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# Electromagnetic quasi-normal modes of rotating black holes


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## Introduction

From the recent developments in the field of gravitational waves detection, it is clear that finding the electromagnetic (EM) counterpart to such events can prove to be very useful. As a way to study the fundamental physics of the central engine of compact massive objects, one may use the EM quasi-normal modes (QNMs). The QNMs are the characteristic complex frequencies with which the linearized perturbations of the Kerr metric are radiated away to infinity. The QNMs find numerous applications in theoretical and numerical astrophysics, including in highly non-linear full general relativistic black hole simulations. Lately, they have been considered in the frame of the so called "multimessenger approach" in which one uses simultaneously the gravitational, electromagnetic and neutrino spectra to compare the predictions of general relativity with astrophysical observations.

## The Heun functions

Under the assumption $\Psi = e^{i(\omega t+m\phi)}S(\theta)R(r)$, the problem reduces to the Teukolsky Angular Equation and the Teukolsky Radial Equation, which can be solved analytically in terms of **confluent Heun functions**.
Using the exact analytical solutions and the properties of the confluent Heun functions, we are able to impose the boundary conditions *directly* to obtain the spectral system. This approach has been pioneered in works by Fiziev and it has been used successfully to find the gravitational QNMs and the EM "primary jets" modes.

## Main results

The key results of our work are as follow:

1. High-precision reproduction of known QNM frequencies and study of the numerical stability of the modes in the radial complex plane.
2. New QBM spectrum, obtained for the inverse boundary conditions, its numerical stability also studied.
3. An additional spurious spectrum found to contradict assumptions of the problem and thus classified as numerical artifact.

## References


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[3] Fiziev P. and Staicova D. *Phys. Rev. D*, 84:127502, 2011.


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## The TAE and the TRE and their solutions

The two second-order linear differential equations which govern the late-time ring-down of the EM perturbation ($s=-1$) of the Kerr metric are: the Teukolsky Angular Equation (TAE):

$$\left((1-u^2)\,S_{lm,u}\right)_{,u} + \left((a\omega u)^2 - 2a\omega u + E_{lm} - 1 - \frac{(m-u)^2}{1-u^2}\right) S_{lm} = 0, \qquad (1)$$

and the Teukolsky Radial Equation (TRE):

$$\frac{d^2 R_{lm}}{dr^2} + \left(\frac{K^2}{(r-r_+)(r-r_-)} + i\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)K - \lambda + 4i\omega r\right)\frac{R_{lm}}{(r-r_+)(r-r_-)} = 0 \qquad (2)$$

where $\Delta = (r-r_-)(r-r_+)$, $K = -\omega(r^2+a^2) - ma$, $\lambda = E - s(s+1) + a^2\omega^2 + 2am\omega$ and $u = \cos(\theta)$. $r_\pm = M \pm \sqrt{M^2-a^2}$ are the two horizons, $a$ – the rotational parameter, $M$ – the mass.
Both differential equations are of the confluent Heun type (two regular singularities: $u=\pm1, r=r_\pm$, one irregular: $\infty$) and their solutions are of the form:

$$S_{1,2}(z_{1,2}) = e^{\alpha_{1,2}z_{1,2}}\, z_{1,2}^{\beta_{1,2}/2}\, z_{2,1}^{\gamma_{1,2}/2}\,\mathrm{HeunC}(\alpha_{1,2},\beta_{1,2},\gamma_{1,2},\delta_{1,2},\eta_{1,2},z_{1,2}),$$

where the parameters differ for the solutions of the TAE and the TRE.
**The boundary conditions** are as follow:
– angular boundary conditions: regularity on the sphere
– radial boundary conditions: black hole boundary conditions ($r_+ \leftarrow R(r) \rightarrow \infty$) or their inverse.
We solve the so obtained spectral system of transcendental equations for $\{\omega, E\}$ using the developed by the team two-dimensional Müller method, implemented in MAPLE.

## The numerical results

We obtain two types of discrete spectra: the QNMs ($\sin(\arg(r)+\arg(\omega))<0$) and QBMs ($\sin(\arg(r)+\arg(\omega))>0$).
For the case when there is no rotation ($a=0$), the complex frequencies can be seen on Fig. 1.

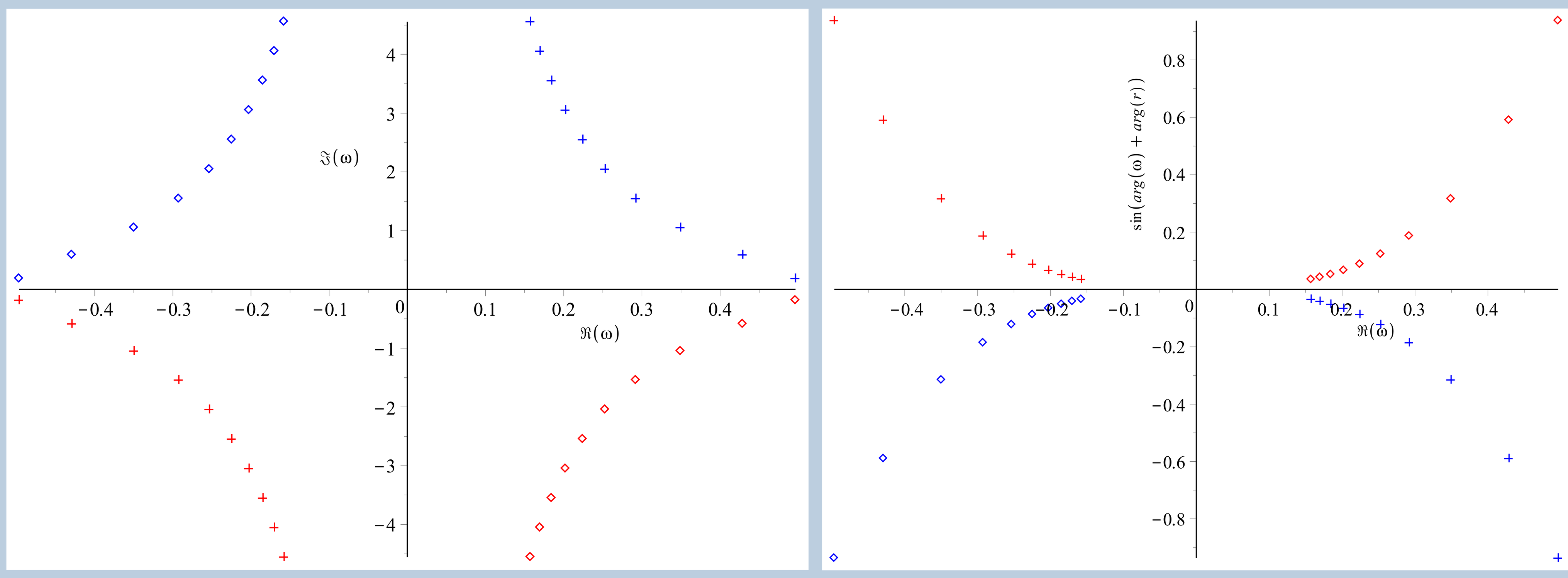


**Figure 1:** a) The complex frequencies for $a=0$, for $m=0, l=1$. (b) the boundary condition $\sin(\arg(\omega)+\arg(r))$ for them. The red diamonds are obtained from $R_1(r)$ with $\arg(r)=1/2\pi$, the red crosses – from $R_1(r)$ with $\arg(r)=3/2\pi$, the blue diamonds – from $R_2(r)$ with $\arg(r)=1/2\pi$, the blue crosses – from $R_2(r)$ with $\arg(r)=3/2\pi$.

When we add rotation to the black hole ($a \in [0, M)$) we obtain the spectra showed on Fig. 2:

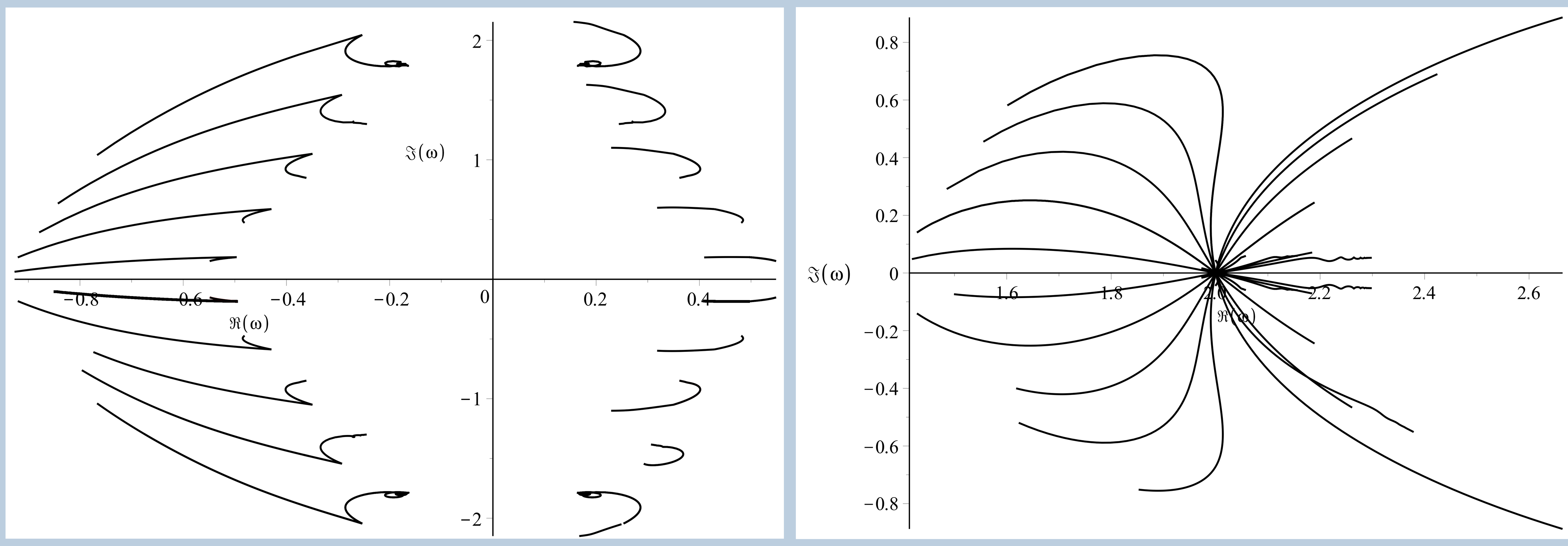


**Figure 2:** A complex plot of all the $\omega_{m,n}(a)$ and $E_{m,n}(a)$ obtained for $a=[0,M)$ for $m=0,1, l=1$ $n=0..4$

By exploring the numerical stability of the so-obtained spectra, we have demonstrated they are stable in wide regions of the complex $r$-plane. Further comparison with results obtained by the well-established continued-fractions method showed they match with precision of more than 9 digits.
In the near-extremal regime ($a/M \in [0.995..0.99998]$), we compared our numerical results with the analytical formula by Hod (2008): $\omega = m\Omega - i2\pi T_{BH}(n+1/2+i\delta)$ and we found very good fit between the theoretical and the numerical results. Here $\delta$ is a complex number, $\Omega = \frac{a}{r_+^2-a^2}$ is the angular velocity of the horizon, $T_{BH} = \frac{r_+-r_-}{4\pi(r_++a^2)}$ – the Hawking temperature.
We have used $|r| = 110$ and $M = 1/2$