{\alpha\,\beta-q}{z-1}+\frac{q}{z}\right)H(z)=0 \quad (3)$$

Series solution:$y=\sum_{r=0}^{\infty}c_r z^n$ defines a 3-term recursion:

$$g_k^{(a)}=k(k-4p+\gamma+\delta-1)-\sigma, f_k^{(a)}=-(k+1)(k+\gamma), h_k^{(a)}=4p(k+\alpha-1)$$

$$f_k^{(a)}c_{k+1}^{(a)}+g_k^{(a)}c_k^{(a)}+h_k^{(a)}c_{k-1}^{(a)}=0, c_{-1}=0, c_0=1$$

In Maple notations, the default solution ofthis ODE is denoted as

$$HeunC(\alpha,\beta,\gamma,\delta,\eta,z)$$

S-homotopic transformations give 16 exact local Frobenius type solutions:

$$H=e^{\sigma_\alpha\frac{\alpha\pm z\pm}{2}}z_\pm^{\sigma_\beta\frac{\beta\pm}{2}}z_\mp^{\sigma_\gamma\frac{\gamma\pm}{2}}HeunC(\sigma_\alpha\alpha_\pm,\sigma_\beta\beta_\pm,\sigma_\gamma\gamma_\pm,\delta_\pm,\eta_\pm,z_\pm)$$

> FunctionAdvisor(HeunC)

The Heun functions in Maple

Numerical problems:

- no known integral representations
- not all the identities known
- converging series solution only inside the radius of convergence

Implementation in Maple:

- Direct numerical integration of the ODE (the default method)
- A sequence of concatenated Taylor series expansions

```
> with(MathematicalFunctions)
    [&Intersect, &Minus, &Union, Assume, Coulditbe, Evalf, Get, Is, SearchFunction, Sequences, Series]    (1)
> Evalf(%HeunC(1.1,2.3,3.1,2.5,0.7,1.5),Z)
    HeunCZ and HeunCZPrime at Z = .500000000000-.559016994375*I using a series expansion around Z = 0
    HeunCZ and HeunCZPrime at Z = 1.06250000000-.559016994375*I using a series expansion around Z = .500000000000-.559016994375*I
    HeunCZ and HeunCZPrime at Z = 1.46784061235-.442066134599*I using a series expansion around Z = 1.06250000000-.559016994375*I
    HeunCZ and HeunCZPrime at Z = 1.93166507175-.308241228683*I using a series expansion around Z = 1.46784061235-.442066134599*I
    HeunCZ and HeunCZPrime at Z = 2.55325018478-.104209905759*I using a series expansion around Z = 1.93166507175-.308241228683*I
    HeunCZ at Z = 3.000000000 using a series expansion around Z = 2.55325018478-.104209905759*I
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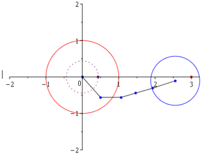


CPU time elapsed during evaluation: .455 seconds

$-1.554731573 - 0.7202406450\,I$ (2)

Black holes

Definition: A black hole is a region of the space-time whose boundary is causaly disconnected from the rest of the Universe. Classically nothing can escape, even light.

Einstein solutions: the Schwarzshild metric (non-rotating) $(r_s = 2M, G = 1, c = 1)$:

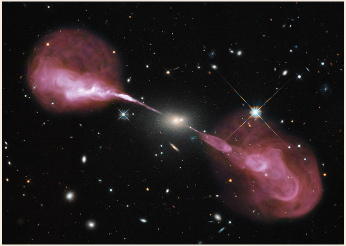
$$ds^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2 - r^2(d\theta^2 - sin(\theta)^2 d\phi^2) \quad (4)$$

and the Kerr metric (rotating) $(\Sigma = r^2 + a^2 cos(\theta)^2, \Delta = r^2 - r_s r + a^2)$.

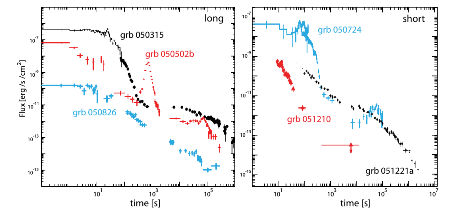
$$ds^2 = (1 - \frac{r_s r}{\Sigma})dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - (r^2 + a^2 + \frac{r_s r a^2}{\Sigma} sin(\theta)^2 d\phi^2 + 2\frac{r_s r a sin(\theta)^2}{\Sigma} dt d\phi \quad (5)$$

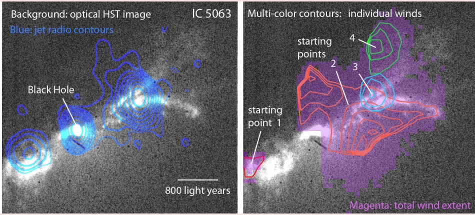
According to observational data:

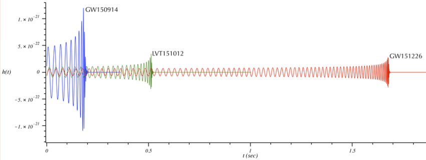
- Stellar black hole: $4 - 15 M_\odot$
- Medium-size black holes: $1.10^3 - 4.10^4 M_\odot$
- Super-massive black holes: $1.10^6 - 9.10^9 M_\odot$
- Double and triple systems of SMBH



Why are they so interesting?



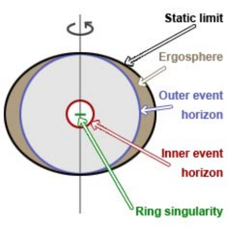




1. Gravitational waves finally observed – so far, only from systems of stellar black holes! What do we know?
2. The mystery of GRB – $E_{iso} \sim 10^{53} erg$, $t \sim sec$, $t_{flares} \sim 10^5 s$, /APJ, 778:54, 2013, ApJ 766:30, 2013/)

– What is the central engine?

– Jets formation – how and why?



The ringing of the black holes /Teukolsky (1972)/

Linear perturbation of the Kerr metric for $\Psi = e^{i(\omega t + m\phi)} S(\theta) R(r)$ is described with the Teukolsky equations.
The Angular Teukolsky Equation (TAE):

$$\left(\left(1-u^2\right) S_{lm,u}\right)_{,u} + \left((a\omega u)^2 + 2a\omega s u + {}_sE_{lm} - s^2 - \frac{(m+su)^2}{1-u^2}\right) S_{lm} = 0, \quad (6)$$

and the Radial Teukolsky Equation(R):

$$\frac{d^2 R_{\omega,E,m}}{dr^2} + (1+s)\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)\frac{dR_{\omega,E,m}}{dr} + + \left(\frac{K^2}{(r-r_+)(r-r_-)} - is\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)K - -\lambda - 4\, is\omega r\right)\frac{R_{\omega,E,m}}{(r-r_+)(r-r_-)} = 0 \quad (7)$$

where $\Delta = r^2 - 2Mr + a^2 = (r-r_-)(r-r_+)$, $K = -\omega(r^2+a^2) - ma$, $\lambda = E - s(s+1) + a^2\omega^2 + 2am\omega$ and $u = \cos(\theta)$.
For EM perturbations: $s = -1$. For GR: $s = -2$.
The two horizons are: $r_\pm = M \pm \sqrt{M^2 - a^2}$. Unknowns:$\omega, E$!!!

Solutions in terms of the confluent Heun function

The solution of TAE:

$$S_{1,2}(\theta) = e^{\alpha_{1,2} z_{1,2}} z_{1,2}^{\beta_{1,2}/2} z_{2,1}^{\gamma_{1,2}/2} \mathrm{HeunC}(\alpha_{1,2}, \beta_{1,2}, \gamma_{1,2}, \delta_{1,2}, \eta_{1,2}, z_{1,2}) \quad (8)$$

where $z_1 = \cos(\theta/2)^2$, $z_2 = \sin(\theta/2)^2$, and the parameters are:

For the case $m = 0$:

$$\alpha_1 = -\alpha_2 = 4\,a\omega,$$
$$\beta_1 = \beta_2 = 1,$$
$$\gamma_1 = -\gamma_2 = -1,$$
$$\delta_1 = -\delta_2 = 4\,a\omega,$$
$$\eta_1(\omega) = \eta_2(-\omega) = 1/2 - E - 2\,a\omega - a^2\omega^2$$

For the case $m = 1$:

$$\alpha_1 = \alpha_2 = -4\,a\omega,$$
$$\beta_1 = \gamma_2 = 2,$$
$$\gamma_1 = \beta_2 = 0,$$
$$\delta_1 = -\delta_2 = 4\,a\omega,$$
$$\eta_1(\omega) = \eta_2(-\omega) = 1 - E - 2\,a\omega - a^2\omega^2$$

The solutions of TRE:

$$R(r) = C_1 R_1(r) + C_2 R_2(r), \text{for} \tag{9}$$
$$R_1(r) = e^{\frac{\alpha z}{2}} (r - r_+)^{\frac{\beta+1}{2}} (r - r_-)^{\frac{\gamma+1}{2}} \mathrm{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$$
$$R_2(r) = e^{\frac{\alpha z}{2}} (r - r_+)^{\frac{-\beta+1}{2}} (r - r_-)^{\frac{\gamma+1}{2}} \mathrm{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, z),$$

where $z = -\frac{r-r_+}{r_+-r_-}$, and the parameters are:

$$\alpha = -2\,i\left(r_+ - r_-\right)\omega, \beta = -\frac{2\,i(\omega\left(a^2 + {r_+}^2\right) + am)}{r_+ - r_-} - 1, \gamma = \frac{2\,i(\omega\left(a^2 + {r_-}^2\right) + am)}{r_+ - r_-} - 1,$$
$$\delta = -2i\left(r_+ - r_-\right)\omega\left(1 - i\left(r_- + r_+\right)\omega\right),$$
$$\eta = \frac{1}{2}\frac{1}{\left(r_+ - r_-\right)^2}\Big[4\omega^2 {r_+}^4 + 4\left(i\omega - 2\omega^2 r_-\right){r_+}^3 + \left(1 - 4a\omega\, m - 2\omega^2 a^2 - 2E\right)\times$$
$$\left({r_+}^2 + {r_-}^2\right) + 4\left(i\omega\, r_- - 2i\omega\, r_+ + E - \omega^2 a^2 - \frac{1}{2}\right) r_-\, r_+ - 4a^2\left(m + \omega\, a\right)^2\Big].$$

Boundary conditions: /Fiziev(2009), Staicova and Fiziev (2010, 2015)/

For the TAE we require regularity on the sphere. This means the Wronskan of the 2 solutions $S_1(\theta)$ and $S_2(\theta)$, should be $W[S_1(\theta), S_2(\theta)] = 0$, or:

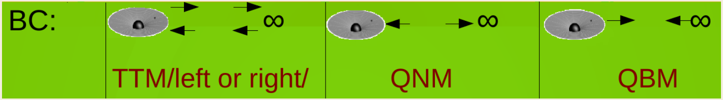
$$W[S_1, S_2] = \frac{\mathrm{HeunC}'(\alpha_1, \beta_1, \gamma_1, \delta_1, \eta_1, (\cos(\pi/6))^2)}{\mathrm{HeunC}(\alpha_1, \beta_1, \gamma_1, \delta_1, \eta_1, (\cos(\pi/6))^2)} + \frac{\mathrm{HeunC}'(\alpha_2, \beta_2, \gamma_2, \delta_2, \eta_2, (\sin(\pi/6))^2)}{\mathrm{HeunC}(\alpha_2, \beta_2, \gamma_2, \delta_2, \eta_2, (\sin(\pi/6))^2)} = 0 \qquad (10)$$

For TRE:

QNM – (black hole boundary conditions): R_2 is valid for $\Re(\omega) \notin (-\frac{ma}{2Mr_+}, 0)$ and $\sin(\arg(\omega) + \arg(r)) < 0$.

QBM – (quasibound boundary conditions): R_1 is valid for $\Re(\omega) \notin (-\frac{ma}{2Mr_+}, 0)$ and $\sin(\arg(\omega) + \arg(r)) > 0$.

TTM modes are missing in the EM case



EM Quasi-Normal Modes ($s = -1$) for a=0 /D.S. and Fiziev (2015)/

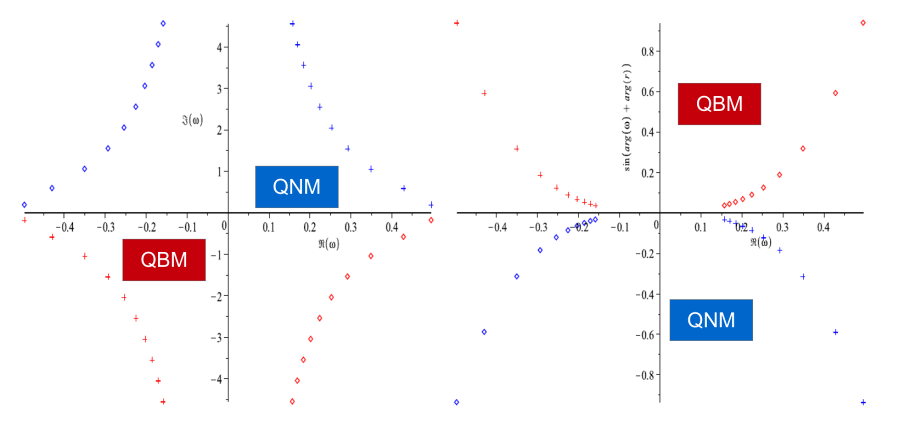


Figure: a) QNM and QBM modes for $m = 0, l = 1$ b) boundary condition for them: $sin(arg(\omega) + arg(r))$

The spectrum for $a \in [0, M]$ /Fiziev and D.S. (2015)/

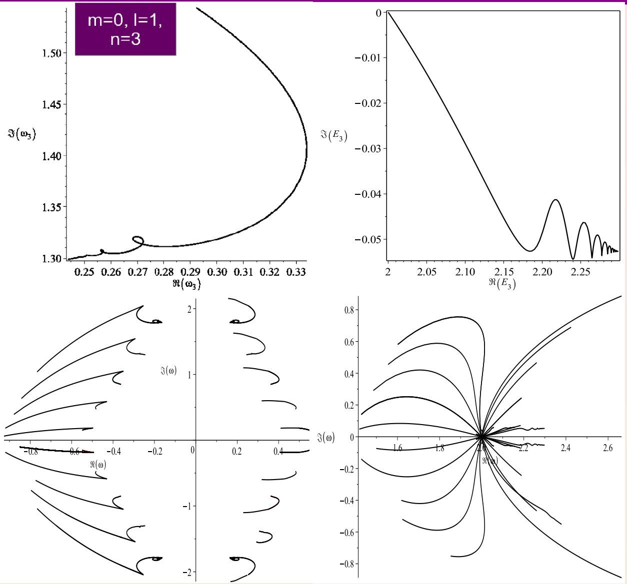


Figure: $\omega_{m,n}(a)$ and $E_{m,n}(a)$ for $a = [0, M)$, $m = 0, 1, l = 1$ $n = 0..4$

Spurious modes /D.S. and Fiziev (2015)/

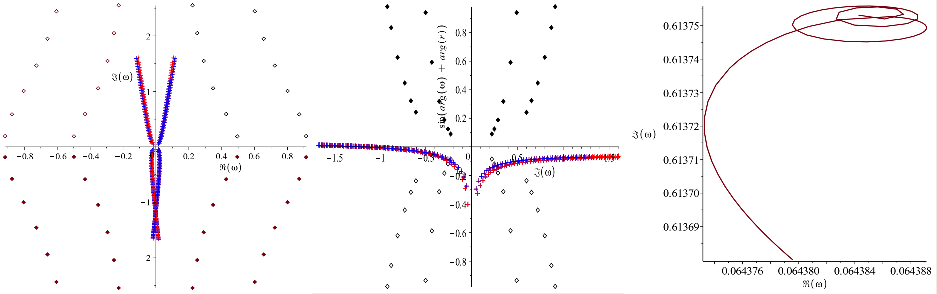


Figure: (a) Unphyisical modes (crosses) with QNM and QBM (diamonds) for $a = 0$, $m = 0, l = 1, 2$. (b) Boundary condition for them $\sin(\arg(\omega) + \arg(r))$ (c) the mode with $n = 3$ for $a \in [0, M)$

In order to distinguish the spurious modes from the physical ones, one needs to test their numerical stability. Physical modes should not depend on r! Our tests showed that the spurious modes indeed depend on r.

Gravitational QNMs: the case $s = -2$

The spectral system:

$$W(\theta) = \frac{\mathrm{HeunC}'\left(-4\,a\omega, 2-m, m+2, -8\,a\omega, \eta^a_+, z_A\right)}{\mathrm{HeunC}\left(-4\,a\omega, 2-m, m+2, -8\,a\omega, \eta^a_+, z_A\right)} + \frac{\mathrm{HeunC}'\left(4\,a\omega, m+2, 2-m, 8\,a\omega, \eta^a_-, 1-z_A\right)}{\mathrm{HeunC}\left(4\,a\omega, m+2, 2-m, 8\,a\omega, \eta^a_-, 1-z_A\right)} \quad (11)$$

where the derivatives are with respect to z_A, $\theta = \pi/3$ and $\eta^a_\pm = 2 + 1/2\,m^2 - E \pm 4\,a\omega - a^2\omega^2$, $z_A = \sin^2(\frac{\theta}{2})$. TRE:

$$R(r) = HeunC\left(\alpha, \beta, \gamma, \delta, \eta, -\frac{r - r_+}{r_+ - r_-}\right) \quad (12)$$

with $B = (r_+ - r_-)/(r_+ + r_-), \Omega = M\omega, C = a/M$ and $r_\pm = M \pm \sqrt{M^2 - a^2}$ and

$$\alpha = -4\,iB\Omega,\ \beta = \frac{\sqrt{Q_+}}{B},\ \gamma = -\frac{\sqrt{Q_-}}{B},\ \delta = -8\,B\Omega\,(\Omega + i) \quad (13)$$

$$\eta = -\frac{1}{2B^2}\Big(-2\,B^4\Omega^2 - 8\,iB^3\Omega - 8\,\Omega^2 B^3 - 8\,iB^2\Omega - 10\,B^2\Omega^2 - B^2m^2 + 2\,AB^2 + 4\,Cm\Omega + 4\,\Omega^2 + m^2$$

$$Q_q\pm = 8\,iB^2\Omega - 4\,B^2\Omega^2 + B^2m^2 \pm \Big(4\,iBCm - 4\,BCm\Omega + 8\,iB\Omega - 8\,B\Omega^2\Big) - 4\,Cm\Omega + 4\,B^2 - 4\,\Omega^2 - m^2$$

The spectrum for the gravitational QNMs

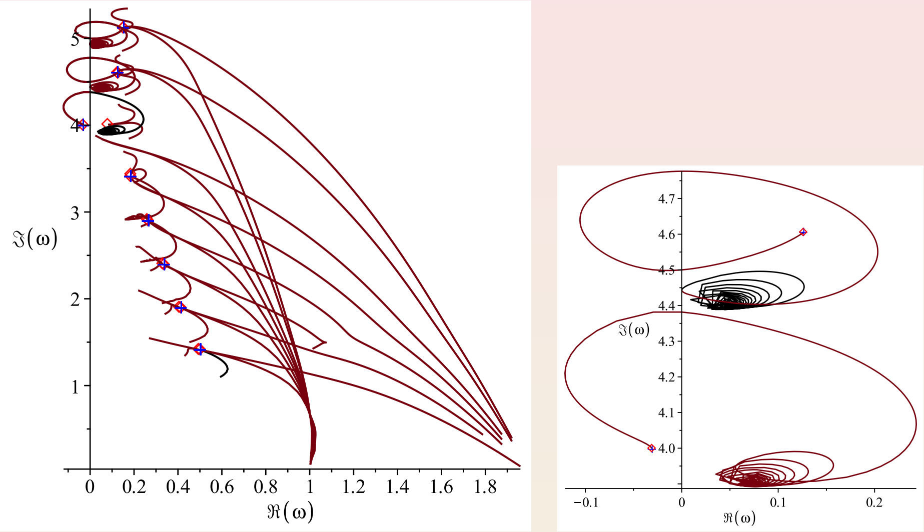


Figure: The Gravitational QNMs for $m = 0, 1, -1, 2, -2$

Jet modes

What happens if we impose qualitatively new boundary conditions?
For the solutions of the the TAE, we use the new requirement that the confluent Heun functions should be polynomial. The polynomiality condition reads:

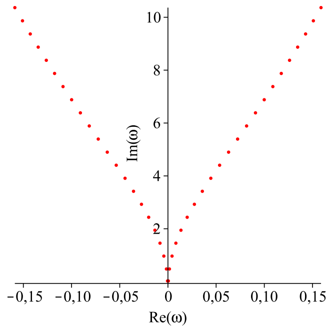
$$\frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} + N + 1 = 0. \qquad \Delta_{N+1}(\mu) = 0.$$

Here, the integer $N \geq 0$ is the degree of the polynomial and $\Delta_{N+1}(\mu)$ is three-diagonal determinant specified in Fiziev 2009
The polynomial requirement for the angular solutions fixes the following relation between E and ω:

$${}_{s=-1}E_m^{\pm}(\omega) = -(a\omega)^2 - 2\,a\omega\,m \pm 2\sqrt{(a\omega)^2 + a\omega\,m}. \tag{14}$$

The spectra compared

Staicova and Fiziev:
Astrophys. Space Sci. (2011) 332   Astrophys. Space Sci. (2015) 358



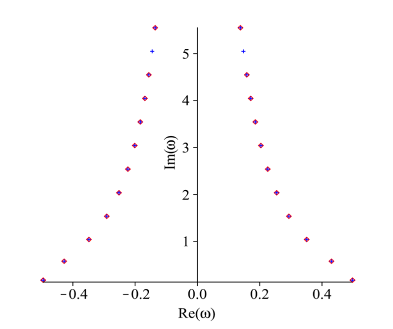


Figure: Left: Jets modes Right: QNM

The new boundary conditions change qualitatively the spectrum!

The analytical fit

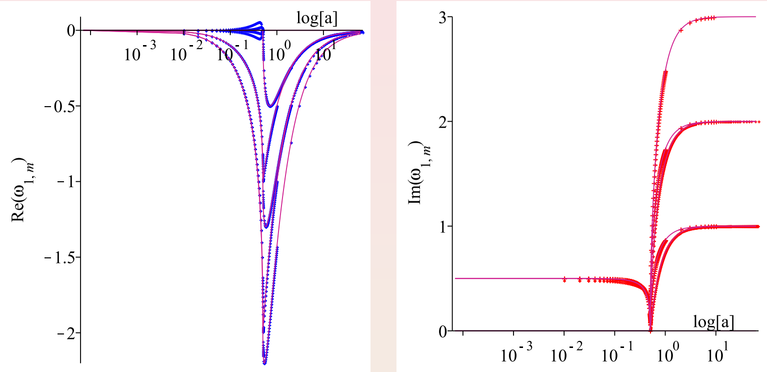


Figure: Comparison between our numerical results plotted with blue (red) crosses and the analytical formula (violet lines) in the cases $N = 0, 1$.

The best fit for our numerical data for the lowest modes is:

$$\omega_{n=0,1,m} = (-m + iN\sqrt{b^2 - 1}\,)\,\Omega_+, \quad N = 0, 1., \Omega_+ = a/2Mr_+, b = M/a \tag{15}$$

Comparison between Jets modes & QNM for $a > 0$

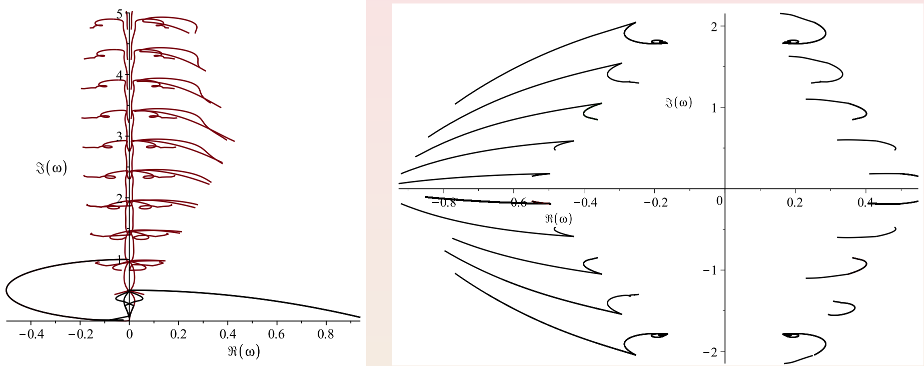
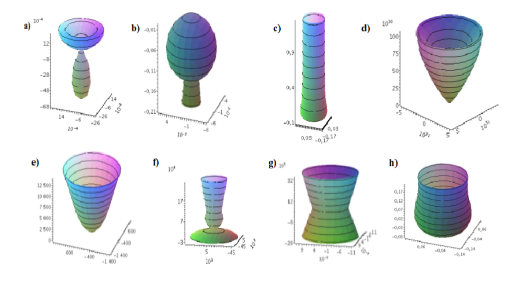


Figure: Left: Jets modes ($m = 0, -1$), Right: QNM ($m = 0, 1$)

While for the Jets modes we are able to reach $a > M$ without finding the symmetrical with respect to the real axis unstable modes, the QNMs and the QBMs are completely symmetrical.

The solutions of the TAE



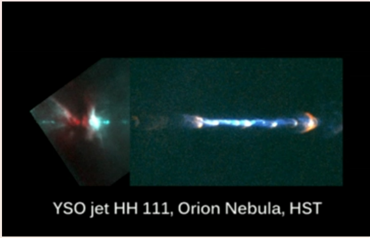


Figure: The angular part of the solution (left) and jets observed in Nature (right)

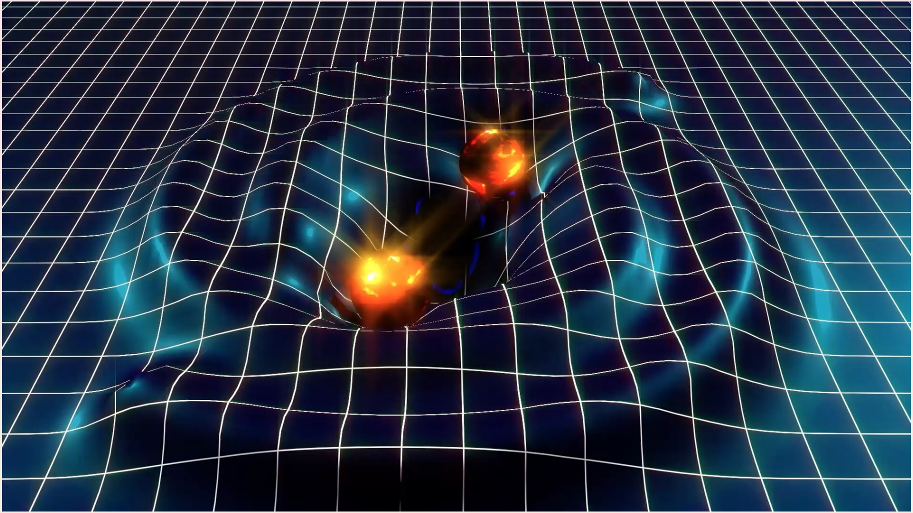
Using the polynomial singular condition instead of the regularity condition offers us a simple though approximate way to model collimated outflows!

We have applied the confluent Heun function to:

- Black holes and naked singularities in the (proper) jets case (EM). We have found a new spectrum and natural mechanism for collimation.
- Black holes QNMs and QBMs (EM). We have found that the spectrum obeys the symmetries of the metric. Also we have found spurious spectrum.
- Black holes QNMs and QBMs (GR) We have found the non-zero imaginary part of the 8th mode.

Using the Heun functions we were able to both reproduce old results independently and qualitatively results.
Despite the numerical challenges, the Heun functions are promising tool in any field of physics.

Thank you for you attention!



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Connection between coefficients in Ronveaux Eq. (1) and the definition in Maple:

$$\alpha = -\epsilon^R, \beta = \gamma^R - 1, \gamma = \delta^R - 1,$$

$$\delta = -\alpha^R\beta^R + \epsilon^R/2(\gamma^R + \delta^R), \eta = -\epsilon^R\gamma^R/2 + q^R - 1/2(\gamma^{R^2} - 1)$$

From Fiziev, arXiv:0902.1277 The $\Delta_n(\mu)$ condition:

$$\begin{vmatrix} \mu-q_1 & 1(1+\beta) & 0 & \dots & 0 & 0 & 0 \\ N\alpha & \mu-q_2+1\alpha & 2(2+\beta) & \dots & 0 & 0 & 0 \\ 0 & (N-1)\alpha & \mu-q_3+2\alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu-q_{N-1}+(N-2)\alpha & (N-1)(N-1+\beta) & 0 \\ 0 & 0 & 0 & \dots & 2\alpha & \mu-q_N+(N-1)\alpha & N(N+\beta) \\ 0 & 0 & 0 & \dots & 0 & 1\alpha & \mu-q_{N+1}+N\alpha \end{vmatrix}, \quad (1.9)$$

$q_n = (n-1)(n+\beta+\gamma)$, $\delta = \mu + \nu - \alpha\frac{\beta+\gamma+2}{2}, \eta = \frac{\alpha(\beta+1)}{2} - \mu - \frac{\beta+\gamma+\beta\gamma}{2}$