Вариационно смятане

1.) $L\Big(\underbrace{(u_A)_{A=1}^{N}}_{u},\ \underbrace{(u_{A,\mu})}_{(u_\mu)},\ \underbrace{(x_\mu)}_{x}{}_{\mu=1}^{D}\Big)$

$$u = \varphi(x) = (\varphi_A(x)) \qquad \frac{\partial}{\partial x^\mu}$$

$$S = \int d^D x \; L\left(\varphi(x), (\partial_{x_\mu}\varphi(x)), x\right)$$

$$\left(\varphi(x;\varepsilon)\right)_{\varepsilon\in(-a,a)} \text{ – вариация}$$

$$\frac{d}{d\varepsilon}\int d^D x \; L\left(\varphi(x;\varepsilon), (\partial_{x_\mu}\varphi), x\right) =$$

$$= \int d^D x \; \partial_\varepsilon\Big(L\left(\varphi(x;\varepsilon), (\partial_{x_\mu}\varphi), x\right)\Big)$$

$$\partial_\varepsilon \Big(L(\cdot\cdot) \Big) = \sum_A \partial_{u_A} L \cdot \partial_\varepsilon \varphi_A$$

$$+ \sum_{A,\mu} \partial_{u_{A,\mu}} L \cdot \partial_\varepsilon \partial_{x^\mu} \varphi_A$$

$$= \partial_{x^\mu} \Big(\sum_{A,\mu} \partial_{u_{A,\mu}} L(\cdots) \cdot \partial_\varepsilon \varphi_A \Big)$$

$$+ \sum_A \mathcal{E}_A (L) \Big(\varphi(x), (\partial_{x^\mu} \varphi), x \Big) \cdot \partial_\varepsilon \varphi_A(x)$$

$$\partial_{x^\mu} \Big(L \big(\varphi(x), (\partial_{x^\mu} \varphi), x \big) \Big) = \frac{\partial L}{\partial x^\mu} +$$

$$+ \sum_A \partial_{u_A} L \cdot u_{A,\mu} + \sum_{A,\nu} \partial_{u_{A,\nu}} L \cdot u_{A,\mu,\nu}$$

$$= \mathcal{D}_{x^\mu} L \Big|_{\substack{u_A = \varphi_A(x),\ u_{A,\mu} = \partial_{x^\mu} \varphi_A, \\ u_{A,\mu,\nu} = \partial_{x^\mu} \partial_{x^\nu} \varphi_A}}$$

$$\mathcal{E}_A(L)(u,(u_\mu),x)$$
$$= \partial_{u_A} L - \sum_\mu \mathcal{D}_{x^\mu}\left(\partial_{u_{A,\mu}} L\right)$$

$$\mathcal{J}_\mu(L)(u,(u_\mu),x;v) = \sum_A \partial_{u_{A,\mu}} L \cdot v_A$$

$$\partial_\varepsilon\Big(L\big(\varphi(x),(\partial_{x^\mu}\varphi),x\big)\Big)$$
$$= \sum_A \mathcal{E}_A(L)(\varphi,(\partial_{x_\mu}\varphi),x)\cdot\partial_\varepsilon\varphi_A(x)$$
$$+ \sum_\mu \partial_{x^\mu}\Big\{\mathcal{J}_\mu(L)\big(\varphi,(\partial_{x_\mu}\varphi),x;\partial_\varepsilon\varphi(x)\big)\Big\}$$

$$\int_W d^D x\ \partial_{x^\mu} F = \int_{\partial W} d\sigma\ n^\mu \cdot \partial_{x^\mu} F$$

Ако $\forall\ \varphi(x,\varepsilon)$, $\partial_\varepsilon\varphi\Big|_{\substack{\text{в околност}\\ \text{на } \partial W}} = 0$

$\varphi(x,0)=\varphi(x)$ и $\dfrac{d}{d\varepsilon}S = 0$

$$\Rightarrow\ \Big|\ \mathcal{E}_A(L)\big(\varphi(x),(\partial_{x^\mu}\varphi),x\big) = 0 \qquad (1.0.)$$

Разпределения / обобщени функции
distributions / generalized functions

Обобщават лин. функционали:

$$f(x) \longmapsto \int_{\mathbb{R}^D} d^D x \; F(x) \, f(x)$$

"тест функция"

тест функция = C^∞-функция, която = 0 извън ограничено множ. в $\mathbb{R}^D$

$\mathcal{D}(\mathbb{R}^D)$ = вект. пространство на $\forall$ тест функц.

$$f_i \xrightarrow[\text{в } \mathcal{D}]{} f \iff \begin{cases} \forall f_i \text{ и } f \text{ са } = 0 \\ \text{извън едно и също} \\ \text{ограничено множ.} \\ \partial^n f_i \rightrightarrows \partial^n f \\ \forall n \end{cases}$$

Опр. $\mathcal{D}(\mathbb{R}^D) \longrightarrow \mathbb{R}$ (или $\mathbb{C}$)

лин. непр. се нарича разпределение

Символичен запис:

$$f \longmapsto F(f) =: \int_{(\mathbb{R}^D)} d^D x \; F(x) f(x)$$

$\mathcal{D}'(\mathbb{R}^D)$ = вект. простр. на $\forall$ разпр.

Пример:

$$f \longmapsto \delta(f) = f(0) = \int d^D x \; \delta(x) f(x)$$

δ-функция на Дирак

$\forall$ $C(\mathbb{R}^D)$ - функция е разпределение

L^1_{loc} (симв. запис - действителен)

Ако F - разпределение, $W \underset{\text{отв.}}{\subseteq} \mathbb{R}^D$

$$F|_W = 0 \iff F(f) = 0$$

за $\forall f$ т.че $f|_{\mathbb{R}^D \setminus W} = 0$

$$\bigcup \{ W\text{ - отв.} \mid F|_W = 0 \} =: \mathbb{R}^D \setminus \operatorname{supp} F \leftarrow$$

носител
support
- затворено множ.

$$\operatorname{supp} \delta(x) = \{0\}$$

$$F(x) \in C(\mathbb{R}^D) \ . \quad \operatorname{supp} F = \overline{F^{-1}(\mathbb{R}^D \setminus \{0\})}$$

Операции над разпределения

0) Лин. комбинации (векторно прост.)

1) Ако $h \in C^\infty(\mathbb{R}^D)$, $F \in \mathcal{D}'(\mathbb{R}^D)$

то $(h \cdot F)(f) := F(hf) \leftarrow$ ще бъде $\mathcal{D}$-непр. по f

тест функц.

Пример: $(h\,\delta)(f) = \delta(hf) = h(0) f(0)$
$= h(0) \cdot \delta(f) \quad \forall f$

$\Rightarrow \; h\delta = h(0)\,\delta$ или $h(x)\,\delta(x) = h(0)\,\delta(x)$.

2.) Ако $F \in \mathcal{D}'(\mathbb{R}^D)$

$$(\partial_{x^\mu} F)(f) := -F(\partial_{x^\mu} f)$$

ще бъде $\mathcal{D}$-непр. по f

тест функц.

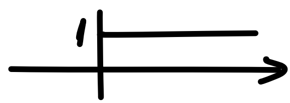
Символично:

$$\int d^D x \; (\partial_{x^\mu} F)\, f = -\int d^D x \; F\, \partial_{x^\mu} f$$

– "интегриране по части"

Съгласувано за $F \in C^1(\mathbb{R}^D)$, понеже $f|_\infty = 0$ и няма гранич. членове

Пример: $\theta(x) \in \mathcal{D}'(\mathbb{R})$



$$\theta(f) = \int_{\mathbb{R}} dx \; \theta(x)\, f(x) = \int_0^\infty dx \; f(x)$$

<u>Тв.</u> $\partial_x \theta(x) = \delta(x)$

$$(\partial_x \theta)(f) = -\int_0^\infty dx \; \partial_x f(x) = -f(x)\Big|_0^\infty$$

$$= f(0) - \underbrace{f(\infty)}_{0} = f(0) = \delta(f).$$

Пример: $\left(\partial_{x^\mu}\delta\right)(f) = -\partial_{x^\mu} f(0)$

Свойство: $\operatorname{supp} \partial_{x^\mu} F \subseteq \operatorname{supp} F$

$\operatorname{supp} h \cdot F \subseteq \operatorname{supp} F$

<u>Теорема</u> Ако $F \in \mathcal{D}'(\mathbb{R}^D)$ т.че $\operatorname{supp} F = \{0\}$, то

$$F = \sum_{n=0}^{N} \sum_{\mu_1 \cdots \mu_n} \underset{\substack{\cap \\ \mathbb{R}}}{c_{\mu_1, \dots, \mu_n}} \partial_{x^{\mu_1}} \cdots \partial_{x^{\mu_n}} \delta$$

Проблем с умножение на разпр.

$$\left(\mathcal{P}\frac{1}{x}\right)(f) := \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} dx \, \frac{1}{x} f(x)$$

$-\ \exists$ за тест функц. и е $\mathcal{D}$-непр. по f

т.че $\mathcal{P}\frac{1}{x} \in \mathcal{D}'(\mathbb{R})$

$$\left(x \cdot \mathcal{P}\left(\frac{1}{x}\right)\right)(f) = \int dx\, f(x) = 1(f)$$

т.е. $x \cdot \mathcal{P}\left(\frac{1}{x}\right) = 1$

$$\underbrace{\left(\delta(x) \cdot x\right)}_{0} \cdot \mathcal{P}\left(\frac{1}{x}\right) \neq \delta(x) \cdot \left(\underbrace{x \cdot \mathcal{P}\left(\frac{1}{x}\right)}_{1}\right)$$

До противоречия водят и изрази като $\delta(x)^2$, $\theta(x)\,\delta(x)$ и т.н.

Операция 3) Субституция

$F(g(x))$, $g: \mathbb{R}^D \to \mathbb{R}^D$ гладко и обратимо

$$\int d^D x \; F(g(x))\, f(x) :=$$

$$= \int d^D x \; \left|\frac{\partial g^{-1}}{\partial x}\right| F(x)\, f(g^{-1}(x))$$

$$(F \circ g)(f) := F\left(\left|\frac{\partial g^{-1}}{\partial x}\right| f(g^{-1}(x))\right)$$

тест функц.

4) Ако supp F – ограничено и $h \in C^{\infty}(\mathbb{R}^D)$

$F(h) := F(f)$ за $f \in \mathcal{D}(\mathbb{R}^D)$ т.че

$$h\big|_{\operatorname{supp} F} = f\big|_{\operatorname{supp} F}$$

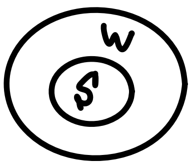
$$f = \chi h \quad \text{при} \quad S = \operatorname{supp} F$$

Основен инструмент:

<u>Теорема</u> Ако $S \underset{\text{комп.}}{\subseteq} \mathbb{R}^D$, $W \underset{\substack{\text{отв.}\\ \text{огр.}}}{\subseteq} \mathbb{R}^D$

$S \subset W$

то $\exists\, \chi \in \mathcal{D}(\mathbb{R}^D)$



$$\chi\big|_S = 1 \,, \quad \chi\big|_{\mathbb{R}^D \setminus \overline{W}} = 0 .$$

Приложение: ако $F(x) \in \mathcal{D}'(\mathbb{R}^D)$

и supp F – комп., то

$$F(x-y) \in \mathcal{D}'(\mathbb{R}^D \times \mathbb{R}^D)$$

$$\int d^Dx\, d^Dy \;\; F(x-y)\, f(x,y)$$

$$= \int d^Dx\, d^Dy \;\; F(x)\, f(x+y, y)$$

$$= \int d^D x \; F(x) \underbrace{\int d^D y \; f(x+y, y)}_{h(x) \in C^\infty(\mathbb{R}^D)}$$

$$= F(h)$$

отново се проверява $\mathcal{D}$-непр. по f.

Частично насищане $F(x,y)$ - разир. по x, y

$\int d^D y \; F(x,y) f(y)$ - разир. по x :

$$\int d^D y \, d^D x \; F(x,y) f(y) f_1(x)$$

$\delta(x-y)$ - обобщава $\delta_{j,k}$

$$\sum_j \delta_{j,k} f_k = f_j$$

$$\int d^D x \; \delta(x-y) f(y) = f(x)$$

$$= \int d^D x \; \delta(y-x) f(y)$$

Операция 5 - тенз. произведение $F_1 \otimes F_2$

$$\int d^D x_1 \, d^D x_2 \; F_1(x_1) F_2(x_2) \; f_1(x_1) f_2(x_2)$$

$$:= \left(\int d^D x_1 \; F_1 f_1 \right) \left(d^D x_2 \; F_2 f_2 \right)$$

Операция б) Конволюция

Нека supp F_1, supp F_2 - комп.

$$\int d^D y \; F_1(x-y)\, F_2(y-z)$$

- определено

$$\int d^D x\, d^D z \left(\int d^D y\, F_1(x-y)\, F_2(y-z) \right) f(x,z)$$

$$=: \int d^D x' d^D y'\, F_1(x')\, F_2(y') \int d^D z\, f(x'+y'+z, z)$$

$$\operatorname{supp} F_1 \otimes F_2 \subseteq \operatorname{supp} F_1 \times \operatorname{supp} F_2$$

- комп.

$$\int d^D y \; \delta(x-y)\,\delta(y-z) = \delta(x-z)$$

Вариац. смятане

$$\frac{\partial x_\mu}{\partial x_\nu} = \delta_{\mu,\nu} \rightsquigarrow \frac{\delta \varphi(x)}{\delta \varphi(y)} = \delta(x-y)$$

$$\frac{\delta\, \partial_{x^\mu} \varphi(x)}{\delta \varphi(y)} = \partial_{x^\mu}\, \delta(x-y).$$

За $\mathcal{F}\{\varphi(x)\}$: $\dfrac{\delta \mathcal{F}}{\delta \varphi(x)} \in \mathcal{D}'(\mathbb{R}^D)$,

$$\frac{\delta \mathcal{F}}{\delta \varphi(x)}(f) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}\{\varphi(x) + \varepsilon f(x)\}$$

$$\frac{\delta}{\delta \varphi(x)} \int d^D y \; L\left(\varphi(y), (\partial_{x_\mu}\varphi), y\right)$$

$$= \int d^D y \; \frac{\delta L\left(\varphi(y), (\partial_{x_\mu}\varphi), y\right)}{\delta \varphi(x)}$$

$$= \int d^D y \left(\frac{\partial L}{\partial u}(y)\, \frac{\delta \varphi(y)}{\delta \varphi(x)} + \frac{\partial L}{\partial u_\mu}(y)\, \frac{\delta\, \partial_{x_\mu} \varphi(y)}{\delta \varphi(x)} \right)$$

$$= \int d^D y \left(\frac{\partial L}{\partial u}(y)\, \delta(x-y) + \frac{\partial L}{\partial u_\mu}\, \partial_{y^\mu} \delta(x-y) \right)$$

$$= \int d^D y \left(\frac{\partial L}{\partial u} - \partial_{y^\mu} \frac{\partial L}{\partial u_\mu} \right) \delta(x-y)$$

$$= \mathcal{E}(L)(x)$$

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