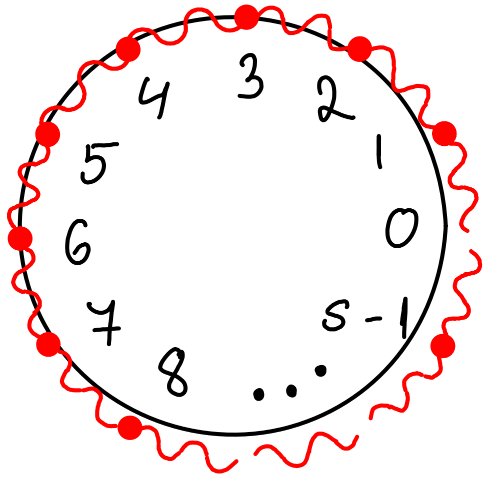
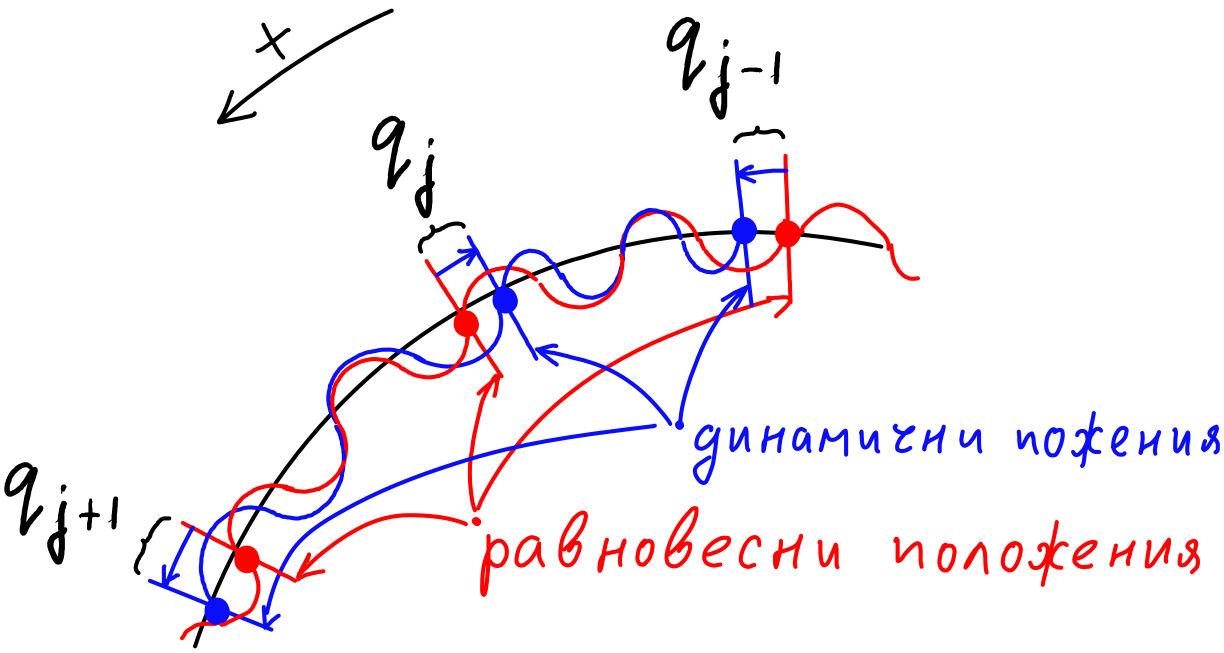
Квантуване на модела на трептящ пръстен, теория на пертурбациите и диаграми на Файнман.



1. Конфигур. пр-во, Фазово пр. и коорд.



$$q_j, \ j \in \{0, \dots, s-1\}, \quad \dot{q}_j = \frac{dq_j}{dt}$$

индексното пр-во $\mathbb{Z}_s = \mathbb{Z}/s\mathbb{Z}$

- цели числа по модул s - пръстен

Пример $\mathbb{Z}_3$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

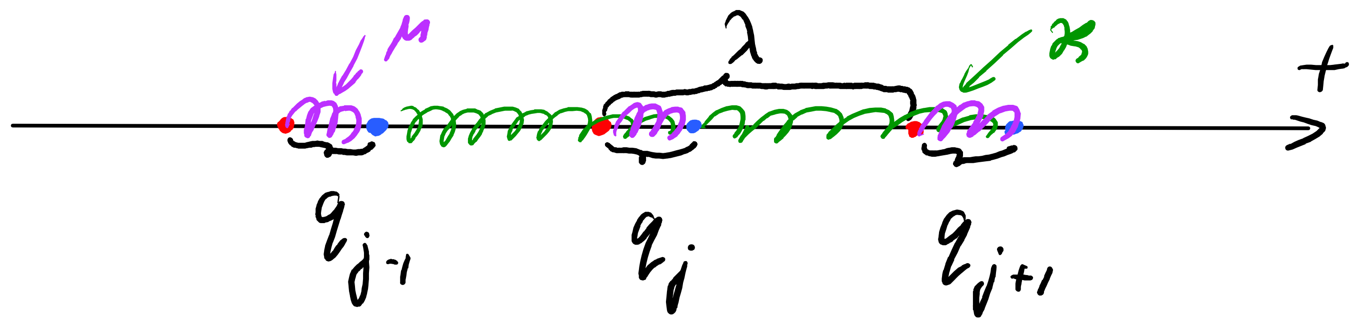
При непрекъснатата граница / continuum limit

$$\left.\begin{aligned} \mathbb{Z}_s &\longmapsto \mathbb{R} \\ \mathbb{Z}_s^3 &\longmapsto \mathbb{R}^3 \end{aligned}\right\} \text{пр-вото}$$

2. Енергия (хамилтониан) и лагранжиан

$$H = E_{kin} + E_{pot}$$

$$E_{kin} = \sum_{j \in \mathbb{Z}_s} \frac{1}{2} m \dot{q}_j^2$$



$$E_{pot} = \sum_{j\in\mathbb{Z}_s} \frac{1}{2}\kappa\left(\lambda - q_j + q_{j+1}\right)^2 \leftarrow \text{между съседи}$$

$$+ \sum_{j\in\mathbb{Z}_s} \frac{1}{2!}\mu\, q_j^2 \leftarrow \text{енергия на еласт. спрямо центровете}$$

$$+ \sum_{j\in\mathbb{Z}_s} \frac{1}{4!} g' \, q_j^4 \; - \text{анхармонична добавка}$$

q_j^3 – добавките биха нарушили положителността на E_{pot}

$\Rightarrow$ нарушаване на устойчивостта на равновесната точка $q_j = 0$.

$$E_{pot} = \frac{1}{2}\varkappa \sum_{j\in\mathbb{Z}_s} \left(\lambda^2 - 2\lambda(q_j - q_{j+1}) + (q_j - q_{j+1})^2\right)$$

+ остальные

$$= \frac{1}{2}\varkappa s \lambda^2 - \frac{\cancel{2}}{\cancel{2}}\lambda\varkappa \cancel{\sum_{j\in\mathbb{Z}_s}(q_j - q_{j+1})}$$

(при $j = s-1$, $j+1 = s = 0 \mod s$)

$$+ \frac{1}{2}\varkappa \sum_{j\in\mathbb{Z}_s}(q_j - q_{j+1})^2 + \ldots$$

$$H = \sum_{j\in\mathbb{Z}_s} \frac{m}{2}\dot{q}_j^2 + \sum_{j\in\mathbb{Z}_s}\frac{\varkappa}{2}(q_j - q_{j+1})^2$$

$$+ \sum_{j\in\mathbb{Z}_s}\frac{\mu}{2}q_j^2 + \sum_{j\in\mathbb{Z}_s}\frac{g'}{4!}q_j^4$$

$$+ \cancel{\text{const}}$$

$$\mathcal{L} = E_{kin} - E_{pot}$$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m \dot{q}_j$$

$$H = \sum_{j \in \mathbb{Z}_s} \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}$$

$$\{p_j, q_{j'}\} = \delta_{j,j'} = \begin{cases} 1 & j = j' \\ 0 & j \neq j' \end{cases}$$

3. Уравнения за движение

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \frac{\partial E_{kin}}{\partial \dot{q}_j} = \frac{1}{m} p_j$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = -\frac{\partial E_{pot}}{\partial q_j}$$

$$= -\varkappa \frac{\cancel{2}}{\cancel{2}} \left(q_j - q_{j+1} - (q_{j-1} - q_j) \right)$$

$$-\mu \frac{2}{2} q_i - \frac{1}{\cancel{4!}} g' \cancel{4}\, q_i^3$$

$3!$

$$\left| \begin{aligned} m\ddot{q}_j &= -\varkappa\left(2q_j - q_{j+1} - q_{j-1}\right) \\ &\quad - \mu q_j - \frac{1}{3!} g' q_j^3 \\ & j \in \mathbb{Z}_s \end{aligned} \right.$$

4. Диагонализация

$$\ddot{\tilde{q}}_j = -\omega_j \tilde{q}_j - \dots$$

членове от по-висока степен по $\tilde{q}_i$

$$\tilde{q}_i = ?$$

$$\tilde{q}_k := \frac{1}{\sqrt{s}} \sum_{j \in \mathbb{Z}_s} e^{2\pi i \frac{jk}{s}} q_j \leftarrow \text{Дискр. Ф.П.}$$

$$k \in \mathbb{Z}_s = \mathbb{Z}/_{s\mathbb{Z}} = \{0, 1, \dots, s-1\}$$

⚠

$$e^{2\pi i \frac{(j+ns)(k+ms)}{s}} = e^{2\pi i \frac{jk}{s}} \cdot \underbrace{e^{2\pi i N \frac{\cancel{s}}{\cancel{s}}}}_{=1}$$

⚠

$$\underbrace{\sum_{j \in \mathbb{Z}_s} e^{2\pi i \frac{j \cdot k}{s}}}_{\substack{\text{геометрическая прогрессия} \\ \text{geometric series}}} = s\,\delta_{k,0} = \begin{cases} s \cdot 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$1 + \xi + \dots + \xi^{s-1} = \frac{1 - \xi^s}{1 - \xi}$$

$$\Rightarrow \quad q_j = \frac{1}{\sqrt{s}} \sum_{k \in \mathbb{Z}_s} e^{-2\pi i \frac{jk}{s}} \tilde{q}_k$$

проверка: r.h.s =

$$= \frac{1}{\sqrt{s}} \sum_{k \in \mathbb{Z}_s} e^{-2\pi i \frac{jk}{s}} \frac{1}{\sqrt{s}} \sum_{j' \in \mathbb{Z}_s} e^{2\pi i \frac{j'k}{s}} q_{j'}$$

$$= \frac{1}{\cancel{s}} \sum_{j' \in \mathbb{Z}_s} \underbrace{\sum_{k \in \mathbb{Z}_s} e^{2\pi i (j'-j)k/s}}_{\cancel{s}\,\delta_{j'-j,0} \,=\, \cancel{s}\,\delta_{j',j}} q_{j'} = q_j$$

ok.

Условието за реалност: $q_j^* = q_j \in \mathbb{R}$

$$\Longleftrightarrow \quad \tilde{q}_k^* = \tilde{q}_{-k}$$

Пб.

$$\tilde{q}_k^* = \frac{1}{\sqrt{s}} \left(\sum_{j \in \mathbb{Z}_s} e^{2\pi i \frac{jk}{s}} q_j \right)^*$$

$$e^{-2\pi i \frac{jk}{s}} \qquad q_j^* = q_j$$

$$= \frac{1}{\sqrt{s}} \sum_{j \in \mathbb{Z}_s} e^{2\pi i \, j(-k)/s} q_j = \tilde{q}_{-k}$$

Ако изх $\tilde{q}_k^* = \tilde{q}_{-k}$

$$q_j^* = \frac{1}{\sqrt{s}} \left(\sum_{k \in \mathbb{Z}_s} e^{-2\pi i \frac{jk}{s}} \tilde{q}_k \right)^* = \cdots = q_j$$

$$e^{2\pi i \frac{jk}{s}} \qquad \tilde{q}_{-k}$$

Скоби на Поасон:

$$\{\tilde{p}_k, \tilde{q}_{k'}\} = ?$$

$$\tilde{p}_k = \frac{1}{\sqrt{s}} \sum_{j \in \mathbb{Z}_s} e^{2\pi i \frac{jk}{s}} p_j$$

$$\{\tilde{p}_k, \tilde{q}_{k'}\} = \frac{1}{\sqrt{s}}\frac{1}{\sqrt{s}} \cdot$$

$$\cdot \underbrace{\sum_{j\in\mathbb{Z}_s}\sum_{j'\in\mathbb{Z}_s}}_{\sum_{j\in\mathbb{Z}_s},\ j'=j} \underbrace{e^{2\pi i\left(\frac{jk}{s}+\frac{j'k'}{s}\right)}}_{= e^{2\pi i\frac{jk}{s}}\, e^{2\pi i\frac{j'k'}{s}}} \underbrace{\{p_j, q_{j'}\}}_{\delta_{j,j'}}$$

$$= \frac{1}{\cancel{s}} \underbrace{\sum_{j\in\mathbb{Z}_s} e^{2\pi i\frac{j(k+k')}{s}} \cdot 1}_{\cancel{s}\,\delta_{k+k',0} \;=\; \cancel{s}\,\delta_{k,-k'}}$$

$$\Rightarrow \{\tilde{p}_k, \tilde{q}_{k'}\} = \delta_{k,-k'}$$

Координатите $\tilde{q}_k, \tilde{p}_k$ – импулсно представяне

$$H = ? \quad , \quad E_{kin} = \frac{m}{2} \sum_{j \in \mathbb{Z}_s} \dot{q}_j^2$$

$$= \frac{1}{2m} \sum_{j \in \mathbb{Z}_s} p_j^2 = \frac{1}{2m} \sum_{j \in \mathbb{Z}_s} \frac{1}{\sqrt{s}\sqrt{s}} \cdot$$

$$\cdot \sum_{k \in \mathbb{Z}_s} \sum_{k' \in \mathbb{Z}_s} e^{2\pi i \left(\frac{jk}{s} + \frac{jk'}{s}\right)} \tilde{p}_k \tilde{p}_{k'}$$

$$= \frac{1}{2m\cancel{s}} \cdot \sum_{\substack{k \in \mathbb{Z}_s \\ k' \in \mathbb{Z}_s}} \cdot \underbrace{\sum_{j \in \mathbb{Z}_s} e^{2\pi i \frac{j(k+k')}{s}}}_{\cancel{s}\,\delta_{k+k',0} = \cancel{s}\,\delta_{k,-k'}} \tilde{p}_k \tilde{p}_{k'}$$

$$= \frac{1}{2m} \sum_{k \in \mathbb{Z}_s} \tilde{p}_k \underbrace{\tilde{p}_{-k}}_{\tilde{p}_k^*} = \frac{1}{2m} \sum_{k \in \mathbb{Z}_s} |p_k|^2$$

$$E_{pot,0} = ?$$

harmonischer Teil

$$= \frac{\kappa}{2} \sum_{j\in\mathbb{Z}_s} \left(\cancel{2} q_j - q_{j+1} - \cancel{q_{j-1}}\right)^2$$

$$+ \frac{\mu}{2} \sum_{j\in\mathbb{Z}_s} q_j^2 = \frac{\mu}{2} \sum_{k\in\mathbb{Z}_s} |\tilde{q}_k|^2$$

$$= \frac{\kappa}{2} \sum_{j\in\mathbb{Z}_s} \frac{1}{\sqrt{s}\sqrt{s}} \sum_{k\in\mathbb{Z}_s} \sum_{k'\in\mathbb{Z}_s}$$

$$\cdot \left(\cancel{2} e^{2\pi i \frac{jk}{s}} \tilde{q}_k - e^{2\pi i \frac{(j+1)k}{s}} \tilde{q}_k - \cancel{e^{2\pi i \frac{(j-1)k}{s}} \tilde{q}_k}\right)$$

$$\cdot \left(\cancel{2} e^{2\pi i \frac{jk'}{s}} \tilde{q}_{k'} - e^{2\pi i \frac{(j+1)k'}{s}} \tilde{q}_{k'} - \cancel{e^{2\pi i \frac{(j-1)k'}{s}} \tilde{q}_{k'}}\right)$$

$$\| $$

$$e^{2\pi i \frac{jk'}{s}} \; e^{2\pi i \frac{k'}{s}}$$

$$= \frac{\varkappa}{2} \sum_{k,k' \in \mathbb{Z}_s} \frac{1}{s} \sum_{j \in \mathbb{Z}_s} e^{2\pi i \left(\frac{jk}{s} + \frac{jk'}{s}\right)}$$

$$\underbrace{\cdot \left(\cancel{2}\,1 - e^{2\pi i k/s} \cancel{- e^{-2\pi i k/s}}\right) \cdot \left(\cancel{2}\,1 - e^{2\pi i k'/s} \cancel{- e^{-2\pi i k'/s}}\right)}_{\overset{!!}{=}\ \omega'^2_k} \cdot \tilde{q}_k \tilde{q}_{k'}$$

при $k = k'$
корекция на
1:07:39

$$= \frac{\varkappa}{2} \sum_{k,k' \in \mathbb{Z}_s} \frac{1}{s} \underbrace{\sum_{j \in \mathbb{Z}_s} e^{2\pi i j (k+k')/s}}_{= \cancel{s}\,\delta_{k+k',0} = \cancel{s}\,\delta_{k,-k'}}$$

$$\cdot\, \omega'^2_k \cdot \tilde{q}_k \tilde{q}_{k'}$$

$$= \frac{\varkappa\, \omega'^2_k}{2} \sum_{k \in \mathbb{Z}_s} |\tilde{q}_k|^2$$

$H_0 = E_{kin} + E_{pot,0}$ – своб. хамилтониан

$$H_0 = \frac{1}{2m} \sum_{k \in \mathbb{Z}_s} |\tilde{p}_k|^2$$

$$+ \sum_{k \in \mathbb{Z}_s} \frac{1}{2} \omega_k''^2 |\tilde{q}_k|^2$$

$$\dot{\tilde{q}}_k = \frac{1}{m} \tilde{p}_k$$

$$\dot{\tilde{p}}_k = -\omega_k''^2 \tilde{p}_k$$

$$\ddot{\tilde{q}}_k = -\underbrace{\frac{\omega_k''^2}{m}}_{= \omega_k^2} \tilde{q}_k$$

$$\omega_k''^2 = \omega_k^2 m$$

$$\omega_k''^2 = \mu + \omega_k'^2 = \cdots$$

$$\omega_k = \sqrt{\frac{8\varkappa}{m}\sin^2\frac{\pi k}{s} + \frac{\mu}{m}}$$

$$= \omega_{-k}$$

5. Величините, които са класич. аналог на операторите на раждане и унищожение

$$a_k, a_k^* = ?$$

$$a_k = \alpha_k \tilde{q}_k + \beta_k \dot{\tilde{q}}_k$$

$$a_k^* \underset{\substack{\uparrow \\ \text{спрегнат}}}{=} (a_k)^* = \bar{\alpha}_k \tilde{q}_{-k} + \bar{\beta}_k \dot{\tilde{q}}_{-k}$$

$\alpha_k, \beta_k = ?$ т.че:

- $$\{a_k, a_{k'}^*\} = i\delta_{k,k'}$$

$$\{a_k, a_{k'}\} = 0 = \{a_k^*, a_{k'}^*\}$$

$$H_0 = \sum_{k\in\mathbb{Z}_s} \varepsilon_k a_k^* a_k$$

$$\{\dot{\tilde{q}}_k, \tilde{q}_k\} = \frac{1}{m}\{\tilde{p}_k, \tilde{q}_{k'}\} = \frac{1}{m}\delta_{k,k'}$$

$\Longrightarrow$

$$\bar{\alpha}_k \beta_k - \alpha_{-k}\bar{\beta}_{-k} = im$$

$$\bar{\alpha}_k \beta_k + \alpha_{-k}\beta_{-k} = 0$$

$$\varepsilon(k)\,|\alpha_k|^2 = \frac{m}{2}\omega_k^2$$

$$\varepsilon(k)\,|\beta_k|^2 = \frac{m}{2}$$

Заб. $A = A^*$, $B = B^*$

$$[A, B]^* = [B^*, A^*] = -[A, B]$$

$$[p, q] = i\hbar$$

$$[a, a^*] = 1$$

$$\Rightarrow \quad a_k := \sqrt{\frac{m\omega_k}{2}}\,\tilde{q}_k + i\sqrt{\frac{m}{2\omega_k}}\,\dot{\tilde{q}}_k$$

$$a_k^* := \sqrt{\frac{m\omega_k}{2}}\,\tilde{q}_{-k} - i\sqrt{\frac{m}{2\omega_k}}\,\dot{\tilde{q}}_{-k}$$

проверава се непосредствено

- $\{a_k, a_{k'}^*\} = i\,\delta_{k,k'}$

  $\{a_k, a_{k'}\} = 0 = \{a_k^*, a_{k'}^*\}$

- $H_0 = \sum_{k\in\mathbb{Z}_s} \omega_k\, a_k^* a_k$

6. Преход към "полеви означения"

$j \mapsto x$ – координата

$k \mapsto p$ – импулс

$q_j(t) \mapsto \varphi(x,t)$

$p_k(t) \mapsto \pi(x,t)$

$x \in \{0, 1.\lambda, 2.\lambda, \dots, (s-1)\lambda\} = R$ – решетка

$p \in \{0, 1.\tilde{\lambda}, 2.\tilde{\lambda}, \dots, (s-1)\tilde{\lambda}\} =: \tilde{R}$ – дуална реш.

$s\lambda = \Lambda$ – дължината на цялата R-реш.

$s\tilde{\lambda} = \tilde{\Lambda}$ – ——— / ——— дуална

$$\tilde{\lambda} := \frac{\hbar}{\Lambda} = \frac{\hbar}{s\lambda}, \quad \lambda\tilde{\lambda} = \frac{\hbar}{s}$$

→ виж още на следващата стр. !

Фурие транс.

$$\{f_j\}_{j=0}^{s-1} \mapsto \{\tilde{f}_k\}_{k=0}^{s-1}$$

Допълнителна страница

фактически

нови означения	стари означ.
$\varphi(x,t)$	$:= q_j(t)$

$$\varphi(x,t) := q_j(t)$$

$$\|$$

за $\quad j\lambda \,, \quad j \in \{0,1,\dots,s-1\}$

$$\|$$

$$\pi(x,t) := p_j(t)$$

$$\tilde{\varphi}(p,t) := \tilde{q}_k(t)$$

$$\|$$

за $\quad k\tilde{\lambda} \,, \quad k \in \{0,1,\dots,s-1\}$

$$\|$$

$$\tilde{\pi}(p,t) := \tilde{p}_k(t)$$

$$\tilde{f}(p) = \frac{1}{\sqrt{s}} \sum_{x \in R} \underbrace{e^{2\pi i \frac{x \cdot p}{\hbar}}}_{\substack{\parallel \\ e^{2\pi i \frac{j\lambda \cdot k\tilde{\lambda}}{\hbar}}}} f(x)$$

$$e^{2\pi i \frac{j\lambda \cdot k\tilde{\lambda}}{\hbar}} = e^{2\pi i \frac{jk}{s}}$$

$$s = \frac{\Lambda}{\lambda}$$

$$\tilde{f}(p) = \sqrt{\frac{\lambda}{\Lambda}} \sum_{x \in R} e^{2\pi i \frac{xp}{\hbar}} f(x)$$

$$\frac{1}{\sqrt{\tilde{\lambda}}} \tilde{f}(p)$$

$$= \frac{1}{\sqrt{\hbar}} \underbrace{\sum_{x \in R} e^{2\pi i \frac{xp}{\hbar}} \frac{1}{\sqrt{\lambda}} f(x) \underbrace{\Delta x}_{\lambda}}_{\text{риманове сума}}$$

$$\sum_{x\in R} e^{2\pi i \frac{xp}{\hbar}} = s\,\underbrace{\delta(p)}$$

"дискр
δ-функц."

$$\delta(p) = \delta_p = \begin{cases} 1 & p=0 \\ 0 & p\neq 0 \end{cases}$$

С подходящо преумножаване

$$\underset{j\lambda}{\underset{\parallel}{\varphi(x},t)} = K(m,\lambda,\mu,\kappa,\hbar)\cdot q_j$$

Уравненията за движение
на дискр. ур-е на Клайн–Гордън
Klein – Gordon

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 \varphi(x,t) - \left(\Delta_x^{dis}\varphi\right)(x,t)$$

$$+ \frac{Mc^2}{\hbar^2}\varphi(x,t) = \frac{1}{3!}\, g\, \varphi(x,t)^3$$

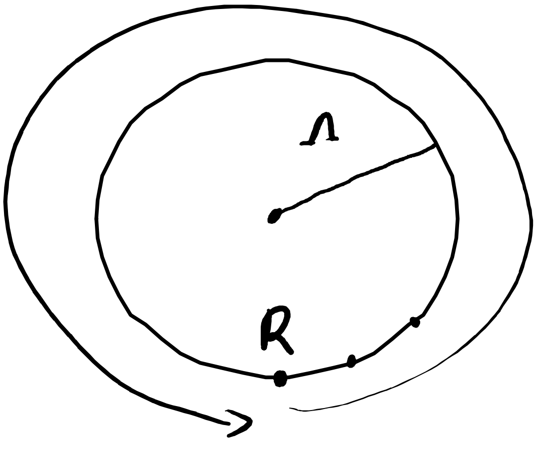
където g, M, c са функции на $\lambda, \kappa, \mu, \hbar, g', m$

число!

$$\Delta_x^{dis} := \frac{1}{\lambda^2}\Big(\varphi(x_+,t)+\varphi(x_-,t)-2\varphi(x,t)\Big)$$

$$x_\pm := x \pm \lambda \mod R$$

$$x = j\lambda \quad , \quad x_\pm = (j \pm 1 \mod s)\,\lambda$$



В исторически началната постановка на К.Т.П.

Квантовите полета са оператори решения на Ч.Д.У.

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 \hat{\varphi}(x,t) - \left(\Delta^{dis}_{x} \hat{\varphi}\right)(x,t)$$

$$+ \frac{Mc^2}{\hbar^2}\hat{\varphi}(x,t) = \frac{1}{3!}\, g\, \hat{\varphi}(x,t)^3$$

със задача на Коши / Cauchy

$$\left[\hat{\varphi}(x,t), \partial_t \hat{\varphi}(x',t)\right] = i\hbar\, \delta(x-x')$$

$$\left[\hat{\varphi}(x,t), \hat{\varphi}(x',t)\right] = 0$$

$$= \left[\partial_t \hat{\varphi}(x,t), \partial_t \hat{\varphi}(x',t)\right]$$

времената са равни

ККС при равни времена

Швебер (Schweber, QFT)

Една от стратегиите е по теория пертурбациите по γ

$$\hat{\varphi}(x,t) := \hat{\varphi}_\gamma(x,t)$$

$$= \hat{\varphi}_0(x,t) + \gamma^1 \varphi_1(x,t) + \cdots$$

$$\hbar = c = 1$$

$$\partial_t^2 \hat{\varphi}_\gamma - \Delta_x^{dis} \hat{\varphi}_\gamma + M^2 \hat{\varphi}_\gamma = \frac{1}{3!} \gamma \hat{\varphi}_\gamma^3$$

$$\partial_t^2 \sum_{n=0}^{\infty} \gamma^n \hat{\varphi}_n - \Delta_x^{dis} \sum_{n=0}^{\infty} \gamma^n \hat{\varphi}_n + M^2 \sum_{n=0}^{\infty} \gamma^n \varphi_n$$

$$= \frac{1}{3!} \sum_{n_1, n_2, n_3 = 0}^{\infty} \gamma^{n_1+n_2+n_3+1} \hat{\varphi}_{n_1} \hat{\varphi}_{n_2} \hat{\varphi}_{n_3}$$

Сравняваме ред по ред.

$n = 0$

$$\partial_t^2 \hat{\varphi}_0 - \Delta_x^{dis} \hat{\varphi}_0 + M^2 \hat{\varphi}_0 = 0$$

— (свободното) ур-е на Клайн-Гордон

$n = 1$

$$\partial_t^2 \hat{\varphi}_1 - \Delta_x^{dis} \hat{\varphi}_1 + M^2 \hat{\varphi}_1 = \underbrace{\frac{1}{3!} \hat{\varphi}_0(x,t)^3}_{\text{вече е решено в теорията на пертурб.}}$$

За $\hat{\varphi}_1$ задачата е нехомогенно ЧДУ. (хиперболично)

с дясна страна (източник) определено от $\hat{\varphi}_0$

Общ случай:

$$\partial_t^2 \hat{\varphi}_n - \Delta_x^{dis} \hat{\varphi}_n + M^2 \hat{\varphi}_n(x,t)$$

$$= \frac{1}{3!} \underbrace{J_n\left(\hat{\varphi}_0(x,t), \ldots, \hat{\varphi}_{n-1}(x,t) \right)}_{\substack{\text{полиномиални функции} \\ \text{от вече определени решения}}}$$

$\hookrightarrow$ решението води до формулата с T-exp.