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Квантова теория на полето и елементарните частици: Лекция 14

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Допълнения и промени спрямо първата версия са поставени в червен цвят.

Конспективни записки.

1. Фурие трансформации

а) За $f : \mathbb{R}^N \to \mathbb{C}$

$\mathcal{F}(f) \equiv \tilde{f} : \mathbb{R}^N \to \mathbb{C}$

върху $\mathbb{R}^N$

$$\mathcal{F}(f)(\vec{p}) = (2\pi)^{-\frac{N}{2}} \int d^N x \, e^{i\vec{p}\cdot\vec{x}} f(\vec{x})$$

$$\mathcal{F}^{-1}(f)(\vec{x}) = (2\pi)^{-\frac{N}{2}} \int d^N p \, e^{-i\vec{p}\cdot\vec{x}} f(\vec{p})$$

Аргумент на f : $\vec{x} \in \underset{\parallel 2}{\mathbb{R}^N}$, $\vec{x} = (x^j)$

Аргумент на $\tilde{f}$: $\vec{p} \in (\mathbb{R}^N)^*$, $\vec{p} = (p_k)$

$$\vec{p}\cdot\vec{x} = p_j x^j$$

б) Област : $\mathcal{F} : \mathcal{S}(\mathbb{R}^N) \xrightarrow{\cong} \mathcal{S}(\mathbb{R}^N)$, където

$$\mathcal{S}(\mathbb{R}^N) := \left\{ f \in C^\infty(\mathbb{R}^N, \mathbb{C}) \,\middle|\, \sup \left| P(\vec{x}, \vec{\partial}_x) f(\vec{x}) \right| < \infty \ \forall \text{ полином } P \right\}$$

– бързо намаляващи тест функции

в) L^2-унитарност: за $\|f\|^2 := \int d^N x \, |f(\vec{x})|^2$

$$\Rightarrow \quad \|f\|^2 = \|\tilde{f}\|^2$$

бавно разстящи обобщени функции

$$\begin{tikzcd}
\mathcal{D}(\mathbb{R}^N) \arrow[r, phantom, "\subseteq"] & \mathcal{S}(\mathbb{R}^N) \arrow[loop above, "\mathcal{F}"] \arrow[r, phantom, "\subseteq"] & L^2(\mathbb{R}^N) \arrow[loop above, "\mathcal{F}"] \arrow[d, phantom, "\text{Теорема на Рис } \cong"] & \\
& & L^2(\mathbb{R}^N)' \arrow[loop below, "\mathcal{F}"] \arrow[r, phantom, "\subseteq"] & \mathcal{S}'(\mathbb{R}^N) \arrow[loop below, "\mathcal{F}"] \arrow[r, phantom, "\subseteq"] & \mathcal{D}'(\mathbb{R}^N)
\end{tikzcd}$$

топологични дуални $\longrightarrow$

г) От формулата за обратимост:

$$f(\vec{y}) = (2\pi)^{-N} \int d^N p \int d^N x \, e^{i\vec{p}\cdot\vec{x} - i\vec{p}\cdot\vec{y}} f(\vec{x})$$

$$= \int d^N x \, (2\pi)^{-N} \int d^N p \, e^{i\vec{p}\cdot\vec{x} - i\vec{p}\cdot\vec{y}} f(\vec{x})$$

$$\Rightarrow \int d^N p \, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = (2\pi)^N \delta(\vec{x}-\vec{y}) \quad \text{или също,}$$

$$\int d^N p \, e^{i\vec{p}\cdot\vec{x}} = (2\pi)^N \delta(\vec{x})$$

Регуляризация:

$$\int d^N p \, e^{i\vec{p}\cdot\vec{x} - \frac{1}{2}\lambda \vec{p}^2} = \int d^N p \, e^{-\frac{1}{2}\lambda\left(\vec{p} - \frac{\vec{x}}{\lambda}\right)^2} e^{-\frac{1}{2\lambda}\vec{x}^2}$$

$$= (2\pi)^{N/2} \underbrace{\lambda^{-N/2} e^{-\frac{1}{2\lambda}\vec{x}^2}}_{\text{при } \lambda \downarrow 0 \ \downarrow \text{ в } \mathcal{S}'}$$

$$\delta(\vec{x})$$

g) $\mathcal{F}(x_j f) = -i\partial_{p_j}\mathcal{F}(f)$, $\mathcal{F}(\partial_{x_j} f) = i p_j \mathcal{F}(f)$

за $f \in \mathcal{S}'(\mathbb{R}^N)$.

$$\mathcal{F}(f_1 \cdots f_n)(\vec{p}) = (2\pi)^{-\frac{N(n-1)}{2}} \int_{\vec{p}_1+\cdots+\vec{p}_n=\vec{p}} \mathcal{F}(f_1)(\vec{p}_1)\cdots\mathcal{F}(f_n)(\vec{p}_n).$$

$$\mathcal{F}(f^*)(\vec{p}) = \mathcal{F}(f)^*(-\vec{p})$$

2. Канонично квантуване на поле на Клайн-Гордон (скаларно поле).

Постановка: търсим операторно-значна функция $\varphi(\underline{x},t)$ такава, че

$$\left(\partial_t^2 - c^2\partial_{\underline{x}}^2 + \frac{m^2c^4}{\hbar^2}\right)\varphi(\underline{x},t) = -\frac{1}{3!}\lambda'\varphi(\underline{x},t)^3,$$

$$\varphi(\underline{x},t)^* = \varphi(\underline{x},t),$$

К.К.С. в началния момент $(t=0)$

$$[\varphi(\underline{x}_1,0), \partial_t\varphi(\underline{x}_2,0)] = i\hbar\,\delta(\underline{x}_1-\underline{x}_2)$$

$$[\varphi(\underline{x}_1,0), \varphi(\underline{x}_2,0)] = 0 = [\partial_t\varphi(\underline{x}_1,0), \partial_t\varphi(\underline{x}_2,0)]$$

$$\tilde{\varphi}(\underline{p},t) := (2\pi\hbar)^{-\frac{3}{2}}\int d^3x\, e^{\frac{i\underline{p}\cdot\underline{x}}{\hbar}}\varphi(\underline{x},t)$$

<u>Твърдение.</u> а) К.К.С. в $t=0$ $\Longleftrightarrow$

$$[\tilde\varphi(\underline{p}_1,0), \partial_t\tilde\varphi(\underline{p}_2,0)] = i\hbar\,\delta(\underline{p}_1+\underline{p}_2)$$

$$[\tilde\varphi(\underline{p}_1,0), \tilde\varphi(\underline{p}_2,0)] = 0 = [\partial_t\tilde\varphi(\underline{p}_1,0), \partial_t\tilde\varphi(\underline{p}_2,0)]$$

б) Полеви уравнения:

$$\left(\partial_t^2 + \frac{\mathcal{E}(\underline{p})^2}{\hbar^2}\right)\tilde\varphi(\underline{p},t) = -\frac{1}{3!}\lambda'\,\widetilde{(\varphi^3)}(\underline{p},t),$$

$$\widetilde{(\varphi^3)}(\underline{p},t) = (2\pi\hbar)^{-3}\int\limits_{\underline{p}_1+\underline{p}_2+\underline{p}_3=\underline{p}} \tilde\varphi(\underline{p}_1,t)\,\tilde\varphi(\underline{p}_2,t)\,\tilde\varphi(\underline{p}_3,t),$$

$$\mathcal{E}(\underline{p}) = (m^2c^4+\underline{p}^2c^2)^{1/2}$$

Проверка на а): $[\tilde\varphi(\underline{p}_1,0), \partial_t\tilde\varphi(\underline{p}_2,0)]$

$$= (2\pi\hbar)^{-3}\int d^3x_1\,d^3x_2\; e^{\frac{i}{\hbar}(\underline{p}_1\cdot\underline{x}_1+\underline{p}_2\underline{x}_2)}\; i\hbar\,\delta(\underline{x}_1-\underline{x}_2)$$

$$= i\hbar^{-2}(2\pi)^{-3}\int d^3x\; e^{\frac{i}{\hbar}\underline{x}\cdot(\underline{p}_1+\underline{p}_2)}\,\frac{\hbar^3}{\hbar^3}$$

$$= i\hbar\,(2\pi)^{-3}\int d^3x\; e^{i\underline{x}\cdot(\underline{p}_1+\underline{p}_2)}$$

$$= i\hbar\,\delta(\underline{p}_1+\underline{p}_2)$$

Решения при $\lambda = 0$

– континуална система от независими осцилатори

$$\tilde{\varphi}_0(\underline{p},t) = \frac{\hbar}{(2\varepsilon(\underline{p}))^{1/2}} \tilde{\Psi}(\underline{p})\, e^{-\frac{i}{\hbar}\varepsilon(\underline{p})t}$$
$$+ \frac{\hbar}{(2\varepsilon(\underline{p}))^{1/2}} \tilde{\Psi}(-\underline{p})^*\, e^{\frac{i}{\hbar}\varepsilon(\underline{p})t}$$

$$\{\tilde{\varphi}(\underline{p},0),\ \partial_t\tilde{\varphi}(\underline{p},0) \mid \underline{p}\} \leftrightarrow \{\tilde{\Psi}(\underline{p}),\ \tilde{\Psi}(\underline{p})^* \mid \underline{p}\}$$

<u>Твърдение.</u> К.К.С. в $t = 0$ $\Longleftrightarrow$

$$[\tilde{\Psi}(\underline{p}_1), \tilde{\Psi}(\underline{p}_2)^*] = \delta(\underline{p}_1 - \underline{p}_2),$$
$$[\tilde{\Psi}(\underline{p}_1), \tilde{\Psi}(\underline{p}_2)] = 0 = [\tilde{\Psi}(\underline{p}_1)^*, \tilde{\Psi}(\underline{p}_2)^*]. \qquad \square$$

<u>Твърдение.</u> $\exists$ самоспрегнат оператор H_0 такъв, че

$$\varphi_0(\underline{x},t) = e^{\frac{i}{\hbar}H_0 t}\, \varphi(\underline{x},0)\, e^{-\frac{i}{\hbar}H_0 t}$$

<u>Доказателство.</u> Ако положим:

$$H_0 := \int d^3\underline{p}\ \varepsilon(\underline{p})\, \tilde{\Psi}(\underline{p})^* \tilde{\Psi}(\underline{p})$$

$$\begin{cases} \frac{i}{\hbar}[H_0, \tilde{\varphi}(\underline{p},t)] = \partial_t \tilde{\varphi}(\underline{p},t) \\ \frac{i}{\hbar}[H_0, \partial_t\tilde{\varphi}(\underline{p},t)] = -\frac{\varepsilon(\underline{p})^2}{\hbar^2}\tilde{\varphi}(\underline{p},t). \end{cases} \qquad \square$$

<u>Твърдение.</u> $\exists$ самоспрегнат оператор $H = H_0 + H_1$,

$$\tilde{\varphi}(\underline{p}, t) = e^{\frac{i}{\hbar}Ht}\,\tilde{\varphi}(\underline{p}, 0)\,e^{-\frac{i}{\hbar}Ht},$$

$$\varphi(\underline{x}, t) = e^{\frac{i}{\hbar}Ht}\,\varphi(\underline{x}, 0)\,e^{-\frac{i}{\hbar}Ht}.$$

– за целта: $H_1 = \frac{\lambda'}{4!}\int d^3x\ \varphi(\underline{x}, 0)^4$. $\square$

Преход към $\varphi(\underline{x}, 0)$ в H_0:

$$\begin{aligned} H_0 &= \frac{1}{2}\int d^3p \Big(\partial_t \tilde{\varphi}(\underline{p}, 0)\, \partial_t \tilde{\varphi}(-\underline{p}, 0) \\ &\quad + \frac{\varepsilon(\underline{p})^2}{\hbar^2}\tilde{\varphi}(\underline{p}, 0)\,\tilde{\varphi}(-\underline{p}, 0)\Big) + \mathcal{E}_0 \\ &= \frac{1}{2}\int d^3x \Big(\big(\partial_t \varphi(\underline{x}, 0)\big)^2 \\ &\quad + c^2 \big(\partial_{\underline{x}} \varphi(\underline{x}, 0)\big)\cdot\big(\partial_{\underline{x}} \varphi(\underline{x}, 0)\big) \\ &\quad + \frac{m^2c^4}{\hbar^2}\varphi(\underline{x}, 0)^2\Big) + \mathcal{E}_0 \end{aligned}$$

$\mathcal{E}_0$ – константа идваща от пренареждане на оператори.

Всъщност, $\mathcal{E}_0 \sim \delta(0) = \infty$ (безсмислено).

– пропускаме я – първата пренормировка $\Leftrightarrow$

постулат за нормално произведение в H

$$H_1 = \frac{\lambda'}{4!}\int d^3x\ {:}\varphi(\underline{x}, 0)^4{:}$$

$\lambda' := \lambda c^3 \hbar^{-1} \Rightarrow [\lambda] = 1$.

Следствие:

$\forall$ формули за $\varphi(\underline{x}, 0)$ са верни и за $\varphi(\underline{x}, t)$.

Лагранжиан:

$$L = \frac{1}{2}\int d^3x \Big(\big(\partial_t \varphi(\underline{x},t)\big)^2 - c^2 \big(\partial_{\underline{x}} \varphi(\underline{x},t)\big)\cdot\big(\partial_{\underline{x}} \varphi(\underline{x},t)\big)$$
$$- \frac{m^2c^4}{\hbar^2}\varphi(\underline{x},t)^2 - \frac{\lambda'}{4!}\varphi(\underline{x},t)^4 \Big)$$

подобен на кинетичната енергия

подобни на потенциалната енергия – обръщат си знака

Действие: $S = \int dt\, L(t)$

$$S = -\frac{c}{2}\int d^4x \left((\partial_{x^\mu}\varphi)(\partial_{x_\mu}\varphi) + \frac{m^2c^2}{\hbar^2}\varphi^2 \right) - \frac{\lambda}{4!}\frac{c}{\hbar}\int d^4x\, \varphi^4$$

$$\equiv -\frac{1}{2}\int d^4x \left((\partial_x \varphi)^2 + m^2\varphi^2 \right) - \frac{\lambda}{4!}\int d^4x\, \varphi^4$$

– ако $c = \hbar = 1$.

3. Функции на Грийн (Green).

За свободното поле ($\lambda = 0$)

$$G_0(x_1, \dots, x_N) := \langle \Omega_0 | T(\varphi_0(x_1) \cdots \varphi_0(x_N)) \Omega_0 \rangle$$

За взаимодействащото поле:

$$G(x_1, \dots, x_N) := \langle \Omega | T(\varphi(x_1) \cdots \varphi(x_N)) \Omega \rangle$$

$H\Omega = 0$ – вакуум при взаимодействие

$H_0 \Omega_0 = 0$ – вакуум без взаимодействие

Ω_0 = вакуумният вектор в пространството на Фок.

$H_1 \Omega_0 \neq 0$ понеже H_1 съдържа $(\psi^*)^4$.

$\Rightarrow$ $\mathbb{C}\Omega \neq \mathbb{C}\Omega_0$ – различни състояния.

Например: ($\hbar = c = 1$)

$$G_0(x_1, x_2) = \langle \Omega_0 | T(\varphi_0(\underline{x}_1, t_1)\, \varphi_0(\underline{x}_2, t_2)) \Omega_0 \rangle$$

$$= (2\pi)^{-3} \int d^3p_1\, d^3p_2\; e^{i \underline{p}_1 \cdot \underline{x}_1 + i \underline{p}_2 \cdot \underline{x}_2}$$

$$\times \underbrace{\langle \Omega_0 | T(\tilde{\varphi}_0(\underline{p}_1, t_1)\, \tilde{\varphi}_0(\underline{p}_2, t_2)) \Omega_0 \rangle}_{\tilde{G}(\underline{p}_1, t_1;\, \underline{p}_2, t_2)}$$

Замествайки $\tilde{\varphi}(\underline{p},t)$ с $\tilde{\Psi}(\underline{p})$ и използвайки

$$\langle \Omega_0 | \tilde{\Psi}(\underline{p}_1) \tilde{\Psi}(\underline{p}_2)^* \Omega_0 \rangle = \delta(\underline{p}_1 - \underline{p}_2)$$

$$\langle \Omega_0 | \tilde{\Psi}(\underline{p}_1) \tilde{\Psi}(\underline{p}_2) \Omega_0 \rangle = 0 = \langle \Omega_0 | \tilde{\Psi}(\underline{p}_1)^* \tilde{\Psi}(\underline{p}_2)^* \Omega_0 \rangle$$

$$= \langle \Omega_0 | \tilde{\Psi}(\underline{p}_1)^* \tilde{\Psi}(\underline{p}_2) \Omega_0 \rangle$$

$$\Rightarrow \quad \tilde{G}(\underline{p}_1, t_1; \underline{p}_2, t_2)$$

$$= \frac{1}{2\varepsilon(\underline{p}_1)} e^{-i\varepsilon(\underline{p}_1)(t_1 - t_2)} \theta(t_1 - t_2)\, \delta(\underline{p}_1 + \underline{p}_2)$$

$$+ \frac{1}{2\varepsilon(\underline{p}_1)} e^{-i\varepsilon(\underline{p}_1)(t_2 - t_1)} \theta(t_2 - t_1)\, \delta(\underline{p}_1 + \underline{p}_2)$$

След допълнителна Фурие-трансформация по t_1 и t_2

$$\tilde{G}(p_1, p_2) \equiv \tilde{G}(p_{1,0}, \underline{p}_1; p_{2,0}, \underline{p}_2)$$

$$:= (2\pi)^{-1} \int dt_1\, dt_2\; e^{i p_{0,1} t_1 + i p_{0,2} t_2}$$

$$\times \tilde{G}(\underline{p}_1, t_1; \underline{p}_2, t_2)$$

се получава

$$\tilde{G}(p_1, p_2) = \mathrm{const} \cdot \frac{1}{-p_1^2 + m^2}\, \delta(p_1 + p_2)$$

Към x_1, x_2:

$$G(x_1, x_2) = (2\pi)^{-2} \int dp_1 dp_2 \, e^{-ip_1 \cdot x_1 - ip_2 \cdot x_2} \, \tilde{G}(p_1, p_2)$$

При маса $m = 0$:

$$G(x_1, x_2) = const \cdot \frac{1}{(x_1 - x_2)^2} \, , \text{ при } (x_1 - x_2)^2 \neq 0$$

– рационална функция. Като разпределение има особеност върху светлинния конус.

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