1. Фурие трансформации

a) За $f : \mathbb{R}^N \to \mathbb{C}$

$$\mathcal{F}(f) \equiv \tilde{f} : \mathbb{R}^N \to \mathbb{C}$$

върху $\mathbb{R}^N$

$$\mathcal{F}(f)(\vec{p}) = (2\pi)^{-\frac{N}{2}} \int d^N x \, e^{i\vec{p}\cdot\vec{x}} f(\vec{x})$$

$$\mathcal{F}^{-1}(f)(\vec{x}) = (2\pi)^{-\frac{N}{2}} \int d^N p \, e^{-i\vec{p}\cdot\vec{x}} f(\vec{p})$$

Арг. на f : $\vec{x} \in \mathbb{R}^N$, $\vec{x} = (x^j)$

$\cong$

Арг. на $\tilde{f}$: $\vec{p} \in (\mathbb{R}^N)^*$, $\vec{p} = (p_k)$

$$\vec{p}\cdot\vec{x} = p_j x^j$$

б) Област:

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^N) \xRightarrow{\simeq} \mathcal{S}(\mathbb{R}^N)$$

$$\mathcal{S}(\mathbb{R}^N) := \{ f \in C^\infty(\mathbb{R}^N, \mathbb{C}) \mid$$

$$\mid \sup | P(\vec{x}) f(\vec{x}) | < \infty \ \forall \text{ пол. } P \}$$

– бързо намаляващи тест функк.

в) L^2 – унитарност

$$\|f\|^2 := \int d^N x \, |f(\vec{x})|^2$$

$$\Rightarrow \quad \|f\|^2 = \|\tilde{f}\|^2$$

бавно
разтящи
обобщени
функции

$$\begin{tikzcd}
\mathcal{S}(\mathbb{R}^N) \arrow[loop above, "\mathcal{F}"] \arrow[r, phantom, "\subseteq"] & L^2(\mathbb{R}^N) \arrow[loop above, "\mathcal{F}"] \arrow[d, phantom, "\cong"] & \\
& L^2(\mathbb{R}^N)' \arrow[loop below, "\mathcal{F}"] \arrow[r, phantom, "\subseteq"] & \mathcal{S}'(\mathbb{R}^N) \arrow[loop below, "\mathcal{F}"]
\end{tikzcd}$$

$$\mathcal{S}'(\mathbb{R}^N) \subsetneq \mathcal{D}'(\mathbb{R}^N)$$

$$\mathcal{D}(\mathbb{R}^N) \subsetneq \mathcal{S}(\mathbb{R}^N)$$

2) $$f(\vec{y}) = (2\pi)^{-N} \int d^N p \int d^N x$$

$$\times\, e^{i\vec{p}\cdot\vec{x} - i\vec{p}\cdot\vec{y}} f(\vec{x})$$

$$f(\vec{y}) = \int d^N x \, (2\pi)^{-N} \int d^N p \, e^{i\vec{p}\cdot\vec{x} - i\vec{p}\cdot\vec{y}} \times f(\vec{x})$$

$$\Rightarrow \int d^N p \, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = (2\pi)^N \delta(\vec{x}-\vec{y})$$

$$\int d^N p \, e^{i\vec{p}\cdot\vec{x}} = (2\pi)^N \delta(\vec{x})$$

Регуляризация

$$\int d^N p \, e^{i\vec{p}\cdot\vec{x} - \frac{1}{2}\lambda \vec{p}^{\,2}} =$$

$$= \int d^N p \, e^{-\frac{1}{2}\lambda\left(\vec{p} - \frac{\vec{x}}{\lambda}\right)^2} e^{-\frac{1}{2\lambda}\vec{x}^{\,2}}$$

$$= (2\pi)^{N/2} \underbrace{\lambda^{-N/2} e^{-\frac{1}{2\lambda}\vec{x}^{\,2}}}_{\substack{\text{при } \lambda \downarrow 0 \ \downarrow \text{ в } S' \\ \delta(\vec{x})}}$$

g) $\mathcal{F}(x_j f) = -i\partial_{p_j}\mathcal{F}(f)$,

$$\mathcal{F}(\partial_{x_j} f) = i p_j \mathcal{F}(f)$$

за $f \in \mathcal{S}'(\mathbb{R}^N)$.

$$\mathcal{F}(f_1 \cdots f_n)(\vec{p}) = \int_{\vec{p}_1 + \cdots + \vec{p}_n = \vec{p}} d^N p_1 \cdots d^N p_n$$

$$\times \mathcal{F}(f_1)(\vec{p}_1) \cdots \mathcal{F}(f_n)(\vec{p}_n) .$$

$$\mathcal{F}(f^*)(\vec{p}) = \mathcal{F}(f)^*(-\vec{p})$$

2. Канонично квантуване на поле на Клайн-Гордон

$$\left(\partial_t^2 - c^2 \partial_{\underline{x}}^2 + \frac{m^2 c^4}{\hbar^2}\right) \varphi(\underline{x}, t)$$

$$= \frac{1}{3!} \lambda \varphi(\underline{x}, t)^3$$

К.К.С. в началния момент $(t=0)$

$$[\varphi(\underline{x}_1, 0), \partial_t \varphi(\underline{x}_2, 0)] = i\hbar\, \delta(\underline{x}_1 - \underline{x}_2)$$

$$[\varphi(\underline{x}_1, 0), \varphi(\underline{x}_2, 0)] = 0$$

$$= [\partial_t \varphi(\underline{x}_1, 0), \partial_t \varphi(\underline{x}_2, 0)]$$

$$\tilde{\varphi}(\underline{p}, t) := (2\pi\hbar)^{-\frac{3}{2}} \int d^3x\, e^{\frac{i\underline{p}\cdot\underline{x}}{\hbar}} \varphi(\underline{x}, t)$$

К.К.С. в $t=0$ $\Longleftrightarrow$

$$[\tilde{\varphi}(\underline{p}_1,0),\ \partial_t\tilde{\varphi}(\underline{p}_2,0)] = i\hbar\,\delta(\underline{p}_1+\underline{p}_2)$$

$$[\tilde{\varphi}(\underline{p}_1,0),\ \tilde{\varphi}(\underline{p}_2,0)] = 0$$

$$= [\partial_t\tilde{\varphi}(\underline{p}_1,0),\ \partial_t\tilde{\varphi}(\underline{p}_2,0)] \longrightarrow$$

Полеви ур-я:

$$\left(\partial_t^2 + \frac{\varepsilon(\underline{p})^2}{\hbar^2}\right)\tilde{\varphi}(\underline{p},t)$$

$$= \frac{1}{3!}\lambda\,\widetilde{\varphi(\underline{x},t)^3}$$

$$\varepsilon(\underline{p}) = \left(m^2c^4 + \underline{p}^2c^2\right)^{1/2}$$

Приложение:

$$[\tilde{\varphi}(\underline{p}_1, 0), \partial_t \tilde{\varphi}(\underline{p}_2, 0)]$$

$$= (2\pi\hbar)^{-3} \int d^3x_1 \, d^3x_2 \, e^{\frac{i}{\hbar}(\underline{p}_1 \cdot \underline{x}_1 + \underline{p}_2 \underline{x}_2)} \times i\hbar \, \delta(\underline{x}_1 - \underline{x}_2)$$

$$= i\hbar^{-2} (2\pi)^{-3} \int d^3x \, e^{\frac{i}{\hbar} \underline{x} \cdot (\underline{p}_1 + \underline{p}_2)} \frac{\hbar^3}{\hbar^3}$$

$$= i\hbar \, (2\pi)^{-3} \int d^3x \, e^{i \underline{x} \cdot (\underline{p}_1 + \underline{p}_2)}$$

$$= i\hbar \, \delta(\underline{p}_1 + \underline{p}_2)$$

Решения при $\lambda = 0$

– континуална система от независими осцилатори

$$\tilde{\varphi}(\underline{p}, t) = \frac{\hbar}{(2\mathcal{E}(\underline{p}))^{1/2}} \tilde{\Psi}(\underline{p}) e^{-\frac{i}{\hbar}\mathcal{E}(\underline{p})t} + \frac{\hbar}{(2\mathcal{E}(\underline{p}))^{1/2}} \tilde{\Psi}(-\underline{p})^{*} e^{\frac{i}{\hbar}\mathcal{E}(\underline{p})t}$$

$$\{\tilde{\varphi}(\underline{p}, 0),\ \partial_t \tilde{\varphi}(\underline{p}, 0) \mid \underline{p}\}$$

$$\Longleftrightarrow \{\tilde{\Psi}(\underline{p}),\ \tilde{\Psi}(\underline{p})^{*} \mid \underline{p}\}$$

К.К.С. в $t = 0 \iff$

$$[\tilde\Psi(\underline{p}_1), \tilde\Psi(\underline{p}_2)^*] = \delta(\underline{p}_1 - \underline{p}_2)$$

$$[\tilde\Psi(\underline{p}_1), \tilde\Psi(\underline{p}_2)] = 0$$

$$= [\tilde\Psi(\underline{p}_1)^*, \tilde\Psi(\underline{p}_2)^*]$$

Ако положим:

$$H_0 := \int d^3\underline{p} \; \varepsilon(\underline{p}) \, \tilde\Psi(\underline{p})^* \, \tilde\Psi(\underline{p})$$

$$\frac{i}{\hbar}[H_0, \tilde\varphi(\underline{p}, t)] = \partial_t \tilde\varphi(\underline{p}, t)$$

$$\frac{i}{\hbar}[H_0, \partial_t \tilde\varphi(\underline{p}, t)]$$

$$= -\frac{\varepsilon(\underline{p})^2}{\hbar^2} \tilde\varphi(\underline{p}, t)$$

$$\tilde{\varphi}(\underline{p},t) = e^{\frac{i}{\hbar}Ht}\,\tilde{\varphi}(\underline{p},0)\,e^{-\frac{i}{\hbar}Ht}$$

$$\varphi(\underline{x},t) = e^{\frac{i}{\hbar}Ht}\,\varphi(\underline{x},0)\,e^{-\frac{i}{\hbar}Ht}$$

$$H = H_0 + H_1$$

$$H_0 = \frac{1}{2}\int d^3p\left(\partial_t\tilde{\varphi}(\underline{p},0)\,\partial_t\tilde{\varphi}(-\underline{p},0) + \frac{\varepsilon(\underline{p})^2}{\hbar^2}\tilde{\varphi}(\underline{p},0)\,\tilde{\varphi}(-\underline{p},0)\right) + \mathcal{E}_0$$

$$= \frac{1}{2}\int d^3x\left(\left(\partial_t\varphi(\underline{x},0)\right)^2 + c^2\left(\partial_{\underline{x}}\varphi(\underline{x},0)\right)\cdot\left(\partial_{\underline{x}}\varphi(\underline{x},0)\right) + \frac{m^2c^4}{\hbar^2}\varphi(\underline{x},0)^2\right) + \mathcal{E}_0$$

$$\mathcal{E}_0 \sim \int d^3p = \infty$$

– пропускаме я – първата пренормировка $\Longleftrightarrow$ постулат за нормално произведение в H

$$H_1 = \frac{\lambda'}{4!} \int d^3x \; :\varphi(\underline{x},0)^4:$$

$$\lambda' := \lambda c^3 \hbar^{-1}, \quad [\lambda] = 1 .$$

<u>Следствие:</u>

$\forall$ формули за $\varphi(\underline{x},0)$ са верни и за $\varphi(\underline{x},t)$

Лагранжиан:

$$L = \frac{1}{2}\int d^3x \Big(\big(\partial_t \varphi(\underline{x},t)\big)^2 - c^2 \big(\partial_{\underline{x}} \varphi(\underline{x},t)\big)\cdot\big(\partial_{\underline{x}} \varphi(\underline{x},t)\big) - \frac{m^2c^4}{\hbar^2}\varphi(\underline{x},t)^2 - \frac{\lambda'}{4!}\varphi(\underline{x},t)^4 \Big)$$

Действие: $S = \int dt\, L(t)$

$$S = -\frac{c}{2}\int d^4x \Big((\partial_{x^\mu}\varphi)(\partial_{x_\mu}\varphi) + \frac{m^2c^2}{\hbar^2}\varphi^2 + \frac{\lambda}{4!}\frac{c}{\hbar}\varphi^4 \Big)$$

$$\equiv -\frac{1}{2}\int d^4x \Big((\partial_x\varphi)^2 + m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 \Big)$$

– ако $c = \hbar = 1$.