Вариационно смятане

1.) $$L\left(\underbrace{(u_A)_{A=1}^{N}}_{u},\ \underbrace{(u_{A,\mu})}_{(u_\mu)},\ \underbrace{(x_\mu)_{\mu=1}^{D}}_{x}\right)$$

$$u = \varphi(x) = (\varphi_A(x))$$

$$\frac{\partial}{\partial x^\mu}$$

$$S = \int d^D x \ L\left(\varphi(x), (\partial_{x_\mu}\varphi(x)), x\right)$$

$$\left(\varphi(x;\varepsilon)\right)_{\varepsilon\in(-a,a)}$$ – вариация

$$\frac{d}{d\varepsilon}\int d^D x \ L\left(\varphi(x;\varepsilon), (\partial_{x_\mu}\varphi), x\right) =$$

$$= \int d^D x \ \partial_\varepsilon\left(L\left(\varphi(x;\varepsilon), (\partial_{x_\mu}\varphi), x\right)\right)$$

$$\partial_\varepsilon\big(L(\cdot\cdot)\big) = \sum_A \partial_{u_A} L(\cdots)\cdot\partial_\varepsilon\varphi_A + \sum_{A,\mu}\partial_{u_{A,\mu}} L(\cdots)\cdot\partial_\varepsilon\partial_{x^\mu}\varphi_A$$

$$= \partial_{x^\mu}\Big(\sum_{A,\mu}\partial_{u_{A,\mu}} L(\cdots)\cdot\partial_\varepsilon\varphi_A\Big) \to$$

по-долу $\partial_{x^\mu}(J)$

$$+ \sum_A \underbrace{\mathcal{E}_A(L)\big(\varphi(x),(\partial_{x^\mu}\varphi),x\big)}_{\text{по опр. остатъка}}\cdot\partial_\varepsilon\varphi_A(x)$$

$$\partial_{x^\mu}\Big(L\big(\varphi(x),(\partial_{x^\mu}\varphi),x\big)\Big) = \frac{\partial L}{\partial x^\mu}(\cdots) + \sum_A \partial_{u_A} L\cdot u_{A,\mu} + \sum_{A,\nu}\partial_{u_{A,\nu}} L\cdot u_{A,\mu,\nu}$$

$$=: \mathcal{D}_{x^\mu} L \;\Big|\; u_A = \varphi_A(x),\ u_{A,\mu} = \partial_{x^\mu}\varphi_A,\ u_{A,\mu,\nu} = \partial_{x^\mu}\partial_{x^\nu}\varphi_A$$

$$\mathcal{D}_{x^\mu} L\big((u_A),(u_{A,\mu}),(x_\mu)\big)$$

$$:= \frac{\partial L}{\partial x^\mu}\big((u_A),(u_{A,\mu}),(x_\mu)\big) +$$

$$+ \sum_A \partial_{u_A} L\cdot u_{A,\mu} + \sum_{A,\nu}\partial_{u_{A,\nu}} L\cdot u_{A,\mu,\nu}$$

$$\boxed{\begin{aligned}&\mathcal{E}_A(L)\left(u,(u_\mu),x\right)\\ &:= \partial_{u_A} L - \sum_\mu \mathcal{D}_{x^\mu}\left(\partial_{u_{A,\mu}} L\right)\end{aligned}}$$

Оператор на Лагранж – Ойлер

$$\boxed{\mathcal{J}_\mu(L)\left(u,(u_\mu),x;v\right) := \sum_A \partial_{u_{A,\mu}} L \cdot v_A}$$

"компесац. ток"

$$\begin{aligned}&\partial_\varepsilon\left(L\left(\varphi(x),(\partial_{x^\mu}\varphi),x\right)\right)\\ &= \sum_A \mathcal{E}_A(L)\left(\varphi,(\partial_{x_\mu}\varphi),x\right)\cdot\partial_\varepsilon\varphi_A(x)\\ &+ \sum_\mu \partial_{x^\mu}\left\{\mathcal{J}_\mu(L)\left(\varphi,(\partial_{x_\mu}\varphi),x;\partial_\varepsilon\varphi(x)\right)\right\}\end{aligned}$$

$$\int_W d^D x\, \partial_{x^\mu} F = \int_{\partial W} d\sigma\; n^\mu \cdot F$$

Ако $\forall\ \varphi(x,\varepsilon)$, $\partial_\varepsilon\varphi\Big|_{\text{в околност на } \partial W} = 0$

$\varphi(x,0) = \varphi(x)$ и $\frac{d}{d\varepsilon} S = 0$

$$\Rightarrow \quad \mathcal{E}_A(L)\left(\varphi(x),(\partial_{x^\mu}\varphi),x\right) = 0$$

(Л.О.)

Основно тъждество на вариац. смятане:

$$
\begin{aligned}
&\partial_\varepsilon \Big(L\big(\varphi(x), (\partial_{x^\mu}\varphi), x\big)\Big) \\
&= \sum_A \mathcal{E}_A(L)\big(\varphi, (\partial_{x_\mu}\varphi), x\big) \cdot \partial_\varepsilon \varphi_A(x) \\
&+ \sum_\mu \partial_{x^\mu} \Big\{ J_\mu(L)\Big(\varphi, (\partial_{x_\mu}\varphi), x\, ;\, \partial_\varepsilon \varphi(x)\Big)\Big\}
\end{aligned}
$$

Двата члена в дяс. стр. са определени еднознач. от следните условия:

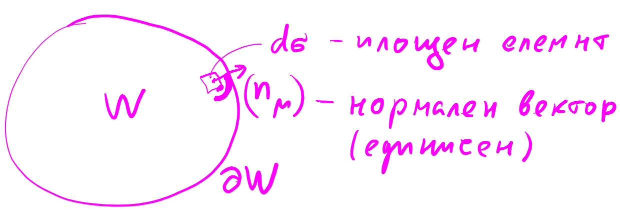
1) В първия член няма $\partial_\varepsilon \partial_{x^\mu}\varphi \equiv \partial_{x^\mu}\partial_\varepsilon \varphi$

2) Втория член има вида

$$\sum_\mu \partial_{x^\mu}(\cdots)$$

Теорема на Стокс / Stokes (частен случай)

$$\int_W d^D x \; \partial_{x^\mu} F = \int_{\partial W} d\sigma \;\; n^\mu \cdot F$$



В $D=1$: $\displaystyle\int_a^b dx \, \frac{dF}{dx}(x) = \underbrace{F(b) - F(a)}_{\int_{\partial W} F}$

В частни случаи на $D=3$ се нарича теорема на Гаус

Коментар за общата теорема на Стокс

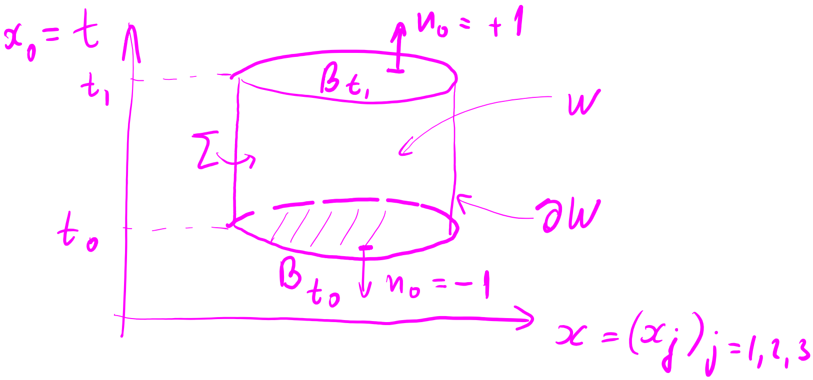
$$\int_N d\alpha = \int_{\partial N} \alpha$$

$N \subset M$ подмн.

външен диф.

α е $((\dim N) - 1)$ - форма

Интерпретация на закон за запазване



$$\partial W = B_{t_0} \cup B_{t_1} \cup \Sigma$$

$$\int_W d^Dx \left(\sum_\mu \partial_{x^\mu} J_\mu(x)\right) = \int_{\partial W} d\sigma \cdot \sum_\mu n^\mu J_\mu$$

$$= \int_{B_{t_0}} d^{D-1}x \underbrace{n^0}_{(-1)} \cdot J_0 + \int_{B_{t_1}} d^{D-1}x \underbrace{n^0}_{+1} J_0$$

$$+ \int_\Sigma d\sigma \sum_\mu n^\mu J_\mu$$

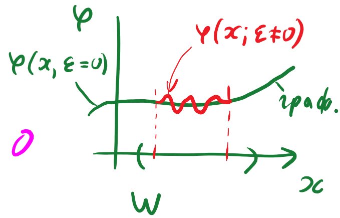
И ако допълнително

$$\sum_\mu \partial_{x^\mu} J_\mu = 0 \quad (\Longleftrightarrow \operatorname{div} J = 0)$$

$$\Rightarrow \int_{B_{t_1}} d^{D-1}x\, J_0 - \int_{B_{t_0}} d^{D-1}x\, J_0$$

$$= -\int_\Sigma d\sigma \cdot \sum_\mu n^\mu J_\mu$$

Теорема на Л. О.



$$\frac{d}{d\varepsilon}\int_W d^Dx\; L\big(\varphi(x;\varepsilon), \partial_x \varphi(x;\varepsilon), x\big) = 0$$

за $\forall\, \varphi(x;\varepsilon)$ т.че $\partial_\varepsilon \varphi(x;\varepsilon)\Big|_{\text{някаква околност на } \partial W} = 0$

и $\varphi(x;\varepsilon=0) = \varphi(x)$ – отнапред фиксирано

$$\Longleftrightarrow \quad \mathcal{E}_A\big(\varphi(x), \partial\varphi(x), \partial^2\varphi(x), x\big) = 0$$

$A = 1, \dots, N$ $\qquad x \in W$ iterior

Ур-е на Л. О.

Док. От осн. тъждество и теор. на Стокс

$$\frac{d}{d\varepsilon}\int_W d^Dx\; L(\varphi, \partial\varphi, x) \equiv 0$$

по условие

$$= \int_W d^Dx\; \mathcal{E}_A(\varphi, \partial\varphi, \partial^2\varphi, x)\cdot \partial_\varepsilon \varphi(x,\varepsilon)$$

произволна

$+\, 0$

от Стокс

Частен случай на теоремата на Ньотер

Теорема Нека $\varphi(x;\varepsilon)$ е такава фамилия от полета за която 1) $\varphi(x;\varepsilon)\big|_{\varepsilon=0} = \varphi(x)$ – отнапред зададено

2) $$S\{\varphi(x;\varepsilon)\} = := \int_W d^D x \, L\left(\varphi(x;\varepsilon), \partial_x \varphi(x;\varepsilon), x\right)$$

= константа от ε (не зависи от ε)

(симетрия на действието)

за $\forall W$

Тогава имаме закон за запазване

$$\sum_\mu \partial_{x^\mu}\left(J_\mu\left(\varphi(x), \partial_x\varphi, x\,;\, \partial_\varepsilon \varphi(x,\varepsilon)\Big|_{\varepsilon=0}\right)\right)$$

$$= 0$$

при условие, че $\varphi(x)$ изпълнява ур. на Л.О.

Док.

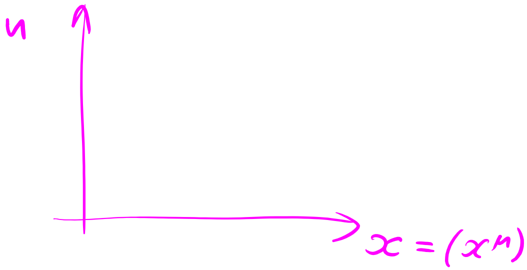
Отново от основното тъждество

$$\partial_\varepsilon\Big(L\big(\varphi(x), (\partial_{x^\mu}\varphi), x\big)\Big) = 0 \text{ по условие}$$

$$= \sum_A \mathcal{E}_A(L)\big(\varphi, (\partial_{x_\mu}\varphi), x\big)\cdot \partial_\varepsilon \varphi_A(x)$$

$$+ \sum_\mu \partial_{x^\mu}\Big\{ J_\mu(L)\Big(\varphi, (\partial_{x_\mu}\varphi), x\,;\, \partial_\varepsilon \varphi(x)\Big)\Big\}$$

При общата теорема на Ньотер



$$(x, u) \mapsto (x', u')$$

$$\begin{cases} x' = F(x; \varepsilon) \\ u' = G(x, u; \varepsilon) \end{cases}$$

закон за трансф. на полетата

$$u' = \varphi_\varepsilon(x')$$

$$\begin{cases} x' = F(x, \varepsilon) \\ \varphi_\varepsilon(x') = G(x, \varphi(x); \varepsilon) \end{cases}$$

$$\int_W d^D x \; L(\varphi_\varepsilon(x), \partial_x \varphi_\varepsilon(x), x)$$

$$= \mathrm{const}(\varepsilon) \quad \text{за } \forall W$$

$$\Leftrightarrow$$

$$\partial_\varepsilon \Big(L(\varphi_\varepsilon, \partial_x \varphi_\varepsilon, x) + \mathrm{div}(X) \Big)$$

$$= \mathcal{E}(\varphi, \partial\varphi, x) \cdot \partial_\varepsilon \varphi$$

$$+ \,\mathrm{div}\Big(\underbrace{\mathcal{Y} + \partial_\varepsilon X}_{\mathcal{Y}^{Noether}} \Big)$$

— Ньотеров ток

нова добавка

http://theo.inrne.bas.bg/~mitov/QFTandEP2020kwe6fo95/NN_QFT_2015_lectures_1-5_A4.pdf

Стр. 115

Забележка:

Истинският, физически закон за запазване на нещо е интегралния закон

$$\int_{\partial W} (J\cdot n)\, d\sigma = 0$$

$$= \int_{B_{t_1}} d\sigma\, J_0 - \int_{B_{t_0}} d\sigma\, J_0 + \int_{\Sigma} d\sigma\, (J\cdot n)$$

$$\Longleftrightarrow \quad \sum_{\mu} \partial_{x^{\mu}} J_{\mu} = 0$$

обичайните частни производни

Но в гравитацията се среща т. нар. "ковариантен закон за запазване"

$$\sum_{\mu} \nabla_{\mu} J_{\mu} = 0$$

Разпределения / обобщени функции

distributions / generalized functions

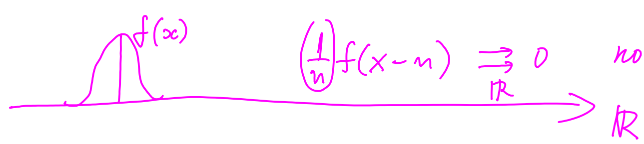
Обобщават лин. функционали:

$$\underset{\text{"тест функция"}}{f(x)} \longmapsto \int_{\mathbb{R}^D} d^D x \; F(x)\, f(x)$$

тест функция = C^∞-функция, която = 0 извън ограничено множ. в $\mathbb{R}^D$

$\mathcal{D}(\mathbb{R}^D)$ = вект. пространство на $\forall$ тест функц.

$$f_i \xrightarrow[\text{в } \mathcal{D}]{} f \iff \begin{cases} \forall f_i \text{ и } f \text{ са } = 0 \text{ извън едно и също ограничено множ.} \\ \partial^n f_i \rightrightarrows \partial^n f \quad \forall n \end{cases}$$



$\frac{1}{n} f(x-n) \underset{\mathbb{R}}{\rightrightarrows} 0$ но не в $\mathcal{D}$

Опр. $\mathcal{D}(\mathbb{R}^D) \longrightarrow \mathbb{R}$ (или $\mathbb{C}$)

лин. непр. се нарича разпределение

Символичен запис:

$$f \longmapsto F(f) =: \int_{(\mathbb{R}^D)} d^D x \; F(x) f(x)$$

$\mathcal{D}'(\mathbb{R}^D)$ = вект. простр. на $\forall$ разпр.

Пример: а)

$$f \longmapsto \delta(f) = f(0) = \int d^D x \; \delta(x) f(x)$$

δ-функция на Дирак

б)

$\forall$ $C(\mathbb{R}^D)$ - функция е разпределение

L^1_{loc} (симв. запис - действителен)

в смисъл на Риман-Лебег

Ако F - разпределение, $W \underset{\text{отв.}}{\subseteq} \mathbb{R}^D$

$$F\big|_W = 0 \iff F(f) = 0$$

за $\forall f$ т.че $f\big|_{\mathbb{R}^D \setminus W} = 0$

$$\bigcup \{ W\text{ - отв.} \mid F|_W = 0 \} =: \mathbb{R}^D \setminus \operatorname{supp} F \leftarrow$$

носител support - затворено множ.

$$\operatorname{supp} \delta(x) = \{0\}$$

$F(x) \in C(\mathbb{R}^D)$. $\operatorname{supp} F = \overline{F^{-1}(\mathbb{R}^D \setminus \{0\})}$

Операции над разпределения

0) Лин. комбинации (векторно прост.)

1) Ако $h \in C^\infty(\mathbb{R}^D)$, $F \in \mathcal{D}'(\mathbb{R}^D)$

то $(h \cdot F)(f) := F(hf) \leftarrow$ ще бъде $\mathcal{D}$-непр. по f

тест функц.

Пример: $(h\,\delta)(f) = \delta(hf) = h(0)f(0)$

$$= h(0) \cdot \delta(f) \quad \forall f$$

$\Rightarrow \; h\delta = h(0)\,\delta$ или $h(x)\,\delta(x) = h(0)\,\delta(x)$.

2.) Ако $F \in \mathcal{D}'(\mathbb{R}^D)$

$$(\partial_{x^\mu} F)(f) := -F(\partial_{x^\mu} f)$$

ще бъде $\mathcal{D}$-непр. по f

тест функц.

Символично:

$$\int d^D x \; (\partial_{x^\mu} F)\, f = -\int d^D x \; F\, \partial_{x^\mu} f$$

– "интегриране по части"

Съгласувано за $F \in C^1(\mathbb{R}^D)$, понеже $f\big|_\infty = 0$ и няма гранич. членове

Пример: $\theta(x) \in \mathcal{D}'(\mathbb{R})$

$$\theta(f) = \int_{\mathbb{R}} dx \; \theta(x)\, f(x) = \int_0^\infty dx \; f(x)$$

Тв. $\partial_x \theta(x) = \delta(x)$

$$(\partial_x \theta)(f) = -\int_0^\infty dx \; \partial_x f(x) = -f(x)\Big|_0^\infty$$

$$= f(0) - \underbrace{f(\infty)}_{0} = f(0) = \delta(f) \,.$$

Пример: $\left(\partial_{x^\mu}\delta\right)(f) = -\partial_{x^\mu} f(0)$

Свойство: $\operatorname{supp} \partial_{x^\mu} F \subseteq \operatorname{supp} F$

$\operatorname{supp} h \cdot F \subseteq \operatorname{supp} F$

Теорема Ако $F \in \mathcal{D}'(\mathbb{R}^D)$ т.че

$\overline{\operatorname{supp} F} = \{0\}$, то

$$F = \sum_{n=0}^{N} \sum_{\mu_1 \cdots \mu_n} \underset{\substack{\cap \\ \mathbb{R}}}{c_{\mu_1,\ldots,\mu_n}} \partial_{x^{\mu_1}} \cdots \partial_{x^{\mu_n}} \delta$$

Проблем с умножение на разпр.

$$\left(\mathcal{P}\frac{1}{x}\right)(f) := \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} dx \, \frac{1}{x} f(x)$$

– $\exists$ за тест функц. и е $\mathcal{D}$-непр. по f

т.че $\mathcal{P}\frac{1}{x} \in \mathcal{D}'(\mathbb{R})$

$$\left(x\cdot \mathcal{P}\left(\frac{1}{x}\right)\right)(f) = \int dx\, f(x) = 1(f)$$

т.е. $x\cdot \mathcal{P}\left(\frac{1}{x}\right) = 1$

$$\Big(\underbrace{\delta(x)\cdot x}_{0}\Big)\cdot \mathcal{P}\left(\frac{1}{x}\right) \neq \delta(x)\cdot \Big(\underbrace{x\cdot \mathcal{P}\left(\frac{1}{x}\right)}_{1}\Big)$$

До противоречия водят и изрази като

$\delta(x)^2$, $\theta(x)\,\delta(x)$ и т. н.

Операция 3) Субституция

$F(g(x))$, $g: \mathbb{R}^D \to \mathbb{R}^D$

гладко и обратимо

$$\int d^D x \; F(g(x))\, f(x) :=$$

$$= \int d^D x \; \left|\frac{\mathcal{D} g^{-1}}{\mathcal{D} x}\right| F(x)\, f(g^{-1}(x))$$

$$(F\circ g)(f) := F\Big(\underbrace{\left|\frac{\mathcal{D} g^{-1}}{\mathcal{D} x}\right| f(g^{-1}(x))}_{\text{тест функц.}}\Big)$$

4) Ако supp F – ограничено и $h \in C^\infty(\mathbb{R}^D)$

$F(h) := F(f)$ за $f \in \mathcal{D}(\mathbb{R}^D)$ т.че

$$h\big|_{\operatorname{supp} F} = f\big|_{\operatorname{supp} F}$$

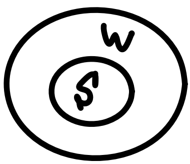
$$f = \chi h \quad \text{при} \quad S = \operatorname{supp} F$$

Основен инструмент:

Теорема Ако $S \underset{\text{комп.}}{\subseteq} \mathbb{R}^D$, $W \underset{\substack{\text{отв.}\\ \text{огр.}}}{\subseteq} \mathbb{R}^D$

$S \subset W$

то $\exists\, \chi \in \mathcal{D}(\mathbb{R}^D)$



$$\chi\big|_S = 1, \quad \chi\big|_{\mathbb{R}^D \setminus \overline{W}} = 0.$$

Приложение: ако $F(x) \in \mathcal{D}'(\mathbb{R}^D)$

и supp F – комп., то

$$F(x-y) \in \mathcal{D}'(\mathbb{R}^D \times \mathbb{R}^D)$$

$$\int d^Dx\, d^Dy \;\; F(x-y)\, f(x,y)$$

$$= \int d^Dx\, d^Dy \;\; F(x)\, f(x+y, y)$$

$$= \int d^D x \; F(x) \underbrace{\int d^D y \; f(x+y, y)}_{h(x) \in C^\infty(\mathbb{R}^D)}$$

$$= F(h)$$

отново се проверява $\mathcal{D}$-непр. по f.

Частично насищане $F(x,y)$ – разпр. по x, y

$\int d^D y \; F(x,y) f(y)$ – разпр. по x :

$$\int d^D y \, d^D x \; F(x,y) f(y) f_1(x)$$

$\delta(x-y)$ – обобщава $\delta_{j,k}$

$$\sum_j \delta_{j,k} f_k = f_j$$

$$\int d^D x \; \delta(x-y) f(y) = f(x)$$

$$= \int d^D x \; \delta(y-x) f(y)$$

Операция 5 – тенз. произведение $F_1 \otimes F_2$

$$\int d^D x_1 \, d^D x_2 \; F_1(x_1) F_2(x_2) \; f_1(x_1) f_2(x_2)$$

$$:= \left(\int d^D x_1 \; F_1 f_1 \right) \left(d^D x_2 \; F_2 f_2 \right)$$

Операция 6) Конволюция

Нека $\mathrm{supp}\, F_1$, $\mathrm{supp}\, F_2$ - комп.

$$\int d^D y \; F_1(x-y)\, F_2(y-z)$$

- определено

$$\int d^D x\, d^D z \left(\int d^D y \; F_1(x-y)\, F_2(y-z) \right) f(x,z)$$

$$=: \int d^D x'\, d^D y' \; F_1(x')\, F_2(y') \int d^D z \; f(x'+y'+z, z)$$

$$\mathrm{supp}\, F_1 \otimes F_2 \subseteq \mathrm{supp}\, F_1 \times \mathrm{supp}\, F_2$$

- комп.

$$\int d^D y \; \delta(x-y)\,\delta(y-z) = \delta(x-z)$$

$$\sum_\nu \delta_{\mu\nu}\,\delta_{\nu\rho} = \delta_{\mu\rho}$$

Вариац. смятане

$$\frac{\partial x_\mu}{\partial x_\nu} = \delta_{\mu,\nu} \;\rightsquigarrow\; \frac{\delta \varphi(x)}{\delta \varphi(y)} = \delta(x-y)$$

$$\frac{\delta\, \partial_{x^\mu} \varphi(x)}{\delta \varphi(y)} = \partial_{x^\mu}\, \delta(x-y).$$

За $\mathcal{F}\{\varphi(x)\}$: $\dfrac{\delta \mathcal{F}}{\delta \varphi(x)} \in \mathcal{D}'(\mathbb{R}^D)$,

$$\frac{\delta \mathcal{F}}{\delta \varphi(x)}(f) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}\{\varphi(x) + \varepsilon f(x)\}$$

$$\frac{\delta}{\delta \varphi(x)} \int d^D y \; L\big(\varphi(y), (\partial_{x_\mu}\varphi), y\big)$$

$$= \int d^D y \; \frac{\delta L\big(\varphi(y), (\partial_{x_\mu}\varphi), y\big)}{\delta \varphi(x)}$$

$$= \int d^D y \left(\frac{\partial L}{\partial u}(y)\, \frac{\delta \varphi(y)}{\delta \varphi(x)} + \frac{\partial L}{\partial u_\mu}(y)\, \frac{\delta\, \partial_{x_\mu} \varphi(y)}{\delta \varphi(x)} \right)$$

$$= \int d^D y \left(\frac{\partial L}{\partial u}(y)\, \delta(x-y) + \frac{\partial L}{\partial u_\mu}\, \partial_{y^\mu} \delta(x-y) \right)$$

$$= \int d^D y \left(\frac{\partial L}{\partial u} - \partial_{y^\mu} \frac{\partial L}{\partial u_\mu} \right) \delta(x-y)$$

$$= \mathcal{E}(L)(x)$$

СЛЕДВАЩИЯ ПЪТ ПРОДЪЛЖАВАМЕ С ТЕМАТА "АКСИОМАТИЧНА КВАНТОВА ТЕОРИЯ НА ПОЛЕТО". ПРЕДВАРИТЕЛНИ ЗАПИСКИ:

http://theo.inrne.bas.bg/~mitov/renormalization2020/all_lectures/renormalization220420v1.pdf
http://theo.inrne.bas.bg/~mitov/renormalization2020/all_lectures/renormalization-230420.pdf

ДОПЪЛНИТЕЛЕН МАТЕРИАЛ ИМА НА СТР. 69 (ТОЧКА 2.1.1) ОТ МОЙТА ДИСЕРТАЦИЯ:
http://theo.inrne.bas.bg/~mitov/renormalization2020/notes/Nikolov_N.M.,_Dissertation_2002.pdf