Mathematical
Surveys
and
Monographs

Volume 96

Operads in Algebra, Topology and Physics

Martin Markl
Steve Shnider
Jim Stasheff

American Mathematical Society

Selected Titles in This Series

96 **Martin Markl, Steve Shnider, and Jim Stasheff,** Operads in algebra, topology and physics, 2002

95 **Seiichi Kamada,** Braid and knot theory in dimension four, 2002

94 **Mara D. Neusel and Larry Smith,** Invariant theory of finite groups, 2002

93 **Nikolai K. Nikolski,** Operators, functions, and systems: An easy reading. Volume 2: Model operators and systems, 2002

92 **Nikolai K. Nikolski,** Operators, functions, and systems: An easy reading. Volume 1: Hardy, Hankel, and Toeplitz, 2002

91 **Richard Montgomery,** A tour of subriemannian geometries, their geodesics and applications, 2002

90 **Christian Gérard and Izabella Łaba,** Multiparticle quantum scattering in constant magnetic fields, 2002

89 **Michel Ledoux,** The concentration of measure phenomenon, 2001

88 **Edward Frenkel and David Ben-Zvi,** Vertex algebras and algebraic curves, 2001

87 **Bruno Poizat,** Stable groups, 2001

86 **Stanley N. Burris,** Number theoretic density and logical limit laws, 2001

85 **V. A. Kozlov, V. G. Maz'ya, and J. Rossmann,** Spectral problems associated with corner singularities of solutions to elliptic equations, 2001

84 **László Fuchs and Luigi Salce,** Modules over non-Noetherian domains, 2001

83 **Sigurdur Helgason,** Groups and geometric analysis: Integral geometry, invariant differential operators, and spherical functions, 2000

82 **Goro Shimura,** Arithmeticity in the theory of automorphic forms, 2000

81 **Michael E. Taylor,** Tools for PDE: Pseudodifferential operators, paradifferential operators, and layer potentials, 2000

80 **Lindsay N. Childs,** Taming wild extensions: Hopf algebras and local Galois module theory, 2000

79 **Joseph A. Cima and William T. Ross,** The backward shift on the Hardy space, 2000

78 **Boris A. Kupershmidt,** KP or mKP: Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems, 2000

77 **Fumio Hiai and Dénes Petz,** The semicircle law, free random variables and entropy, 2000

76 **Frederick P. Gardiner and Nikola Lakic,** Quasiconformal Teichmüller theory, 2000

75 **Greg Hjorth,** Classification and orbit equivalence relations, 2000

74 **Daniel W. Stroock,** An introduction to the analysis of paths on a Riemannian manifold, 2000

73 **John Locker,** Spectral theory of non-self-adjoint two-point differential operators, 2000

72 **Gerald Teschl,** Jacobi operators and completely integrable nonlinear lattices, 1999

71 **Lajos Pukánszky,** Characters of connected Lie groups, 1999

70 **Carmen Chicone and Yuri Latushkin,** Evolution semigroups in dynamical systems and differential equations, 1999

69 **C. T. C. Wall (A. A. Ranicki, Editor),** Surgery on compact manifolds, second edition, 1999

68 **David A. Cox and Sheldon Katz,** Mirror symmetry and algebraic geometry, 1999

67 **A. Borel and N. Wallach,** Continuous cohomology, discrete subgroups, and representations of reductive groups, second edition, 2000

66 **Yu. Ilyashenko and Weigu Li,** Nonlocal bifurcations, 1999

65 **Carl Faith,** Rings and things and a fine array of twentieth century associative algebra, 1999

(Continued in the back of this publication)

Operads in Algebra, Topology and Physics

Mathematical
Surveys
and
Monographs

Volume 96

Operads in Algebra, Topology and Physics

Martin Markl
Steve Shnider
Jim Stasheff

American Mathematical Society

2000 *Mathematics Subject Classification.* Primary 18D50, 55P48.

ABSTRACT Operads were originally studied as a tool in homotopy theory, specifically for iterated loop spaces Recently the theory of operads has received new inspiration from and applications to homological algebra, category theory, algebraic geometry and mathematical physics Many of the theoretical results and applications, scattered in the literature, are brought together here along with new results and insights as well as some history of the subject

---

Library of Congress Cataloging-in-Publication Data

Markl, Martin, 1960–

Operads in algebra, topology and physics / Martin Markl, Steve Shnider, Jim Stasheff
p cm — (Mathematical surveys and monographs, ISSN 0076-5376 , v 96)
Includes bibliographical references
ISBN 0-8218-2134-2 (alk paper)
1. Operads I. Shnider, S (Steven), 1945– II Stasheff, James D III Title IV Mathematical surveys and monographs ; no. 96.

QA169 M356 2002
511 3—dc21 2002016342

---

10 9 8 7 6 5 4 3 2 1 07 06 05 04 03 02

Contents

Preface

Operads are mathematical devices which describe algebraic structures of many varieties and in various categories. Operads are particularly important/useful in categories with a good notion of 'homotopy' where they play a key role in organizing hierarchies of higher homotopies. Significant examples first appeared in the 1960's though the formal definition and appropriate generality waited for the 1970's. These early occurrences were in algebraic topology in the study of (iterated) loop spaces and their chain algebras. In the 1990's, there was a renaissance and further development of the theory inspired by the discovery of new relationships with graph cohomology, representation theory, algebraic geometry, derived categories, Morse theory, symplectic and contact geometry, combinatorics, knot theory, moduli spaces, cyclic cohomology and, not least, theoretical physics, especially string field theory and deformation quantization. The generalization of quadratic duality (e.g. Lie algebras as dual to commutative algebras) together with the property of Koszulness in an essentially operadic context provided an additional computational tool for studying homotopy properties outside of the topological setting.

The aim of this book is to exhibit operads as tools for this great variety of applications, rather than as a theory pursued for its own sake. Most of the results presented are scattered throughout the literature (some of them belonging to the current authors). At times the exposition goes beyond the original sources so that some results in the book are more general than the ones in the literature. Also a few gaps in the available proofs are filled. Some items, such as the construction of various free operads, are given with all the bells and whistles for the first time here.

In an extensive introduction, we review the history (and prehistory) and hope to provide some feeling as to what operads are good for, both in a topological context and a differential graded algebraic context. The basic examples of the endomorphism operad and tree operads are presented. Just as group theory without representations is rather sterile, so operads are best appreciated by their representations, known as (varieties of) algebras, especially 'strong homotopy' algebras. We introduce the most common types: A_∞, $E_\infty, L_\infty, C_\infty$. We also consider generalizations such as cyclic operads, modular operads and partial operads.

Next we present a technical part, reviewing basic definitions in the full glory of the symmetric monoidal category setting and relating operads to associated structures: triples (monads).

We then review classical results (mostly in topology) without going into great detail in the proofs since most results exist in a well-established literature. We emphasize the guiding principles of 'recognition,' 'approximation,' homotopy invariance and computational consequences (homology operations).

In a more algebraic section, we establish certain key constructions and properties: bar and cobar constructions, free operads, Koszul duality and cohomology of operad algebras. An application is made to providing minimal models of homotopy algebras.

The remainder of the book is devoted to providing access to some of the myriad of results of the 'renaissance of operads' in which operads have proved their worth in contexts quite different from those of their birth. We emphasize algebraic constructions for operads, geometric examples related to configuration spaces and moduli spaces, generalizations such as cyclic and modular operads. Such generalizations are motivated by applications to deformation quantization, string field theory, quantum cohomology and Gromov-Witten invariants. We have had to be somewhat selective in our choice of topics influenced by our own personal tastes. We are confident that we have failed to include all the latest applications since the field is progressing so rapidly; see also the Epilog.

The book is intended for researchers and students as well as anyone who wishes to get the flavor of operads and their application. We have tried to provide overviews and introductions as well as technical machinery for the reader's use. Particularly technical details have been sequestered to appendices; in Frank Adams' language, to operate the machine, it is not necessary to raise the bonnet (look under the hood).

ACKNOWLEDGMENTS: We wish to thank the following people, with the standard reservation that although they helped us with this book in many ways, they are not responsible for the mistakes, which are solely our own sins. We thank J.M. Boardman, J. Bernstein, Y.-Z. Huang, M.M. Kapranov, T. Kimura, P. van der Laan, J.-L. Loday, J.P. May, D.H. Van Osdol, P. Salvatore, P. Somberg, R. Vogt and A.A. Voronov.

Many others who have contributed to our research in this field also deserve our appreciation. The senior author wishes to give particular thanks to John Coleman Moore and J. Frank Adams who started him on the road to operads. The junior author wishes to express his thanks to Jiří Bok who pried the door to mathematics open for him.

Our collaboration has benefited from facilities at the University of North Carolina at Chapel Hill, USA, at the University of Pennsylvania, Philadelphia, USA, at the Mathematical Institute of the Academy of Sciences of the Czech Republic at Prague and at Bar Ilan University, Ramat Gan, Israel.

The first author was supported by grant GA ČR 201/99/0675, grant ME 333 and grant DMS-9803435. The second author was partially supported by the Israel Science Foundation – Center of Excellence Program founded by the Israel Academy of Sciences and Humanities. The third author was supported by grant DMS-9803435.

– – – – –

CONVENTION: The book consists of two parts. When referring to theorems, formulas, etc., we write the roman numeral of the part explicitly only when the item we refer to is not in the current part of the book.

Part I

CHAPTER 1

Introduction and History

A prehistory

Operads involve an abstraction of a family of composable functions of several variables $Map(X^n, X)$ together with an action of permutations of variables. As such, they have appeared in various ways over the last century and have recently had a great variety of applications which have provided the impetus for this book.

Operads as such were originally studied as a tool in homotopy theory, specifically for iterated loop spaces, but the theory of operads has recently received new inspiration from homological algebra, category theory, algebraic geometry and mathematical physics. The name *operad* and the formal definition appear first in the early 1970's in J. Peter May's "The Geometry of Iterated Loop Spaces" [**May72**], but a year or more earlier, Boardman and Vogt described the same concept under the name '*categories of operators in standard form*,' inspired by PROPs and PACTs of Adams and Mac Lane. In fact, there is an abundance of prehistory. Weibel [**Wei**] points out that the concept first arose a century ago in A.N. Whitehead's "A Treatise on Universal Algebra" [**Whi98**], published in 1898.

Whitehead was actually describing quadratic operads in the category of vector spaces, specifically those generated by a single binary product. In sections 20-22, under the title '*complete algebraic system*,' he fixes a product and refers to the order of higher products determined by iterated compositions thereof. In section 92 (p. 172) he defines '*invariant equations of condition*'; in modern language these are exactly the relations on the free operad on a single binary operation needed to produce the operad in question. He goes on in section 93 to add associativity explicitly and studies the resulting theory of symmetric and exterior algebras.

Operads (or, more precisely, 'systems with compositions') arose again in the mid-1950's in the work of Lazard on formal groups.

1.1. Lazard's formal group laws

In 1954, M. Lazard, studying formal group laws, introduced the structure of 'analyseur' to axiomatize composition laws. This structure is a refinement of a more basic structure, 'a family of polynomial functions,' which is probably the earliest explicit example of an operad in the category of modules over a ring. Of course the Hochschild cochain complex, which had been introduced a decade earlier, has an operadic structure, but this was not really explicit until the work of Gerstenhaber [**Ger63**] some nine years later. The description we give here follows Lazard's review of his work in Séminaire Bourbaki 1954-1955 [**Laz59**] and Chevalley's extensive review [**Che56**] of Lazard's full length paper [**Laz55**].

Let $\mathbf{k}$ be a commutative ring with unity and E a unitary $\mathbf{k}$-module. A not necessarily linear function $f \in Map(E^{\oplus n}, E)$ will be called a function of n variables and will be denoted $f(x_1, \dots, x_n)$. The set of such functions has the structure of a $\mathbf{k}$-module. An argument x_i of $f(x_1, \dots, x_n)$ is called neutral if $f(a_1, \dots, a_n) = f(b_1, \dots, b_n)$ whenever $a_j = b_j$ for all $j \neq i$; the function is then said to be neutral in that argument. Composition of a function of n arguments with any one of the natural projections $E^{\oplus(n+k)} \to E^{\oplus n}$ 'adds neutral arguments,' that is, defines a function of $n+k$ arguments which does not depend on k of the arguments. Similarly we can reduce a function by deleting its neutral arguments; i.e. by factoring it through the projection that eliminates those arguments. There are then several conditions that describe what Lazard calls a *family $\mathcal{P}$ of polynomial functions* on a $\mathbf{k}$-module E.

A family is a graded $\mathbf{k}$-module,

$$\mathcal{P} = \bigoplus_{0<n} \mathcal{P}^n, \quad \text{where } \mathcal{P}^n \subset Map(E^{\oplus n}, E),$$

which is closed under composition of one function f of m variables with m 'input' functions g_i, $1 \leq i \leq n$, each depending on the same number n of arguments. We also assume that $\mathcal{P}$ contains the identity function $id_E : E \to E$.

The next step is to introduce the concept of homomorphism and equivalence, so that one can eventually define an abstract object independent of the particular $\mathbf{k}$-module which appears as the domain of definition of the functions.

A *homomorphism* $\phi : \mathcal{P} \to \mathcal{P}'$ from a family $\mathcal{P}$ of polynomial functions on a module E to a family $\mathcal{P}'$ of polynomial functions on a module E' respects the number of variables, composition and the corresponding identity maps. An isomorphism of families is then an invertible homomorphism.

An *'analyseur incomplet'* is defined as an equivalence class of families of polynomial functions, under the equivalence relation given by isomorphism. A (complete) *analyseur* is defined by a completion process analogous to the completion of a polynomial ring to a formal power series ring.

A basic example of analyseur is given by the polynomial functions without constant term on an infinite field. A *group law* in an analyseur $\mathcal{A}$ is an element of $\mathcal{A}^2$ satisfying

$$f(x,y) = x + y \quad \text{modulo terms of homogeneity} \geq 2, \text{ and}$$
$$f(f(x,y),z) - f(x, f(y,z)) = 0.$$

Lazard defines a q-germ of a group law as one that satisfies the associativity axiom modulo terms of homogeneity $q+1$. He then introduces a cohomological obstruction to the extension of a q-germ to a $q+1$ germ which anticipates the Gerstenhaber deformation theory of algebras; see [**Ger64**].

1.2. PROPs and PACTs

Towards the end of the 1950's, Adams and Mac Lane were motivated by problems in algebraic topology, namely situations such as coproducts in iterated bar constructions or simplicial chain complexes which were commutative only 'up to higher homotopies.' There are, to quote Mac Lane [**Mac65**], 'so many of them that general conceptual methods are needed for their treatment.' They therefore invented PROPs and PACTs, although the first publication from this collaboration did not appear until 1963 [**Mac63b**].

PROPs are categories **H** (an abbreviation of PROduct and Permutation category) with objects the natural numbers $\{0, 1, 2, \ldots\}$ and 'Hom-sets' of maps from m to n denoted

$$\mathbf{H}\binom{m}{n}.$$

There is a strictly associative 'monoidal structure' (denoted by $\otimes$) given by composition on 'Hom-sets'

$$\otimes : \mathbf{H}\binom{m}{n} \times \mathbf{H}\binom{n}{p} \to \mathbf{H}\binom{m}{p} \tag{1.1}$$

which is suitably symmetric. In particular, $\mathbf{H}\binom{n}{n}$ contains the symmetric group Σ_n as a specified subgroup; thus the 'tensor product' (1.1) induces a left Σ_m-, right Σ_n-action on $\mathbf{H}\binom{m}{n}$.

Let us point out that the convention given here is opposite to the convention we use in the book and follows the 'categorial' composition rule for arrows, not for maps. Intuitively, the space $\mathbf{H}\binom{m}{n}$ parameterizes operations with m inputs and n outputs.

PACTs were PROPs in which the Hom-sets $\mathbf{H}\binom{m}{n}$ were dg modules over a commutative ring with the 'tensor' product (1.1) and symmetric group actions being compatible with the dg structure.

Of special importance is the endomorphism PROP $\mathbf{End}_X\binom{m}{n}$ of maps $X^{\odot m} \to X^{\odot n}$ for any object X in a monoidal category with a product $\odot$. An *action* of a PROP **H** on an object X is then a map of PROPs $\mathbf{H} \to \mathbf{End}_X$. In this situation we also say that X is an *algebra over* **H** or that X is an **H**-*space*.

1.3. Non-Σ operads and operads

In addition to the publication of PACTs and PROPs, 1963 was a good year for operads with the publication of Gerstenhaber 'comp algebras' in his study of Hochschild cohomology [**Ger63**] and Stasheff's 'associahedra' for his homotopy characterization of loop spaces [**Sta63a**]. With hindsight, we can say that these constructions are examples of what are now often called 'non-Σ operads,' the 'non-Σ' indicating that we do not assume any symmetric group action.

In 1966/67, Mac Lane ran a seminar on PACTs and PROPs at the University of Chicago. Among the participants were Adams, Boardman, Stasheff and Vogt [**Vog98**]. Boardman's linear isometries PROP (later reincarnated as an operad) provided the key to developing an understanding of infinite loop spaces, as was his 'little cubes PROP' the key to understanding iterated loop spaces in general. The following year, May built upon Boardman and Vogt's structures to finalize the notion he dubbed 'operad.'

The model is the set $Map(X^n, X)$ of all functions of n variables in a set X for all $n \geq 1$. Let

$$\circ_i : Map(X^n, X) \times Map(X^m, X) \longrightarrow Map(X^{n+m-1}, X) \tag{1.2}$$

be given, for $1 \leq i \leq n$, by

$$(f \circ_i g)(x_1, \ldots, x_{m+n-1}) = f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m-1}), x_{i+m}),$$

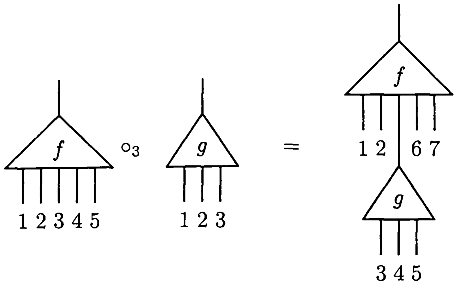


FIGURE 1. Viewing elements of operads as abstract operations.

A *non-Σ operad* $(\mathcal{P}, \circ_i)$ in the category of sets consists of a sequence of sets $\{\mathcal{P}(n)\}_{n\geq 1}$, a unit map $1 \in \mathcal{P}(1)$ and products $\circ_i : \mathcal{P}(m) \times \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$ for $m, n \geq 1$, $1 \leq i \leq m$, satisfying the relations manifest in the example $\mathcal{P}(n) := Map(X^n, X)$. Analogous objects for the category of graded modules were introduced by Gerstenhaber [**Ger63**] under the name *comp algebra*.

Non-Σ operads can be defined in any monoidal category (apparently first remarked by Malraison in the mid-1970's [**Mal74**]) and operads in any symmetric monoidal category (see Section II.1.2). May's original definition, motivated by the topological applications, was in the category of compactly generated topological spaces.

Passing to associated chain complexes and hence to the category of differential graded modules was implicit already in Stasheff's work, but apparently was not formalized until the early 1980's by Hinich and Schechtman [**HS87**].

An *operad* is a non-Σ operad $(\mathcal{P}, \circ_i)$ with an action of the symmetric group Σ_n on $\mathcal{P}(n)$ for each n. These actions are compatible with the $\circ_i$-operations as manifest in the example of the *endomorphism operad* $\mathcal{E}nd_X := \{Map(X^n, X)\}_{n\geq 1}$. Observe that each non-$\Sigma$ operad can be made an operad replacing, for each $n \geq 1$, $\mathcal{P}(n)$ by $\mathcal{P}(n) \times \Sigma_n$ and extending the structure operations $\circ_i$ to be compatible with the action of the symmetric groups; see also Remark II.1.15.

It is sometimes useful to visualize elements of $\mathcal{P}(n)$ as abstract 'operations' with n inputs and one output; $f \circ_i g$ can be then interpreted as the result of inserting the output of g into the ith input of f. In Figure 1, $f \in \mathcal{P}(5)$, $g \in \mathcal{P}(3)$ and $f \circ_3 g \in \mathcal{P}(7)$.

We will notationally distinguish non-Σ operads from operads by underlining. Thus, for example, the non-Σ operad describing topological monoids will be denoted by $\underline{\mathcal{M}on}$ and its symmetrized version by $\mathcal{M}on$. There is, however, one exception to this rule in the book: the operad $\underline{\mathcal{N}}$ discussed in Section 1.16 is an ordinary Σ-operad.

May's original definition (see Section II.1.2) includes $\mathcal{P}(0)$ which is a single point. More importantly, rather than deal with the individual $\circ_i$-operations, May's definition combines them into a single operation

$$\gamma : \mathcal{P}(k) \times \mathcal{P}(j_1) \times \cdots \times \mathcal{P}(j_k) \longrightarrow \mathcal{P}(j_1 + \cdots + j_k) \tag{1.3}$$

satisfying upon iteration the relations of associativity and symmetry manifest in the example of the endomorphism operad $\{Map(X^n, X)\}_{n\geq 1}$. In general, γ can be obtained, by iterating appropriate $\circ_i$'s, as

$$\gamma(f \times g_1 \times \cdots \times g_n) := (\cdots(((f \circ_1 g_1) \circ_{l_1+1} g_2) \circ_{l_1+l_2+1} g_3) \cdots) \cdots ,$$

where l_i is the arity of g_i. In turn, using the identity $1 \in \mathcal{P}(1)$, the composition $f \circ_i g$ can be reconstructed as

$$f \circ_i g = \gamma(f \ \times 1 \times \cdots \times g \times \cdots \times 1) \ (g \text{ at the } i\text{th position}).$$

Thus, under the presence of a unit, both approaches are equivalent. The advantage of the $\circ_i$-approach is that the defining operations are binary, thus giving rise to a natural grading of free objects (see also discussion in II.1.9). The relation between the γ- and $\circ_i$-formalisms will be discussed in Section II.1.7.

Operads are important not in and of themselves but, like PROPs, through their representations, more commonly called 'algebras over operads.' An *algebra A over an operad $\mathcal{P}$* is a map of operads $\mathcal{P} \to \mathcal{E}nd_A$. This is just a fancy way of saying that an algebra A is a set (an object in the category) with a coherent system of maps $\mathcal{P}(n) \times A^n \to A$.

Each operad $\mathcal{P}$ generates a PROP $\mathbf{H}$ with $\mathbf{H}\binom{m}{1} := \mathcal{P}(m)$ for each $m \geq 1$ such that algebras over the PROP $\mathbf{H}$ are the same as algebras over the operad $\mathcal{P}$; see also Remark II.1.57. But it is not true that each PROP is generated by an operad. An example is provided by bialgebras (= Hopf algebras without antipode). They are objects consisting of a vector space V together with an associative multiplication $\mu : V \otimes V \to V$ and coassociative comultiplication $\Delta : V \to V \otimes V$ such that Δ is a morphism of algebras (equivalently, μ is a morphism of coalgebras).

While bialgebras are algebras over a certain PROP [**Mar96a**], operads cannot accommodate both the multiplication μ and the comultiplication Δ.

1.4. Theories

Another formalization of the concept of a variety of algebras given by operations and laws without existential quantifiers was introduced in 1963 by Lawvere [**Law63**].

Let $\mathfrak{S}$ be the category whose objects are finite sets $[n] := \{1, \dots, n\}$ and morphisms all set maps between these finite sets. The dual category $\mathfrak{S}^{\mathrm{op}}$ is a category with finite products in which every object $[m]$ is the n-fold product $[1]^{\times n}$ of the distinguished object $[1]$.

A (finitary) *algebraic theory* is then a category Θ with the same objects as $\mathfrak{S}^{\mathrm{op}}$, together with a faithful functor $\mathfrak{S}^{\mathrm{op}} \to \Theta$ that preserves objects and products. A *Θ-space*, or a *Θ-algebra*, in a category $\mathcal{C}$ is a functor $A : \Theta \to \mathcal{C}$ preserving the products. The object $X := A([1]) \in \mathcal{C}$ is the *underlying space* of the algebra A.

The existence of the functor $\mathfrak{S}^{\mathrm{op}} \to \Theta$ means that each Θ-structure on a space X necessarily contains operations

$$\sigma^* : X^n \to X^m, \ \sigma^*(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(m)}),$$

for $\sigma : \{1, \dots, m\} \to \{1, \dots, n\} \in \mathfrak{S}([m], [n])$. In particular, each Θ-structure contains projections and permutations of variables.

Each operad $\mathcal{P}$ generates a theory Θ such that $\Theta([m], [1])$ is obtained from $\mathcal{P}(m)$ by adjoining all projections $[m] = [1]^{\times m} \to [1]$. Algebras in a category $\mathcal{C}$ over

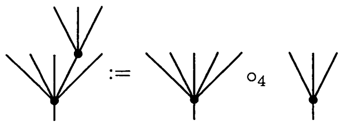


FIGURE 2. Grafting planar trees.

the theory Θ are the same as algebras over the operad $\mathcal{P}$ provided the monoidal structure of $\mathcal{C}$ is given by products. This construction was studied in detail by Boardman and Vogt in [**BV73**]; they call $\mathcal{P}$ the *spine* of the theory Θ.

On the other hand, it is not true that all theories are of this form. For example, Boolean rings where we require $x^2 = x$ cannot be described as algebras over an operad. Roughly speaking, a Lawvere theory is generated by an operad (spine) if and only if the defining axioms of the theory involve no repetitions and each axiom is homogeneous in the number of variables.

Since $\Theta([m],[n]) = \Theta([m],[1])^{\times n}$ for each $m, n \geq 1$, Lawvere's theories are intrinsically 'algebraic,' thus they, like operads, cannot accommodate objects with 'coalgebraic' operations, like coalgebras or Hopf algebras.

1.5. Tree operads

In the early days, three examples were particularly important in the development of the subject; the linear isometries operad, the little cubes operad and the associahedra non-Σ operad (see respectively Sections II.2.7, II.2.2 and II.1.6). The underlying *tree operads* have received more attention as the subject has developed.

To quote Boardman and Vogt [**BV73**], 'the trees are inspired by the attempt to obtain a general composite operation from a collection of indecomposable operations.'

Let $\underline{\mathcal{T}ree}(n)$ be the set of (isomorphism classes of) planar trees with 1 root at the bottom and n leaves at the top, regarded as labeled from left to right: 1 through n. The sequence $\underline{\mathcal{T}ree} = \{\underline{\mathcal{T}ree}(n)\}_{n\geq 1}$ forms a non-Σ operad in the category of sets: Given trees $S \in \underline{\mathcal{T}ree}(k)$, $T \in \underline{\mathcal{T}ree}(j)$, for each $1 \leq i \leq k$, let $S \circ_i T$ be the tree obtained by grafting the root of T to the leaf of S labeled i. This is indicated in Figure 2.

Similarly, let $\mathcal{T}ree(n)$ be the set of isomorphism classes of non-planar, abstract trees with 1 root and n leaves labeled (arbitrarily) 1 through n. The collection $\mathcal{T}ree = \{\mathcal{T}ree(n)\}_{n\geq 1}$ of tree spaces forms an operad by the same rule as for planar trees using the labeling: Given trees $S \in \mathcal{T}ree(k), T \in \mathcal{T}ree(j)$, for each $1 \leq i \leq k$, let $S \circ_i T$ be the tree obtained by grafting the root of T to the leaf of S labeled i. The symmetric groups act by permuting the labels.

We will see later in Section II.1.9 that $\underline{\mathcal{T}ree} = \{\underline{\mathcal{T}ree}(n)\}_{n\geq 1}$ and $\mathcal{T}ree = \{\mathcal{T}ree(n)\}_{n\geq 1}$ are respectively a free non-Σ operad and a free operad and play a central role in many applications.

A brief note regarding the 'direction of gravity' is in order here. We talk about trees, grafting, etc., and this botanical terminology naturally suggests to draw these objects as in Figure 2, with the root at the bottom.

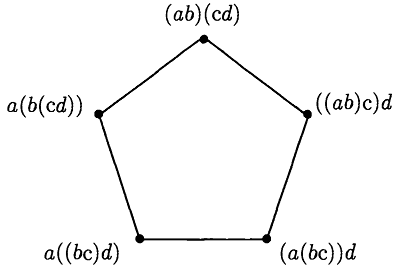


FIGURE 3. The pentagon. The sides represent a single application of a specific associating homotopy $h(u,v,w)$ from $u(vw)$ to $(uv)w$. For example, the bottom edge above from left to right is given by $h(a,bc,d)$.

On the other hand, we should always keep in mind that these pictures symbolize operations, that is, they are in fact a kind of flow chart, so it is more natural and convenient to have inputs at the bottom rather on the top and use 'up-rooted' trees instead.

The authors' personal inclination is the second convention, while a traditional one is the first. So in this historical overview we use botanical trees, and then we switch to up-rooted ones. The only exceptions from the up-rooted convention will be Sections II.3.4 and II.3.5 with trees associated to surjections and trees with levels which are more natural to draw with the root at the bottom.

1.6. A_∞-spaces and loop spaces

In 1957, Sugawara began to publish a series of papers [**Sug57a, Sug57b, Sug61**] characterizing H-spaces and loop spaces. One of his most important results was a *recognition principle* in homotopy invariant terms identifying which H-spaces were of the homotopy type of a loop space. This led in 1961 to Stasheff's homotopy characterization of connected based loop spaces in terms of what we would now call the non-Σ operad of associahedra [**Sta63a**].

In the course of simplifying Sugawara's recognition principle for loop spaces (by eliminating the use of homotopy inverses), Stasheff was led to consider higher homotopies for associativity, e.g. the pentagon in Figure 3

While Stasheff continued this work on H-spaces as a student in Oxford, Frank Adams came to visit the Mathematical Institute and discussed his work with Mac Lane on PACTs and PROPs. With a key idea from Adams, Stasheff created the polytopes now known as *associahedra* [**Sta61, Sta63a**]. The name is due to Gil Kalai, a geometric combinatorialist [**Lee89, Lee85**].

The associahedron K_n can be described now as a convex polytope with one vertex for each way of associating n ordered variables, that is, ways of inserting parentheses in a meaningful way in a word of n letters. For $n = 5$, a portrait due to Masahico Saito is in Figure 4.

To describe all the cells of K_n, the language of planar rooted trees is helpful, as Adams indicated to Stasheff. The dimension d-cells of K_n are all of the form $\prod_{i=1}^k K_{n_i}$, where $d = n_1 + \cdots + n_k - 2k$. In particular all the facets (cells of

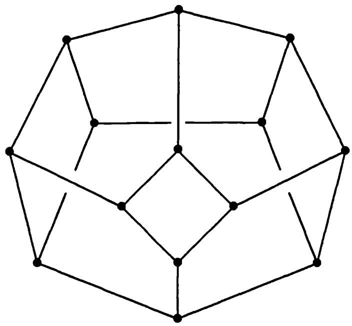


FIGURE 4. Saito's portrait of K_5.

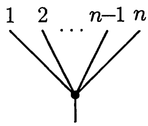


FIGURE 5. The corolla.

codimension one) are of the form $K_r \times K_s$, $r + s = n + 1$. If we let the top dimensional cell of K_n be indexed by the *corolla*, the planar tree with n leaves (see Figure 5), then the facets of e.g. K_4 are labeled as in Figure 6.

In general, the facets are labeled by grafting the s-corolla to leaf i of the r-corolla. The corresponding inclusion of facets

$$\circ_i : K_r \times K_s \hookrightarrow K_{r+s-1}$$

makes the set of associahedra $\underline{\mathcal{K}} = \{K_n\}_{n\geq 1}$ into a non-Σ operad.

Jumping ahead in time, there was an 'open' problem in combinatorial geometry (posed around 1978 and solved in 1984 by Haiman) as to whether there was a *convex polytope* realizing the associahedron. As originally constructed, the K_n were convex but curvilinear, although linearized early on by Milnor (unpublished).

Thanks to Kapranov, who bridged between the topological and combinatorial communities, the combinatorialists realized that the solution predated the question. A particular realization as a truncation of the standard simplex and related to symplectic moment maps and toric varieties was presented by Shnider and Sternberg in their book ([**SS94**], second edition). The cover illustration describes K_5 as a truncated 3-simplex. The general case is described in more detail in an appendix to [**Sta97**].

An A_∞-*space* Y is a topological space Y together with a family of maps

$$m_n : K_n \times Y^n \to Y$$

which fit together to make Y an algebra over the non-Σ operad $\underline{\mathcal{K}} = \{K_n\}_{n\geq 1}$. The main result is the following theorem.

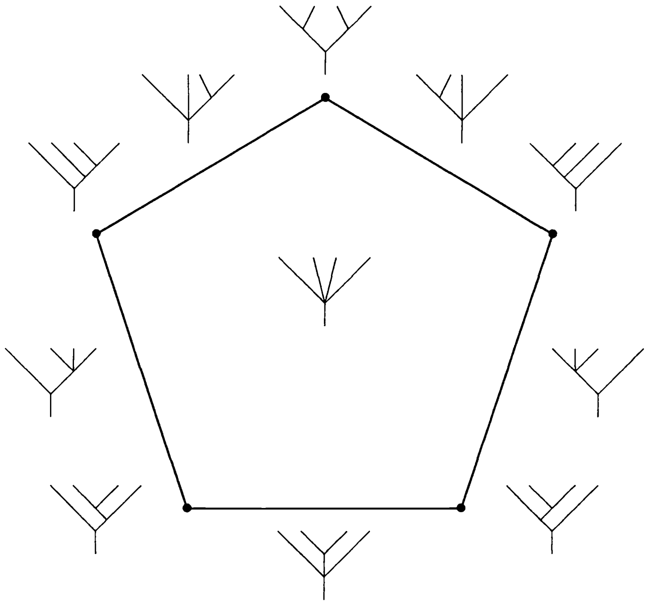


FIGURE 6. K_4 with facets labeled by trees.

THEOREM. *A connected space Y (of the homotopy type of a CW-complex with a nondegenerate base point) has the homotopy type of a based loop space ΩX for some X if and only if Y admits the structure of an A_∞-space.*

This result, like Sugawara's, is the archetype of results in the theory of homotopy invariant structures on topological spaces. To show that a connected A_∞-space Y has the homotopy type of a loop space involves constructing a 'classifying space' BY, defined as a quotient

$$BY := \coprod_{n\geq 0} K_{n+2} \times Y^n / \sim$$

where the identification $\sim$ involves the structure maps m_n. One then shows that Y has the homotopy type of the loop space ΩBY.

After his work categorizing loops spaces homotopy theoretically, Sugawara considered 'strongly homotopy commutative' strictly associative H-spaces as a step toward identifying two fold loop spaces $\Omega^2 X$. The loop multiplication on $\Omega^2 X$ must be homotopy commutative (cf. the standard proof that $\pi_1(\Omega X)$ is abelian), but the commuting homotopy must satisfy a whole family of coherence relations with regard to products of an arbitrary number of elements. In fact, for an associative H-space Y, the existence of an H-space structure $BY \times BY \to BY$ requires a coherent family of higher homotopies $h_n : I^{n-1} \times Y^n \to Y$ of which h_2 is a commuting homotopy and h_3 is related to Mac Lane's hexagon condition for coherence of a monoidal category.

Such a family $\{h_n\}$ is a special case of the A_∞-maps between A_∞-spaces X and Y whose homotopy classes correspond precisely to homotopy classes of maps $BX \to BY$ (see Proposition II.2.23). Sugawara's strong homotopy commutativity was a homotopy invariant description for a connected associative H-space having the homotopy type of the loop space on an H-space; such a characterization of a double loop space (the loop space of a loop space) awaited the invention of operads.

1.7. E_∞-spaces and iterated loop spaces

A major force in the development of this sort of higher homotopy structure was the study of infinite loop spaces, originating with Bott periodicity which showed that the classifying space BU of the unitary K-theory had the homotopy type of the connected component of the double loop space $\Omega^2 BU$ and hence can be regarded as an iterated loop space $\Omega^k X_k$ for all k.

A space X_0 is called an *infinite loop space* if there is a sequence of spaces X_k and homotopy equivalences $X_k \sim \Omega X_{k+1}$ for $k \geq 0$. The early work of Boardman and Vogt on this problem dealt with characterizing such spaces as 'homotopy everything' spaces or what are now called E_∞-spaces. Homotopy everything is a bit of an exaggeration (as they acknowledge in [**BV73**]) since an E_∞-operad action implies that, up to homotopy, we are dealing with a single binary operation.

In May's terminology [**May72**], an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ is an E_∞*-operad* if and only if each $\mathcal{P}(n)$ is Σ_n-free and contractible. Similarly, a non-Σ operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ is an A_∞*-operad* if and only if each $\mathcal{P}(n)$ is contractible.

THEOREM. *A connected space Y (of the homotopy type of a CW-complex with a nondegenerate base point) has the homotopy type of a based infinite loop space, i.e. has the homotopy type of $\Omega^k X_k$ for some spaces X_k for all k, if Y admits the structure of an E_∞-space.*

Boardman and Vogt were able to identify many infinite loop spaces, e.g. the classifying space BPL for PL bordisms, by use of the linear isometries operad (see Section II.2.7), which has recently been used to great advantage in a very powerful approach to the stable homotopy category [**May98**].

Homotopy characterization of iterated loop spaces $\Omega^k X_k$ for some space X_k for some particular k required the full power of the theory of operads (see Section II.2.2). The relevant operad is one of the most basic topological examples, the *little k-cubes operad* originally due to Boardman and Vogt [**BV68**], $\mathcal{C}_k = \{\mathcal{C}_k(n)\}_{n\geq 1}$, where $\mathcal{C}_k(n)$ consists of an ordered collection of n k-cubes linearly embedded in the standard k-dimensional unit cube I^k with disjoint interiors and axes parallel to those of I^k.

THEOREM. *A connected space Y (of the homotopy type of a CW-complex with a nondegenerate base point) has the homotopy type of a based k-fold loop space $\Omega^k X_k$ for some space X_k if Y admits the structure of a $\mathcal{C}_k$-space.*

For $k = 1$, $\mathcal{C}_1$ is an A_∞-operad of which the earlier associahedron operad $\mathcal{K}$ forms a small 'model.' Small models for the $\mathcal{C}_k$ for $k > 1$ came much later; see [**GJ94, MS99**].

For some applications, particularly to mathematical physics, the *little k-disks operad* $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n\geq 1}$ has some advantages. Let $B^k \subset \mathbb{R}^k$ denote the standard unit ball. The space $\mathcal{D}_k(n)$ consists of an ordered collection of n embeddings of B^k in B^k by translation and dilation with disjoint interiors.

Since an element of $\Omega^k X_k$ can be regarded as a based map $(S^k, *) \to (X_k, *)$, there is an action of the rotation group $SO(k)$ on $\Omega^k X_k$. The operad that reflects this symmetry is the *framed little k-disks operad* $\mathfrak{f}\mathcal{D}_k = \{\mathfrak{f}\mathcal{D}_k(n)\}_{n\geq 1}$. The space $\mathfrak{f}\mathcal{D}_k(n)$ consists of an ordered collection of k embeddings of B^k in B^k by translation, rotation and dilation with disjoint interiors. This extra $SO(k)$-symmetry is relevant to the study of free loop spaces and mapping spaces $X^{S^{k-1}}$ [**SW01**], and other applications (see Section II.4.1).

For A_∞-spaces and E_∞-spaces, we have the simple description of A_∞-non-Σ- and E_∞-operads as having contractible components; for $\mathcal{C}_n$-spaces, the characterization of an operad homotopy type is more subtle; see also Section II.2.1.

1.8. A_∞-algebras

Although introduced originally in the category of topological spaces, (non-Σ) operads were implicit almost immediately in the differential graded category. Thanks to the cellular structure of the associahedra K_n, the cellular chain complexes $\{CC_*(K_n)(n)\}_{n\geq 1}$ form a non-Σ operad $\underline{\mathcal{A}ss}_\infty$ in the category of dg $\mathbb{Z}$-modules. Algebras over $\underline{\mathcal{A}ss}_\infty$ (called A_∞-algebras) generalize dg associative algebras and can be defined directly.

An A_∞-*algebra* (or strongly homotopy associative algebra) consists of a graded module V with maps

$$m_n : V^{\otimes n} \to V \text{ of degree } n-2$$

satisfying suitable compatibility conditions $(A_n)_{n\geq 1}$. In particular,

(A_1) $m_1 = d$ is a differential,
(A_2) $m = m_2 : V \otimes V \to V$ is a chain map, that is, d is a derivation with respect to $m = m_2$,
(A_3) $m_3 : V^{\otimes 3} \to V$ is a chain homotopy for associativity of the multiplication m, i.e.

$$m_3 d^{[3]} + dm_3 = m(m \otimes 1\!\!1) - m(1\!\!1 \otimes m),$$

where $d^{[3]}$ denotes $d \otimes 1\!\!1^{\otimes 2} + 1\!\!1 \otimes d \otimes 1\!\!1 + 1\!\!1^{\otimes 2} \otimes d$,
(A_4) m_4 is a 'higher homotopy' such that $m_4 d^{[4]} - dm_4$ has five terms, corresponding to the edges of the pentagon K_4:

$$m_4 d^{[4]} - dm_4 = m_3(m_2 \otimes 1\!\!1^{\otimes 2} - 1\!\!1 \otimes m_2 \otimes 1\!\!1 + 1\!\!1^{\otimes 2} \otimes m_2) - m_2(m_3 \otimes 1\!\!1 + 1\!\!1 \otimes m_3).$$

A precise formulation of axioms for A_∞-algebras is given in Example II.3.132.

An alternative formulation generalizes the bar construction for an associative differential graded algebra: An A_∞-algebra structure on a graded vector space A is equivalent to a coderivation differential δ of degree -1 with respect to the total grading on the tensor coalgebra $\mathbb{T}^c(\uparrow A)$ on the suspension of the graded vector space A (see Example II.3.90). As a coderivation, δ is determined by the formula $\delta = \delta_1 + \delta_2 + \cdots$, where

$$\delta_n(\uparrow a_1 \otimes \cdots \otimes \uparrow a_n) := \epsilon \cdot \uparrow m_n(a_1 \otimes \cdots \otimes a_n), \text{ for } a_1, \ldots, a_n \in A,$$

where ϵ is an appropriate sign; see also [**Mar92**].

1.9. Partiality and A_∞-categories

The definition of an operad postulates the operations γ or $\circ_i$ being defined for all elements of $\mathcal{P}(k) \times \mathcal{P}(j_1) \times \cdots \times \mathcal{P}(j_k)$, respectively all elements of $\mathcal{P}(k) \times \mathcal{P}(j)$, just as the n-ary operations of an algebra are usually specified to be defined for all n-tuples of elements. There are, however, many naturally occurring examples in which the relevant operations are defined only on specified subsets (compare relaxing the definition of a monoid to that of a category). Such *partial* structures are of interest in a variety of ways.

A *partial algebra* [**KM93**] X over an operad $\mathcal{P}$ denotes a structure in which there are specified Σ_n-invariant *domains* $X_n \subset X^n$ and maps $\mathcal{P}(n) \times X_n \to X$ satisfying the appropriately restricted version of the usual axioms.

The definition of a *partial operad* similarly relaxes the definition by specifying domains of *composable* elements in $\mathcal{P}(k)\times\mathcal{P}(j_1)\times\cdots\times\mathcal{P}(j_k)$ and requiring the symmetries and associativity relations to hold only on such composable domains. A significant example is provided by vertex operator algebras (VOAs) [**Hua94, HL93**]. Another kind of partial operad structure is considered in [**Mar99a**] on configuration spaces of points in a Riemannian manifold. The Axelrod-Singer compactification [**AS94**] of the configuration space is then shown to be an appropriate operadic completion of this partial operad structure; see Section II.4.3.

A_∞-CATEGORIES. Just as a category can be considered as a partial algebra over the non-Σ associative operad $\underline{\mathcal{A}ss}$, so an A_∞-category is a partial algebra over the non-Σ operad $\underline{\mathcal{A}ss}_\infty$ for A_∞-algebras.

As the archetype of an A_∞-space is the space of based loops, so the archetype of a topological A_∞-category is the free path space $X^I = Map(I, X)$ considered as the space of morphisms for the space of objects X. Since the paths are of fixed parameter length, composition is associative only up to (higher) homotopy. However, the trivial path at a particular point acts as a unit only up to higher homotopy, an aspect we will avoid. To generalize in other contexts, think of a monoid as a category with one object and then proceed to a multi-object version. This was first done by Smirnov in 1987 [**Smi92**] to handle functorial homology operations and their dependence on choices (cf. indeterminacy).

More recently, Fukaya [**FS97**] reinvented A_∞-categories with remarkable applications to Morse theory and Floer homology. He [**FS97**] did this first in a graded module context where an *A_∞-category* C consists of

(i) a class of objects $Ob(C)$,
(ii) for any two objects $X, Y \in C$, a $\mathbb{Z}$-graded abelian group of morphisms $Hom(X, Y)$,
(iii) for all $n \geq 1$, composition maps

$$m_n : Hom(X_1, X_2) \otimes Hom(X_2, X_3) \otimes \cdots \otimes Hom(X_n, X_{n+1}) \to Hom(X_1, X_{n+1})$$

of degree $2 - n$,

such that the composition maps satisfy obvious modifications of the A_∞-identities recalled in Section 1.8 and Example II.3.132.

Since $m_1^2 = 0$, the modules of morphisms are in fact dg modules with a differential $d = m_1$ of degree $+1$. For morphisms f, g, h for which $(fg)h$ and $f(gh)$ are defined, these composites are required to be homotopic in the space of morphisms, the pentagon condition must be satisfied up to homotopy, etc. There is also a topological version of the above notion. Namely, a *topological A_∞-category $\mathcal{C}$* consists of

(i) a space of objects $Ob(\mathcal{C})$,
(ii) for any two objects $X, Y \in \mathcal{C}$, a space of morphisms $Hom(X,Y)$, with continuous target and source maps,
(iii) for all $n \geq 1$, continuous composition maps

$$m_n : K_n \times Hom(X_1, X_2) \times Hom(X_2, X_3) \times \cdots \times Hom(X_n, X_{n+1}) \to Hom(X_1, X_{n+1})$$

such that the composition maps satisfy obvious modifications of the axioms for A_∞-spaces (Section 1.6). Observe that in the above definitions we do not assume any kind of (homotopy) units. A very similar concept is that of Δ-category introduced by Vogt and Schwänzl in [**SV92b**].

Fukaya introduced A_∞-categories to study *Floer homology* in terms of *Morse theory*. He shows how Morse theory on a manifold M gives rise to an A_∞-category whose objects are smooth functions $f \in C^\infty(M)$. Let us briefly recall, following [**FS97**, Lecture 2], his construction. To avoid sign issues, we will assume that the coefficients are $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

For an n-dimensional smooth manifold M and a Morse function $f \in C^\infty(M)$, the *Morse homology* of M is defined as follows. Let $C_i(M,f)$ be the free $\mathbb{Z}_2$-module generated by the (finite) set of critical points p of f of index $\mu(p) = i$, $0 \leq i \leq n$. For two critical points p and q of f, let $\mathcal{M}(p,q)$ be the set of the negative (downhill) gradient flow lines which flow from p to q. It can be shown that, for $\mu(q) = \mu(p) - 1$, the set $\mathcal{M}(p,q)$ is 'generically' finite. The differential $\partial : C_i(M,f) \to C_{i-1}(M,f)$ is defined by

$$C_i(M,f) \ni p \longmapsto \partial(p) := \sum_{\mu(q)=i-1} \#\mathcal{M}(p,q)q \in C_{i-1}(M,f),$$

where $\#\mathcal{M}(p,q)$ denotes the number of elements of $\mathcal{M}(p,q)$. The homology of this complex coincides with the ordinary homology of M,

$$H_*(C_*(M,f),\partial) \cong H_*(M;\mathbb{Z}_2),$$

see also [**Sch93**, Definition 4.2]. Morse cochains are introduced as the vector space duals, $C^i(M,f) := (C_i(M,f))^*$; let $d : C^i(M,f) \to C^{i+1}(M,f)$ be the dual of ∂. Observe that

$$C_i(M,f) \cong C^{n-i}(M,-f).$$

For $f, g \in C^\infty(M)$ such that $f - g$ is a Morse function, Fukaya defines $Hom^i(f,g)$ to be the cochain complex

$$Hom^i(f,g) := C^i(M, f-g).$$

The differential $d = m_1$ is the differential of the Morse cochain complex recalled above.

The higher order m_n's are determined by graphs whose edge interiors correspond to negative (downhill) gradient flow lines relating $n+1$ critical points of $n+1$ functions.

For example, the product m_2 is determined using the graph Γ_3 with one vertex and three semi-infinite edges $e_i, i \in \{1,2,3\}$ (which are identified with the non-negative reals) as follows. Given $f_i \in C^\infty(M)$ such that $f_1 + f_2 + f_3 = 0$, let p_i be

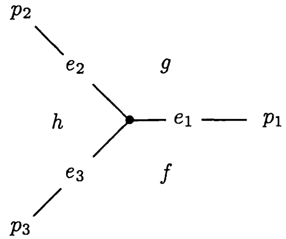


FIGURE 7. A scheme for computing m_2. If we read anticlockwise starting with the domain denoted by f, we see that e_1 is the gradient line of $g - f$, e_2 is the gradient line of $h - g$ and e_3 of $f - h$.

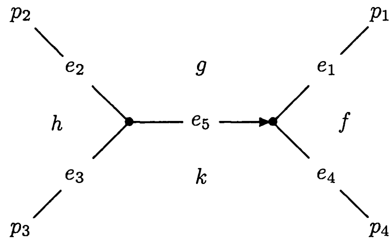


FIGURE 8. A graph used in Fukaya's definition of m_3.

critical points of f_i respectively. Define $\mathcal{M}(p_1, p_2, p_3) \subset Map(\Gamma_3, M)$ as those maps u such that $u|e_i$ is a gradient line for f_i converging to p_i. For $\mu(p_1)+\mu(p_2)+\mu(p_3) = 2n$, the space $\mathcal{M}(p_1, p_2, p_3)$ is 'generically' a finite set of points. Fukaya then defines, for $i + j + k = 2n$,

$$z_3 : C_i(M, f_1) \otimes C_j(M, f_2) \otimes C_k(M, f_3) \to \mathbb{Z}_2$$

by

$$z_3(p_1 \otimes p_2 \otimes p_3) := \#\mathcal{M}(p_1, p_2, p_3).$$

Dualization gives

$$m_2 : C^i(M, -f_1) \otimes C^j(M, -f_2) \to C^{i+j}(M, f_3).$$

Now for three maps f, g, h, put $f_1 := g - f$, $f_2 = h - g$ and $f_3 = f - h$ to get the structure map

$$m_2 : Hom^i(f, g) \otimes Hom^j(g, h) \to Hom^{i+j}(f, h).$$

The situation is schematically illustrated in Figure 7.

The higher m_n's are defined in a similar way, but using the moduli space of graphs (which in this case means planar unrooted trees with all vertices of valence ≥ 3) with $n + 1$ semi-infinite legs and internal edges of finite length.

An example of such a graph Γ used in the definition of m_3 is given in Figure 8. It has four 'semi-infinite' legs, $e_1, \dots, e_4$, and one oriented internal edge e_5 of a finite length $t \geq 0$. For smooth functions f, g, h, k such that $f_1 := g - f$, $f_2 := h - g$, $f_3 := k - h$, $f_4 := f - k$ and $f_5 := k - g$ are Morse functions, consider the space of all maps $u : \Gamma \to M$ such that $u|_{e_i}$ is the gradient flow of f_i, $1 \leq i \leq 5$. It can be shown that, for $\mu(p_1)+\mu(p_2)+\mu(p_3)+\mu(p_4)=3n-1$, the set of such maps is 'generically' finite. We may define, as before, for $i+j+k+l=3n-1$,

$$z_4 : C_i(M,f_1)\otimes C_j(M,f_2)\otimes C_k(M,f_3)\otimes C_l(M,f_4)\to \mathbb{Z}_2.$$

A dualization gives the structure operation

$$m_3 : Hom^i(f,g)\otimes Hom^j(g,h)\otimes Hom^k(h,l)\to Hom^{i+j+k-1}(f,l).$$

Fukaya's A_∞-categories regarded critical points of a Morse function as the objects and gradient flows as morphisms. Morse-Bott theory [**Bot88**] allows for critical sets which are not points, e.g. closed loops or paths with fixed boundaries. In the setting of symplectic and contact geometry, the morphisms can be taken to be special pseudo-holomorphic disks [**Eli00, ENS01, Dei00**]. In fact, the moduli spaces of such disks can be developed into an operad or the Swiss-cheese generalization [**Vor99b**]; see also Section 1.19.

Inspired by Fukaya's work, Nest and Tsygan [**NT97**] have proposed an A_∞-category with automorphisms of an associative algebra as objects and for the space of morphisms, a correspondingly 'twisted' version of the Hochschild complex of the corresponding endomorphism algebras.

Categories can be related to classifying spaces via the realization of the nerve of the category. As A_∞-maps can be (up to homotopy) identified with maps of the corresponding classifying spaces (see Proposition II.2.23), so lax functors (which are the name for A_∞-maps between categories) can be identified with maps of the corresponding realized nerves. Iterated classifying spaces appear in relation to monoidal categories, braided monoidal categories, etc.; see [**BFSV**].

1.10. L_∞-algebras

Perhaps because the very definition of a Lie algebra involves an additive structure and thus has no analog for spaces, the Lie analog L_∞-algebras of A_∞-algebras did not appear until the early 1980's in the work of Schlessinger and Stasheff [**SS85, SS84**] on perturbations of rational homotopy types, though examples abound in Sullivan's minimal models [**Sul77**] in the 1970's.

Just as an A_∞-algebra can be described either in terms of a family of multilinear operations or in terms of a coderivation differential on a tensor coalgebra, so an *L_∞-algebra* can be described either in terms of suitably symmetric n-ary brackets or in terms of a coderivation differential on the corresponding cocommutative coassociative coalgebra $\bigwedge^c$.

Since the Jacobi identity, consisting of three terms, does not correspond to a commutative diagram, one could not expect the axioms of L_∞-algebras to be modeled on a nice sequence of polyhedra as A_∞-algebras are modeled by the chain complex of the associahedra, the reason being roughly that there are no segments with three ends. But there is still a sequence of bipartite graphs which reflects the axioms of L_∞-algebras, introduced in [**MS01**] and called the *Lie-hedra*. The fourth term of this sequence is a bipartite graph derived from the Peterson graph in Figure 9. A precise definition of L_∞-algebras is given in Example II.3.133.

These L_∞-algebras became explicit in the context of deformation theory where the guiding principle (going back to Grothendieck but championed by Deligne and Schlessinger-Stasheff) is that a deformation problem is 'controlled' by a dg Lie algebra or, rather, by a homotopy type of dg Lie algebra.

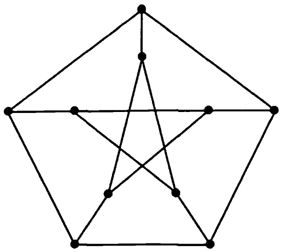


FIGURE 9. The Peterson graph.

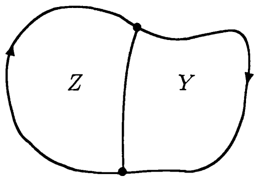


FIGURE 10. Two closed strings forming a third one.

In the early 1990's, L_∞-algebras appeared in various contexts, the most novel being that of closed string field theory as constructed by the physicist Zwiebach, where the relation to moduli spaces of punctured Riemann spheres is crucial.

As part of his closed string field theory [**Zwi93**], Zwiebach introduced a bracket on the differential forms on the free loop space of a manifold. The idea is that two closed strings (free loops) Y and Z join to form a third $Y * Z$ if a semicircle of one agrees with a reverse oriented semicircle of the other, as indicated in Figure 10. The loop $Y * Z$ is formed from the complementary semicircles of each.

On functions or differential forms on the free loop space, the convolution bracket then corresponds to all ways of decomposing a free loop in the form $Y * Z$. To be more precise, the convolution product of two *fields* ϕ, ψ, i.e. complex-valued functions on the space of strings, is defined by the formula

$$(\phi * \psi)(U) := \int_{U=Z*Y} \phi(Z)\psi(Y)d\mu, \tag{1.4}$$

where the integration is taken over all decompositions of the string U into Z and Y as in Figure 10, and μ is a certain measure which is in fact known to people who deal with stochastic integrals as a *Brownian bridge*, an analog of Brownian motion with both ends of the path fixed.

It can be shown that the product (1.4) is a bracket of a certain L_∞-structure whose higher homotopies can be described by similar explicit formulas over some more complicated configuration spaces.

The operad $\mathcal{L}ie_\infty$ for L_∞ algebras appeared independently about the same time [**GK94, HS93**]. Surprisingly, the operad for ordinary Lie algebras appears to have occurred first in [**HS93**], though it is clearly implicit in [**Coh88**]. Hinich and Schechtman give a description of the operad for L_∞-algebras in the language of trees, but attribute the discovery to Ginzburg and Kapranov, who moreover give the operad a very nice geometric interpretation which clarifies the relation to physics via (a stratification of) a real compactification of moduli spaces of Riemann spheres with punctures (see Section II.4.2). Ginzburg and Kapranov also put it in the context of the quadratic duality for operads they developed [**GK94**] (see Section II.3.2).

1.11. C_∞-algebras

Just as we have both A_∞-spaces and A_∞-algebras, so too we have E_∞-spaces and E_∞-algebras. These are algebras over an operad $\mathcal{P}$ for which the components $\mathcal{P}(n)$ are Σ_n-free and contractible dg modules, which are homotopy commutative and have all sorts of higher homotopies.

For algebras over an A_∞-operad, the small representative provided by the associahedra or their cellular chains has been very useful. We have no such small representatives of E_∞-operads and so no nice small set of higher homotopies for commutativity, though some explicit constructions of algebraic E_∞-operads were given [**Smi94**].

An interesting alternative is to consider strictly commutative A_∞-algebras. In that case, what symmetry should we require of the homotopies for associativity? There are 'standard constructions' which provide adjoint functors from dg Lie algebras to cocommutative dg coalgebras and from commutative dg algebras to dg Lie coalgebras ([**Qui69, Moo71**], see also the tables in Section 1.13). For dg Lie algebras L, the standard construction is the graded symmetric coalgebra on the suspension of L and thus the n-ary brackets of an L_∞-algebra have corresponding symmetry. The standard construction for dg commutative algebras is the Harrison chain complex, which is a dg Lie coalgebra (see Section II.3.8). This can be described, using a result of Ree [**Ree58**], as the quotient of the tensor coalgebra by the 'shuffle decomposables.' Thus an appropriate specification for a 'strictly commutative' A_∞-algebra A, to be called a *C_∞-algebra* (also known as a *balanced A_∞-algebra*), is via a coderivation on the graded Lie coalgebra on the suspension of A or equivalently in terms of a coherent set of n-ary products which vanish on shuffles.

1.12. n-ary algebras

Although we have emphasized that operads parameterize families of operations of various 'arities,' all the examples considered so far start with binary operations. On the other hand, there are generalizations of binary algebras which are generated by a single n-ary operation. Remarkably, in the case of Lie algebras, such generalizations have received considerable attention, beginning with Filipov in 1985 and continuing, independently, by Hanlon and Wachs [**HW95**] (combinatorial algebraists), by Azcárraga and Bueno [**APB97**] (physicists), and by Gnedbaye [**Gne96**] who is the only one to treat them operadically. *n-Lie algebras* have an alternating (or graded skew symmetric in the graded case) n-multilinear bracket $[-, \dots, -]$ which satisfies

$$\sum_{\sigma\in\Sigma_{2n-1}} (-1)^{\mathrm{sgn}(\sigma)}[v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n-1)}, [v_{\sigma(n)}, \dots, v_{\sigma(2n-1)}]] = 0.$$

For $n = 2$, the above equation is the standard Jacobi identity, thus 2-Lie algebras are ordinary Lie algebras. The above axiom is, up to degree and signs, that of an L_∞-algebra with only a single n-ary operation nonzero. See also [**VV98**].

On the other hand, Nambu invented, as a 'toy model' for quarks considered as triples, a trilinear bracket given, for smooth functions f_1, f_2, f_3 on $\mathbb{R}^3$, by the *Jacobian*:

$$\{f_1, f_2, f_3\} := \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} = \sum_{\sigma\in\Sigma_3} (-1)^{\mathrm{sgn}(\sigma)} \frac{\partial f_1}{\partial x_{\sigma(1)}} \frac{\partial f_2}{\partial x_{\sigma(2)}} \frac{\partial f_3}{\partial x_{\sigma(3)}}$$

This skew-symmetric trilinear product satisfies another generalization of Jacobi identity, namely

$$\{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4, \{f_1, f_2, f_5\}\} = \{f_1, f_2, \{f_3, f_4, f_5\}\}.$$

This was further generalized by Takhtajan [**Tak94**] to the *fundamental identity* for the generalized Nambu n-linear 'bracket.' This identity was known also to Flato and Fronsdal in 1992, though unpublished, and to Sahoo and Valsakumar [**SV92a**].

For a comparison of these two distinct generalizations of the Jacobi identity for n-ary brackets, see [**VV98**]. Other examples of n-ary algebras are introduced in [**Gne96**].

1.13. Operadic bar construction and Koszul duality

Since, as we will see in Section II.1.8, a dg operad $\mathcal{P}$ is itself a monoid in a symmetric monoidal category, the bar construction expressed in the appropriate generality applies to $\mathcal{P}$, producing a dg cooperad $\mathcal{B}(\mathcal{P})$. The linear dual of $\mathcal{B}(\mathcal{P})$ is then a dg operad denoted by $\mathbf{C}(\mathcal{P})$ and called the *cobar complex* of the operad $\mathcal{P}$, see Section II.3.1. We will also need the *dual dg operad* $\mathbf{D}(\mathcal{P})$, which is just $\mathbf{C}(\mathcal{P})$ suitably regraded. The operads $\mathbf{C}(\mathcal{P})$ and $\mathbf{D}(\mathcal{P})$ were introduced by Ginzburg and Kapranov in [**GK94**] while the operad $\mathcal{B}(\mathcal{P})$ was introduced by Getzler and Jones in [**GJ94**].

If $\mathcal{A}ss$ is the operad for associative algebras, then $\mathbf{D}(\mathcal{A}ss)$ is the operad $\mathcal{A}ss_\infty$ for A_∞-algebras and the homology $H_*(\mathbf{D}(\mathcal{A}ss))$ is again the operad $\mathcal{A}ss$ for associative algebras. If $\mathcal{L}ie$ is the operad for Lie algebras, then $\mathbf{D}(\mathcal{L}ie)$ is the operad $\mathcal{C}om_\infty$ for C_∞-algebras and $H_*(\mathbf{D}(\mathcal{L}ie)) = \mathcal{C}om$, the operad for commutative algebras. Similarly, the dual dg operad of the operad $\mathcal{C}om$ is the operad $\mathcal{L}ie_\infty$ for L_∞-algebras and the cohomology $H_*(\mathbf{D}(\mathcal{C}om))$ is the operad $\mathcal{L}ie$. We may systemize the above facts as:

$$\begin{aligned}
&\text{if } \mathcal{P} = \mathcal{A}ss, \text{ then } \mathbf{D}(\mathcal{P}) = \mathcal{A}ss_\infty \text{ and } H_*(\mathbf{D}(\mathcal{P})) = \mathcal{A}ss,\\
&\text{if } \mathcal{P} = \mathcal{L}ie, \text{ then } \mathbf{D}(\mathcal{P}) = \mathcal{C}om_\infty \text{ and } H_*(\mathbf{D}(\mathcal{P})) = \mathcal{C}om \text{ and}\\
&\text{if } \mathcal{P} = \mathcal{C}om, \text{ then } \mathbf{D}(\mathcal{P}) = \mathcal{L}ie_\infty \text{ and } H_*(\mathbf{D}(\mathcal{P})) = \mathcal{L}ie.
\end{aligned} \tag{1.5}$$

A similar pattern was observed in rational homotopy theory, where the following correspondence has been folklore:

associative algebra products $- \cdot -$ on a vector space A	$\longleftrightarrow$	square zero coderivations δ on the tensor coassociative coalgebra $\mathbb{T}^c(\uparrow A)$ with $\delta(\uparrow a \otimes \uparrow b) = (-1)^{\deg(a)+1} \uparrow (a \cdot b)$

commutative associative algebra products $-\cdot-$ on a graded vector space A	$\longleftrightarrow$	square zero coderivations δ on the Lie coalgebra $\mathbb{L}^c(\uparrow A)$ with $\delta([\uparrow a, \uparrow b]) = (-1)^{\deg(a)+1} \uparrow (a \cdot b)$
Lie algebra brackets $[-,-]$ on a graded vector space A	$\longleftrightarrow$	square zero coderivations δ on the cocommutative coassociative coalgebra $\wedge^c(\uparrow V)$ with $\delta(\uparrow a \wedge \uparrow b) = (-1)^{\deg(a)+1} \uparrow [a, b]$

On a more sophisticated level, the above pattern manifests itself in the $\mathcal{L}_*$ - $\mathcal{C}^*$ duality between Quillen and Sullivan models of a topological space or, even more abstractly, in the equivalence between the homotopy category of differential graded Lie algebras and the homotopy category of differential graded commutative algebras [**Tan83**].

A conceptual, operadic explanation for this pattern was provided by Ginzburg and Kapranov by their concept of quadratic (sometimes also called Koszul) duality for operads which defines, for any quadratic operad $\mathcal{P}$, its *quadratic dual* $\mathcal{P}^!$.

Quadratic operads are defined as those having a presentation of a special type (see Section II.3.2). As we will see later, all three operads $\mathcal{A}ss$, $\mathcal{L}ie$ and $\mathcal{C}om$ are quadratic, thus their duals are defined and they are

$$\mathcal{A}ss^! = \mathcal{A}ss,\ \mathcal{C}om^! = \mathcal{L}ie \text{ and } \mathcal{L}ie^! = \mathcal{C}om.$$

Ginzburg and Kapranov went further and introduced a certain homological property of quadratic operads. Operads sharing this property are called *Koszul* (quadratic) operads and are characterized by the isomorphism $H(\mathbf{D}(\mathcal{P})) = \mathcal{P}^!$. So table (1.5) above expresses the fact that all three operads $\mathcal{A}ss$, $\mathcal{L}ie$ and $\mathcal{C}om$ are Koszul.

The concept of quadratic dual and of Koszulness was inspired by a similar classical concept of Priddy for associative algebras, which is in accord with the opening remark of this section that operads are associative monoids in a suitable category.

Of course, mention of the bar construction should bring to mind the possibility of defining the 'homology of an operad' by analogy with Hochschild homology of an associative algebra. This was first considered by Markl in the context of PROPs [**Mar96a**] as a tool for the study of deformations of general algebraic structures, but the theory was not fully developed until the advent of Koszulness for operads as special cases of PROPs, which made explicit calculations possible.

As suggested by [**MS01**], the homology of an operad is closely related to the coherence of corresponding algebraic or categorial structures. A closed model category structure on the category of operads was studied by Hinich [**Hin97**]. The work of Baues, Jibladze and Tonks [**BJT97**] then introduced a homology theory for monoids in a monoidal category of which operads are special cases.

1.14. Cyclic operads

Algebras with invariant inner products $\langle -,- \rangle$ are of considerable importance in mathematics and especially in mathematical physics; invariance means that $\langle ab, c\rangle = \langle a, bc\rangle$ in the associative case and $\langle [a,b], c\rangle = \langle a, [b,c]\rangle$ in the Lie case. Using the inner product, the n-ary operations $A^{\otimes n} \to A$ can be converted to operations $A^{\otimes n+1} \to \mathbf{k}$ with cyclic symmetry. To handle such algebras via operads, the notion of a cyclic operad was introduced by Getzler and Kapranov [**GK95**] following ideas of Kontsevich [**Kon94**]. The formal definitions and applications are given in Sections II.5.1 and II.5.2, but the basic idea is well illustrated by the planar tree operad and by the associahedra. If we treat the root of a planar tree as just another leaf, then the order of leaves is now cyclic rather than linear. We no longer have a grafting operation $\circ_i$ but rather can define ${}_i\circ_j$ by grafting the ith leaf of the first tree to the jth leaf of the second tree and renumbering accordingly. It is obvious that the pentagon (see Figure 3) admits an action by the cyclic group $\mathbb{Z}/5\mathbb{Z}$ and an alternate view of K_5 shows a 6-fold symmetry. Recall (Section 1.6) that the associahedra form a non-Σ operad $\underline{\mathcal{K}} = \{K_n\}_{n\geq 1}$ which generates an operad $\mathcal{K} = \{\mathcal{K}(n)\}_{n\geq 1}$ freely: $\mathcal{K}(n) = \Sigma_n \times K_n$. Corresponding to the cyclic symmetry of unrooted planar trees with $n+1$ leaves and the visible cyclic symmetry of K_4 and K_5, the associahedra can be regarded as forming a cyclic non-Σ operad.

Now consider an A_∞-algebra with an inner product and convert the n-ary operations $A^{\otimes n} \to A$ to operations $A^{\otimes n+1} \to \mathbf{k}$ with cyclic symmetry. These *cyclic* A_∞*-algebras*, as well as their Lie and commutative analogs, play a key role in Kontsevich's formal noncommutative symplectic geometry via his graph cohomology (see Section II.5.5). They also make an appearance in the algebra of string field theory.

1.15. Moduli spaces and modular operads

One reason for the explosive development of operad theory in the 1990's was the introduction of operadic structures in topological field theories, e.g. CFT's (conformal field theories) and SFT's (string field theories), which in turn was inspired by the importance of moduli spaces of Riemann surfaces with punctures or boundaries (or other decorations) in these physical theories. A major tool in the study of such moduli spaces is the combinatorial structure of decompositions of Riemann surfaces into elementary pieces, such as a Riemann sphere with holes. The combinatorics can be described simply in terms of a graph so that a tubular neighborhood of the graph is topologically the surface. If the graph is a tree, the corresponding surface has genus 0; physicists will speak of a theory at tree level when it is governed by a moduli space of Riemann spheres with boundaries.

Since trees play such a fertile role in the theory of operads, it is perhaps not surprising that there is a generalization called a *modular operad* for which graphs provide the basic combinatorial structure [**GK98**].

To be more precise, while the 'flow chart' for an operad is a tree, the flow chart for a modular operad is given by a *stable labeled graph*, that is, by a graph equipped with a map g from the vertices to the natural numbers. In other words, g labels each vertex of the graph by a natural number, called the *genus* of the vertex. The labels are subject to the *stability condition*: at each vertex v, we have

$$2(g(v)-1)+val(v) > 0,$$

where val denotes the valence of the vertex, the number of (half) edges incident with the vertex.

The analog of 'leaves' for trees are the legs of a graph, i.e. half-edges with a vertex at only one end. If the legs of a labeled graph are labeled by a finite set S, the grafting operation for trees can be extended to graphs, with the additional subtlety that two legs of a single graph are allowed to be grafted to each other.

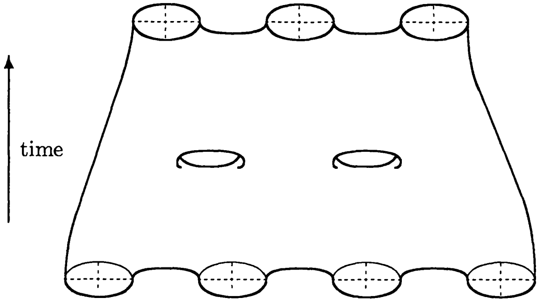


FIGURE 11. A word sheet representing four closed strings evolving into three via some nontrivial interactions.

The formal definition of a modular operad based on these operations is given in Section II.5.3. They play a central role in the physical theories mentioned above and considered in detail in Section II.5.7. That same 'down to earth' combinatorial quality reflected in the study of moduli spaces appeared earlier in Grothendieck's 'Esquisse' (see Section 1.1).

1.16. Operadic interpretation of closed string field theory

String theory deals with particles as maps of an interval into space (open strings) or of a circle into space (closed strings). In contrast, *string field theory* deals with *fields* which can be thought of as functions on or sections of a bundle over the space of such strings, i.e. a path space or a free loop space. The algebra of such fields is quite subtle since it is *not* given by pointwise multiplication of functions but rather is a convolution algebra derived from a (partially defined) product/composition of strings. Further, as strings evolve in space-time, they trace out world sheets, that is, maps of a Riemann surface with boundary into space-time. The surfaces are not just cylinders as the strings compose or decompose, as indicated in Figure 11. Such a surface can be regarded as a tubular neighborhood of a graph which in physics is a Feynman diagram. *Tree level* refers to tubular neighborhoods of trees or, equivalently, to Riemann surfaces of genus 0.

The algebra of a *closed string field theory (CSFT)* can be formalized while retaining the importance of the Riemann surfaces as generating the relevant operad. The structure is quite elaborate, so we start with a *conformal field theory (CFT).* We then extend it to a *topological conformal field theory (TCFT)* or *'string background'* which is a certain complex known as a *BRST complex* involving the Lie algebra of vector fields on the circle. Finally we use 'string vertices' which form a map of operads $s : \underline{\mathcal{N}} \to \widehat{\mathcal{M}}_0$, where the nth component $\underline{\mathcal{N}}(n)$ of $\underline{\mathcal{N}}$ is a compactification of the moduli space of Riemann spheres with $n+1$ marked and 'decorated' points while the nth component $\widehat{\mathcal{M}}_0(n)$ of $\widehat{\mathcal{M}}_0$ is the moduli space of Riemann spheres with $n+1$ punctures and disjoint holomorphic disks at each puncture. More precise definitions of these spaces are given below. Combining all the above data and passing to singular chains, we arrive at a CSFT at tree level as an algebra over the operad $C_*(\underline{\mathcal{N}}) = \{C_*(\underline{\mathcal{N}})(n)\}_{n\geq 1}$ of singular chains.

Somewhat more formally, a string field theory is formulated in terms of a 'state space' $\mathcal{H}$ which has the structure of a *conformal field theory* (CFT). Physicists usually have a (possibly indefinite) complex Hilbert space as their state space $\mathcal{H}$. We will initially need only a vector space still denoted $\mathcal{H}$. A CFT can be described, at tree level, as giving $\mathcal{H}$ the structure of an algebra over an operad $\widehat{\mathcal{M}}_0 = \{\widehat{\mathcal{M}}_0(n)\}_{n\geq 1}$ constructed from the moduli spaces $\widehat{\mathcal{M}}_0(n)$ of nondegenerate Riemann spheres Σ with $n+1$ punctures (labeled $1, \dots, n, \infty$) and disjoint holomorphic disks at each puncture (disjoint holomorphic embeddings of the standard disk $|z| < 1$ into Σ centered at the punctures). The spaces $\widehat{\mathcal{M}}_0(n)$, $n \geq 1$, form an operad under *sewing* Riemann spheres at punctures (cutting out the disks $|z| \leq r$ and $|w| \leq r$ for some $r = 1 - \epsilon$ at sewn punctures i and ∞ and identifying the annuli $r < |z| < 1/r$ and $r < |w| < 1/r$ via $w = 1/z$). The symmetric group Σ_n interchanges punctures along with the holomorphic disks, as usual. The action of the operad $\widehat{\mathcal{M}}_0$ is usually written as $\widehat{\mathcal{M}}_0(n) \ni \Sigma \mapsto |\Sigma\rangle \in Hom(\mathcal{H}^{\otimes n}, \mathcal{H})$.

The physics of closed strings requires that the theory be invariant under reparameterization of the strings, that is, invariant under diffeomorphisms of the circle. The corresponding Lie algebra, the algebra of smooth vector fields on the circle, is known as the *Virasoro algebra* Vir with central charge 0. Let us denote by $\mathcal{V}$ the complexification of this algebra, $\mathcal{V} := \mathrm{Vir} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Vir} \oplus \overline{\mathrm{Vir}}$. It is an infinite dimensional topological Lie algebra generated by the elements $L_m := z^{m+1}\partial/\partial z$, $m \in \mathbb{Z}$, with the commutators given by the formula $[L_m, L_n] = (n-m)L_{m+n}$, and by their complex conjugates $\overline{L}_m$.

A CFT has an implicit action T of $\mathcal{V}$ on $\mathcal{H}$ induced from the action of Vir on $\widehat{\mathcal{M}}_0$ given as follows: For $m \geq 0$, the vector field L_m extends to the disk inside the circle and exponentiates to a diffeomorphism of the disk. The corresponding twice punctured Riemann sphere has the disk at 0 given by this diffeomorphism, the disk at ∞ being standard. For $m < 0$, the roles of 0 and ∞ are reversed.

To qualify as a *string background*, a CFT must have the following additional *data*:

(i) a $\mathbb{Z}$-grading by the *ghost number*, $\mathcal{H} = \bigoplus \mathcal{H}^i$,
(ii) a differential (of square 0) of degree 1, denoted Q,
(iii) a representation $T : \mathcal{V} \to \mathcal{E}nd(\mathcal{H})$ of degree 0,
(iv) a map of degree -1, $b : \mathcal{V} \to \mathcal{E}nd(\mathcal{H})$ such that $b(v)^2 = 0$ for all $v \in \mathcal{V}$.

For $v \in \mathrm{Vir}$, $b(v)$ is called an *antighost operator*. The reason for the 'ghost-antighost' terminology will appear below. These data must satisfy certain *compatibility axioms*:

(v) as operators, for $v_1, v_2, v \in \mathrm{Vir}$,
(a) $[T(v_1), b(v_2)] = b([v_1, v_2])$, (b) $[Q, T(v)] = 0$, (c) $[Q, b(v)] = T(v)$ and
(vi) the operators $|\Sigma\rangle$ are homogeneous of ghost degree 0.

Notice that $[Q, b(v)] = T(v)$ indicates that $b(v)$ is a homotopy operator showing $T(v)$ is homotopic to zero, reflecting the fact that the action of $\mathcal{V}$ induces the trivial action on cohomology.

The graded space $\mathcal{H}$ with the operator Q is called a *BRST complex*. The homology of $(\mathcal{H}, Q)$ is called the *BRST cohomology*, emphasizing the special role of the Lie algebra in the complex. For a more detailed exposition of this approach to TCFT, see [**Vor94**].

There is a special realization of such a state space $\mathcal{H}$, the Batalin-Vilkovisky or BV-complex, which contains the Chevalley-Eilenberg complex for the $\mathcal{V}$-module $\mathcal{H}$. This is the structure utilized by Zwiebach [**Zwi93**]. In addition to being a dg commutative algebra, $\mathcal{H}$ has a *Gerstenhaber bracket* $\{-,-\} : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p+q-1}$, which is also called an *antibracket*. This means, after a shift in degree, $(\mathcal{H}, \{-,-\})$ is a graded Lie algebra and

$$\{x, yz\} = \{x, y\}z + (-1)^{(\deg(x)+1)\deg(y)} y\{x, z\}.$$

In other words, $\mathcal{H}$ forms a *Gerstenhaber algebra*, see Section 1.17.

This is achieved by starting with a space of fields Φ with basis ϕ^a and adjoining a free graded commutative algebra generated by *antighosts* $b_m, \bar{b}_m$ of degree 2 and *ghosts* $c_m, \bar{c}_m$ of degree -1 as well as *antifields* ϕ^*_a of degree 1. The antighost operators are now implemented as inner derivations: $b(L_m) = [b_m, -]$. The ghosts implement the Lie algebra differential as inner derivations $[c_m, -]$; see [**BV81, Zwi93**].

One of the nicest implications of the above structure of a string background is the construction of a morphism of dg operads from the operad of singular chains $C_*(\widehat{\mathcal{M}}_0) = \{C_*(\widehat{\mathcal{M}}_0(n))\}_{n\geq 1}$:

$$C_*(\widehat{\mathcal{M}}_0(n)) \to Hom(\mathcal{H}^{\otimes n}, \mathcal{H}),$$

extending linearly the action

$$\Sigma \in C_0(\widehat{\mathcal{M}}_0(n)) \mapsto |\Sigma\rangle$$

of the operad $\mathcal{M}_0$. This makes $\mathcal{H}$ into an algebra over the operad $C_*(\widehat{\mathcal{M}}_0)$ which, in turn, makes the BRST cohomology into an algebra over the operad $H_*(\widehat{\mathcal{M}}_0) = \{H_*(\widehat{\mathcal{M}}_0(n))\}_{n\geq 1}$. The operad $\widehat{\mathcal{M}}_0$ is homotopy equivalent to the framed little disks operad (Section 1.7), therefore, by Getzler's theorem [**Get94a**], $\mathcal{H}$ has an additional operator Δ which, together with the Gerstenhaber algebra structure, yields the structure of a BV (Batalin-Vilkovisky) algebra (see also Theorem II.4.7).

Let us consider the moduli space $\mathcal{N}(n)$ of Riemann spheres with $n+1$ decorated marked points, where a decoration is a choice of a real tangent direction at the marked point. In phyzspeak, these decorations are called 'phase parameters' at the point. There is a natural Σ_n-equivariant map $\widehat{\mathcal{M}}_0(n) \to \mathcal{N}(n)$ given by assigning to each holomorphic coordinate $u : U \to \Sigma$, where $U = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the unit disk, its *'phase parameter'* or argument $\arg(u)$ defined by

$$\arg(u) := arg\ \frac{d}{dt}u(t)\bigg|_{t=0},\ t \in \mathbb{R}.$$

The space $\mathcal{N}(n)$ has a natural compactification $\underline{\mathcal{N}}(n)$ which can be described in terms of stable n-punctured complex curves of genus 0 decorated with 'relative phase parameters' at double points and 'phase parameters' at punctures. The collection $\underline{\mathcal{N}} = \{\underline{\mathcal{N}}(n)\}_{n\geq 2}$ forms a pseudo-operad, with $\circ_i$-operations defined by identifying the relevant marked points and introducing nodal singularities as for the configuration pseudo-operad $\overline{\mathcal{M}}$ in Section II.4.2.

Then a *closed string field theory* (CSFT) over the string background as above involves a choice of a smooth map of operads $s : \underline{\mathcal{N}} \to \widehat{\mathcal{M}}_0$. The components $s(n) : \underline{\mathcal{N}}(n) \to \widehat{\mathcal{M}}_0(n)$ of s may be constructed as 'inverses' of the natural mappings $\widehat{\mathcal{M}}_0(n) \to \mathcal{N}(n) \hookrightarrow \underline{\mathcal{N}}(n)$. The construction, due to Zwiebach and Wolf [**WZ94**], is highly nontrivial. It is convenient to add the space $\underline{\mathcal{N}}(1) := (S^1)^2$ (the space of phase parameters θ_1 and θ_2 at 0 and ∞), defining the composition $\circ_i(\theta_1, \theta_2)$ for $(\theta_1, \theta_2) \in \underline{\mathcal{N}}(1)$ just by changing the phase parameter at the corresponding puncture by $\theta_1 + \theta_2$. This encodes the action of the rotation group on the spaces $\underline{\mathcal{N}}(1)$. The images $s(\underline{\mathcal{N}}(n)) \subset \widehat{\mathcal{M}}_0(n)$ of these mappings are called *string vertices*.

Finally, the map of operads $s : \underline{\mathcal{N}} \to \widehat{\mathcal{M}_0}$ allows us to define a *CSFT at tree level* as an algebra over the operad $C_*(\underline{\mathcal{N}}) = \{C_*(\underline{\mathcal{N}})(n)\}_{n\geq 1}$ of singular chains.

A CSFT implies the existence of an L_∞-algebra structure on $\mathcal{H}_{\mathrm{rel}}$, a BRST subcomplex (which is called semirelative in the physical literature and relative in the mathematics literature). The space $\mathcal{H}_{\mathrm{rel}}$ is defined as the subcomplex annihilated by the operators corresponding to rigid rotations of the disk: $b_0^- := b(L_0 - \overline{L}_0)$ and $L_0^- := L_0 - \overline{L}_0$ (see, for example, [**KSV95**, Section 4]).

We believe that the reader will find useful the following 'road map' which summarizes the constructions above:

$$\begin{aligned} \text{CFT} &:= \text{an } \widehat{\mathcal{M}}_0\text{-algebra } \mathcal{H}\,, \\ \text{string background} &:= \text{CFT} + \text{BRST differential} + \text{action of Vir}, \\ \text{CSFT} &:= \text{string background} + \text{string vertices}. \end{aligned}$$

A full-fledged closed string field theory (CSFT) requires an extension of the above structure to Riemann surfaces of arbitrary genera. In Section II.5.7 we take up this considerably more complicated structure which depends on the notion of modular operad due to Getzler and Kapranov [**GK98**] discussed in Section II.5.3.

1.17. From topological operads to dg operads

For a topological operad $\mathcal{P}$, chain complexes which respect products (functors from topological spaces to dg modules taking products to products) provide a *chain operad*, an operad in the category of dg modules over the ground ring, with the monoidal structure given by the tensor product. For example, the singular chain complex always works. Given a chain operad $\mathcal{P}$, the corresponding homologies $H_*(\mathcal{P}(n))$ form an operad $H_*(\mathcal{P}) = \{H_*(\mathcal{P}(n)\}_{n\geq 1}$ in the category of graded modules.

Since the associahedra are themselves cells or, rather, regular cell complexes with the operad structure given by the cellular inclusions $\circ_i : K_r \times K_s \to K_{r+s-1}$, their cellular chain complexes $\{CC_*(K_n\}_{n\geq 1}$ form a non-Σ chain operad, which is precisely the non-Σ operad $\underline{\mathcal{A}ss}_\infty$ for A_∞-algebras, while the homology $H_*(\underline{\mathcal{K}})$ is the non-Σ operad $\underline{\mathcal{A}ss}$ for associative algebras.

For any E_∞-operad, the homology operad is that for commutative associative algebras. The calculation for the little cubes operad is not so simple, but is known, thanks to the calculations of Fred Cohen [**Coh76**]. As a space, $\mathcal{C}_k(n)$ has the homotopy type of the configuration space $\mathrm{Con}(\mathbb{R}^k, n)$ of ordered n-tuples of points in $\mathbb{R}^k$.

THEOREM. *Let $k \geq 2$. An algebra over the operad $H_*(\mathcal{C}_k)$ is a graded algebra with two operations:*

(i) *a graded commutative associative product, $a \otimes b \mapsto ab$, and*
(ii) *a graded anticommutative 'bracket' of degree $k-1$, $a \otimes b \mapsto [a, b]$*

such that, after regrading, the bracket satisfies the usual graded Jacobi identity and the two operations are related by a graded Leibniz rule:

$$[a, bc] = [a, b]c + (-1)^{(\deg(a)+k+1)\deg(b)} b[a, c].$$

Some authors refer to this bracket structure as a 'k-Lie algebra'; unfortunately the same phrase is used for generalizations of Lie algebra in terms of a single k-ary 'bracket' [**VV98**], as mentioned above.

An algebra over the full homology $H_*(\mathcal{C}_k)$ is known as a *k-braid algebra* because of the relation of the homology to the braid group. It is a graded analog of a *Poisson algebra* (the $k = 1$ case of the above algebraic structure).

The original work on operads and comp algebras came together unexpectedly in 1994 when it was pointed out by Getzler [**Get94a**] that the Gerstenhaber algebra structure on the Hochschild cohomology of an associative algebra A with coefficients in itself was an algebra over $H_*(\mathcal{C}_2)$. Nowadays any algebra over $H_*(\mathcal{C}_2)$ (or sometimes over the homology $H_{-*}(\mathcal{C}_2)$ with the opposite grading), is called a *Gerstenhaber algebra.* The situation at the chain level is much more subtle and will be discussed in Section 1.19.

There is another way to construct dg operads from topological ones. Axelrod and Singer constructed in [**AS94**] a real compactification $\mathsf{F}_k(n)$ of the moduli space of configurations of n distinct points in $\mathbb{R}^k$. The collection $\mathsf{F}_k = \{\mathsf{F}_k(n)\}_{n\geq 1}$ is known to be an operad in the category of manifolds-with-corners [**Mar99a**]. It has a natural stratification with strata labeled by trees, and the E_1-term of the spectral sequence of this stratification is isomorphic to $\mathbf{D}(\mathcal{B}_k^!)$, the dual dg operad of the quadratic dual $\mathcal{B}_k^!$ of the operad $\mathcal{B}_k$ for k-braid algebras. The dual dg operad $\mathbf{D}(\mathcal{B}_k^!)$ contains the operad $\mathcal{L}ie_\infty$ as a suboperad generated by the quadratic dual of $\mathcal{C}om \subset \mathcal{B}_k$.

Thus, although there is no known topological operad with chains that give the $\mathcal{L}ie_\infty$ operad, the operad $\mathcal{L}ie_\infty$ is still a suboperad of a dg operad with a geometrical origin.

1.18. Homotopy invariance in algebra and topology

As mentioned in Section 1.6, operads were prefigured in the early investigations of loop spaces, topological groups and monoids in homotopy invariant terms. Boardman and Vogt's groundbreaking book "Homotopy Invariant Algebraic Structures on Topological Spaces" [**BV73**] begins with a brief history starting (in their Introduction) with Mac Lane's seminar at the University of Chicago in 1967. A detailed account of this seminar and its seminal influence has been given by Vogt [**Vog98**], who kept excellent notes. They remark:

> *'The disadvantage of topological groups and monoids is that they do not live in homotopy theory,'*

meaning that a space homotopy equivalent to a topological monoid need not admit a strictly associative multiplication (with unit) recognized by the equivalence. Stasheff's A_∞-spaces did live in homotopy theory; indeed, this was a major reason for studying them.

Boardman and Vogt developed their theory initially to establish various stable groups as infinite loop spaces, but soon expanded their research to more general homotopy invariant algebraic structures. Their book is written in the language of 'categories of operators' called PROPs. Operads give rise to a special class of PROPs and serve much the same purpose for the main applications.

Although the initial emphasis was on homotopy theory for topological spaces, from the very beginning there was implicit the study of homotopy invariant algebraic structures in other contexts. The appropriate setting for describing algebraic structures is now acknowledged to be that of a monoidal category (see Section II.1.1), while for homotopy theory it is that of a (closed) model category (CMC); see [**Qui67**]. We are content to work with (compactly generated) topological spaces or dg modules (chain complexes).

Since there is a very simple operad $\mathcal{M}on$ describing topological monoids, with $\mathcal{M}on(n) = \Sigma_n$, $n \geq 1$, (except for $\mathcal{M}on(0)$ which is a singleton) and the more complicated $\mathcal{K}$ for the homotopy invariant version of monoids, called A_∞-spaces, an operad must have some special properties to capture an algebraic structure homotopy invariantly. Somewhat analogously, for an associative algebra (without unit) the corresponding operad is $\mathcal{A}ss$ with $\mathcal{A}ss(n) = k[\Sigma_n]$, $n \geq 1$, with the more complicated $\mathcal{A}ss_\infty$ for A_∞-algebras; see Section II.3.10.

Given an operad $\mathcal{P}$ for an (ordinary) algebraic theory, a *strongly homotopy $\mathcal{P}$-algebra* is to be a homotopy invariant concept. By this we mean that each 'space' (which may be either a topological space or a chain complex, depending on the category we work in), homotopy equivalent to a $\mathcal{P}$-space, admits a strongly homotopy $\mathcal{P}$-structure recognized by the equivalence. In particular, each $\mathcal{P}$-space is of course also a strongly homotopy $\mathcal{P}$-space. From our modern perspective, a conceptual understanding of homotopy invariance is the following.

There exists, at least 'philosophically,' a closed model category (CMC) structure on the category of operads, both on the topological and on the algebraic side, though rigorous constructions of these closed model structures have been given only in some special cases. A strongly homotopy $\mathcal{P}$-algebra is now understood to be an algebra over a *cofibrant model* of $\mathcal{P}$. With respect to such a CMC structure, a cofibrant model of $\mathcal{P}$ is, by definition, a cofibrant operad $\mathcal{C}$ together with a weak equivalence (again in the CMC structure above) $\alpha : \mathcal{C} \to \mathcal{P}$. Homotopy invariance of a structure means that an object possesses the structure if and only if any homotopy equivalent object possesses the same structure induced by the homotopy equivalence. The importance of cofibrant operads is in this context expressed by the principle:

'Algebras over cofibrant operads are homotopy invariant.'

On the topological side, this principle is implicit in the above mentioned book by Boardman and Vogt [**BV73**], where a cofibrant resolution was provided by the W-construction $W\mathcal{P}$. The cofibrant nature of $W\mathcal{P}$ was clarified much later, in [**Vog**]. On the algebraic side, the analogous results were worked out in [**Mar99c**]. While a general cofibrant model is unique only up to a weak equivalence, in the category of chain complexes there exist special cofibrant operads, called *minimal operads*, having the wonderful property that they are *isomorphic* if and only if they are weakly equivalent; see Proposition II.3.120. A cofibrant model of $\mathcal{P}$ using this special kind of cofibrant operads is called the *minimal model* of $\mathcal{P}$ [**Mar96c**].

In the category of chain complexes (= differential graded modules), all classical strong homotopy algebras are algebras over these minimal cofibrant operads, so they are homotopy invariant in this category. See also Section II.3.10.

1.19. Formality, quantization and Deligne's conjecture

From the earliest days, operads have played a central role in the study of H-spaces with algebraic structures up to homotopy, but, in contrast, the role and importance of operads as a tool to describe and work with more general strong homotopy algebras was realized comparatively lately.

As explained in detail for topological spaces in Section II.2.9, a 'strongly homotopy $\mathcal{P}$-algebra' is defined so as to be a concept independent of the representative of the given homotopy type. As an extreme case, over a field, the homology $H(V)$ of a dg vector space (V, d) has the same chain homotopy type as (V, d) itself. If (V, d) has the structure of a dg $\mathcal{P}$-algebra, so does $H(V)$ with $d = 0$, but they are not necessarily equivalent as dg $\mathcal{P}$-algebras. A choice of homotopy equivalence of $H(V)$ with V (for example, for dg vector spaces over a field, a 'Hodge' decomposition $V = H \oplus X \oplus dX$) induces a structure of strongly homotopy $\mathcal{P}$-algebra on $H(V)$, which is equivalent to the original dg $\mathcal{P}$-algebra structure on V as strongly homotopy $\mathcal{P}$-algebra. This was first shown by Kadeishvili [**Kad80**], though implicit in Gugenheim [**Gug82**]: For the case of dg associative algebras A, the homology $H(A)$ is a (strictly) associative algebra, but still may have induced nontrivial homotopies forming an A_∞-structure (see Section II.2.6).

Sometimes the induced structure has trivial homotopies, e.g. $m_i = 0$ for $i > 2$ in the A_∞-case. This is formalized as follows (following the initial concept in rational homotopy theory):

DEFINITION. For a dg operad $\mathcal{P}$, an $\mathcal{P}$-algebra A is *formal* if there is a $\mathcal{P}$-algebra X with $\mathcal{P}$-algebra morphisms $A \leftarrow X \rightarrow H(A)$ inducing isomorphisms in homology.

For the associative commutative case in characteristic zero, this is often expressed informally by saying that 'vanishing of Massey products implies formality'; see the discussion of Massey products in Section II.2.6. Under some mild assumptions and over a field, formality is equivalent to the existence of a 'strongly homotopy $\mathcal{P}$-algebra map' $H(A) \rightarrow A$ which induces an isomorphism in homology.

For differential graded commutative algebras, examples abound in rational homotopy theory. Recall that a smooth manifold M is *formal* if the commutative associative dg algebra $\Omega^*_{\mathrm{DR}}(M)$ of de Rham forms is formal in the sense of the above definition. Examples of formal manifolds are compact Kähler manifolds, Lie groups and complete intersections [**DMGS75, Hal83**].

For differential graded Lie algebras, the most striking example is the dg Lie algebra of the Hochschild complex for the algebra $A = C^\infty(M)$ of smooth functions on M, where M is a Poisson manifold [**Kon97**]. This example occurs in the theory of deformation quantization, which refers to deforming a Poisson algebra to a noncommutative algebra with the deformation being given, to first order, by the Poisson bracket. The deformation theory of any associative algebra is controlled by the Lie algebra which is the Hochschild cochain complex $CH^*(A; A)$ of A with coefficients in itself, the bracket being the one defined by Gerstenhaber [**Ger63**]; see also Section II.3.9. When A is a Poisson algebra, the Poisson bracket represents a class θ in the second Hochschild cohomology $HH^2(A; A)$ which corresponds to an 'infinitesimal' deformation of the commutative product on A. The primary obstruction to extending the deformation, given by the Gerstenhaber bracket $[\theta, \theta] \in HH^3(A; A)$, vanishes but there are higher obstructions which do not, in general, vanish [**Mat97**].

Formality of the Hochschild complex $CH^*(A;A)$ as differential graded Lie algebra implies the vanishing of all the higher obstructions. Kontsevich proves the formality for $A = C^\infty(M)$ in the case of a Poisson manifold by constructing an L_∞-map from $HH^*(A;A)$ to $CH^*(A;A)$, thus exhibiting the higher order terms in the deformation. (Mathieu [**Mat97**] has an example to show that not every Poisson algebra can be deformation quantized.) Cattaneo and Felder [**CF99**] have given a Σ-model derivation of Kontsevich's quantization; it relates to the 'Swiss-cheese' operad of Voronov [**Vor98**].

Since operads themselves can be regarded as generalized algebras, it is not hard to define 'formality' for dg operads [**Mar96c**]:

DEFINITION. A dg operad $\mathcal{P}$ is *formal* if there is a dg operad $\mathcal{X}$ with dg operad morphisms $\mathcal{P} \leftarrow \mathcal{X} \rightarrow H(\mathcal{P})$ which induce isomorphisms in homology.

Here the important example is the operad variously known as e_2 or the G-operad describing Gerstenhaber algebras. The formality is an essential ingredient in some proofs of the Deligne conjecture, as we now explain.

DELIGNE'S CONJECTURE. *The Hochschild complex of an arbitrary associative algebra is naturally an algebra over a chain model of the little disks operad $\mathcal{D}_2$.*

This has now been verified by several researchers. The answer turns out to be somewhat less exciting than the original conjecture as many of the homotopies visible in the chains of the little disks operad act trivially on the Hochschild cochain complex. However, the variety of techniques developed in attacking the conjecture amply justify the effort.

Gerstenhaber's original work included specific homotopies for the commutativity of the cup product and for the Leibniz relation. In [**GV95**], Gerstenhaber and Voronov applied to the Hochschild complex multi-variable *brace* operations, introduced by Kadeishvili [**Kad88**] and later by Getzler [**Get94a**]. Gerstenhaber and Voronov interpreted these operations as homotopies (and higher homotopies) for the G-algebra identities. The relations of these operations together with the cup product on the Hochschild complex are given explicitly in [**GV95**]. It was clear in their work that the G-operad e_2 had a natural morphism to the homology of the operad describing the braces and the cup product. The latter leads to yet another interpretation of a '*homotopy G-algebra*,' with an operad to be denoted $\mathcal{HG}$ which acts naturally on the Hochschild complex via the braces and the dot product. This particular operad, $\mathcal{HG}$, was first described as such by McClure-Smith [**MS99**].

Deligne's conjecture is verified in a variety of ways, but all first provide a map of dg operads $C_*(\mathcal{D}_2) \rightarrow \mathcal{HG}$, where $C_*(\mathcal{D}_2)$ is a chain model of the little discs operad $\mathcal{D}_2$.

Getzler and Jones [**GJ94**] showed that the brace operations further implied the structure of a B_∞-*algebra* on the Hochschild complex. B_∞-algebras were introduced by Baues in his study of the double bar construction [**Bau81**]. (N.B. There is *no* operad B of which B_∞ is the higher homotopy version.) The B_∞-structure is given by identifying some of the Baues operations with braces and the cup product and sending the others to 0.

There is another set of higher homotopies defined in terms of the G_∞-operad which is the minimal model, in the sense of Markl [**Mar96c**], of the G-operad $e_2 = H_*(\mathcal{D}_2)$; see also Section II.3.10. The operad G_∞ can also be described as the dual dg operad of the quadratic dual of the G-operad as introduced in Ginzburg-Kapranov [**GK94**].

Notice that, compared to $\mathcal{HG}$, there are different choices here as to which relations are to be relaxed up to higher homotopy.

By a very involved construction, Tamarkin [**Tam98a**] provided an operad map $G_\infty \to B_\infty$. Then he [**Tam98b**] showed that the singular chain operad $C_*(\mathcal{D}_2)$ is formal, i.e. quasi-isomorphic to its homology $e_2 = H_*(\mathcal{D}_2)$. Kontsevich then found a simpler geometric proof of this result. The quasi-isomorphism in turn implies a map of operads $G_\infty \to C_*(\mathcal{D}_2)$. Later Tamarkin and Tsygan [**TT00**] found a rather simple algebraic construction of $G_\infty \to B_\infty$.

Then Voronov gave a solution of Deligne's Conjecture which involves a morphism $G_\infty \to B_\infty$ which comes from his realization of B_∞ as a quotient of another cofibrant model of e_2 obtained directly from the topology of moduli space of points on $\mathbb{CP}^1$.

In contrast, the recent proof by Kontsevich and Soibelman [**KS00**] constructs a combinatorial model of $\mathcal{HG}$ which has a map to a cell model of e_2 which they prove is a quasi-isomorphism. Their combinatorial model for $\mathcal{HG}$ is particularly interesting since it involves the 'insertion' of one tree at a vertex of another; see Section 1.20. Perhaps the following will keep track of the relations among the several operads:

$$\begin{tikzcd} & G_\infty \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & & \\ C_*(\mathcal{D}_2) & & B_\infty \arrow[r, "f_3"] & \mathcal{HG} \arrow[r, "f_4"] & \mathcal{E}nd(CH^*(A;A)) \end{tikzcd}$$

The operad map f_1 was constructed/implicit in Tamarkin's work [**Tam98b**] and given a new proof by Kontsevich [**Kon99**] who also extended it to higher dimensions (n-algebras). Similarly f_2 was first constructed by Tamarkin and then simplified by Tamarkin and Tsygan [**TT00**] and given an alternate geometric construction by Voronov [**Vor98**]. The map f_3 was implicit in Getzler and Jones [**GJ94**] who give $f_4 \circ f_3$ while Gerstenhaber-Voronov [**GV95**, Lemma 8] describe f_3 and f_4 explicitly. (We could add an arrow from $\mathcal{HG}$ to $C_*(\mathcal{D}_2)$ if we take the chains to be the cellular chains of the McClure-Smith cellular model. Also Kontsevich-Soibelman [**KS00**] construct one from the W-resolution of $\mathcal{HG}$ to another cell model of $\mathcal{D}_2$.)

1.20. Insertion operads

Insertion of one tree or more generally one graph in another leads to a variety of operads corresponding to at least three different types of insertion.

Inside the $\mathcal{HG}$-operad mentioned in Section 1.19 is the operad $\mathcal{B}race$ for an abstract brace algebra (forgetting the cup product), first described as such by Chapoton [**Cha00**] using the insertion operations of Kontsevich and Soibelman. We follow the exposition by Chapoton, of [**Cha00**, Section 2], together with the pictures in [**Cha00, CL00**] somewhat adapted.

Again we are dealing with rooted planar trees but where each leaf has two vertices while the root edge has only one. All the vertices are labelled (say by a finite set I). Define $\mathcal{B}race(I)$ to be the $\mathbb{Z}$-module generated by the set of such trees. Depict the tree T as lying in the unit upper semidisk centered at the origin

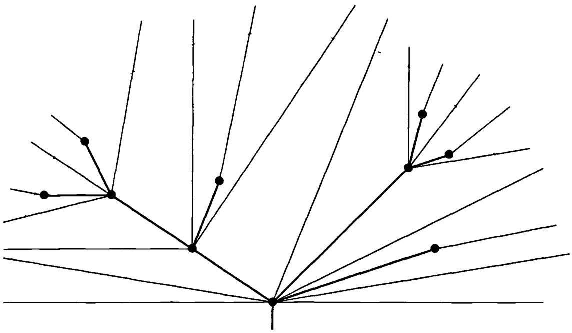


FIGURE 12. Angles (lines ending outside the semicircle) of a planar tree.

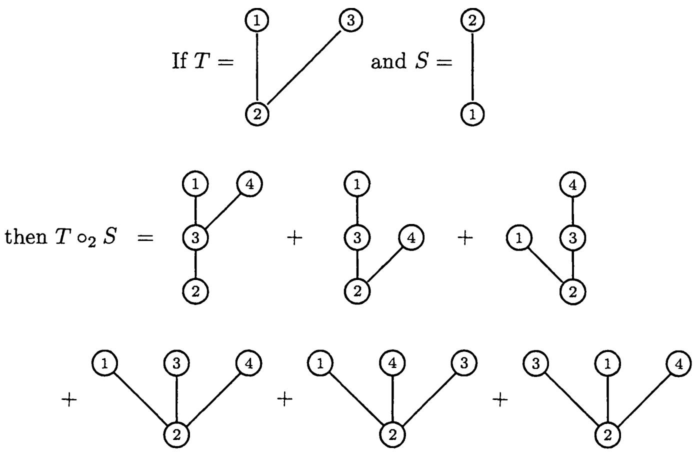


FIGURE 13. Example of composition operation $T \circ_j S$.

except for the root which extends below; see Figure 12. Define *Angles*(T) as non-intersecting arcs issuing from each vertex, one lying in each angle formed by adjacent edges at the vertex and ending at the upper semicircle boundary. These angles are ordered from left to right along the semicircle as are the edges of *In*(j) entering each vertex j.

The insertion of a tree S into a tree T at a vertex j will be defined in terms of a monotone nondecreasing function f from the ordered set of edges *In*(j) entering j to the ordered set *Angles*(S). Given such an f, define $T \circ_j^f S$ by excising a neighborhood of j leaving some dangling edges. The incoming edges are then grafted to the vertices of S along the angles indicated by the function f. The root edge of S is then identified with the outgoing edge from j The composition operation $T \circ_j S$ is just the sum over all such f; see Figure 13.

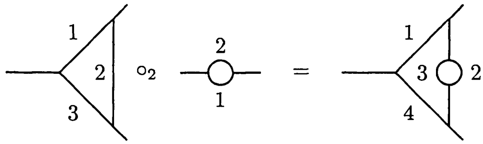


FIGURE 14. An edge insertion.

For the full $\mathcal{HG}$-operad, the trees are slightly more elaborate to accommodate the cup product, but the insertion idea is the same.

Physicists, too, are fond of insertions, leading to other operads and algebras over them. The $\circ_i$-operations correspond to inserting one tree or more generally one graph into another, either into an edge or into a vertex, in a way quite different from that of Kontsevich and Soibelman.

WARNING: Although terminology within graph theory is well established, in applications to physics and related parts of mathematics, terminology is not fixed. In particular watch out for the following: Leaves of a tree or legs of a graph may have one vertex at one end only or may have two vertices, one at each end. Rooted trees may have just a root vertex or may have a root edge ending in a root vertex. Trees may be oriented toward the root or away from it; cf. Section II.1.5.

Insertion of a graph with two external legs into a graph with labeled edges plays a very important role in 'renormalization of Feynman diagrams' and the Hopf algebras of Connes and Kreimer [**CK00**]. One edge is cut and the dangling edges are identified with the external legs of the graph being inserted. Either the edges are oriented or there is a sum over the two possibilities. See Figure 14.

The collection of all graphs with two external legs and numbered internal edges forms an operad under such insertions according to the numbers.

Various classes of graphs form operads under vertex insertion. For example, consider the vector space spanned by all k-valent graphs with the same number k of external legs and all vertices numbered. Given two such graphs T and S, the composition $T \circ_i S$ is obtained by 'inserting' S into vertex i, that is, removing the vertex i of T and identifying the dangling edges with the leaves of S, averaging over all such identifications. Repeated insertions lead to graphs such as, for $k = 3$, in Figure 15, which is very reminiscent of the little squares operad. For renormalization, there will be a need to consider edges decorated with 'momenta'; there's more to come!

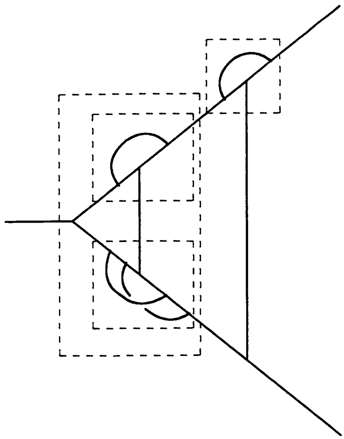


FIGURE 15. Vertex insertions.

Part II

CHAPTER 1

Operads in a Symmetric Monoidal Category

1.1. Symmetric monoidal categories

In this section we develop some of the tools necessary for the study of operads in full generality, beginning with some basic definitions and results from category theory [**Mac63b, Mac65**].

A *category with multiplication* is a category $\mathcal{C}$ and a covariant bifunctor $\odot$: $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. The multiplication is associative if there is an invertible natural transformation called the *associator* or *associativity constraint*,

$$\begin{aligned} a : \odot(\mathbb{1} \times \odot) &\to \odot(\odot \times \mathbb{1}) \\ a_{A,B,C} : A \odot (B \odot C) &\to (A \odot B) \odot C, \quad \text{for } A, B, C \in \mathcal{C}, \end{aligned} \tag{1.1}$$

where, as usual, $\mathbb{1}$ denotes the identity functor, $A\odot(B\odot C) = \odot(id\times\odot)(A, B, C)$, etc.

Having such an associator, we can consider diagrams whose vertices are iterates of $\odot$ and whose edges are applications of instances of a. Then $(\mathcal{C}, \odot)$ is called *coherent* if all these diagrams commute, the simplest being the pentagon (see Figure 1). Mac Lane [**Mac65**] proved that commutativity of the pentagon is a necessary and sufficient condition for coherence.

A category $\mathcal{C}$ is called *monoidal* if it has a multiplication $\odot$ with an associator a satisfying the Mac Lane pentagon condition and there is an object $\mathbf{1} \in \mathcal{C}$ called the *unit object* together with natural isomorphisms

$$l = l_A : \mathbf{1} \odot A \to A, \quad r = r_A : A \odot \mathbf{1} \to A$$

such that, in addition to the pentagon condition on a, the triangles in Figure 2 commute. It is an easy exercise (see [**Kel64**]) that commutativity of the triangle at the bottom together with the naturality of l and r imply the commutativity of the other two triangles. A monoidal category is called *strict* when all the $a_{A,B,C}$, l_A and r_A are identity morphisms. One formulation of Mac Lane's coherence theorem is that any monoidal category is equivalent to a strict monoidal category. Stated in this form, Mac Lane's theorem is sometimes called a rectification theorem.

A monoidal category is called *symmetric* if there exists a natural transformation

$$s_{A,B} : A \odot B \longrightarrow B \odot A,$$

which is of order two, $s_{B,A}s_{A,B} = id_{A\odot B}$, and satisfies the hexagon identity, which is equivalent to the commutativity of the diagram in Figure 3.

REMARK 1.1. There is an inconsistency in the notation in the literature, as to whether the associator should go from $A \odot (B \odot C) \to (A \odot B) \odot C$ or vice versa, but by the invertibility assumption, it does not matter which we chose. However, the different choices imply alternative forms of the pentagon identity. Mac Lane uses the symbol a for associativity and by this means that the parentheses are to

$$
\begin{tikzcd}
 & (A \odot B) \odot (C \odot D) \arrow[dr] & \\
A \odot (B \odot (C \odot D)) \arrow[ur] \arrow[d] & & ((A \odot B) \odot C) \odot D \\
A \odot ((B \odot C) \odot D) \arrow[rr] & & (A \odot (B \odot C)) \odot D \arrow[u]
\end{tikzcd}
$$

FIGURE 1. The pentagon.

$$
\begin{tikzcd}
A \odot (B \odot \mathbf{1}) \arrow[rr, "a"] \arrow[dr, "id \odot r"'] & & (A \odot B) \odot \mathbf{1} \arrow[dl, "r"] \\
 & A \odot B &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \odot (A \odot B) \arrow[rr, "a"] \arrow[dr, "l"'] & & (\mathbf{1} \odot A) \odot B \arrow[dl, "l \odot id"] \\
 & A \odot B &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \odot (\mathbf{1} \odot B) \arrow[rr, "a"] \arrow[dr, "id \odot l"'] & & (A \odot \mathbf{1}) \odot B \arrow[dl, "r \odot id"] \\
 & A \odot B &
\end{tikzcd}
$$

FIGURE 2. The coherence conditions on the unit. Actually, the condition at the bottom suffices.

be moved to the left. Drinfel'd uses the symbol Φ to mean the operation of moving the parentheses to the right.

The structure of *braided monoidal category* generalizes the structure of symmetric monoidal category by allowing a symmetry which is not necessarily of order two. In this case there is another hexagon identity (which is redundant in the symmetric monoidal case) given by commutativity of a diagram similar to Figure 3 with $s_{A\odot B,C}, s_{B,C}$ and $s_{A,C}$ replaced by $s^{-1}_{C,A\odot B}, s^{-1}_{C,B}$ and $s^{-1}_{C,A}$, respectively.

In the following discussion we will focus on two main examples. The first example is the category $\mathtt{Mod}_{\mathbf{k}}$ of $\mathbf{k}$-modules for a commutative ring $\mathbf{k}$ and, more generally, differential graded $\mathbf{k}$-modules. When $\mathbf{k}$ is a field, we will sometimes write $\mathtt{Vec}$ instead of $\mathtt{Mod}_{\mathbf{k}}$, $\mathtt{gVec}$ for the category of graded $\mathbf{k}$-modules and $\mathtt{dgVec}$ for the category of differential graded $\mathbf{k}$-modules.

The second example is the category $\mathtt{Set}$ of sets and its specialization to topological spaces. The category $\mathtt{Mod}_{\mathbf{k}}$ has a monoidal structure arising from the standard tensor product over $\mathbf{k}$,

$$A \odot B := A \otimes_{\mathbf{k}} B \text{ for } A, B \in \mathtt{Mod}_{\mathbf{k}} \text{ and } \mathbf{1} := \mathbf{k}.$$

$$\begin{tikzcd}
& A \odot (B \odot C) \arrow[dl, "id_A \odot s_{B,C}"'] \arrow[dr, "a"] & \\
A \odot (C \odot B) \arrow[d, "a"'] & & (A \odot B) \odot C \arrow[d, "s_{A \odot B, C}"] \\
(A \odot C) \odot B \arrow[dr, "s_{A,C} \odot id_A"'] & & C \odot (A \odot B) \arrow[dl, "a"] \\
& (C \odot A) \odot B &
\end{tikzcd}$$

FIGURE 3. The hexagon.

The monoidal structure on Set is given by the cartesian product

$$X \odot Y := X \times Y \text{ for } X, Y \in \mathtt{Set} \text{ and } \mathbf{1} := \{x\}, \text{ the set with one element.}$$

The subcategory of the category of sets consisting of Hausdorff spaces with compactly generated topology and continuous maps has a monoidal structure given by the cartesian product with the compactly generated topology. This was the setting in which operads were first defined by May, [**May72**].

In all the cases in the previous paragraph, the associator and the isomorphisms in the definition of the left and right units are given by the standard identifications. The symmetry is given by transposing factors. In the graded case, there is a sign factor, as described below. We point out that although these examples are not strictly monoidal, the identifications are canonical and the categories are coherent. An example of a strict monoidal category is given by the category of endofunctors on a given category with natural transformations as morphisms and composition as the monoidal structure.

To define a symmetric monoidal structure on the category of differential graded $\mathbf{k}$-modules, $\mathtt{dgMod}_{\mathbf{k}}$, extend the tensor product of $\mathbf{k}$-modules to a tensor product of complexes in the standard way:

$$(1.2) \qquad \begin{aligned} (A^* \otimes B^*)^k &:= \bigoplus_{i+j=k} A^i \otimes_{\mathbf{k}} B^j, \\ d_{A \otimes B}(a \otimes b) &:= d_A(a) \otimes b + (-1)^{deg(a)} a \otimes d_B(b), \\ s_{A,B}(a \otimes b) &:= (-1)^{deg(a) deg(b)} b \otimes a, \end{aligned}$$

for $(A^*, d_A), (B^*, d_B)$ objects of $\mathtt{dgMod}_{\mathbf{k}}$, $a \in A^*$, $b \in B^*$ homogeneous elements. We will assume that all differentials have degree 1 unless explicitly stated otherwise The unit object $\mathbf{1} = (\mathbf{1}^*, d_{\mathbf{1}} = 0)$ is given by

$$\mathbf{1}^j := \begin{cases} \mathbf{k} \text{ when } j = 0 \text{ and} \\ 0 \text{ when } j \neq 0. \end{cases}$$

A monoidal category is precisely the setting for the categorical concept of a monoid or "algebra" in Mac Lane's original terminology, [**Mac63b**]. Given a monoidal category $\mathcal{C}$, an object A is called a $\mathcal{C}$-*monoid* if there are $\mathcal{C}$-morphisms

$\mu : A \odot A \to A$, $\eta : \mathbf{1} \to A$ (respectively called the "multiplication" and "unit" for A) which satisfy associativity and unitarity conditions:

$$(1.3) \qquad \mu(id_A \odot \mu) = \mu(\mu \odot id_A)a_{A,A,A}, \quad \mu(id_A \odot \eta) = r_A, \quad \mu(\eta \odot id_A) = l_A.$$

The definition of morphisms of $\mathcal{C}$-monoids is the usual one; they must be compatible with all of the morphisms in the definition of a $\mathcal{C}$-monoid. We use $\mathtt{Mon}(\mathcal{C})$ to denote the resulting category.

1.2. Operads

Let $\mathtt{Set}_f$ be the category of nonempty finite sets and bijections. The category $\mathtt{Set}_f$ is equivalent to its skeleton, a small category Σ, called the *symmetric groupoid*. The objects of Σ are the natural numbers identified with the sets $[n] = \{1, \cdots, n\}$, and the morphisms are naturally identified with the symmetric groups

$$\Sigma_n = End_\Sigma([n]) = End_{\mathtt{Set}_f}([n]), \ n \geq 1.$$

In general, for a finite set X, define

$$\Sigma_X := End_{\mathtt{Set}_f}(X).$$

The category $\mathtt{Fun}(\mathtt{Set}_f^{op}, \mathcal{C})$ of contravariant functors from $\mathtt{Set}_f$ to $\mathcal{C}$ is called the category of $\mathtt{Set}_f$-modules and denoted $\mathtt{Set}_f$-$\mathtt{Mod}$. The category $\mathtt{Fun}(\Sigma^{op}, \mathcal{C})$ of contravariant functors from Σ to $\mathcal{C}$ is called the category of Σ-modules and will be denoted Σ-$\mathtt{Mod}$.

Note that for each $\mathtt{Set}_f$-module A and $X \in \mathtt{Set}_f$, the object $A(X)$ in $\mathcal{C}$ has a right Σ_X-action. A Σ-module A is represented by a sequence of objects, $\{A([n])\}_{n \geq 1}$ in $\mathcal{C}$ with a right Σ_n-action on $A([n])$. We will use the notation $A(n)$ in place of $A([n])$.

May's original definition of an operad [**May72**] was for the category of topological spaces, with compactly generated topology, but it generalizes with a few minor assumptions to an arbitrary symmetric monoidal category $\mathcal{C}$ with multiplication denoted by $\odot$. In Section 1.7 we will reformulate the definition for $\mathtt{Set}_f$-modules, but it is simpler to begin with a definition for Σ-modules.

One of the axioms in the definition of an operad involves an equivariance condition. The precise formulation is based on the following definition.

DEFINITION 1.2. For an ordered partition $\mathbf{m} = (m_1, \dots, m_n)$ of $m = \sum_{i=1}^n m_i$ and $\sigma \in \Sigma_n$, define the *block permutation* $\sigma_{m_1, \quad, m_n} \in \Sigma_m$ to be the permutation which acts on the set $[m]$ by permuting n intervals of lengths $m_1, m_2, \dots, m_n$ in exactly the same way that σ permutes the numbers $1, \dots, n$. More precisely, if

$$(m'_1, \dots, m'_n) := \sigma(m_1, \dots, m_n) = (m_{\sigma^{-1}(1)}, \dots, m_{\sigma^{-1}(n)}),$$

then $\sigma_{m_1, \quad, m_n}$ is defined to be one-to-one monotonic from the ith subinterval

$$\{j |\ m_1 + \cdots + m_{i-1} < j \leq m_1 + \cdots + m_i\}$$

of the partition (m_i) onto the $\sigma(i)$th subinterval

$$\{k |\ m'_1 + \cdots + m'_{\sigma(i)-1} < k \leq m'_1 + \cdots + m'_{\sigma(i)}\}$$

of the partition (m'_i).

Block permutations satisfy the composition rule

$$(\sigma\tau)_{m_1,\ ,m_n} = \sigma_{m_{\tau^{-1}(1)},\ ,m_{\tau^{-1}(n)}}\tau_{m_1,.\ ,m_n}.$$

EXAMPLE 1.3. For the purposes of this example, we will represent an element $\sigma \in \Sigma_n$ by the $2 \times n$ matrix

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}.$$

If $n = 3$, $m = 7$, $(m_1, m_2, m_3) = (2, 2, 3)$ and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

then $(m'_1, m'_2, m'_3) = (3, 2, 2)$.

The subintervals determined by m and m' are respectively $(12 \mid 34 \mid 567)$ and $(123 \mid 45 \mid 67)$. The block permutation $\sigma_{2,2,3}$ permutes subintervals according to the permutation σ, that is, it sends the first and third intervals of $(12 \mid 34 \mid 567)$ to the third and first intervals, respectively, of $(123 \mid 45 \mid 67)$:

$$\sigma_{2,2,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 5 & 1 & 2 & 3 \end{pmatrix}.$$

DEFINITION 1.4. (Operad – May's original version) An *operad* in a (strict) symmetric monoidal category $\mathcal{C}$ is a Σ-module $\mathcal{P}$ together with a family of structure morphisms

$$\gamma_{n,m_1,\ ,m_n} : \mathcal{P}(n) \odot \mathcal{P}(m_1) \odot \cdots \odot \mathcal{P}(m_n) \longrightarrow \mathcal{P}(m_1 + \cdots + m_n),$$

for $n \geq 1$, $m_1, \dots, m_n \geq 1$, satisfying the axioms:

1. Associativity. Given natural numbers m_i for $1 \leq i \leq n$ and $l_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$, define

$$m := \textstyle\sum_{1\leq i\leq n} m_i,\ l_i := \sum_{1\leq j\leq m_i} l_{i,j},\ l := \sum_{1\leq i\leq n} l_i$$

and

$$\mathbf{m} := (m_1, \dots, m_n),\ \mathrm{l} := (l_{1,1}, \dots, l_{n,m_n}),$$
$$\mathrm{l}_i := (l_{i,1}, \dots, l_{i,m_i}),\ \mathrm{l}' := (l_1, \dots, l_n).$$

Let

$$\mathcal{P}[\mathbf{m}] := \mathcal{P}(m_1) \odot \cdots \odot \mathcal{P}(m_n),\quad \mathcal{P}[\mathrm{l}_i] := \mathcal{P}(l_{i,1}) \odot \cdots \odot \mathcal{P}(l_{i,m_i}),$$
$$\mathcal{P}[\mathrm{l}] := \mathcal{P}(l_{1,1}) \odot \cdots \odot \mathcal{P}(l_{n,m_n}) = \mathcal{P}(\mathrm{l}_1) \odot \cdots \odot \mathcal{P}(\mathrm{l}_n)$$

and

$$\mathcal{P}[\mathrm{l}'] := \mathcal{P}(l_1) \odot \cdots \odot \mathcal{P}(l_n).$$

Then the following diagram commutes

(1.4)
$$\begin{tikzcd}
\mathcal{P}(n) \odot \mathcal{P}[\mathbf{m}] \odot \mathcal{P}[\mathbf{l}] \arrow[r, "\rho"] \arrow[dd, "\gamma_{n,\mathbf{m}} \odot \mathbb{1}"'] & \mathcal{P}(n) \odot (\mathcal{P}(m_1) \odot \mathcal{P}[\mathbf{l}_1]) \odot \cdots \odot (\mathcal{P}(m_n) \odot \mathcal{P}[\mathbf{l}_n]) \arrow[d, "\mathbb{1} \odot \bigodot_{i=1}^{n} \gamma_{m_i;\mathbf{l}_i}"] \\
 & \mathcal{P}(n) \odot \mathcal{P}[\mathbf{l}'] \arrow[d, "\gamma_{n;\mathbf{l}'}"] \\
\mathcal{P}(m) \odot \mathcal{P}[\mathbf{l}] \arrow[r, "\gamma_{m;\mathbf{l}}"] & \mathcal{P}(l)
\end{tikzcd}$$

where ρ applies the symmetry in $\mathcal{C}$ to permute the factors in $\mathcal{P}[\mathbf{m}] \odot \mathcal{P}[\mathbf{l}]$ to give a product of the factors $(\mathcal{P}(m_i) \odot \mathcal{P}[\mathbf{l}_i])$ for $1 \leq i \leq n$.

2. <u>Equivariance</u>. Given $\sigma \in \Sigma_n$ and an n-tuple $\mathbf{m} = (m_1, \ldots, m_n)$, define $\sigma\mathbf{m} := (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)})$. Let $\overline{\sigma} : \mathcal{P}[\mathbf{m}] \to \mathcal{P}[\sigma\mathbf{m}]$ be the permutation of the factors given by the symmetry in $\mathcal{C}$ and $\sigma_{\mathbf{m}}$ the block permutation described in Definition 1.2. Then the following diagram commutes:

(1.5)
$$\begin{tikzcd}
\mathcal{P}(n) \odot \mathcal{P}[\mathbf{m}] \arrow[r, "\mathbb{1} \odot \overline{\sigma}"] \arrow[dd, "\sigma \odot \mathbb{1}"'] & \mathcal{P}(n) \odot \mathcal{P}[\sigma\mathbf{m}] \arrow[d, "\gamma_{n;\sigma\mathbf{m}}"] \\
 & \mathcal{P}(m) \arrow[d, "\sigma_{\mathbf{m}}"] \\
\mathcal{P}(n) \odot \mathcal{P}[\mathbf{m}] \arrow[r, "\gamma_{n;\mathbf{m}}"] & \mathcal{P}(m).
\end{tikzcd}$$

3. <u>Unit</u>. If $\mathbf{1}$ is the unit object of $\mathcal{C}$, then there is an $\eta : \mathbf{1} \to \mathcal{P}(1)$ such that the composite morphisms

$$\mathcal{P}(n) \odot \mathbf{1}^{\odot n} \xrightarrow{\mathbb{1} \odot \eta^{\odot n}} \mathcal{P}(n) \odot \mathcal{P}(1) \odot \cdots \odot \mathcal{P}(1) \xrightarrow{\gamma_{n,1,\ ,1}} \mathcal{P}(n)$$

and

$$\mathbf{1} \odot \mathcal{P}(m) \xrightarrow{\eta \odot \mathbb{1}} \mathcal{P}(1) \odot \mathcal{P}(m) \xrightarrow{\gamma_{1,m}} \mathcal{P}(m)$$

are respectively the iterated right unit morphism and the left unit morphism for the underlying monoidal category $\mathcal{C}$.

Operads form a subcategory $\mathsf{Op}_{\mathcal{C}}$, or simply Op when $\mathcal{C}$ is understood, of the category of Σ-modules. Morphisms of operads are required to respect the structure morphisms in the source and target.

EXAMPLE 1.5. A trivial example of an operad in the category of sets is $\imath := \{\imath(n)\}_{n \geq 1}$, where

$$\imath := \begin{cases} [1] \text{ for } n = 1 \text{ and} \\ \emptyset \ \text{ for } n > 1, \end{cases}$$

with the obvious structure morphisms.

EXAMPLE 1.6. A slightly less trivial example is $C := \{C(n)\}_{n \geq 1}$ with

$$C(n) := [1] \text{ for } n \geq 1,$$

with the trivial action of Σ_n and the obvious structure morphisms.

One of the most important examples of an operad is the endomorphism operad.

DEFINITION 1.7. Let $(\mathcal{C}, \odot)$ be a symmetric monoidal category with internal Hom functor, $\underline{Hom}_{\mathcal{C}}$. The *endomorphism operad* $\mathcal{E}nd_X$ for an object $X \in \mathcal{C}$ is defined by

$$\mathcal{E}nd_X(n) := \underline{Hom}_{\mathcal{C}}(X^{\odot n}, X). \tag{1.6}$$

The structure morphism

$$\gamma_{n;m_1, \quad ,m_n} : \mathcal{E}nd_X(n) \odot (\mathcal{E}nd_X(m_1) \odot \cdots \odot \mathcal{E}nd_X(m_n)) \longrightarrow \mathcal{E}nd_X(m_1 + \cdots + m_n)$$

is defined by

$$\gamma_{n;m_1, \quad ,m_n}(\alpha, \beta_1, \ldots, \beta_n) := \alpha \circ (\beta_1 \odot \cdots \odot \beta_n).$$

The right Σ_n-action on $\mathcal{E}nd_X(n)$ is defined by composition with the left Σ_n-action on $X^{\odot n}$ induced by the symmetric monoidal structure on $\mathcal{C}$.

The operad $\mathcal{E}nd_X$ is particularly important because the theory of operads was developed as a tool in understanding algebraic or topological structures involving a sequence of n-ary operations on an object of a symmetric monoidal category. Such a sequence can be described as a morphism of Σ-modules $\mathcal{P} \to \mathcal{E}nd_X$, or under additional compatibility conditions, a morphism of operads. In the latter case, X is called a $\mathcal{P}$-algebra. A formal definition is given in Section 1.4 below.

EXAMPLE 1.8. To illustrate the equivariance axiom in Definition 1.4, consider the following data from the endomorphism operad $\mathcal{E}nd_X$ in the category $\mathtt{Set}$: $f, g : X^{\times 2} \to X \in \mathcal{E}nd_X(2)$ and $h : X^{\times 3} \to X \in \mathcal{E}nd_X(3)$. Let $\sigma \in \Sigma_2$ be the generator. The composition in $\mathcal{E}nd_X$ of the vertical arrow on the left of diagram (1.5) with the lower horizontal arrow applied to the 5-tuple $(a, b, c, d, e) \in X^{\times 5}$ is

$$\Big((f\sigma) \circ (h, g)\Big)(a, b, c, d, e) = (f\sigma)(h(a, b), g(c, d, e)) = f(g(c, d, e), h(a, b)).$$

On the other hand, the result of composing the upper horizontal arrow of the diagram with the first vertical arrow on the left applied to the same 5-tuple is

$$\Big(f \circ (\bar{\sigma}(h, g))\sigma_{2,3}\Big)(a, b, c, d, e) = f \circ (g, h)(c, d, e, a, b) = f(g(c, d, e), h(a, b)).$$

We see that these two expressions coincide.

There is also a dual concept of $\mathcal{P}$-coalgebra based on the coendomorphism operad.

DEFINITION 1.9. Let $(\mathcal{C}, \odot)$ be a symmetric monoidal category with internal Hom functor, $\underline{Hom}_{\mathcal{C}}$. The *coendomorphism operad* $\mathcal{C}o\mathcal{E}nd_X$ for an object $X \in \mathcal{C}$ is defined by

$$\mathcal{C}o\mathcal{E}nd_X(n) := \underline{Hom}_{\mathcal{C}}(X, X^{\odot n}). \tag{1.7}$$

The structure morphism

$$\mathcal{C}o\mathcal{E}nd_X(n) \odot \Big(\mathcal{C}o\mathcal{E}nd_X(m_1) \odot \cdots \odot \mathcal{C}o\mathcal{E}nd_X(m_n)\Big) \xrightarrow{\gamma_{n,m_1, \ldots, m_n}} \mathcal{C}o\mathcal{E}nd_X(m_1 + \cdots + m_n)$$

is defined by the diagram

$$\begin{tikzcd}
Co\mathcal{E}nd_X(n) \odot (Co\mathcal{E}nd_X(m_1) \odot \cdots \odot Co\mathcal{E}nd_X(m_n)) \arrow[rr, "\gamma"] \arrow[dr, "S"'] & & Co\mathcal{E}nd_X(m_1 + \cdots + m_n) \\
& (Co\mathcal{E}nd_X(m_1) \odot \cdots \odot Co\mathcal{E}nd_X(m_n)) \odot Co\mathcal{E}nd_X(n) \arrow[ur, "(\beta_1 \odot \cdots \odot \beta_n) \circ \alpha"'] &
\end{tikzcd}$$

The right Σ_n-action on $Co\mathcal{E}nd_X(n)$ is defined by composition with the right Σ_n-action on $X^{\odot n}$ induced by the symmetric monoidal structure on $\mathcal{C}$.

Another important example of an operad is given by the sequence of permutation groups $\Sigma := \{\Sigma_n\}_{n\geq 1}$ with structure morphisms defined with the help of Definition 1.2.

PROPOSITION 1.10. *The sequence of permutation groups $\Sigma := \{\Sigma_n\}_{n\geq 1}$ with morphisms*

$$\gamma_{n;m_1,\ldots,m_n} : \Sigma_n \times \Sigma_{m_1} \times \cdots \times \Sigma_{m_n} \to \Sigma_{m_1+\cdots+m_n},$$
$$\gamma_{n;m_1,\ldots,m_n}(\sigma, \rho_1, \ldots, \rho_n) := \sigma_{m_1,\ldots,m_n} \circ (\rho_1 \times \cdots \times \rho_n)$$

defined using the block permutations from Definition 1.2 has the structure of an operad in the category $\mathtt{Set}_f$.

PROOF. The proof is left as an exercise for the reader. ∎

DEFINITION 1.11. Given an operad $\mathcal{P}$ in the category of sets, there is an associated operad $\mathbf{k}[\mathcal{P}]$ in the category $\mathtt{Mod}_{\mathbf{k}}$ of $\mathbf{k}$-modules called the $\mathbf{k}$-*linearization* of $\mathcal{P}$ which is defined by

$$\mathbf{k}[\mathcal{P}](n) := \mathbf{k}[\mathcal{P}(n)],\ n \geq 1.$$

That is, the arity n component is the free $\mathbf{k}$-module with basis given by the set $\mathcal{P}(n)$.

DEFINITION 1.12. The *associative operad $\mathcal{A}ss$* in $\mathtt{Mod}_{\mathbf{k}}$ is defined by

$$\mathcal{A}ss := \mathbf{k}[\Sigma],$$

where Σ is the operad in $\mathtt{Set}$ defined in Proposition 1.10. The *commutative operad $\mathcal{C}om$* in $\mathtt{Mod}_{\mathbf{k}}$ is defined by

$$\mathcal{C}om := \mathbf{k}[C],$$

where C is the operad in $\mathtt{Set}$ defined in Example 1.6.

REMARK 1.13. We will see in Section 1.4 that the operad $\mathcal{A}ss$ is the operad describing associative algebras and the operad $\mathcal{C}om$ is the operad defining commutative algebras. The strange composition law in Proposition 1.10 comes from a natural composition law in the operad describing associative algebras. See Proposition 1.27 below.

It is sometimes useful to drop the equivariance axiom for an operad. This leads to the concept of a non-Σ operad.

DEFINITION 1.14. A *nonsymmetric operad* or *non-Σ operad* in a (not necessarily symmetric) monoidal category $\mathcal{C}$ is a sequence $\{\mathcal{P}(n)\}_{n\geq 1}$ of objects in $\mathcal{C}$ satisfying the associativity axiom and the unit axiom in Definition 1.4.

REMARK 1.15. Each operad can be considered as a non-Σ operad by forgetting the Σ_n-actions. On the other hand, given a non-Σ operad $\underline{\mathcal{P}}$, there is an associated operad $\mathcal{P}$ with $\mathcal{P}(n) := \underline{\mathcal{P}}(n) \times \Sigma_n$, $n \geq 1$, with the structure operations induced by structure operations of $\underline{\mathcal{P}}$ and the operad Σ introduced in Proposition 1.10. Here we mean by $\underline{\mathcal{P}}(n) \times \Sigma_n$ the coproduct of copies of $\underline{\mathcal{P}}(n)$ indexed by Σ_n. We call this process the *symmetrization* of a non-Σ operad.

For example, the operad $\mathcal{A}ss$ from Definition 1.12 is the symmetrization of the non-Σ operad $\underline{\mathcal{A}ss}$ given by $\underline{\mathcal{A}ss}(n) := \mathbf{k}$ for each $n \geq 1$. Other examples of non-Σ operads can be found in Section 1.5, Definition 1.41 and Section 1.6.

1.3. Pseudo-operads

Pseudo-operads are a variation on the theme of operads where the "many variable" operad composition laws are replaced with a family of binary composition laws. The axioms for these composition laws are the same as those for the "$\circ_i$-operations" introduced by Gerstenhaber [**Ger63**] in his study of the Hochschild cochain complex of an associative algebra, with the difference that in [**Ger63**] there was no symmetric group action, therefore no equivariance condition. For an operad there are analogous $\circ_i$-operations defined by composing the operad structure morphisms with the unit as follows:

$$\circ_i : \mathcal{P}(m) \odot \mathcal{P}(n) \xrightarrow{\cong} \mathcal{P}(m) \odot (\mathbf{1} \odot \cdots \odot \mathcal{P}(n) \odot \cdots \odot \mathbf{1}) \xrightarrow{\gamma_{m;1,\ldots,n,\ldots,1}} \mathcal{P}(m+n-1)$$

where $\mathcal{P}(n)$ is placed in position i of the factor in parentheses. The structure map $\gamma_{m;n_1,\ldots,n_m}$ is recovered as a composition of these $\circ_i$-product maps. Also, using this approach one can avoid the multi-indexing required in the standard description of the operad composition laws.

Markl in [**Mar96c**] defined 'pseudo-operads' or 'non-unitary' operads as Σ-modules with $\circ_i$-operations which satisfy axioms equivalent to the associativity and equivariance axioms of Definition 1.4 but without a unit axiom. In the definition below, operation $f \circ_i g$ should be thought of as 'substitution of g in position i of f.' This reverses the convention in [**Mar96c**], so the axioms are slightly different.

To deal with permutations, we use the composition law for permutations defined in Proposition 1.10. Let $\sigma \in \Sigma_m$, $\rho \in \Sigma_n$, and define $\sigma \circ_i \rho \in \Sigma_{m+n-1}$ by

$$\sigma \circ_i \rho := \sigma_{1,\ldots,1,n,1,\ldots,1} \circ (1 \times \cdots \times \rho \times \cdots \times 1), \tag{1.8}$$

where $\sigma_{1,\ldots,1,n,1,\ldots,1}$ is defined in Definition 1.2.

DEFINITION 1.16. (Markl's pseudo-operads) A *pseudo-operad* in a monoidal category $\mathcal{C}$ is a Σ-module $\mathcal{P}$ together with composition operations

$$\circ_i : \mathcal{P}(m) \odot \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1),\ m,n \geq 1,\ 1 \leq i \leq m,$$

satisfying the axioms:

1. Associativity. For the iterated compositions of $\mathcal{P}(m) \odot \mathcal{P}(n) \odot \mathcal{P}(p)$, the following associativity holds:

$$\circ_i(\circ_j \odot \mathbb{1}) = \begin{cases} \circ_{j+p-1}(\circ_i \odot \mathbb{1})(\mathbb{1} \odot \tau), & \text{for } 1 \leq i \leq j-1, \\ \circ_j(\mathbb{1} \odot \circ_{i-j+1}), & \text{for } j \leq i \leq j+n-1, \text{ and} \\ \circ_j(\circ_{i-n+1} \odot \mathbb{1})(\mathbb{1} \odot \tau) & \text{for } j+n \leq i, \end{cases} \tag{1.9}$$

where $\tau : \mathcal{P}(n) \odot \mathcal{P}(p) \to \mathcal{P}(p) \odot \mathcal{P}(n)$ is the transposition given by the symmetry.

2. Equivariance. The operation $\circ_i$ is equivariant in the sense:

$$\circ_i(\sigma \odot \rho) = (\sigma \circ_i \rho) \circ_{\sigma(i)} \text{ on } \mathcal{P}(m) \odot \mathcal{P}(n), \tag{1.10}$$

where $(\sigma \circ_i \rho)$ is defined in equation (1.8).

REMARK 1.17. The associativity axiom can be understood by considering the endomorphism operad $\mathcal{E}nd_X$ for $X \in \mathtt{Mod}_{\mathbf{k}}$; cf. Definition 1.7. In this case the strange terms displayed on the right of (1.9) can be seen to arise from the relabeling of the positions of the arguments after a $\circ_i$-operation. The symmetry τ is simply transposition. For example, for $\alpha \in \mathcal{E}nd_X(m)$, $\beta \in \mathcal{E}nd_X(n)$ and $\gamma \in \mathcal{E}nd_X(p)$ and $1 \leq i \leq j-1$,

$$\begin{aligned} &\circ_i(\circ_j \odot \mathbb{1})(\alpha \odot \beta \odot \gamma) \\ &= (\alpha \circ_j \beta) \circ_i \gamma = \big(\alpha(\mathbb{1}^{\odot j-1} \odot \beta \odot \mathbb{1}^{\odot m-j})\big)\,(\mathbb{1}^{\odot i-1} \odot \gamma \odot \mathbb{1}^{\odot m+n-1-i}) \\ &= \alpha(\mathbb{1}^{\odot i-1} \odot \gamma \odot \mathbb{1}^{\odot j-i-1} \odot \beta \odot \mathbb{1}^{\odot m-j}) \\ &= \big(\alpha(\mathbb{1}^{\odot i-1} \odot \gamma \odot \mathbb{1}^{\odot m-i})\big)\,(\mathbb{1}^{\odot j+p-1} \odot \beta \odot \mathbb{1}^{\odot m-j-1}) \\ &= (\alpha \circ_i \gamma) \circ_{j+p-1} \beta = \circ_{j+p-1}(\circ_i \odot \mathbb{1})(\mathbb{1} \odot \tau)(\alpha \odot \beta \odot \gamma). \end{aligned}$$

Just as there is a non-Σ version of May's operads, there is also a non-Σ version of pseudo-operads.

DEFINITION 1.18. A *nonsymmetric pseudo-operad* or *non-Σ pseudo-operad* in a monoidal category $\mathcal{C}$ is a sequence $\{\mathcal{P}(n)\}_{n\geq 1}$ of objects in $\mathcal{C}$ with $\circ_i$-operations satisfying the associativity axioms (1.9).

REMARK 1.19. There are examples of pseudo-operads that are not 'natural' operads (that is, we need to add the unit artificially if we wish to have an operad), such as the operad of stable marked genus 0 surfaces; see Section 4.2. The relation between operads and pseudo-operads will be made more precise in Section 1.7.

1.4. Operad algebras

As mentioned above, operads arose as a tool for studying algebraic or topological structures defined by a sequence of n-ary operations on an object in a suitable symmetric monoidal category. The following definition gives a more precise formulation.

DEFINITION 1.20. Let $\mathcal{P}$ be an operad in $\mathcal{C}$ a category with internal Hom functor as in Definition 1.7 and X an object in $\mathcal{C}$. A *$\mathcal{P}$-algebra* structure on the object X is a morphism of operads $\alpha_X : \mathcal{P} \longrightarrow \mathcal{E}nd_X$, that is, a family of Σ_n-equivariant morphisms

$$\alpha_X(n) : \mathcal{P}(n) \longrightarrow \mathcal{E}nd_X(n),\ n \geq 1,$$

compatible with the structure maps of $\mathcal{P}$ and $\mathcal{E}nd_X$. The $\alpha_X(n)$ are called *$\mathcal{P}$-algebra structure* morphisms or simply *$\mathcal{P}$-structure* morphisms.

Equivalently, a $\mathcal{P}$-algebra structure on X is a family of Σ_n-equivariant morphisms in $\mathcal{C}$

$$\hat{\alpha}_X(n) : \mathcal{P}(n) \odot X^{\odot n} \longrightarrow X,$$

where the Σ_n-action on the target X is trivial. We also say that X is *an algebra over $\mathcal{P}$*.

The equivalence of the two descriptions follows immediately from the adjoint functor isomorphism for $\underline{Hom}_{\mathcal{C}}$,

$$Hom_{\mathcal{C}}(\mathcal{P}(n), \underline{Hom}_{\mathcal{C}}(X^{\odot n}, X)) \cong Hom_{\mathcal{C}}(\mathcal{P}(n) \odot X^{\odot n}, X),$$

which preserves equivariance.

If we assume the category $\mathcal{C}$ has finite colimits, hence finite coequalizers, we can define a $\odot$-product of $\mathcal{P}(n)$ and $X^{\odot n}$ over Σ_n by

$$(1.11)\quad \mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n} := \underset{\sigma\,\in\,\Sigma_n}{\text{coequalizer}} \left\{\sigma^{-1} \odot \sigma : \mathcal{P}(n) \odot X^{\odot n} \longrightarrow \mathcal{P}(n) \odot X^{\odot n}\right\}.$$

Expressed in terms of the $\odot$-product over Σ_n, a $\mathcal{P}$-algebra structure on X is a compatible family of morphisms in $\mathcal{C}$

$$\hat{\alpha}_X(n) : \mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n} \longrightarrow X.$$

DEFINITION 1.21. Let X and Y be $\mathcal{P}$-algebras. A morphism $\varphi \in Hom_{\mathcal{C}}(X, Y)$ is a *morphism of $\mathcal{P}$-algebras* if the following diagram commutes for all $n \geq 1$:

$$\begin{tikzcd}
\mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n} \arrow[r, "\hat{\alpha}_X(n)"] \arrow[d, "1 \odot_{\Sigma_n} \varphi^{\odot n}"'] & X \arrow[d, "\varphi"] \\
\mathcal{P}(n) \odot_{\Sigma_n} Y^{\odot n} \arrow[r, "\hat{\alpha}_Y(n)"] & Y
\end{tikzcd}$$

Let $\mathcal{P}$-$\mathtt{alg}$ denote the category of $\mathcal{P}$-algebras.

REMARK 1.22. The definition of $\mathcal{P}$-algebra is formulated so that $\mathcal{A}ss$-algebras and $\mathcal{C}om$-algebras coincide with the standard definitions, where $\mathcal{A}ss$ and $\mathcal{C}om$ are defined in Definition 1.12. The definition of the operad $\mathcal{L}ie$ describing Lie algebras is not as easy to define as $\mathcal{A}ss$ and $\mathcal{C}om$, but there is a shortcut based on the free operad algebra.

DEFINITION 1.23. For $\mathcal{P}$ an operad in a symmetric monoidal category $\mathcal{C}$, the *free $\mathcal{P}$-algebra functor $\mathcal{F}_{\mathcal{P}}(-)$* is the (unique up to isomorphism) left adjoint to the forgetful functor $U_{\mathcal{P}} : \mathcal{P}\text{-}\mathtt{alg} \longrightarrow \mathcal{C}$, that is, for X an object in $\mathcal{C}$, $\mathcal{F}_{\mathcal{P}}(X)$ satisfies the defining relation for a left adjoint functor:

$$Hom_{\mathcal{P}\text{-}\mathtt{alg}}(\mathcal{F}_{\mathcal{P}}(X), Y) \cong Hom_{\mathcal{C}}(X, U_{\mathcal{P}}(Y)).$$

DEFINITION 1.24. Let $\mathcal{C}$ be a symmetric monoidal category such that $\odot$ is distributive over coproducts. Given an object $X \in \mathcal{C}$, define a functor on $\mathcal{C}$ called the *Schur functor of $\mathcal{P}$*:

$$\mathcal{S}_{\mathcal{P}}(X) := \coprod_{n \geq 1} \mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n}.$$

Observe that there are, for $n \geq 1$, obvious isomorphisms:

$$\mathcal{P}(n) \odot \mathcal{S}_{\mathcal{P}}(X)^{\odot n} \cong \coprod_{m_1, \dots, m_n} \mathcal{P}(n) \odot \big((\mathcal{P}(m_1) \odot_{\Sigma_{m_1}} X^{\odot m_1}) \odot \cdots \odot (\mathcal{P}(m_n) \odot_{\Sigma_{m_n}} X^{\odot m_n})\big).$$

PROPOSITION 1.25. *If we define*

$$\begin{aligned}\alpha(n; m_1, \dots, m_n) : \mathcal{P}(n) &\odot \big((\mathcal{P}(m_1) \odot_{\Sigma_{m_1}} X^{\odot m_1}) \odot \cdots \odot (\mathcal{P}(m_n) \odot_{\Sigma_{m_n}} X^{\odot m_n})\big) \\ &\cong \mathcal{P}(n) \odot (\mathcal{P}(m_1) \odot \cdots \odot \mathcal{P}(m_n)) \odot_{\Sigma_{m_1} \times \dots \times \Sigma_{m_n}} (X^{\odot m_1} \odot \cdots \odot X^{\odot m_n}) \\ &\xrightarrow{\gamma_{n; m_1, \dots, m_n} \odot \mathbb{1}} \mathcal{P}(m_1 + \cdots + m_n) \odot_{\Sigma_{m_1 + \dots + m_n}} X^{\odot (m_1 + \dots + m_n)},\end{aligned}$$

then the morphisms

$$\alpha_{\mathcal{S}_{\mathcal{P}}(X)}(n) := \coprod_{m_1, \dots, m_n} \alpha(n; m_1, \dots, m_n) : \mathcal{P}(n) \odot \mathcal{S}_{\mathcal{P}}(X)^{\odot n} \longrightarrow \mathcal{S}_{\mathcal{P}}(X),$$

induce a $\mathcal{P}$-algebra structure on $\mathcal{S}_{\mathcal{P}}(X)$. The free $\mathcal{P}$-algebra functor is isomorphic to the Schur functor $\mathcal{S}_{\mathcal{P}}$,

$$\mathcal{S}_{\mathcal{P}}(X) \cong \mathcal{F}_{\mathcal{P}}(X),$$

with the structure morphisms defined above.

PROOF. The structure morphisms for $\mathcal{S}_{\mathcal{P}}(X)$ are induced by the symmetries of the symmetric monoidal category and the composition morphisms for the operad $\mathcal{P}$, therefore the $\mathcal{P}$-algebra axioms for $\mathcal{S}_{\mathcal{P}}(X)$ follow from the naturality of the symmetries and the equivariance axioms for the operad $\mathcal{P}$.

In order for $\mathcal{S}_{\mathcal{P}}(-)$ to be a free $\mathcal{P}$-algebra functor, it should be a left adjoint to the forgetful functor $U_{\mathcal{P}}$ from the category of $\mathcal{P}$-algebras to $\mathcal{C}$. According to [**HS71**, Theorem II.7.2], the adjoint relation (adjunction) $\mathcal{S}_{\mathcal{P}} \dashv U_{\mathcal{P}}$ is equivalent to the existence of two natural transformations:

$$\zeta : \mathbb{1} \longrightarrow U_{\mathcal{P}} \mathcal{S}_{\mathcal{P}} \text{ and } \chi : \mathcal{S}_{\mathcal{P}} U_{\mathcal{P}} \longrightarrow \mathbb{1},$$

establishing a bijective correspondence

$$\begin{array}{rccl} & Hom_{\mathcal{P}\text{-alg}}(\mathcal{S}_{\mathcal{P}}(X), Y) & \cong & Hom_{\mathcal{C}}(X, U_{\mathcal{P}}(Y)), \\ (1.12) & \varphi & \longmapsto & U_{\mathcal{P}}(\varphi) \circ \zeta_X, \\ & \chi_Y \circ \mathcal{S}_{\mathcal{P}}(\psi) & \longleftarrow & \psi, \end{array}$$

for each $X \in \mathcal{C}$ and $Y \in \mathcal{P}\text{-alg}$. Define $\zeta_X \in Hom_{\mathcal{C}}(X, U_{\mathcal{P}} \mathcal{S}_{\mathcal{P}}(X))$,

$$\zeta_X : X \cong 1 \odot X \longrightarrow \mathcal{P}(1) \odot X \longrightarrow \coprod_{n \geq 1} \mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n} = U_{\mathcal{P}} \mathcal{S}_{\mathcal{P}}(X).$$

On the other hand, if X is a $\mathcal{P}$-algebra, then the structure morphisms $\widehat{\alpha}_X(n)$: $\mathcal{P}(n) \odot X^{\odot n} \longrightarrow X$ determine

$$\chi_X : \mathcal{S}_{\mathcal{P}} U_{\mathcal{P}}(X) = \coprod_{n \geq 1} \mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n} \longrightarrow X.$$

This defines the adjunctions. Bijectivity of the correspondence (1.12) follows immediately from the commutative diagram for a $\mathcal{P}$-algebra morphism:

$$\begin{tikzcd}
\mathcal{P}(n) \odot_{\Sigma_n} \mathcal{S}_{\mathcal{P}}(X)^{\odot n} \supset \mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n} \arrow[r, "\hat{\alpha}_{\mathcal{S}_{\mathcal{P}}(X)}(n)"] \arrow[d, "1 \odot_{\Sigma_n} \varphi_1^{\odot n}"'] & \mathcal{P}(n) \odot_{\Sigma_n} X^{\odot n} \subset \mathcal{S}_{\mathcal{P}}(X) \arrow[d, "\varphi_n"] \\
\mathcal{P}(n) \odot_{\Sigma_n} Y^{\odot n} \arrow[r, "\hat{\alpha}_Y(n)"] & Y
\end{tikzcd}$$

which shows that all the higher degree components φ_n of a $\mathcal{P}$-algebra morphism $\varphi : \mathcal{S}_{\mathcal{P}}(X) \to Y$ are determined uniquely by the 'linear term'

$$\varphi_1 : X \cong \mathbf{1} \odot X \to \mathcal{P}(1) \odot X \xrightarrow{\mathbb{1} \odot U_{\mathcal{P}}(\varphi) \circ \zeta_X} \mathcal{P}(1) \odot Y \longrightarrow Y. \qquad \blacksquare$$

EXAMPLE 1.26. In the symmetric monoidal category $(\mathtt{Mod}_{\mathbf{k}}, \odot := \otimes_{\mathbf{k}})$, the free $\mathcal{A}ss$-algebra is the tensor algebra and the free $\mathcal{C}om$-algebra is the symmetric algebra,

$$\begin{aligned}
\mathcal{F}_{\mathcal{A}ss}(X) &:= \bigoplus_{n=1}^{\infty} \mathbf{k}[\Sigma_n] \otimes_{\mathbf{k}[\Sigma_n]} X^{\otimes n} = \bigoplus_{n=1}^{\infty} X^{\otimes n} \quad \text{and} \\
\mathcal{F}_{\mathcal{C}om}(X) &:= \bigoplus_{n=1}^{\infty} \mathbb{1} \otimes_{\mathbf{k}[\Sigma_n]} X^{\otimes n} = \bigoplus_{n=1}^{\infty} S^n(X).
\end{aligned}$$

The following proposition shows how to reconstruct the operad from the free algebra functor.

PROPOSITION 1.27. *Let $\mathcal{P}$ be an operad in $\mathtt{Mod}_{\mathbf{k}}$ and X_n the free $\mathbf{k}$-module with basis $\{x_1, \dots, x_n\}$ for indeterminates x_i. Let $\mathcal{F}_{\mathcal{P}}(n)$ be the $\mathbf{k}$-submodule of $\mathcal{F}_{\mathcal{P}}(X_n)$ spanned by those tensor products with each x_i appearing precisely once,*

$$\mathcal{F}_{\mathcal{P}}(n) := \mathrm{Span}\,\{\alpha \otimes_{\Sigma_n} (x_{i_1} \otimes \cdots \otimes x_{i_n})|\ i_j \neq i_k \text{ for } j \neq k\} \subset \mathcal{P}(n) \otimes_{\Sigma_n} X_n^{\otimes n}.$$

Define a right action of Σ_n on $\mathcal{F}_{\mathcal{P}}(n)$ by

$$(\alpha \otimes (x_{i_1} \otimes \cdots \otimes x_{i_n}))\,\sigma := \alpha \otimes (x_{\sigma^{-1}(i_1)} \otimes \cdots \otimes x_{\sigma^{-1}(i_n)}).$$

Then the map

$$\rho : \mathcal{P}(n) \to \mathcal{F}_{\mathcal{P}}(n), \quad \rho : \alpha \mapsto \alpha \otimes (x_1 \otimes \cdots \otimes x_n)$$

is a Σ_n-equivariant isomorphism of $\mathbf{k}$-modules.

The $\circ_i$-operations in $\mathcal{P}$ and $\mathcal{P}$-algebra composition in $\mathcal{F}_{\mathcal{P}}$ are connected by the following equation:

$$\begin{aligned}
(1.13)\quad & (\alpha \circ_i \beta)(x_1 \otimes \cdots \otimes x_{n+m-1}) \\
&= \alpha\,(x_1 \otimes \cdots \otimes x_{i-1} \otimes \beta(x_i \otimes \cdots \otimes x_{i+m-1}) \otimes x_{i+m} \otimes \cdots \otimes x_{n+m-1}),
\end{aligned}$$

where x_i is identified with $\mathbf{1} \otimes x_i \in \mathcal{P}(1) \otimes X_{m+n-1}$.

PROOF. Using the Σ_n-action any element of $\mathcal{F}_{\mathcal{P}}(n)$ can be represented in the form $\alpha \otimes (x_1 \otimes \cdots \otimes x_n)$ for some $\alpha \in \mathcal{P}(n)$, therefore, ρ is onto. On the other hand $\alpha \otimes (x_1 \otimes \cdots \otimes x_n) = 0 \in \mathcal{P}(n) \otimes_{\Sigma_n} (X_n)^{\otimes n}$ implies $\alpha = 0$ so ρ is one-to-one. Since

$$\begin{aligned}
\rho(\alpha)\sigma &= (\alpha \otimes_{\Sigma_n} (x_1 \otimes \cdots \otimes x_n))\,\sigma \\
&= \alpha \otimes_{\Sigma_n} (x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}) = \alpha \otimes_{\Sigma_n} \sigma(x_1 \otimes \cdots \otimes x_n) \\
&= \alpha\sigma \otimes_{\Sigma_n} (x_1 \otimes \cdots \otimes x_n) = \rho(\alpha\sigma),
\end{aligned}$$

ρ is Σ_n-equivariant. The proof of the last assertion is an exercise in manipulating the definition of the composition law. ■

If there is an alternative description of the free algebra associated to a known algebraic structure, one can use Proposition 1.27 and the uniqueness of the free operad functor to describe the associated operad. An important example is that of Lie algebras, for which Proposition 1.27 allows us to define the Lie operad in terms of the free Lie algebra.

DEFINITION 1.28. Let $\mathtt{Free}_{\mathcal{L}ie}(x_1, \dots, x_n)$ be the free Lie algebra generated by $\{x_1, \dots, x_n\}$ in $\mathtt{Mod}_{\mathbf{k}}$. The *Lie operad* $\mathcal{L}ie := \{\mathcal{L}ie(n)\}_{n \geq 1}$ in $\mathtt{Mod}_{\mathbf{k}}$ has an arity n component isomorphic to the $\mathbf{k}$ submodule of $\mathtt{Free}_{\mathcal{L}ie}(x_1, \dots, x_n)$ generated by the Lie products which are linear in each of the x_i. The $\circ_i$-operations are defined by a substitution of the type appearing on the right side of (1.13).

For $\mathbf{k} = \mathbb{C}$, there is a beautiful formula due to Klyatchko [**Kly74**] describing $\{\mathcal{L}ie(n)\}_{n \geq 1}$ as induced representations of Σ_n induced up from the cyclic subgroup $C_n := \{(12 \dots n)^k |\ k = 0, \dots, n-1\} \subset \Sigma_n$:

$$\mathcal{L}ie(n) = Ind \uparrow_{C_n}^{\Sigma_n} \zeta_n,$$

where ζ_n is the one-dimensional complex representation of C_n

$$\zeta_n((12 \dots n)^m) = \exp \frac{2\pi m \sqrt{-1}}{n}.$$

1.5. The pseudo-operad of labeled rooted trees

The operad of trees which we will now describe is not just an interesting example, but is also a very useful tool in the general theory of operads. We begin this section with some basic definitions and terminology.

A tree is a finite connected contractible graph. We will modify the standard convention according to which all edges in a graph have two adjacent vertices and delete the vertices with only one adjacent edge. This means that some edges will have only one adjacent vertex and we call these edges *external edges or legs*. The edges which are adjacent to two vertices will be called *internal edges*. Occasionally it will be convenient to use the standard convention with two vertices adjacent to every edge, in which case we call a vertex adjacent to just one edge an *external vertex*. The remaining vertices will be called *internal vertices*. A more formal definition of a graph will be given in Section 5.3 on modular operads.

All trees are assumed to have at least one edge; the tree with just one edge (and no vertices) is called the *trivial tree*. A *corolla* is a tree with no internal edges. A *rooted tree* is a tree with a distinguished external edge, called the *root*. The remaining external edges are called *leaves*. An external vertex adjacent to a leaf will be called a *leaf vertex*, and the external vertex adjacent to the root, the *root vertex*. A rooted tree has a natural orientation with each edge oriented in the direction of the vertex closest to the root. The root edge is oriented toward the root vertex, but in the case of the trivial tree, this is ambiguous so we have to choose an orientation. In any rooted tree, every vertex is adjacent to a single outgoing edge. The *arity* of the vertex is the number of incoming edges. A tree with no vertices of arity one is called *reduced*. Unless otherwise indicated, we will assume all trees are rooted and oriented in the above sense.

We denote the set of (internal) vertices of a tree T by $Vert(T)$ and the set of edges coming into a vertex v by $In(v)$. When we need to include external vertices, $\overline{Vert}(T)$ will denote the full set of vertices. The set of all edges of T, including the root edge, will be denoted $Edge(T)$, the set of internal edges by $edge(T)$, and the set of leaves by $Leaf(T)$. We will also need the set $Edg(T)$ of all edges without the root (that is, '$Edge = Edg + \mathrm{e}$'). Thus

$$Edg(T) = edge(T) \sqcup Leaf(T) \text{ and } Edge(T) = Edg(T) \sqcup \{\text{the root edge of } T\}.$$

We will occasionally use the notation (v, v') to represent an edge, where $v, v' \in \overline{Vert}(T)$.

Trees form a subcategory of the category of graphs, but not a full subcategory. A morphism $T \to S$ in the category of graphs is an *epimorphism* $f : \overline{Vert}(T) \to \overline{Vert}(S)$ such that

$$\text{if } (v, v') \in Edge(T), \text{ then either } \begin{cases} (i) & (f(v), f(v')) \in Edge(S) \text{ or} \\ (ii) & f(v) = f(v'). \end{cases}$$

In case (ii), we say that f *collapses* or *contracts* the edge $\mathrm{e} = (v, v')$ to the vertex $f(v)$. Morphisms of trees are more restricted than morphisms of graphs. In the case of trees, a morphism may collapse only internal edges. We also assume that morphisms preserve roots of rooted trees. A morphism of trees is clearly an isomorphism if and only if no edge is contracted.

The *height* of an internal vertex is the number of edges separating it from the root of the tree. If we include external vertices, then we define the root vertex to be at height -1 (under ground). The height of a tree is the maximum height of its vertices including leaf vertices. Thus a corolla has height one.

DEFINITION 1.29. An *elementary morphism* (or an *elementary contraction*) $\pi_{\mathrm{e}} : T \to T/\mathrm{e}$ collapses one internal edge e and preserves the complement of e. We use the notation $\pi_{\{\mathrm{e}_1, \dots, \mathrm{e}_n\}} : T \to T/\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ to denote the composition of elementary morphisms collapsing the edges $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$.

The above definition is, of course, informal – we did not even say what the vertices of T/e are. Let $e = (u, v)$ for some $v, w \in Vert(T)$, then

$$Vert(T/e) = (Vert(T) - \{v, w\}) \sqcup \{\mathrm{e}\},$$

with the symbol $\{\mathrm{e}\}$ labeling the vertex of $Vert(T/\mathrm{e})$ obtained by collapsing the edge e and identifying v and w. The corresponding elementary contraction is then given by $f : \overline{Vert}(T) \to \overline{Vert}(T/\mathrm{e})$ defined

$$f(u) := \begin{cases} u, & \text{for } u \notin \{v, w\} \text{ and} \\ \mathrm{e}, & \text{for } u = v \text{ or } u = w. \end{cases}$$

We will use these informal definitions quite often in the book. It will always be clear how to make them formally correct.

REMARK 1.30. Each morphism of trees $f : T \to S$ induces a map $f^* : Edge(S) \to Edge(T)$ – notice the unexpected direction! For example, if $f : T \to T/e$ is an elementary contraction as in Definition 1.29, then $Edge(T/\mathrm{e}) = Edge(T) - \{\mathrm{e}\}$ and $f^* : Edge(T/\mathrm{e}) \hookrightarrow Edge(T)$ is the inclusion.

Each morphism of trees is clearly a composition of elementary contractions and isomorphisms.

A *planar imbedding* of a tree T is a monomorphism $f : \overline{Vert}(T) \to \mathbb{R}^2$ such that, as (v, v') runs over $Edge(T)$, all the line segments $]f(v)f(v')[$ (excluding endpoints $f(v)$ and $f(v')$) are disjoint. The *geometric realization* of T associated to a planar imbedding is the subset of the plane formed by the union of the line segments $[f(v)f(v')]$ (including endpoints). Two realizations are called equivalent if they are related by a continuous isotopy of the plane. A *planar tree* is a (rooted) tree together with an equivalence class of geometric realizations.

An *X-labeled tree* is a pair (T, ℓ) where T is a (rooted) tree and $\ell : Leaf(T) \xrightarrow{\cong} X$ is a bijection from the set of leaves of T to the set X. We use the term *n-labeled tree* for the case when $X = [n]$ and *leaf-labeled tree* when no particular labeling set is specified. There are no nontrivial automorphisms of a leaf-labeled tree which preserve the labels. A *fully-labeled tree* has labels assigned to all the vertices in $\overline{Vert}(T)$. There need be no relation between the labeling sets for the leaf vertices and for the internal vertices.

Each morphism $f : T \to S$ induces an isomorphism (denoted by the same symbol) $f : Leaf(T) \xrightarrow{\cong} Leaf(S)$. We say that f *preserves X-labelings* $\ell : Leaf(T) \xrightarrow{\cong} X$ and $\lambda : Leaf(S) \xrightarrow{\cong} X$ if $\lambda \circ f = \ell$. Thus for each labeling ℓ of T, f induces a morphism $f : (T, \ell) \to (S, \ell \circ f^{-1})$ of labeled trees.

DEFINITION 1.31. The category $\mathtt{Tree}$ has as objects leaf-labeled rooted trees. Morphisms of $\mathtt{Tree}$ are morphisms of trees preserving labelings.

For a finite set X we denote by $\mathtt{Tree}(X)$ or $\mathtt{Tree}_X$ the full subcategory of $\mathtt{Tree}$ whose objects are X-labeled trees. We also denote $\mathtt{Tree}_n := \mathtt{Tree}(\{1, \dots, n\})$ and call elements of $\mathtt{Tree}_n$ *n-trees*. The group Σ_X of bijections of the set X acts on the left on the set of X-labeled trees by permuting the labels,

$$\sigma : (T, \ell) \mapsto (T, \sigma \circ \ell).$$

The above correspondence can be extended to an endofunctor of the category $\mathtt{Tree}_X$ which sends a morphism $f : (T, \ell) \mapsto (S, \lambda)$ to $f : (T, \sigma \circ \ell) \mapsto (S, \sigma \circ \lambda)$. We will call this action of Σ_X on the category $\mathtt{Tree}_X$ the *leaf relabeling action*. We will also use the opposite action

$$\sigma : (T, \ell) \mapsto (T, \sigma^{-1}\ell)$$

which we call the *operadic action*.

There is also an action from the right of the automorphism group of the underlying tree on the X-labelings: An automorphism $\psi : T \to T$ defines a new labeling $\ell \circ \psi$. If ψ is an automorphism of T and $\ell : Leaf(T) \to X$ is a labeling of the leaves, then there exists a $\sigma_\ell(\psi) \in \Sigma_X$ such that

$$\ell \circ \psi = \sigma_\ell(\psi) \circ \ell. \tag{1.14}$$

In the sequel, for brevity, we will delete the symbol $\circ$ for composition. Observe that

$$\sigma_\ell(\psi'\psi'') = \sigma_{\ell\psi'}(\psi'')\sigma_\ell(\psi')$$

and that

$$\sigma_\ell(1\!\!1) = 1\!\!1 \quad \text{and} \quad \sigma_\ell(\psi^{-1}) = \sigma_\ell(\psi)^{-1}.$$

EXAMPLE 1.32. Let $X = \{1, 2, 3\}$ and let T be the 3-tree depicted in Figure 4. Let ψ be the unique automorphism of T such that $\psi(u) = v$ and $\psi(v) = u$. Then

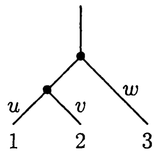


FIGURE 4.

$\sigma_\ell(\psi)$ is clearly the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in \Sigma_3.$$

Equation (1.14) defines an imbedding $\sigma_\ell : Aut(T) \to \Sigma_X$ whose image will be denoted by $\Sigma(T,\ell)$. Given any two X-labelings ℓ and λ of T, there is a $\rho \in \Sigma_X$ such that $\rho\ell = \lambda$. Then, for $\psi \in Aut(T)$,

$$\sigma_\lambda(\psi) = \sigma_{\rho\ell}(\psi) = \rho\sigma_\ell(\psi)\rho^{-1}$$

and

$$\Sigma(T,\lambda) = \Sigma(T,\rho\ell) = \rho\Sigma(T,\ell)\rho^{-1}.$$

The following proposition is obvious.

PROPOSITION 1.33. *The relation $\ell \sim \ell\psi$ for $\psi \in Aut(T)$ defines an equivalence relation on the X-labelings of T. Two X-labeled trees (T,ℓ) and (T,λ) are isomorphic in* $\mathtt{Tree}_X$ *if and only if $\ell \sim \lambda$.*

DEFINITION 1.34. Let $[T,\ell]$ denote the isomorphism class of an X-labeled tree (T,ℓ) in $\mathtt{Tree}_X$. We denote by $\mathcal{T}ree(X)$ the set of isomorphism classes of X-labeled trees,

$$\mathcal{T}ree(X) := \{[T,\ell] \mid (T,\ell) \in X\text{-labeled trees}\}.$$

LEMMA 1.35. *Fix an X-labeling ℓ of T. The assignment $\rho \mapsto \rho\ell$ establishes a one-to-one correspondence between the right cosets of $\Sigma(T,\ell)$ in Σ_X and the equivalence classes of X-labelings of a tree T.*

PROOF. Two labelings $\rho'\ell$ and $\rho\ell$ of the tree T are equivalent if and only if there is an automorphism ψ of T such that $\rho'\ell = \rho\ell\psi = \rho\sigma_\ell(\psi)\ell$. Since ℓ is a bijection, this is equivalent to $\rho' = \rho\sigma_\ell(\psi)$. ■

The following proposition is obvious.

PROPOSITION 1.36. *Define a (right) action of Σ_X on $\mathcal{T}ree(X)$ by*

$$\mathcal{T}ree(\sigma) : [T,\ell] \mapsto [T,\sigma^{-1}\ell].$$

Then

$$X \mapsto \mathcal{T}ree(X), \quad \Sigma_X \ni \sigma \mapsto \mathcal{T}ree(\sigma) \in Hom_{\mathtt{Set}_f}(\mathcal{T}ree(X),\mathcal{T}ree(X))$$

defines a contravariant functor

$$\mathcal{T}ree : \mathtt{Set}_f \longrightarrow \mathtt{Set}_f.$$

Proposition 1.36 says that $\mathcal{T}ree$ is a $\mathtt{Set}_f$-module. In the case $X = [n]$, we omit the brackets and write $\mathcal{T}ree(n)$. The operation which we now define endows $\mathcal{T}ree := \{\mathcal{T}ree(n)\}_{n\geq 1}$ with a pseudo-operad structure.

DEFINITION 1.37. Given an n-labeled tree (T, ℓ), an m-labeled tree (S, λ) and an integer i, $1 \leq i \leq n$, there is a tree, denoted by $T \circ_i S$, called the *grafting of S to T along leaf i*, given by identifying the root edge of S with the leaf i of T. Label the leaves of the new tree as follows:

(i) a leaf coming from T which carried a label j, $1 \leq j \leq i-1$, retains its label,
(ii) a leaf coming from S having label j is relabeled $j + i - 1$ and
(iii) a leaf coming from T which carried a label j, $i+1 \leq j \leq n$, is relabeled $j + m - 1$.

This labeling will be denoted $\ell \circ_i \lambda$.

Note that the set of internal vertices of the new tree is the disjoint union

$$Vert(T \circ_i S) = Vert(T) \sqcup Vert(S).$$

REMARK 1.38. There is an obvious extension of the $\circ_i$-operation on n-labeled trees to an operation $(T, S) \mapsto T \circ_x S$ defined for arbitrary leaf-labeled trees T and S with leaves labeled by finite sets X and Y, respectively, and x the label on the leaf of T to which S is grafted. The set of leaves for the new tree is

$$X \sqcup_x Y := (X - x) \sqcup Y \tag{1.15}$$

and

$$Vert(T \circ_x S) = Vert(T) \sqcup Vert(S).$$

PROPOSITION 1.39. *The $\circ_i$-compositions of Definition 1.37 are well defined on the isomorphism classes $[T, \ell]$ by the equation*

$$[T, \ell] \circ_i [S, \lambda] := [T \circ_i S, \ell \circ_i \lambda]$$

and they endow $\mathcal{T}ree = \{\mathcal{T}ree(n)\}_{n\geq 1}$ with the structure of a pseudo-operad in the category of sets.

PROOF. It is easy to see that arbitrary isomorphisms $f : (T, \ell) \longrightarrow (T, \ell')$ and $g : (S, \lambda) \longrightarrow (S', \lambda')$ of labeled rooted trees extend to an isomorphism

$$f \circ_i g : (T \circ_i S, \ell \circ_i \lambda) \xrightarrow{\cong} (T' \circ_i S', \ell' \circ_i \lambda').$$

Thus the operation $\circ_i$ is well defined on isomorphism classes of labeled trees. The associativity and equivariance conditions of Definition 1.16 for the $\circ_i$-operations are obvious. ∎

REMARK 1.40. The isomorphism class of the trivial tree with the unique 1-labeling defines a unit element, making $\mathcal{T}ree$ a pseudo-operad with unit in the category of sets. By Theorem 1.61, it is an operad.

There is a non-Σ analog of the operad $\mathcal{T}ree$ based on planar trees, defined as follows:

DEFINITION 1.41. Let $\underline{\mathcal{T}ree} := \{\underline{\mathcal{T}ree}(n)\}_{n\geq 1}$, where $\underline{\mathcal{T}ree}(n)$ is the set of isomorphism classes of planar rooted trees with n-leaves and each such tree is given the n-labeling coming from numbering the leaves from left to right relative to the standard (counterclockwise) orientation of the plane.

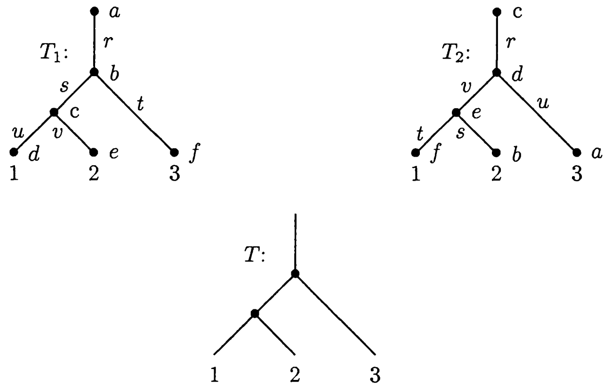


FIGURE 5. The difference between trees and isomorphism classes of trees Both $T_1 \in \mathtt{Tree}_3$ and $T_2 \in \mathtt{Tree}_3$ use the same sets of names for their edges and vertices: $\mathit{Edge}(T_i) = \{r, s, t, u, v\}$ and $\mathit{Vert}(T_i) = \{a, b, c, d, e, f\}$, $i = 1, 2$, but $T_1 \neq T_2$ because the names are permuted. On the other hand, they both belong to the same isomorphism class depicted by the 'naive' labeled tree $T \in \mathcal{T}\mathit{ree}(3)$.

The following statement is a non-Σ version of Proposition 1.39.

PROPOSITION 1.42. *The $\circ_i$-operations defined by grafting at the ith leaf endow $\underline{\mathcal{T}\mathit{ree}} = \{\underline{\mathcal{T}\mathit{ree}}(n)\}_{n \geq 1}$ with the structure of a non-Σ pseudo-operad.*

It is obvious that the component $\mathcal{T}\mathit{ree}(n)$ of the pseudo-operad $\mathcal{T}\mathit{ree}$ is, for each $n \geq 1$, an infinite set. We will consider the sub-pseudo-operad $\mathcal{R}\mathit{tree}$ consisting of isomorphism classes of *reduced* trees (trees with no vertices of arity one). For this pseudo-operad, $\mathcal{R}\mathit{tree}(1) = \emptyset$ and $\mathcal{R}\mathit{tree}(n)$ is finite for each $n \geq 2$.

There is a similar sub-pseudo-operad of the operad $\underline{\mathcal{T}\mathit{ree}}$, the pseudo-operad $\underline{\mathcal{R}\mathit{tree}}$ of reduced *planar trees*. As we will see in Section 1.6, the cells of the Stasheff associahedron K_n are labeled by the nth component $\underline{\mathcal{R}\mathit{tree}}(n)$ of this pseudo-operad.

REMARK 1.43. Although the operads $\mathcal{T}\mathit{ree}$, $\underline{\mathcal{T}\mathit{ree}}$, $\mathcal{R}\mathit{tree}$ and $\underline{\mathcal{R}\mathit{tree}}$ introduced above consist of isomorphism classes of labeled rooted trees, we will sometimes simply call their elements trees and omit explicit notations for labelings.

The difference between trees and isomorphism classes of trees can be informally described as follows. A tree T is, by definition, a graph, that is, each vertex of T and also each edge of T has its 'name' – an element of $\mathit{Vert}(T)$ for each vertex and an element of $\mathit{Edge}(T)$ for each edge. Working with isomorphism classes means forgetting these names. An isomorphism class is then a 'shape' of a tree, without explicit names for edges and vertices. These shapes are in fact what is usually meant by a tree in human language. In other words, the difference between – for example – $\mathtt{Tree}_n$ and $\mathcal{T}\mathit{ree}(n)$ is that elements of $\mathtt{Tree}_n$ are trees considered as graphs (that is, graphs with no loops), while elements of $\mathcal{T}\mathit{ree}(n)$ are 'naive' trees. See also Figure 5.

Since there are no nontrivial automorphisms of labeled trees, labeling edges and vertices of a representative of an isomorphism class of trees defines a unique labeling of *each* representative of this isomorphism class, therefore it makes sense to speak about labeling edges and vertices of an 'isomorphism class.'

1.6. The Stasheff associahedra

One of the earliest and most important examples of a topological operad is derived from the sequence of Stasheff associahedra $K = \{K_n\}_{n\geq 1}$ which were discussed briefly in Section I.1.6. The Stasheff associahedra form a cellular non-Σ pseudo-operad, that is, a sequence of topological spaces with $\circ_i$-operations which are cellular maps satisfying the associativity conditions in Definition 1.16. In fact, Stasheff's realization of the associahedron K_n was as a convex subspace of the cube I^{n-2} designed so that specific maps $\circ_i : K_r \times K_s \longrightarrow K_n$ could be given in terms of coordinate formulas. The cells of dimension k in K_n are labeled by planar rooted trees with $n-2-k$ internal edges. See Figure 6 in Section I.1.6.

Although convex, this realization was curvilinear in part, not a polytope in the classical sense. Here we will sketch an alternative realization as a convex polytope; it bears some similarity to the first such realization due to Mark Haiman [**Hai84**] in 1984. For the present construction, notice there is another labeling of the cells of K_n using as labels bracketings of a sequence of n $\bullet$'s (as position markers); this is more immediately related to associativity in terms of moving parentheses; see Section I.1.6.

To be more precise, there is a bijection between the set of planar rooted trees with n leaves and s internal edges and the set of bracketings of sequence of n $\bullet$'s with s pairs of parentheses – every internal edge introduces a parentheses around the terms corresponding to the input edges. If the tree T corresponds to a bracketing b and the tree T' corresponds to b', then the tree created by grafting T' to T at the leaf labeled i corresponds to the bracketing created by replacing the ith $\bullet$ in b with (b'), that is, placing b' inside a set of parentheses in position i. For example, if $b = b' = \bullet\bullet$ (corresponding to the 2-corolla with no internal edge), then $b \circ_1 b' = (\bullet\bullet)\bullet$. See also Figure 6. Figure 7 shows the cells of the pentagon K_4 indexed by bracketings of four $\bullet$'s.

This section is organized as follows. In Theorem 1.44 we state that there exists, for each $n \geq 2$, a convex polytope whose poset of faces is isomorphic to the poset of bracketings of n $\bullet$'s. This polytope will be constructed as a truncation of the $(n-2)$-dimensional simplex by hyperplanes. This truncation depends on a choice of functions $\{c_n\}_{n\geq 2}$, but the cellular isomorphism class of the result does not depend on these functions (Proposition 1.47). The facial structure of the polytope is reflected by a certain inequality of the functions defining the truncation.

The material in this section is very technical. Since we will not need it in the rest of the book, our proofs will be merely sketched.

THEOREM 1.44. [**Hai84**] *There is a convex polytope $L_n \subset \mathbb{R}^{n-1}$ whose lattice of faces (= cells) represents the lattice of bracketings of n $\bullet$'s and thus is a* realization *of the associahedron.*

The polytope L_n is constructed by truncating an $(n-2)$-dimensional simplex in $\mathbb{R}^{n-1}$,

$$\Delta_c^{n-2} = \{t = (t_1, \dots, t_{n-1}) \mid t_i \geq 0,\ \sum_{k=1}^{n-1} t_i = c\},$$

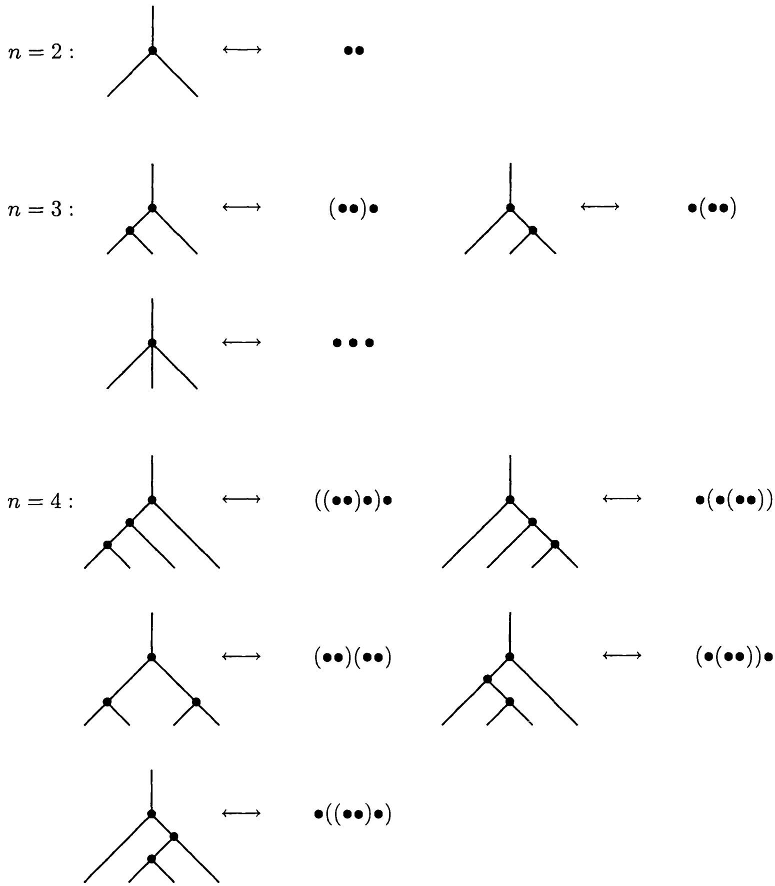


FIGURE 6. Correspondence between trees and parenthesized strings. For $n = 4$, only binary trees are shown.

where c is a positive number. In order to keep track of the truncation procedure, we introduce a family of positive real-valued functions $c = \{c_n\}_{n \geq 2}$ with the domain of definition of c_n being the set of subintervals I of $(1, \dots, n-1)$

$$c_n : \{I \mid I \text{ is a subinterval of } (1, \dots, n-1)\} \longrightarrow \mathbb{R}^+,$$

and satisfying, for $n \geq 2$ and subintervals $I, J \subset (1, \dots, n-1)$, the condition

$$c_n(I) + c_n(J) < c_n(I \cup J) \tag{1.16}$$

if $I \cup J$ is an interval properly containing both I and J.

EXAMPLE 1.45. A possible choice of the function c_n satisfying (1.16) is $c_n(I) := a \cdot 3^{card(I)}$ for any positive number a, where $card(I)$ is the number of elements in I.

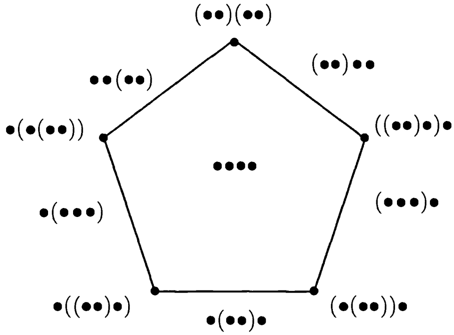


FIGURE 7. Labeling the cells of K_4 by bracketings of four •'s.

This function depends only on the size of the subinterval; however, the analysis of the facial structure of L_n in Proposition 1.47 and the construction of $\circ_i$-operations in Corollary 1.49 requires the use of functions which are not of this simple type.

Let $\bar{c}_n := c_n((1, \dots, n-1))$, and for a subinterval $I := (j+1, \dots, j+i) \subset (1, \dots, n-1)$, let

$$t_I := (t_{j+1}, \dots, t_{j+i}) \text{ and } |t_I| := t_{j+1} + \cdots + t_{j+i}.$$

DEFINITION 1.46. Let $c = \{c_n\}_{n \geq 2}$ be a family of functions satisfying (1.16). Define an $(n-2)$-dimensional convex polytope L_n^c as the truncation of the simplex

$$\Delta_{\bar{c}}^{n-2} := \{t \mid t_1 + \cdots + t_{n-1} = \bar{c}_n,\ t_i \geq 0\}$$

by the affine hyperplanes P_I^c defined by $|t_I| = c_n(I)$, that is,

$$L_n^c := \{t = (t_1, \dots, t_{n-1}) \mid t_1 + \cdots + t_{n-1} = \bar{c}_n,\ |t_I| \geq c_n(I)\},$$

for $I \subset (1, \dots, n-1)$.

Polytope L_4^c for $c = \{c\}_{n \geq 2}$ as in Example 1.45 is depicted in Figure 8. Note that the affine hyperplanes P_I^c are all transversal so that the codimension of a cell of L_n^c is the number of intersecting hyperplanes. Moreover, as part of the proof of Theorem 1.44 (see the second edition of [**SS94**]) one shows that the construction of Definition 1.46 does not depend on the family $c = \{c_n\}_{n \geq 2}$ and that all the L_n^c are isomorphic as cell complexes to K_n:

PROPOSITION 1.47. *The cell complex L_n^c does not depend on c, that is, for any two families c' and c'' satisfying (1.16) there exists a cellular isomorphism $\rho : L_n^{c'} \cong L_n^{c''}$. The poset of faces of L_n^c is isomorphic to that of K_n.*

SKETCH OF PROOF. Let us analyze the facial structure of L_n^c. We define a family of embeddings

$$\circ_i : L_r^{c'} \times L_s^{c''} \to L_n^c$$

for $r + s = n + 1$, $1 \leq i \leq r$, with suitable families c' and c'' of functions satisfying (1.16). In general, the families c' and c'' will not be of the type in Example 1.45, but to keep the discussion from becoming too technical, we are not going to give their precise definition here. In coordinates, the embedding is the obvious one:

$$\circ_i : (t') \times (t'') \mapsto (t'_1, \dots, t'_{i-1}, t''_1, \dots, t''_{s-1}, t'_i, \dots, t'_{r-1}). \tag{1.17}$$

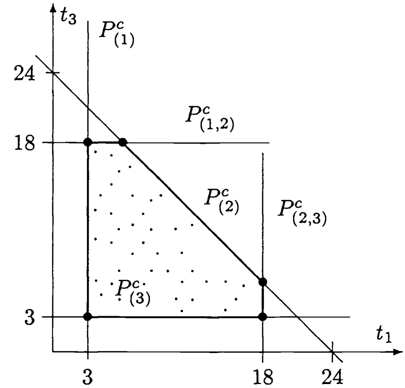


FIGURE 8. The (t_1, t_3)-projection of L^c_4 with $c_4(I) = 3^{card(I)}$.

The trick is to show that the topological boundary of L^c_n decomposes as a union of the images of the $\circ_i$:

$$\partial L^c_n = \bigcup_{\substack{2 \leq s \leq n-1 \\ 1 \leq i \leq r \leq n-s+1}} \circ_i(L^{c'}_r \times L^{c''}_s). \tag{1.18}$$

Furthermore, for the new functions c' and c'' and the labeling of the faces by bracketings as described below, the closed faces $\circ_i(L^{c'}_r \times L^{c''}_s)$ intersect for distinct i, r, s if and only if the corresponding bracketings labeling the unique open cell in the face $\circ_i(L^{c'}_r \times L^{c''}_s)$ enclose either disjoint intervals or strictly nested intervals. See Figure 7 for $n = 4$.

The open cell in the face $\circ_i(L^{c'}_r \times L^{c''}_s)$ is labeled by the bracketing b_I of s $\bullet$'s in the positions $(i, \ldots, i+s-1)$. Note that an interval of length $s-1$ in the indices of the t_i corresponds to an interval of length s in the $\bullet$'s, a single variable t_i corresponding to $(\bullet\bullet)$ in positions $i, i+1$.

EXAMPLE 1.48. The faces of L^c_4 in Figure 8 are labeled as follows.

The face $P^c_{(1,2)} \cap L^c_4 = \iota_1(L^{c'}_2 \times L^{c''}_3)$ is labeled by $(\bullet\bullet\bullet)\bullet$,
the face $P^c_{(2,3)} \cap L^c_4 = \iota_2(L^{c'}_2 \times L^{c''}_3)$ is labeled by $\bullet(\bullet\bullet\bullet)$,
the face $P^c_{(1)} \cap L^c_4 = \iota_1(L^{c'}_3 \times L^{c''}_2)$ is labeled by $(\bullet\bullet)\,\bullet\bullet$,
the face $P^c_{(2)} \cap L^c_4 = \iota_2(L^{c'}_3 \times L^{c''}_2)$ is labeled by $\bullet(\bullet\bullet)\,\bullet$ and
the face $P^c_{(3)} \cap L^c_4 = \iota_3(L^{c'}_3 \times L^{c''}_2)$ is labeled by $\bullet\bullet\,(\bullet\bullet)$.

The above correspondence shows how to identify L^c_4 and pentagon K_4 in Figure 7.

The incidence relations of the faces of L^c should be such that the intersection $P^c_I \cap P^c_J \cap L^c$ is nonempty precisely when the bracketings b_I and b_J are nested or disjoint. This follows from condition (1.16) which implies that if the intersection $P^c_I \cap P^c_J \cap L^c$ is nonempty, then $I \cup J$ cannot be an interval properly containing both I and J. Indeed, (1.16) implies that if $I \cup J$ were an interval properly containing both I and J, then for $t \in P^c_I \cap P^c_J$ and $t_i \geq 0$,

$$|t_{I \cup J}| \leq |t_I| + |t_J| = c(I) + c(J) < c(I \cup J) \tag{1.19}$$

Thus, either

$$(1.20) \qquad 1)\ \ J \subset I \quad \text{or} \quad 2)\ \ I \subset J \quad \text{or} \quad 3)\ \ I \cup J \ \text{ is not an interval.}$$

Either of the conditions 1) or 2) in (1.20) implies that the bracketings b_I and b_J are nested. Condition 3) implies that b_I and b_J are disjoint, since the intervals I and J are disjoint and not adjacent. By induction on the number of parentheses or, equivalently, the number of internal edges of the trees, the full isomorphism of the lattices of faces of K_n and L_n^c is established. ∎

In the following corollary which follows from the above considerations, we write L_n instead of L_n^c because, as we know from Proposition 1.47, the cellular isomorphism class of L_n^c does not depend on the family c as long as it satisfies (1.16).

COROLLARY 1.49. *The sequence* $\{L_n\}_{n\geq 2}$ *with*

$$\circ_i : L_r \times L_s \longrightarrow L_n$$

defined by the $\circ_i$ *in (1.17) form a cellular (topological) non-*Σ *pseudo-operad with the cells of dimension* s *in* L_n *indexed by the planar trees with* n *leaves and* $n-s-2$ *internal edges.*

We will identify $\{L_n\}_{n\geq 2}$ with the operad of Stasheff associahedra $\{K_n\}_{n\geq 2}$. Let us close this section with a couple of general remarks.

It is well known [**Sta95**] that K_n is the real compactification of the configuration space of $n-2$ distinct labeled points in the unit interval. As such, it is a real manifold-with-corners and the tubular neigborhood theorem for manifolds-with-corners implies that it is a truncation of its interior, which is the simplex Δ^{n-2}; see [**Mar99a**, Proposition 6.1] for details.

Devadoss in [**Dev01**] argues that the truncation scheme discussed above corresponds to connected subdiagrams of the Coxeter diagram of A_n. But none of these general arguments give an explicit truncation as described in this section.

1.7. Operads defined in terms of arbitrary finite sets

In the discussion of free operads in Section 1.9 and in the operadic cobar construction in Section 3.9, it will be useful to evaluate an operad (considered as a functor) on an arbitrary finite set, that is, we want to define operads in terms of $\mathtt{Set}_f$-modules as opposed to the Σ-module formulation in Definition 1.4. If we make fairly weak assumptions on the underlying category $\mathcal{C}$, namely, that it has small colimits and that for any object X, the endofunctor $X \odot -$ preserves colimits, the extension of a Σ-module to a $\mathtt{Set}_f$-module presents no problem.

In fact, it will be useful to extend further the category $\mathtt{Set}_f$ to the category of labeled unordered partitions of finite sets, a finite set being identified with the trivial partition. In this context, expressions such as $A(m_1) \odot \cdots \odot A(m_n)$ for a Σ-module A represent the value of the extended functor associated to A on the ordered partition $(m_1, \ldots, m_n)$ of $m = m_1 + \cdots + m_n$.

An equivalent formulation uses surjections of finite sets instead of partitions. A $\mathtt{Set}_f$-Mod extends naturally to this enlarged category $\mathtt{Surj}_f$ whose objects are surjections of finite sets and whose morphisms are commutative diagrams in which f, f' are surjections and ρ, σ are bijections:

(1.21)
$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\sigma"] & Y'. \end{tikzcd}$$

In terms of surjections, we can represent an ordered partition $(m_1, \ldots, m_n)$ by a surjection mapping a sequence of subintervals of lengths $(m_1, \ldots, m_n)$ on the sequence of integers $(1, \ldots, n)$:

$$f : [m] \twoheadrightarrow [n], \quad f(j) := i, \text{ if } m_1 + \cdots + m_{i-1} < j \leq m_1 + \cdots + m_i.$$

Then if we define

(1.22) $$A[f] := A(f^{-1}(1)) \odot \cdots \odot A(f^{-1}(n)) \cong A(m_1) \odot \cdots \odot A(m_n),$$

the operad composition associates a morphism $A[g] \odot A[f] \to A[g \circ f]$ to each sequence $[m] \xrightarrow{f} [n] \xrightarrow{g} [1]$. More generally, the operad composition law is a natural transformation

$$\gamma_{g;f} : A[g] \odot A[f] \to A[g \circ f]$$

defined for any sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. The associativity axiom (see below) takes the form of the equality of two composite natural transformations arising from a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. See Theorem 1.60 below. The usefulness of the formulation in terms of surjections will be clear when we compare different resolutions of operads in Sections 3.5 and 3.6.

DEFINITION 1.50. There is a symmetric monoidal structure on $\mathtt{Surj}_f$ with multiplication $\sqcup$ given by concatenation of surjections

$$(X \xrightarrow{f} Y) \sqcup (X' \xrightarrow{f'} Y') := (X \sqcup X' \xrightarrow{f \sqcup f'} Y \sqcup Y')$$

and symmetry coming from the natural identification of $X \sqcup X'$ and $Y \sqcup Y'$ with $X' \sqcup X$ and $Y' \sqcup Y$, respectively. The unit object is defined purely formally as the 'emptyset surjection' $\overline{\emptyset} : \emptyset \twoheadrightarrow \emptyset$.

Note that any surjection is the $\sqcup$-product of several surjections onto singleton sets, and that, in general,

(1.23) $$\text{if } f = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_n, \text{then } A[f] \cong A[f_1] \odot A[f_2] \odot \cdots \odot A[f_n].$$

Equation (1.23) says that the functor $f \mapsto A[f]$ is a monoidal functor once we add the condition $A[\overline{\emptyset}] = \mathbf{1}$. In fact, (1.23) shows that there is a unique extension of a $\mathtt{Set}_f$-module to a monoidal functor on $\mathtt{Surj}_f$.

A related description of pseudo-operads is presented later in the section, and at the same time, we will make more precise the distinction between operads and pseudo-operads.

There are two functors R (restriction) and E (extension) which allow us to pass back and forth between the categories $\mathtt{Set}_f$-Mod and Σ-Mod. The compositions RE and ER are isomorphic to the respective identity functors, therefore we have the following proposition.

PROPOSITION 1.51. *The categories* $\mathtt{Set}_f$-Mod *and* Σ-Mod *are isomorphic.*

The restriction functor is defined in the obvious way. The extension functor $E : \Sigma\text{-}\mathtt{Mod} \to \mathtt{Set}_f\text{-}\mathtt{Mod}$ uses a standard coequalizer construction. If n is the cardinality of the finite set X, then define the *set of orderings of* X:

$$Ord(X) := \{f | f : X \xrightarrow{\sim} [n]\},$$

consisting of all bijections from X to $[n]$, and let $A'(X)$ be the coproduct of $A(n)$ over the orderings of X,

$$A'(X) := \coprod_{f \in Ord(X)} A(n).$$

For each $\sigma \in \Sigma_n$ there is an endomorphism $A'(\sigma)_X : A'(X) \to A'(X)$ induced by the isomorphisms $A(\sigma^{-1}) : A(n)_f \to A(n)_{\sigma \circ f}$ between the factors of the coproduct. Let $EA(X)$ be the coequalizer of the family of morphisms $A'(\sigma)_X$,

$$EA(X) := \operatorname*{coequalizer}_{\sigma \in \Sigma_n} \left\{ A'(\sigma)_X : A'(X) \to A'(X) \right\}. \tag{1.24}$$

EXAMPLE 1.52. In the case of $\mathtt{Mod}_{\mathbf{k}}$, $EA(X)$ is the module of Σ_n-coinvariants:

$$EA(X) := \left(\bigoplus_{f \in Ord(X)} A(n) \right)_{\Sigma_n}. \tag{1.25}$$

In the case of $\mathtt{Set}$, $EA(X)$ is the quotient of the disjoint union of the $A(n)$ by the equivalence relation $A(n)_f \ni x \sim A(\sigma)x \in A(n)_{\sigma \circ f}$, $\sigma \in \Sigma_n$:

$$EA(X) := \left(\bigsqcup_{f \in Ord(X)} A(n) \right) / \sim . \tag{1.26}$$

CONVENTION. *From now on, to simplify notation, we will denote $EA(X)$ by $A(X)$ and make no distinction between Σ- and $\mathtt{Set}_f$-modules.*

Before giving the definition of an operad in terms of surjections, we need to introduce some more notation and definitions. Define

$$Surj[Y, X] := \{h | X \overset{h}{\twoheadrightarrow} Y\} \text{ and } Surj[n, m] := Surj[[n], [m]]. \tag{1.27}$$

The strange reversal of order Y, X instead of X, Y is explained in Remark 1.56 below.

DEFINITION 1.53. Given a Σ-module A and a surjection $f : X \twoheadrightarrow [n]$, define

$$A[f] := A(f^{-1}(1)) \odot \cdots \odot A(f^{-1}(n)).$$

Note that for $f : X \twoheadrightarrow [1]$, $A[f] = A(X)$, and for any permutation $\sigma \in \Sigma_n$, $A[\sigma] = A(1)^{\odot n}$.

In the discussion above we have been assuming that even if the monoidal structure is not strict, the associativity constraint for the symmetric monoidal category $(\mathcal{C}, \odot)$ is simple enough that there is no need to specify brackets in the multiple product. Define

$$A[n, X] := \coprod_{f \in Surj[n, X]} A[f]. \tag{1.28}$$

The morphisms $\widehat{\sigma} : A[f] \to A[\sigma \circ f]$,

$$\widehat{\sigma} : A(f^{-1}(1)) \odot \cdots \odot A(f^{-1}(n)) \to A\left(f^{-1}(\sigma^{-1}(1))\right) \odot \cdots \odot A\left(f^{-1}(\sigma^{-1}(n))\right), \tag{1.29}$$

given by the symmetric monoidal structure on the category $\mathcal{C}$, define a left Σ_n-action on $A[n, X]$.

To extend the definition of $A[f]$ to surjections $f : X \twoheadrightarrow Y$ with arbitrary finite codomain Y but with no preferred ordering, we need to define an unordered tensor product. This can be done using a coequalizer construction analogous to that used in the definition of the extension functor (1.24).

Let $f : X \twoheadrightarrow Y$ be a surjection onto a finite set Y of cardinality n. For any ordering $g \in Ord(Y)$, $A[g \circ f]$ has been defined in Definition 1.53. Let

$$A'[f] := \coprod_{g \in Ord(Y)} A[g \circ f].$$

For $\sigma \in \Sigma_n$, the morphism $\widehat{\sigma} : A[g \circ f] \to A[\sigma \circ g \circ f]$ defined in (1.29) extends to a morphism of the coproduct

$$A'(\sigma) : A'[f] \to A'[f].$$

DEFINITION 1.54. Define $A[f]$ to be the coequalizer of the morphisms $A'(\sigma)$:

$$A[f] := \operatorname*{coequalizer}_{\sigma \in \Sigma_n} \left\{ A'(\sigma) : A'[f] \to A'[f] \right\}.$$

For any $\rho \in \Sigma_Y$, $f \in Surj[Y, X]$ and $g \in Ord(Y)$, $g \circ \rho \circ g^{-1} \in \Sigma_n$ and the morphisms $\widehat{g \circ \rho \circ g^{-1}} : A[g \circ f] \to A[g \circ \rho \circ f]$ give rise to a morphism from the coproduct $A'[f]$ to $A'[\rho \circ f]$ compatible with the morphisms defining the coequalizer $A[f]$. Therefore, there is a morphism of coequalizers denoted simply $\hat{\rho} : A[f] \to A[\rho \circ f]$. If

$$A[Y, X] := \coprod_{f \in Surj[Y,X]} A[f], \tag{1.30}$$

then $\Sigma_Y \ni \rho \mapsto \hat{\rho}$ defines a left Σ_Y-action on $A[Y, X]$.

For any $h \in \Sigma_X$ there is a restricted bijection of finite sets $h : h^{-1}(f^{-1}(y)) \to f^{-1}(y)$ and by the contravariance of A, there is a morphism $A(h) : A(f^{-1}(y)) \to A(h^{-1}(f^{-1}(y)))$. By functoriality of the $\odot$-product and the coproduct, we get $A(h) : A'[f] \to A'[f \circ h]$ compatible with the morphisms defining the coequalizer $A[f]$, therefore, there is a well-defined morphism $A(h) : A[f] \to A[f \circ h]$.

DEFINITION 1.55. The right action of $h \in \Sigma_X$ on $A[Y, X]$ is defined as the endomorphism of the coproduct (1.30) induced by the morphisms $A(h) : A[f] \to A[f \circ h]$.

In summary, for any two finite sets X, Y and for a Σ-module A we have constructed a left Σ_Y- right Σ_X-module $A[Y, X]$. An example of this construction appears in Section 1.10, Example 1.95.

REMARK 1.56. We use the notation $[Y, X]$ corresponding to $X \twoheadrightarrow Y$ to emphasize that the Σ_Y-action on $A[Y, X]$ is a left action on the target Y and the Σ_X-action is a right action on the source X. Another reason for the choice is in order to have the composition $A[g] \odot A[f] \to A[g \circ f]$ for $g \in Surj[Z, Y]$ and $f \in Surj[Y, X]$ take the same form as the composition law in the associativity axiom in Definition 1.4

REMARK 1.57. The above constructions make the relation between operads and PROPs, mentioned in Section I.1.3, more precise: Given an operad $\mathcal{P}$, then the $\mathcal{P}[m,n]$ are the components of the PROP generated by $\mathcal{P}$.

In general, given any set Y of cardinality n and an assignment of an object A_y in $\mathcal{C}$ to every element of Y, we can define for each $g \in Ord(Y)$ an $\odot$-product

$$\bigodot_g A_y := A_{g^{-1}(1)} \odot \cdots \odot A_{g^{-1}(n)}.$$

The symmetry in $\mathcal{C}$ determines a permutation morphism $\overline{\sigma} : \bigodot_g A_y \to \bigodot_{\sigma\circ g} A_y$ for each $\sigma \in \Sigma_n$.

DEFINITION 1.58. The *unordered $\odot$-product* is defined by

$$(1.31) \quad \bigodot{}_Y A_y := \underset{\sigma\,\in\,\Sigma_n}{\text{coequalizer}} \left\{ \overline{\sigma} : \coprod_{g\in Ord(Y)} \bigodot{}_g A_y \longrightarrow \coprod_{g\in Ord(Y)} \bigodot{}_g A_y \right\}.$$

As with $A[Y,X]$, there is a left Σ_Y-action on $\bigodot{}_Y A_y$.

The association of an object $A[f]$ to a surjection $f : X \twoheadrightarrow Y$ can be considered as determining a functor on $\mathtt{Surj}_f$ with values in $\mathcal{C}$. For the next theorem it will be convenient to introduce a new category $\mathtt{Surj}_f^2$.

DEFINITION 1.59. Let $\mathtt{Surj}_f^2$ be the category with surjection sequences $X \overset{f}{\twoheadrightarrow} Y \overset{g}{\twoheadrightarrow} Z$ as the objects and the obvious commutative diagrams (generalizing diagram (1.21)) as morphisms.

THEOREM 1.60. (Definition of an operad – Finite sets and surjections version) *An* operad *in a symmetric monoidal category $\mathcal{C}$ is equivalent to a contravariant monoidal functor*

$$(f : X \twoheadrightarrow Y) \mapsto \mathcal{P}[f],$$

from the symmetric monoidal category $(\mathtt{Surj}_f, \sqcup)$ *(Definition 1.50), together with a natural transformation*

$$\gamma_{g,f} : \mathcal{P}[g] \odot \mathcal{P}[f] \to \mathcal{P}[g \circ f]$$

of functors on the category of $\mathtt{Surj}_f^2$ *of surjection sequences $X \overset{f}{\twoheadrightarrow} Y \overset{g}{\twoheadrightarrow} Z$. The natural transformation γ (which encodes the set of operad composition laws) satisfies the following axioms:*

1. *Associativity. Given a sequence $X \overset{f}{\twoheadrightarrow} Y \overset{g}{\twoheadrightarrow} Z \overset{h}{\twoheadrightarrow} W$, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{P}[h] \odot \mathcal{P}[g] \odot \mathcal{P}[f] & \xrightarrow{\;\mathbb{1} \odot \gamma_{g,f}\;} & \mathcal{P}[h] \odot \mathcal{P}[g \circ f] \\ {\scriptstyle \gamma_{h;g} \odot \mathbb{1}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma_{h;g\circ f}} \\ \mathcal{P}[h \circ g] \odot \mathcal{P}[f] & \xrightarrow{\;\gamma_{h\circ g;f}\;} & \mathcal{P}[h \circ g \circ f] \end{array}$$

2. *Unit. If $\mathbf{1}$ is the unit object of $\mathcal{C}$, then there is a morphism $\eta : \mathbf{1} \to \mathcal{P}(1)$ such that for any surjection $f : X \twoheadrightarrow Y$, the composite morphisms*

$$\mathcal{P}[f] \odot \mathbf{1}^{\odot n} \xrightarrow{\mathbb{1} \odot \eta^{\odot n}} \mathcal{P}[f] \odot \mathcal{P}[\mathbb{1}_X] \xrightarrow{\gamma_{f, \mathbb{1}_X}} \mathcal{P}[f]$$

and

$$\mathbf{1}^{\odot m} \odot \mathcal{P}[f] \xrightarrow{\eta \odot \mathbb{1}} \mathcal{P}[\mathbb{1}_Y] \odot \mathcal{P}[f] \xrightarrow{\gamma_{\mathbb{1}_Y, f}} \mathcal{P}[f]$$

(X has n elements and Y has m elements) are respectively the iterated left unit morphism and the right unit morphism for the underlying monoidal category $\mathcal{C}$. Here we use the natural identification of $\mathcal{P}(1)$ and $\mathcal{P}(\{x\})$ for any singleton set $\{x\}$.

PROOF. Given $\mathcal{P} \in \mathtt{Set}_f\text{-}\mathtt{Mod}$ satisfying the conditions in Theorem 1.60, if we apply the restriction functor, the Σ-module $R\mathcal{P}$ satisfies the axioms in Definition 1.4. If the target of a surjection is ordered, we can identify it with $[n]$, $f : X \twoheadrightarrow [n]$ and decompose $f = f_1 \sqcup \ldots \sqcup f_n$ where $f_i := f|_{f^{-1}(i)}$. By definition of a monoidal functor, there is an isomorphism $\mathcal{P}[f] \cong \mathcal{P}(f^{-1}(1)) \odot \cdots \odot \mathcal{P}(f^{-1}(n))$. The natural transformation $\gamma_{g,f}$ for the sequence $[m] \overset{f}{\twoheadrightarrow} [n] \overset{g}{\twoheadrightarrow} [1]$ is the operad composition as given in Definition 1.4 and the associativity axiom as given above applied to $[l] \overset{f}{\twoheadrightarrow} [m] \overset{g}{\twoheadrightarrow} [n] \overset{h}{\twoheadrightarrow} [1]$ is the associativity axiom of Definition 1.4. The equivariance axiom in that definition is a consequence of the fact that the morphisms $\gamma_{g;f}$ are well defined on the $\odot$-product of coequalizers $\mathcal{P}[g] \odot \mathcal{P}[f]$.

Conversely, given $\mathcal{P} \in \Sigma\text{-}\mathtt{Mod}$ satisfying the axioms of Definition 1.4, the corresponding $E\mathcal{P} \in \mathtt{Set}_f\text{-}\mathtt{Mod}$ extends to a functor on $\mathtt{Surj}_f$ which satisfies the axioms in Theorem 1.60. First of all, since symmetry for $\sqcup$ identifies $\{y_1\} \sqcup \ .. \sqcup \{y_n\}$ and $\{y_{i_1}\} \sqcup \ldots \sqcup \{y_{i_n}\}$, $\mathcal{P}[f]$ coequalizes the maps

$$\mathcal{P}(f^{-1}(y_1)) \odot \cdots \odot \mathcal{P}(f^{-1}(y_n)) \to \mathcal{P}(f^{-1}(y_{i_1})) \odot \cdots \odot \mathcal{P}(f^{-1}(y_{i_n}))$$

induced by permuting the y_i. Hence the unordered $\odot$-product $\bigodot_Y \mathcal{P}(f^{-1}(y))$ maps onto $\mathcal{P}[f]$. There is an obvious inverse,

$$\mathcal{P}[f] \to \mathcal{P}(f^{-1}(y_1)) \odot \cdots \odot \mathcal{P}(f^{-1}(y_n)) \to \bigodot_Y \mathcal{P}(f^{-1}(y)),$$

so $\mathcal{P}[f] \cong \bigodot_Y \mathcal{P}(f^{-1}(y))$. The associativity and unit axioms follow immediately from the extension procedure. ∎

In this description, operads form a subcategory $\mathtt{Op}_{\mathcal{C}}$ of $\mathtt{Set}_f\text{-}\mathtt{Mod}$ in $\mathcal{C}$. When $\mathcal{C}$ is understood, we denote it simply $\mathtt{Op}$.

Given a pseudo-operad $\mathcal{P}$ considered as a Σ-module with $\circ_i$-operations satisfying the axioms in Definition 1.16, it is possible to use the methods of the proof of Theorem 1.60 to define $\circ_x$-operations on the $\mathtt{Set}_f$-module extension $\mathcal{P}$. We will use the same notation for the extension.

The definition of the $\circ_x$-operations on a pseudo-operad $\mathcal{P}$ is straightforward. Given $f : [n] \xrightarrow{\sim} X$ and $g : [m] \xrightarrow{\sim} Y$, with $f(i) = x$ define $f \sqcup_x g : [m+n-1] \to X \sqcup_x Y$, where $X \sqcup_x Y$ was introduced in (1.15), by

$$(f \sqcup_x g)(j) = \begin{cases} f(j), & 1 \leq j \leq i-1, \\ g(j-i+1), & i \leq j \leq i+m-1, \text{ and} \\ f(j-m+1), & i+m \leq j \leq m+n-1. \end{cases}$$

Then define $\circ_x$ as the composition

$$(1.32)\quad \mathcal{P}(X)\odot\mathcal{P}(Y)\xrightarrow{\mathcal{P}(f)\odot\mathcal{P}(g)}\mathcal{P}(n)\odot\mathcal{P}(m)\xrightarrow{\circ_1}\mathcal{P}(n+m-1)\xrightarrow{\mathcal{P}((f\sqcup_x g)^{-1})}\mathcal{P}(X\sqcup_x Y).$$

The pseudo-operad equivariance axiom implies that $\circ_x$ is well defined independent of the choice of f, g.

The associativity axioms for the $\circ_x$-operations are

$$(1.33)\quad \begin{aligned} \circ_x(\mathbb{1}\odot\circ_y) &= \circ_y(\circ_x\odot\mathbb{1}) : \mathcal{P}(X)\odot\mathcal{P}(Y)\odot\mathcal{P}(Z) \to \mathcal{P}(X\sqcup_x(Y\sqcup_y Z)) \\ \circ_{x_2}(\circ_{x_1}\odot\mathbb{1}) &= \circ_{x_1}(\circ_{x_2}\odot\mathbb{1})(\mathbb{1}\odot\tau) : \mathcal{P}(X)\odot\mathcal{P}(Y)\odot\mathcal{P}(Z)\to\mathcal{P}((X\sqcup_{x_1}Y)\sqcup_{x_2}Z). \end{aligned}$$

Here we have used the equalities of sets,

$$X\sqcup_x(Y\sqcup_y Z) = (X\sqcup_x Y)\sqcup_y Z \text{ and } (X\sqcup_{x_1} Y)\sqcup_{x_2} Z = (X\sqcup_{x_2} Z)\sqcup_{x_1} Y,$$

where $x, x_1, x_2 \in X$ and $y \in Y$.

THEOREM 1.61. ($\mathtt{Set}_f$-definition of a pseudo-operad) *A pseudo-operad $\mathcal{P}$ with values in C as defined in Definition 1.16 extends to a $\mathtt{Set}_f$-module, also denoted $\mathcal{P}$, together with operations $\circ_x : \mathcal{P}(X)\odot\mathcal{P}(Y) \longrightarrow \mathcal{P}(X\sqcup_x Y)$ satisfying the associativity axioms (1.33). Conversely, a $\mathtt{Set}_f$-module with operations $\circ_x$ satisfying the associativity axioms (1.33) restricts to a Σ-module which satisfies the associativity and equivariance axioms of Definition 1.16.*

The proof is an exercise, left to the reader, in the use of the extension and restriction functors.

1.7.1. Operads versus pseudo-operads. The relation between the structure maps of operads and pseudo-operads (both considered as $\mathtt{Set}_f$-modules) can be described as follows. First, given a pseudo-operad $\mathcal{P}$, define the natural transformation

$$\gamma_{g,f} : \mathcal{P}[g]\odot\mathcal{P}[f] \longrightarrow \mathcal{P}[g\circ f],$$

i.e. the operad composition law, by

$$(1.34)\quad \gamma_{g,f} : \mathcal{P}[g]\odot\mathcal{P}[f] = \mathcal{P}[g]\odot\mathcal{P}(f^{-1}(y_1))\odot\cdots\odot\mathcal{P}(f^{-1}(y_n)) \xrightarrow{\circ_{y_n}(\circ_{y_{n-1}}\odot\mathbb{1})\cdot(\circ_{y_1}\odot\ \odot\mathbb{1})} \mathcal{P}[g\circ f].$$

The associativity conditions (1.33) imply that the $\gamma_{g,f}$ satisfy the associativity axiom of Theorem 1.60. The unit axiom in that theorem is fulfilled if there exist morphisms $\eta : \mathbf{1} \to \mathcal{P}(1)$ such that, when extended to $\mathcal{P}(\{x\})$, for all finite sets $X \ni x$, the following diagrams commute

$$(1.35)\quad \begin{tikzcd} \mathcal{P}(X)\odot\mathbf{1} \arrow[r, "r"] \arrow[d, "\mathbb{1}\odot\eta_x"'] & \mathcal{P}(X) \\ \mathcal{P}(X)\odot\mathcal{P}(\{x\}) \arrow[ur, "\circ_x"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1}\odot\mathcal{P}(X) \arrow[r, "l"] \arrow[d, "\eta_x\odot\mathbb{1}"'] & \mathcal{P}(X) \\ \mathcal{P}(\{x\})\odot\mathcal{P}(X) \arrow[ur, "\circ_x"'] & \end{tikzcd}$$

Conversely, if $\mathcal{P}$ is a $\mathtt{Set}_f$-module with an operad structure, we can define $\circ_x$-operations giving $\mathcal{P}$ the structure of pseudo-operad as follows. Given two finite sets X, Y, first define a surjection

$$(1.36)\quad f : X\sqcup_x Y \twoheadrightarrow X\ ,\ f(y) := x \text{ for } y\in Y, \text{ and } f(x') := x' \text{ for } x\neq x'\in X.$$

Let $\eta : \mathbf{1} \to \mathcal{P}(\{x\})$ be the unit for the operad structure and define $j \cdot \mathcal{P}(Y) \to \mathcal{P}[f]$ by

$$\mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(Y) \odot \mathbf{1}^{\odot(n-1)} \xrightarrow{\mathbb{1}\odot\eta^{\odot(n-1)}} \mathcal{P}[f],$$

where $n := \#(X)$. Using the natural identification $\iota : \mathcal{P}(X) \cong \mathcal{P}[g]$ for $g : X \twoheadrightarrow [1]$, define $\circ_x$ by the following diagram:

(1.37)
$$\begin{tikzcd}
\mathcal{P}(X) \odot \mathcal{P}(Y) \arrow[r, "\circ_x"] \arrow[d, "\iota \odot j"'] & \mathcal{P}[g \circ f] \cong \mathcal{P}(X \sqcup_x Y) \\
\mathcal{P}[g] \odot \mathcal{P}[f] \arrow[ur, "\gamma_{g,f}"'] &
\end{tikzcd}$$

The associativity conditions (1.33) for these $\circ_x$ operations follow directly from the associativity axiom in Theorem 1.60.

NOTE: *For the rest of the book we will freely use the alternative definitions of operads and pseudo-operads and not distinguish between Σ-modules and* $\mathtt{Set}_f$-*modules.*

REMARK 1.62. It is clear from the above exposition that the full subcategory of the category $\Psi\mathtt{Op}$ of pseudo-operads $\mathcal{P}$ in $\mathcal{C}$ such that $\mathcal{P}(1) = \mathbf{0}$ is isomorphic to the full subcategory of $\mathtt{Op}$ consisting of operads $\mathcal{Q}$ with $\mathcal{Q}(1) = \mathbf{1}$. So we will usually make *no distinction* between pseudo-operads with $\mathcal{P}(1) = \mathbf{0}$ and operads with $\mathcal{Q}(1) = \mathbf{1}$.

The $\circ_x$-operations reappear in Theorem 1.73 below where we define a functor from the category of trees to the category $\mathcal{C}$. In fact, we will prove that there is an equivalence between pseudo-operad structures on a $\mathtt{Set}_f$-module $\mathcal{P}$ in a symmetric monoidal category $\mathcal{C}$ and functors from the category of trees to $\mathcal{C}$. Various constructions using trees, such as free functors and cobar resolutions, are more appropriate to pseudo-operads than operads, although they are used in both contexts.

1.8. Operads as monoids

There is yet another definition of operads using a monoidal structure on Σ-$\mathtt{Mod}$ (the $\square$-product defined below) which combines the data for all the composition operations. Relative to this monoidal structure, an operad is a monoid and the construction of the free operad can be understood as a modified free monoid construction. See Section 1.9.

Smirnov [**Smi82**] introduced the $\square$-product in the context of operads. Joyal had studied it in the context of general category theory [**Joy81**] where $\mathtt{Set}_f$ is known as the permutation category.

A *species* is a contravariant symmetric monoidal functor from $\mathtt{Set}_f$ to a category $\mathcal{C}$, which satisfies some auxiliary conditions not of interest here. The monoidal structure on the category of species is called the substitution product and is essentially the same as the $\square$-product defined below.

DEFINITION 1.63. Define the $\Box$*-product on* Σ-Mod by

$$(A\Box B)(n) := \coprod_{m=1}^{n} (A(m) \odot_{\Sigma_m} B[m,n]).$$

REMARK 1.64. One motivation for the definition of the $\Box$-product is the following relation with Schur functors (see Definition 1.24)

$$\mathcal{F}_A(\mathcal{F}_B(X)) \cong \mathcal{F}_{A\Box B}(X).$$

To show that the $\Box$-product defines a monoidal structure, we need to check the axioms for the unit and for associativity. The associativity constraint involves a rather complicated re-shuffling of factors and the crucial identity is given in the following proposition which will prove useful later when we compare different characterizations of the Koszul property.

PROPOSITION 1.65. *For each $A, B \in \Sigma$-Mod and natural numbers $n, p \geq 1$,*

$$(A\Box B)[n,p] \cong \coprod_{n\leq m\leq p} A[n,m] \odot_{\Sigma_m} B[m,p],$$

where $A[n,m]$, $B[m,p]$, etc., were introduced in equation (1.28).

In the case of $n = 1$, this reduces to the definition of $A\Box B$ on Σ-Mod. The details of the proof are rather technical and will be deferred to the appendix to this section.

The associativity constraint for $\Box$ comes from the isomorphism in Proposition 1.65:

$$\begin{aligned} ((A\Box B)\Box C)(p) &= \coprod_{1\leq m\leq p} (A\Box B)(m) \odot_{\Sigma_m} C[m,p] \\ &\cong \coprod_{1\leq n\leq m\leq p} (A(n) \odot_{\Sigma_n} B[n,m]) \odot_{\Sigma_m} C[m,p] \\ &\cong \coprod_{1\leq n\leq m\leq p} A(n) \odot_{\Sigma_n} (B[n,m] \odot_{\Sigma_m} C[m,p]) \\ &\cong \coprod_{1\leq n\leq p} A(n) \odot_{\Sigma_n} (B\Box C)[n,p] = (A\Box(B\Box C))(p). \end{aligned} \tag{1.38}$$

Regarding the unit object, we have the following proposition.

PROPOSITION 1.66. *If the monoidal category C has an initial object $\mathbf{0}$, then a unit object for the $\Box$-product on Σ-modules is given by the functor defined by*

$$\mathbf{1}_{\Sigma\text{-Mod}}(X) := \begin{cases} \mathbf{1}_C, & \textit{when } card(X) = 1, \textit{ and} \\ \mathbf{0}, & \textit{otherwise.} \end{cases}$$

The definition for morphisms follows from the properties of the unit and the initial object.

PROOF. Both isomorphisms

$$(\mathbf{1}\Box A)(n) \cong A(n) \quad \text{and} \quad (A\Box \mathbf{1})(n) \cong A(n)$$

follow immediately from the unitarity axioms and the isomorphisms

$$\odot X \cong \mathbf{0} \cong X \odot \mathbf{0} \tag{1.39}$$

and

$$0 \coprod X \cong X \cong X \coprod 0. \tag{1.40}$$

Since the initial object is the colimit over an empty index set, the isomorphisms in (1.39) follow from our assumption that the multiplication $\odot$ on $\mathcal{C}$ commutes with colimits. The isomorphisms in (1.40) are implied by the Yoneda lemma (see [**Mac71**, Lemma III.2.2] or [**HS71**, Proposition II.4.1]) and the following isomorphisms:

$$Hom_{\mathcal{C}}(X \coprod Y, Z) \cong Hom_{\mathcal{C}}(X, Z) \times Hom_{\mathcal{C}}(Y, Z) \text{ and } Hom_{\mathcal{C}}(\mathbf{0}, Z) \cong \{1\}. \quad \blacksquare$$

DEFINITION 1.67. (Operad as monoid) An *operad* is a monoid for the $\square$-product.

THEOREM 1.68. *Definition 1.4, and hence also the definition in Theorem 1.60, are equivalent to Definition 1.67.*

PROOF. Let $\mathcal{P}$ be a monoid for the $\square$-product. Since $\mathcal{P}\square\mathcal{P}$ is a coproduct of terms $\mathcal{P}(n) \odot_{\Sigma_n} (\mathcal{P}(m_1) \odot \cdots \odot \mathcal{P}(m_n))$, a monoid structure $\mu : \mathcal{P}\square\mathcal{P} \to \mathcal{P}$ defines a set of operations $\gamma_{n,m_1, \quad ,m_n}$ of the type appearing in Definition 1.4. The equivariance axiom for these operations follows from the fact that the $\square$-product is defined using the reduced $\odot$-product $\odot_{\Sigma_n}$; see equation (1.11). The commutative diagram expressing the associativity axiom in Definition 1.4 is equivalent to

$$\mu(\mu \square \mathbb{1}) = \mu(\mathbb{1} \square \mu) a_{\mathcal{P},\mathcal{P},\mathcal{P}} : (\mathcal{P}\square\mathcal{P})\square\mathcal{P} \longrightarrow \mathcal{P}.$$

This follows from the description of $a_{\mathcal{P},\mathcal{P},\mathcal{P}}$ as the composition of the isomorphisms in (1.38) which have the effect of shuffling terms in the same way as the morphism ρ at the top of the commutative diagram (1.4). The unit axiom in May's definition is equivalent to the fact that $1_{\Sigma\text{-}\mathsf{Mod}}$ is a unit for μ.

Conversely, let $\mathcal{P}$ be an operad as in Definition 1.4. Then the structure operations $\gamma_{n;m_1, \quad ,m_n}$ define on $\mathcal{P}$ a $\square$-monoid structure in an obvious manner. $\blacksquare$

1.8.1. Appendix. In this appendix we prove Proposition 1.65. By definition,

$$(A\square B)[n,p] = \coprod_{f \in \mathcal{S}urj[n,p]} (A\square B)[f].$$

A given surjection $f : [p] \twoheadrightarrow [n]$ partitions $[p]$ into n subsets and then $A\square B$ runs over all partitions of each of these subsets. By definition

$$\begin{aligned}
(A\square B)[f] &= \bigodot_{1\le i\le n} (A\square B)(f^{-1}(i)) \\
&= \bigodot_{1\le i\le n} \left(\coprod_{m_i=1}^{card(f^{-1}(i))} A(m_i) \odot_{\Sigma_{m_i}} \left[\coprod_{g_i \in \mathcal{S}urj[[m_i], f^{-1}(i)]} B[g_i] \right] \right) \\
&= \bigodot_{1\le i\le n} \left(\coprod_{m_i=1}^{card(f^{-1}(i))} A(m_i) \odot_{\Sigma_{m_i}} \left[\coprod_{g_i \in \mathcal{S}urj[[m_i], f^{-1}(i)]} \left\{ \bigodot_{1\le j\le m_i} B(g_i^{-1}(j)) \right\} \right] \right).
\end{aligned}$$

The rather complicated indexing describes what one might call 'a nested partition,' the subsets $f^{-1}(i)$ partition $[p]$ into n parts and the subsubsets $g_i^{-1}(j)$ for $1 \le j \le m_i$ partition each $f^{-1}(i)$ into m_i parts.

We can distribute the first $\odot$ product (indexed by i) over the adjacent coproduct and permute factors so that all the $A(m_i)$ terms are together and all the $B(g_i^{-1}(j))$ terms are together. When $(A\Box B)[f]$ is rewritten in this form, we have a coproduct over all nested partitions of $[p]$ given by the $f^{-1}(i)$ and $g_i^{-1}(j)$,

$$(A\Box B)[f] \cong \coprod_{\substack{\mathbf{m}\in\mathbf{N}^n \\ 1\le m_i\le \#(f^{-1}(i))}} \left(\bigodot_{1\le i\le n} A(m_i)\right) \odot_{\Sigma_{m_1}\times\ \times\Sigma_{m_n}} \left(\coprod_{g=\sqcup g_i}\left[\bigodot_{j\in\sqcup[m_i]} B(g^{-1}(j))\right]\right).$$

Let $m = |\mathbf{m}| := \sum_{i=1}^n m_i$ whenever $\mathbf{m} = (m_1,\dots,m_n) \in \mathbf{N}^n$ and define a 'standard partition' via the nondecreasing surjection $h_{\mathbf{m}} : [m] \twoheadrightarrow [n]$ by

$$h_{\mathbf{m}}(j) = i \text{ for } m_1+\cdots+m_{i-1} < j \le m_1+\cdots+m_i \tag{1.41}$$

Then we can write the preceding sum as:

$$\coprod_{\{\mathbf{m}|\ 1\le m_i\le\#(f^{-1}(i))\}} A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\ \times\Sigma_{m_n}} \left(\coprod_{\substack{g\in Surj[m,p] \\ h\circ g=f}} B[g]\right)$$

$$\cong \coprod_{\{\mathbf{m}|\ 1\le m_i\le\#(f^{-1}(i))\}} (A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\ \times\Sigma_{m_n}} \mathbf{1}[m,m]) \odot_{\Sigma_m} \left(\coprod_{\substack{g\in Surj[m,p] \\ h\circ g=f}} B[g]\right).$$

If we now take the coproduct over the surjections $f : [p] \twoheadrightarrow [n]$, this removes the conditions on the individual m_i and allows for arbitrary g, so that

$$\begin{aligned} A\Box B[n,p] &= \coprod_{f\in Surj[n,p]} (A\Box B)[f] \\ &\cong \coprod_{\substack{\mathbf{m}\in\mathbf{N}^n \\ |\mathbf{m}|\le p}} (A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\ \times\Sigma_{m_n}} \mathbf{1}[m,m]) \odot_{\Sigma_m} \left(\coprod_{g\in Surj[m,p]} B[g]\right). \end{aligned} \tag{1.42}$$

We need to show that this is isomorphic to $\coprod_{n\le m\le p} A[n,m] \odot_{\Sigma_m} B[m,p]$, and since the factor on the right of the $\odot$-product in (1.42) equals $B[m,p]$, we need only identify the term on the left of the $\odot$-product. The following lemma, whose proof is left as an exercise, completes the proof.

LEMMA 1.69. *There is a right Σ_m-module left Σ_n-module isomorphism:*

$$A[n,m] \cong \coprod_{\{\mathbf{m}|\sum m_i=m\}} (A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\ \times\Sigma_{m_n}} \mathbf{1}[m,m]).$$

The index set for the coproduct is the set of ordered partitions of m into n nonempty parts.

Since the map $\alpha\odot\sigma \longmapsto \alpha\cdot\sigma$ is also a left Σ_n-module morphism, this completes the proof of Proposition 1.65. ∎

1.9. Free operads and free pseudo-operads

In this section we give an explicit construction of the free pseudo-operad generated by a Σ-module. Adjoining a unit gives the free operad. To help the reader keep track of the steps in the construction, we will give a general description of the method before going into the details.

Given a Σ-module A, the first step in the construction is to assign to each labeled tree (T, ℓ) the unordered $\odot$-product $A(T,\ell) := \bigodot_{Vert(T)} A(In(v))$, where $In(v)$ is the set of input edges at the vertex v. The unordered $\odot$-product was defined in Definition 1.58. The free operad generated by A is then the colimit of $A(T,\ell)$ as (T,ℓ) runs over the category $\mathtt{Tree}$ of labeled rooted trees; see Definition 1.77. A more explicit construction of the free operad can be given as follows.

The automorphism group of T acts on the coproduct of the $A(T,\ell)$ as ℓ runs over an equivalence class of labelings and we denote by $A[T,\ell]$ the coequalizer of this action. The free operad generated by A is then the coproduct of the $A[T,\ell]$ as $[T,\ell]$ runs over the set of isomorphism classes of labeled rooted trees (see Proposition 1.82). As objects in $\mathcal{C}$, for a fixed T, all the $A(T,\ell)$ and $A[T,\ell]$ are isomorphic, but we need to assign a labeling in order to define a representation of Σ_n (see Definition 1.81 and equation (1.50)). In Proposition 1.87 we describe this representation as an induced representation and then give some examples.

One can also use $A(T,\ell)$ to define a functor from the category $Iso(\mathtt{Tree})$ of isomorphisms of $\mathtt{Tree}$ to the symmetric monoidal category $\mathcal{C}$ where the Σ-module A take its values; see Proposition 1.72. Theorem 1.73 and Corollary 1.74 state that this functor extends to all morphisms of $\mathtt{Tree}$ if and only if A is a pseudo-operad. This shows that the category of trees is actually better suited to the description of pseudo-operads than to operads and explains why we first describe the free pseudo-operad functor Ψ and then describe the free operad functor Γ by adjoining a unit to Ψ; see (1.58).

To motivate the construction and explain the reason why the construction is as complicated as it is, consider two familiar examples of a free monoid in a monoidal category:

$$\mathcal{F}^{\otimes}_{ree}(X) = \bigoplus_{n\geq 0} X^{\otimes n} \text{ for } X \in \mathtt{Mod_k}$$

and

$$\mathcal{F}^{\times}_{ree}(X) = \bigsqcup_{n\geq 0} X^{\times n} \text{ for } X \in \mathtt{Set},$$

where $X^{\times 0}$ equals some fixed singleton set.

More generally, if $\mathcal{C}$ has countable coproducts and the multiplication commutes with coproducts in both arguments, then the free monoid on $X \in \mathcal{C}$ has a form analogous to the two examples above:

$$\mathcal{F}^{\odot}_{ree}(X) = \coprod_{n\geq 0} X^{\odot n}.$$

This form was used by Barr [**Bar66**] to define a cohomology theory for monoids in any category satisfying the above requirements. Similarly, the homological algebra of operads uses in an essential way the free operad.

For the Smirnov monoidal structure defined in Section 1.2, the multiplication $\square$ in the category of Σ-modules is distributive over coproducts on the left but not on the right. For example, $(E\square(F\coprod G))(2)$ contains a component $E(2)\odot F(1)\odot G(1)$ which does not appear in $(E\square F)(2)$ or $(E\square G)(2)$ and therefore, in general

$$E\square(F\coprod G)\neq(E\square F)\coprod(E\square G).$$

This creates a problem with the naive construction of the free monoid given above. A more sophisticated construction valid for multiplications which are left distributive over coproducts and commute with filtered colimits on the right (as in the case of the $\square$-product) has been described by Baues, Jibladze and Tonks in [**BJT97**]. In this section we describe a different construction of the free operad due to Ginzburg and Kapranov [**GK94**] using trees. It will appear in Section 3.1 as the cobar complex of an operad. In Section 3.3 we introduce another complex closely related to free monoid construction of Baues et al.

The idea underlying the model $\mathcal{F}^{\odot}_{\mathit{ree}}(X)=\coprod_{n\geq 0}X^{\odot n}$ for the free monoid generated by X is that there is a subobject $X^{\odot n}$ corresponding to the input data for the n-fold iterated composition law for a monoid structure on X. By analogy, we expect the free operad to contain a subobject corresponding to the input data for each possible sequence of operadic compositions. Since the objects of the category $\mathtt{Tree}_m$ of rooted trees provide combinatorial objects representing all m-ary operations coming from sequences of k-ary operations, it is natural to use $\mathtt{Tree}_m$ to describe the arity m component of the free operad generated by a Σ-module.

Given a labeled tree (T,ℓ) and a Σ-module A, in order to describe the input data for a sequence of operad compositions of various $A(k)$, we form the unordered $\odot$-product (defined in (1.31)) of the $A(\mathit{In}(v))$ as v runs over the vertices of T.

DEFINITION 1.70. Let A be a Σ-module in $\mathcal{C}$. For a labeled tree (T,ℓ), define

$$A(T,\ell):=\bigodot_{\mathit{Vert}(T)}A(\mathit{In}(v)),$$

where $A(\mathit{In}(v))$ is the object of $\mathcal{C}$ corresponding to the finite set $\mathit{In}(v)$ of input edges of v.

Note that $A(T,\ell)$ does not depend on ℓ as an object in $\mathcal{C}$. If no confusion is possible, we will write simply $A(T)$ instead of $A(T,\ell)$. An example of $A(T)$ is given in Figure 9

REMARK 1.71. Elements of $A(T)$ will often be written as sequences

$$\{a_v\}_{v\in \mathit{Vert}(T)}\ \text{ with }\ a_v\in A(\mathit{In}(v)).$$

We may interpret a_v as a *color of the vertex* v and $A(T)$ as the set of A-colorings of the vertices of T.

In the main definition of this section, Definition 1.77, we will use a certain functoriality of the correspondence $A\mapsto A(T,\ell)$ described in the following proposition. For a category $\mathcal{D}$, let $\mathit{Iso}(\mathcal{D})$ denote the *subcategory of isomorphisms*, that is, the subcategory with the same objects but whose morphisms are the isomorphisms of $\mathcal{D}$.

PROPOSITION 1.72. *The correspondence $(T,\ell)\mapsto A(T,\ell)$ determines, for each Σ-module A, a covariant functor $\mathit{Iso}(\mathtt{Tree})\to\mathcal{C}$.*

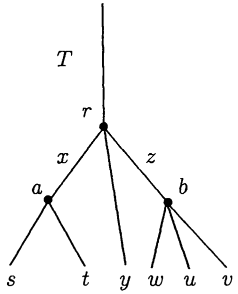


FIGURE 9. An example of a rooted tree with root vertex r, with $A(T) = A(\{x,y,z\}) \odot A(\{s,t\}) \odot A(\{u,v,w\})$.

PROOF. By definition, a Σ-module in the category $\mathcal{C}$ is a contravariant functor from Σ to $\mathcal{C}$, therefore a bijection $g : X \to Y$ of finite sets determines an isomorphism $A(g) : A(Y) \to A(X)$. An isomorphism $f : (T,\ell) \to (S, \ell f^{-1})$ of labeled rooted trees defines a bijection $f^* : Edge(S) \to Edge(T)$ (see Remark 1.30) which induces, for each $v \in Vert(T)$, a bijection $In(f(v)) \xrightarrow{\cong} In(v)$ and thus also an isomorphism $f_v : A(In(v)) \longrightarrow A(In(f(v)))$. Define

$$\overline{f} := \bigodot_{Vert(T)} f_v : \bigodot_{Vert(T)} A(In(v)) \longrightarrow \bigodot_{Vert(T)} A(In(f(v)))$$

and, finally, $A(f) : A(T) \to A(S)$ as the composition

$$\begin{aligned} A(f) : A(T,\ell) = \bigodot_{v \in Vert(T)} A(In(v)) &\xrightarrow{\overline{f}} \bigodot_{v \in Vert(T)} A(In(f(v))) \\ &\cong \bigodot_{w \in Vert(S)} A(In(w)) = A(S, \ell f^{-1}), \end{aligned} \tag{1.43}$$

where the last isomorphism is induced by the bijection $Vert(T) \xrightarrow{\cong} Vert(S)$. The functoriality of the above construction is evident. ■

For an operad or pseudo-operad A, it is possible to extend the construction of Proposition 1.72 to define a functor from `Tree` to $\mathcal{C}$ by extending the definition of $A(f)$ to all morphisms f in `Tree`.

Because each morphism in `Tree` is a composition of isomorphisms and elementary morphisms, it is enough to define the value of the functor A on elementary morphisms. If e_x is an edge oriented from v to w, $X = In(w)$, $Y = In(v)$, then the following composite morphism $A(\pi_x)$ is associated to an elementary morphism of trees $\pi_x : T \to T/e_x$ of the type shown in Figure 10:

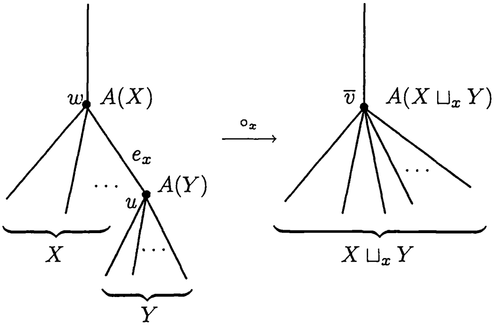


FIGURE 10. The morphism $A(\pi_x) : A(T) \to A(T/e_x)$ where π_x collapses edge e_x with label x to a vertex $\overline{v}$.

$$
(1.44) \qquad \bigodot_{u \in Vert(T)} A(In(u)) \cong A(X) \odot A(Y) \odot \bigodot_{\substack{u \in Vert(T) \\ u \neq v, w}} A(In(u))
$$

$$
\xrightarrow{\circ_x \odot \mathbb{1}} A(X \sqcup_x Y) \odot \bigodot_{\substack{u \in Vert(T/e_x) \\ u \neq \overline{v}}} A(In(u)) \cong \bigodot_{u \in Vert(T/e_x)} A(In(u)).
$$

where the $\circ_x$-operation for a pseudo-operad is defined in (1.32). So we have defined, for any elementary contraction, a morphism:

$$A(\pi_x) : A(T) \longrightarrow A(T/e_x).$$

The functoriality of the above construction is characterized by the following proposition.

THEOREM 1.73. *Let A be a Σ-module in C. Assume that there are, for each pair of finite sets X and Y and each $x \in X$, well-defined operations*

$$\circ_x : A(X) \odot A(Y) \longrightarrow A(X \sqcup_x Y).$$

For a composition of elementary morphisms

$$f = \pi_{x_1} \circ \cdots \circ \pi_{x_n} : T \to T/\{e_{x_1}, \dots, e_{x_n}\}$$

we define

$$
(1.45) \qquad A(f) := A(\pi_{x_1}) \cdots A(\pi_{x_n}) : A(T) \to A(T/\{e_{x_1}, \dots, e_{x_n}\}).
$$

Then a necessary and sufficient condition that definition (1.45) be independent of the representation of f and further that $A(f'f'') = A(f')A(f'')$ for any two composable elementary morphisms f' and f'' is that the composition operations $\circ_x$ satisfy the associativity axiom (1.33).

PROOF. The associativity axiom is precisely the assertion that the composition of $A(\pi_x)$ and $A(\pi_y)$ is independent of the order. The generalization to the composition of an arbitrary number of $\circ_x$-operations is an elementary exercise. ∎

The following corollary easily follows from Theorem 1.73.

COROLLARY 1 74. *Let $\mathcal{P}$ be a pseudo-operad in $\mathcal{C}$. Then the correspondence $T \mapsto \mathcal{P}(T)$ extends to a covariant functor* $\mathtt{Tree} \to \mathcal{C}$.

REMARK 1.75. Theorem 1.73 and Corollary 1.74 show that the category of trees is more suited to the description of pseudo-operads than to operads. The unit axiom in the definition of an operad is an independent piece of data not expressed in terms of morphisms $A(T) \to A(S)$. Therefore we will use trees to describe the free pseudo-operad before describing the free operad.

Let us recall the following standard notions. Given a category $\mathcal{D}$ and a covariant functor $F : \mathcal{D} \to \mathcal{C}$, let

$$\operatorname*{colim}_{x \in \mathcal{D}} F(x)$$

be an element of $\mathcal{C}$ together with a system

$$\left\{ \iota_y : F(y) \to \operatorname*{colim}_{x \in \mathcal{D}} F(x) \right\}_{y \in Ob(\mathcal{D})}$$

of morphisms of $\mathcal{C}$ such that $\iota_z F(f) = \iota_y$ for each $y, z \in Ob(\mathcal{D})$ and for each $f \in Hom_{\mathcal{D}}(y, z)$, having the obvious universal property. We will assume that in $\mathcal{C}$ *appropriate colimits exist.*

EXAMPLE 1.76. If $\mathcal{C} = \mathtt{Mod}_{\mathbf{k}}$, then the colimit of a covariant functor $F : \mathcal{D} \to \mathcal{C}$ is the quotient

$$\operatorname*{colim}_{x \in \mathcal{D}} F(x) = \bigoplus_{x \in \mathcal{D}} F(x) / \sim, \tag{1.46}$$

where $\sim$ is the equivalence generated by

$$F(y) \ni a \sim F(f)(a) \in F(z),$$

for each $a \in F(y)$, $y, z \in Ob(\mathcal{D})$ and $f \in Hom_{\mathcal{D}}(y, z)$. If $\mathcal{C}$ is the category of topological spaces, the formula for the colimit is formally the same as (1.46), with the direct sum replaced by the disjoint union.

We are ready to define the free pseudo-operad functor.

DEFINITION 1.77. For a Σ-module A in $\mathcal{C}$ and a finite set X define

$$\Psi(A)(X) := \operatorname*{colim}_{(T,\ell) \in Iso(\mathtt{Tree}_X)} A(T, \ell). \tag{1.47}$$

Let A and B be Σ-modules and $\alpha : A \to B$ a morphism. For each labeled tree T, there are morphisms $A(In(v)) \to B(In(v))$ for each vertex v of T, therefore, by functoriality of the $\odot$-product, there is also a morphism $\alpha(T) : A(T) \to B(T)$. Then define

$$\Psi(\alpha)(X) := \operatorname*{colim}_{T \in Iso(\mathtt{Tree}_X)} \alpha(T) : \Psi(A)(X) \to \Psi(B)(X).$$

This shows that $\Psi(A)(X)$ is 'functorial in A.' It is equally easy to show that the 'opposite leaf relabeling' $(T, \ell) \mapsto (T, \sigma^{-1}\ell)$ of objects of $Iso(\mathtt{Tree})$ determines, for each bijection $\sigma : X \xrightarrow{\cong} Y$, a morphism $\Psi(A)(\sigma) : \Psi(A)(Y) \to \Psi(A)(X)$ of the colimit, making $\Psi(A)$ a Σ-module in $\mathcal{C}$.

PROPOSITION 1.78. *The Σ-module $\Psi(A)$ defined in Definition 1.77 is, for each Σ-module A in $\mathcal{C}$, a pseudo-operad in $\mathcal{C}$, called the* free pseudo-operad *generated by A.*

PROOF. Recall that in Remark 1.38 we defined, for each X-tree T, each Y-tree S and each $x \in X$, the $X \sqcup_x Y$-labeled tree $T \circ_x S$ with $\mathit{Vert}(T \circ_x S) = \mathit{Vert}(T) \sqcup \mathit{Vert}(S)$. The colimit of the natural identifications

$$\begin{aligned}(1.48)\qquad \circ_x : A(T)\odot A(S) = & \bigodot_{v \in \mathit{Vert}(T)} A(In(v)) \odot \bigodot_{w \in \mathit{Vert}(S)} A(In(w)) \\ \cong & \bigodot_{u \in \mathit{Vert}(T)\sqcup \mathit{Vert}(S)} A(In(u)) \cong \bigodot_{u \in \mathit{Vert}(T \sqcup_x S)} A(In(u)) \cong A(T \circ_x S)\end{aligned}$$

induces the required structure operations (denoted again $\circ_x$)

$$(1.49)\qquad \circ_x : \Psi(A)(X) \odot \Psi(A)(Y) \longrightarrow \Psi(A)(X \sqcup_x Y).$$

By Theorem 1.61, to check that $\Psi(A)$ is a pseudo-operad, it is enough to check the associativity conditions (1.33). The commutativity of the diagram

$$\begin{tikzcd} A(T_1) \odot A(T_2) \odot A(T_3) \arrow[r, "\circ_{x_2}(\circ_{x_1} \odot 1\!\!1)"] \arrow[d, "(1\!\!1 \odot \tau)"'] & A((T_1 \circ_{x_1} T_2) \circ_{x_2} T_3) \\ A(T_1) \odot A(T_3) \odot A(T_2) \arrow[ur, "\circ_{x_1}(\circ_{x_2} \odot 1\!\!1)"'] & \end{tikzcd}$$

follows from the definition of the unordered $\odot$-product. The second equation in the associativity axioms (1.33) is then obtained as the colimit of these diagrams over $T_1, T_2, T_3 \in \mathit{Iso}(\mathtt{Tree})$. The first equation in the associativity axioms (1.33) is true for a similar reason. ∎

The proof that $\Psi(A)$ is indeed the free pseudo-operad on A is postponed to the end of this section (Theorem 1.91).

EXAMPLE 1.79. Let $\mathcal{C} = \mathtt{Set}$ and A the Σ-module with $A(n) = \{\text{point}\}$ for each $n \geq 1$. Then $\Psi(A) = \mathcal{T}ree$, the operad of isomorphism classes of labeled rooted trees introduced in Section 1.5.

To verify this, observe that, for an arbitrary X-labeled tree, $A(T) = \{\text{point}\}$, thus the colimit (1.47) is the disjoint union of one-point sets indexed by isomorphism classes of X-labeled trees, which is, of course, $\mathcal{T}ree(X)$.

Similarly, let B be the Σ-module with $B(n) = \{\text{point}\}$ for each $n \geq 2$ and $B(1) = \emptyset$. Then $\Psi(B)(X) = \mathcal{R}tree(X)$, the set of isomorphism classes of reduced rooted labeled trees.

We are going to give a more explicit description of $\Psi(A)$ as a coproduct of simpler elements introduced in the following definition.

DEFINITION 1.80. Let A be a Σ-module with values in a symmetric monoidal category $\mathcal{C}$ and $[T, \ell]$ an equivalence class of X-labeled trees. For any automorphism

$\psi \in Aut(T)$, let $A(\psi) : A(T,\ell) \longrightarrow A(T,\ell\psi^{-1})$ be the isomorphism in (1.43) and define

$$A[T,\ell] := \underset{\psi \in Aut(T)}{\text{coequalizer}} \left\{ A(\psi) : \coprod_{(T,\lambda)\in[T,\ell]} A(T,\lambda) \longrightarrow \coprod_{(T,\lambda)\in[T,\ell]} A(T,\lambda) \right\}$$

For any $\sigma \in \Sigma_X$, there is a 'leaf relabeling' sending the labeling ℓ to $\sigma\ell$. Since both $A(T,\ell)$ and $A(T,\sigma\ell)$ are copies of the same object in the category $\mathcal{C}$, we have a 'leaf relabeling' morphism

$$\hat{\sigma} : A(T,\ell) = \bigodot_{Vert(T)} A(In(v)) \xrightarrow{\mathbb{1}} \bigodot_{Vert(T)} A(In(v)) = A(T,\sigma\ell).$$

Leaf relabeling clearly commutes with the action of automorphisms of the tree T and therefore defines an isomorphism of the coequalizers

$$\hat{\sigma} : A[T,\ell] \longrightarrow A[T,\sigma\ell].$$

It is clear from the above formula that $A[T,\ell]$ is not, in general, Σ_X-closed. The Σ_X-closure of $A[T,\ell]$ is described in the following definition.

DEFINITION 1.81. For a Σ-module A, finite set X and a rooted tree T, let

$$\begin{aligned}\mathcal{A}[T]_X &:= \coprod_{\substack{\text{equivalence classes}\\ \text{of } X\text{-labelings of } T}} A[T,\ell] \\ &\cong \underset{\psi \in Aut(T)}{\text{coequalizer}} \left\{ A(\psi) : \coprod_{\text{all } X\text{-labelings of } T} A(T,\ell) \to \coprod_{\text{all } X\text{-labelings of } T} A(T,\ell) \right\}.\end{aligned}$$

In the case $X = [n]$, we write $\mathcal{A}[T] := \mathcal{A}[T]_{[n]}$. Leaf relabeling defines a representation of Σ_X on $\mathcal{A}[T]_X$:

$$\rho_T : \Sigma_X \to Aut_{\mathcal{C}}(\mathcal{A}[T]_X). \tag{1.50}$$

Let $\mathcal{T}_X$ denote the set of isomorphism classes of rooted trees with $Leg(T) \cong X$ but with no concrete X-labeling specified. The following proposition follows immediately from definitions.

PROPOSITION 1.82. *Under the assumptions of Definition 1.77, there are decompositions*

$$\Psi(A)(X) \cong \coprod_{[T]\in\mathcal{T}_X} \mathcal{A}[T]_X \cong \coprod_{[T,\ell]\in\mathcal{T}ree(X)} A[T,\ell]. \tag{1.51}$$

In the first decomposition we choose a representative T for each isomorphism class in $\mathcal{T}_X$ and take the coproduct over these representatives. The second decomposition has a similar obvious meaning. Decompositions (1.51) are canonical up to these choices.

Proposition 1.82 gives a very explicit description of the free pseudo-operad. The operadic, right Σ_X-action on $\Psi(A)(X)$ is compatible with the opposite action of the leaf relabeling action (1.50) on the components of the first decomposition in (1.51).

By definition, there is a canonical morphism $j_\rho : A(T,\rho) \to A[T,\ell]$ for any labeling ρ in the equivalence class $[T,\ell]$. The morphisms

$$A(T,\rho\psi) \ni \alpha \mapsto A(\psi)\alpha \in A(T,\rho)$$

determine a morphism from the coproduct

$$\pi : \coprod_{(T,\lambda)\in[T,\ell]} A(T,\lambda) \cong \coprod_{\psi\in Aut(T)} A(T,\rho\psi) \longrightarrow A(T,\rho)$$

such that $\pi A(\psi) = \pi$ for all $\psi \in Aut(T)$. By the universal property of the coequalizer defining $A[T,\ell]$, π induces a morphism from $A[T,\ell]$ to $A(T,\rho)$ which is inverse to j_ρ. Therefore,

$$j_\rho : A(T,\rho) \xrightarrow{\cong} A[T,\ell].$$

Let us formulate the particular case $\rho = \ell$ as a corollary:

COROLLARY 1.83. *The canonical morphism $j_\ell : A(T,\ell) \to A[T,\ell]$ is an isomorphism for any labeled tree (T,ℓ).*

REMARK 1.84. Because $A(T,\ell)$ is canonically isomorphic to $A[T,\ell]$, one could as well write $\Psi(A)(X) = \coprod_{[T,\ell]\in Tree_X} A(T,\ell)$ or even

$$\Psi(A)(X) = \coprod_{T\in Tree(X)} A(T), \tag{1.52}$$

which is certainly simpler than Definition 1.77 based on the colimit. The drawback of (1.52) is that it assumes a choice of a representative T of each isomorphism class in $Tree(X)$. This would complicate the definition of the Σ_X-action; see Remark 1.85. Also defining the operad structure would be difficult – if (T,ℓ) and (S,λ) are our representatives of isomorphism classes $[T,\ell]$ and $[S,\lambda]$, then $(T \circ_i S, \ell \circ_i \lambda)$ need not be our chosen representative of the isomorphism class $[T \circ_i S, \ell \circ_i \lambda]$.

To simplify the exposition, we will, however, often make no distinction between $A(T,\ell)$ and $A[T,\ell]$ and represent $\Psi(A)$ (and similar objects) as in (1.52), but we must always be aware of the subtleties explained above.

REMARK 1.85. If we choose a representative labeling ℓ and identify $A[T,\ell]$ with $A(T,\ell)$ as in Corollary 1.83, the induced action of $\sigma_\ell(\psi) \in \Sigma(T,\ell)$ on $A(T,\ell)$ is given by the composite map

$$\begin{aligned} A(T,\ell) \ni \alpha \mapsto \hat{\sigma}_\ell(\psi)\alpha &\in A(T,\sigma_\ell(\psi)\ell) \\ = A(T,\ell\psi) \ni \hat{\sigma}_\ell(\psi)\alpha \mapsto A(\psi)\hat{\sigma}_\ell(\psi)\alpha &\in A(T,\ell); \end{aligned}$$

see Section 1.5 for the notation.

REMARK 1.86. It follows from the definition of the colimit that, given a labeled tree (T,ℓ), there exists a canonical map $\iota = \iota_{(T,\ell)} : A(T,\ell) \to \Psi(A)(X)$. This map, in fact, identifies $A(T,\ell)$ with $A[T,\ell]$ in the last decomposition of (1.51) and coincides with the identification described in Corollary 1.83, as indicated by the diagram

$$\begin{tikzcd} A(T,\ell) \arrow[r, "\iota_{(T,\ell)}"] \arrow[d, "j_\ell"'] & \Psi(A)(X) \arrow[d, "="] \\ A[T,\ell] \arrow[r] & \Psi(A)(X), \end{tikzcd}$$

where the bottom arrow is the canonical map given by the decomposition (1.51).

Particularly important is the case when the tree T is the X-labeled corolla $c(X)$. Then

$$A(c(X)) \cong A[c(X)] \cong \mathcal{A}[c(X)] \cong A(X)$$

and we denote the map $\iota_{c(X)}$ by

$$\zeta_A(X) : A(X) \to \Psi(A)(X).$$

In the case of the symmetric monoidal category $(\mathtt{Vec}_{\mathbf{k}}, \otimes_{\mathbf{k}})$, we can describe the leaf relabeling representation ρ_T of (1.50) as an induced representation.

PROPOSITION 1.87. *Let $\rho_{(T,\ell)}$ be the leaf relabeling representation ρ_T described in equation (1.50), restricted to the subgroup $\Sigma(T,\ell) \subset \Sigma_X$. The composition of $\rho_{(T,\ell)}$ with the isomorphism $\sigma_\ell : Aut(T) \to \Sigma(T,\ell)$ (equation (1.14)) coincides with the action of the automorphism group of T on $A(T,\ell)$ as described in Remark 1.85 and*

$$\rho_T \cong Ind \uparrow_{\Sigma(T,\ell)}^{\Sigma_X} \rho_{(T,\ell)}$$

as representations of Σ_X on $\mathcal{A}[T]_X$.

PROOF. In category $\mathtt{Vec}_{\mathbf{k}}$ the coequalizer $\mathcal{A}[T]_X$ defined in Definition 1.81 is isomorphic to the quotient of $\bigoplus A(T,\ell)$ modulo the subspace spanned by $\alpha - A(\psi)\alpha$, where $\alpha \in A(T,\ell)$, $A(\psi)\alpha \in A(T,\ell\psi^{-1})$ and $\psi \in Aut(T)$. Define a morphism $\chi : A(T,\ell) \otimes \mathbf{k}[\Sigma_X] \to \mathcal{A}[T]_X$ by

$$\chi : \alpha \otimes \gamma \mapsto \alpha \in A(T,\gamma^{-1}\ell) \mapsto [\alpha] \in A[T,\gamma^{-1}\ell] \hookrightarrow \mathcal{A}[T]_X.$$

Then for $\psi \in Aut(T)$,

$$\begin{aligned} \chi : \alpha \otimes \sigma_\ell(\psi)\gamma &\mapsto \alpha \in A(T,\gamma^{-1}\sigma_\ell(\psi)^{-1}\ell) = A(T,\gamma^{-1}\ell\psi^{-1}), \\ \chi : A(\psi^{-1})\alpha \otimes \gamma &\mapsto A(\psi^{-1})\alpha \in A(T,\gamma^{-1}\ell). \end{aligned}$$

Since

$$A(T,\gamma^{-1}\ell\psi^{-1}) \oplus A(T,\gamma^{-1}\ell) \ni \alpha - A(\psi^{-1})\alpha \mapsto 0 \in \mathcal{A}[T]_X,$$

χ defines a morphism

$$\bar{\chi} : Ind \uparrow_{\Sigma(T,\ell)}^{\Sigma_n} \rho_{(T,\ell)} \cong A[T,\ell] \otimes_{\mathbf{k}[\Sigma(T,\ell)]} \mathbf{k}[\Sigma_X] \longrightarrow \mathcal{A}[T]_X.$$

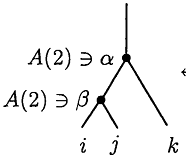
It is clear that $\bar{\chi}$ is surjective, and since the dimensions of the domain and range both equal $\dim A(T,\ell) \times [\Sigma_X : \Sigma(T,\ell)]$, χ is an isomorphism. One checks immediately that it is equivariant with respect to Σ_X. ■

The fact that ρ_T is well defined as a representation of Σ_X independent of the choice of labeling ℓ follows from the next proposition.

PROPOSITION 1.88. *Let G be a finite group, $H \subset G$ a subgroup, $H' = aHa^{-1}$ and (V,ρ) and (V',ρ') representations of H and H' respectively, such that there is an isomorphism $\varphi : V \to V'$ satisfying*

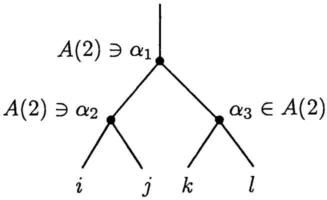
$$\varphi(v\rho(h)) = \varphi(v)\rho'(aha^{-1}) \text{ for } v \in H \text{ and } h \in H.$$

Then the induced representations $Ind \uparrow_H^G \rho$ and $Ind \uparrow_{H'}^G \rho'$ are isomorphic.



$\longleftrightarrow \quad (\alpha \otimes \beta) * (i, j, k) \in A(T, (i, j, k)).$

FIGURE 11.



$\longleftrightarrow (\alpha_1 \otimes \alpha_2 \otimes \alpha_3) * (i, j, k, l) \in A(T, \ell)$

FIGURE 12.

PROOF. If the induced representation is given in terms of group algebras $Ind \uparrow_H^G \rho = V \otimes_{\mathbf{k}[H]} \mathbf{k}[G]$ and $Ind \uparrow_{H'}^G \rho' = V' \otimes_{\mathbf{k}[H']} \mathbf{k}[G]$, then the isomorphism $\Phi : V \otimes_{\mathbf{k}[H]} \mathbf{k}[G] \to V' \otimes_{\mathbf{k}[H']} \mathbf{k}[G]$ is defined by $\Phi(v \otimes g) := \varphi(v) \otimes ag$. ■

EXAMPLE 1.89. If T is the binary tree with three leaves, then the direct sum $\bigoplus_\ell A(T, \ell)$ has a basis corresponding to fully-labeled trees in Figure 11, where α and β run over a basis of $A(2)$ and (i, j, k) represents the labeling of the leaves of T. In this form the equivalences given by the coequalizer in Definition 1.80 are

$$(\alpha \otimes \beta) * (i, j, k) \sim (\alpha \otimes \beta \cdot \tau) * (j, i, k)$$

for τ the generator of Σ_2. The isomorphism

$$(A(2) \otimes A(2)) \otimes_{\mathbf{k}[\Sigma_2]} \mathbf{k}[\Sigma_3] \cong Ind \uparrow_{\Sigma_2}^{\Sigma_3} (A(2) \otimes A(2)) \xrightarrow{\cong} \mathcal{A}[T]$$

(where Σ_2 acts on the first factor of $A(2) \otimes A(2)$) is given by

$$(\alpha \otimes \beta) \otimes \sigma \mapsto (\alpha \otimes \beta) * (\sigma^{-1}(1), \sigma^{-1}(2), \sigma^{-1}(3)).$$

EXAMPLE 1.90. For the tree pictured in Figure 12, $Aut(T) \cong D_4$, the dihedral group of symmetries of the square. The direct sum $\bigoplus_\ell A(T, \ell)$ has a basis indexed by fully-labeled trees as shown in Figure 12. The equivalence relation given by the action of $Aut(T)$ on $\bigoplus_\ell A(T, \ell)$ is generated by the following relations

$$\begin{array}{lcl} (\alpha_1 \otimes \alpha_2 \otimes \alpha_3) * (i, j, k, l) & \sim & (\alpha_1 \otimes (\alpha_2 \cdot \tau) \otimes \alpha_3) * (j, i, k, l), \\ (\alpha_1 \otimes \alpha_2 \otimes \alpha_3) * (i, j, k, l) & \sim & (\alpha_1 \otimes \alpha_2 \otimes (\alpha_3 \cdot \tau)) * (i, j, l, k), \\ (\alpha_1 \otimes \alpha_2 \otimes \alpha_3) * (i, j, k, l) & \sim & (-1)^{\alpha_2 \alpha_3} ((\alpha_1 \cdot \tau) \otimes \alpha_3 \otimes \alpha_2) * (k, l, i, j), \end{array} \tag{1.53}$$

and the isomorphism $\mathcal{A}[T] \cong Ind \uparrow_{D_4}^{\Sigma_4} (A(2) \otimes A(2) \otimes A(2))$ is described by a formula analogous to the one given in Example 1.89.

Let us prove that $\Psi(A)$ is indeed the free pseudo-operad on the Σ-module A.

THEOREM 1.91. *Suppose that $\odot$ is distributive over coproducts on the right and the left, then $\Psi(A)$ with compositions*

$$\circ_x : \Psi(A)(X) \odot \Psi(A)(Y) \longrightarrow \Psi(A)(X \sqcup_x Y)$$

defined in the proof of Proposition 1.78 is the free pseudo-operad, that is, the functor $\Psi : \Sigma\text{-}\mathtt{Mod} \to \Psi\mathtt{Op}$ is left adjoint to the forgetful functor $\mathcal{F}_{or} : \Psi\mathtt{Op} \to \Sigma\text{-}\mathtt{Mod}$.

PROOF. To verify that Ψ is left adjoint to $\mathcal{F}_{or}$, we need to show that for any $A \in \Sigma\text{-}\mathtt{Mod}$ and a pseudo-operad $\mathcal{P} \in \Psi\mathtt{Op}$, there is a bijection

$$(1.54) \qquad Hom_{\Psi\mathtt{Op}}(\Psi(A), \mathcal{P}) \underset{\xi}{\overset{\rho}{\rightleftarrows}} Hom_{\Sigma\text{-}\mathtt{Mod}}(A, \mathcal{F}_{or}(\mathcal{P})).$$

Note that both sets of morphisms consist of natural transformations between contravariant functors from the category $\mathtt{Set}_f$ to $\mathcal{C}$, the only difference being that in $\Psi\mathtt{Op}$ there is a condition of compatibility with the pseudo-operad composition operations which is not required in $\Sigma\text{-}\mathtt{Mod}$. The adjoint relation (adjunction) $\Psi \dashv \mathcal{F}_{or}$ is equivalent to the existence of two natural transformations:

$$(1.55) \qquad \zeta : \mathbb{1}_{\Sigma\text{-}\mathtt{Mod}} \longrightarrow \mathcal{F}_{or}\Psi \text{ and } \chi : \Psi\mathcal{F}_{or} \longrightarrow \mathbb{1}_{\Psi\mathtt{Op}},$$

called, respectively, the unit and counit of the adjunction, satisfying

$$(1.56) \qquad \chi\Psi \circ \Psi\zeta = \mathbb{1}_{\Psi} \text{ and } \mathcal{F}_{or}\chi \circ \zeta\mathcal{F}_{or} = \mathbb{1}_{\mathcal{F}_{or}},$$

where $\chi\Psi(A) := \chi_{\Psi(A)}$, $\Psi\zeta(A) := \Psi(\zeta_A)$ and $\mathbb{1}_{\Psi}$ is the identity natural transformation of Ψ to itself. The terms in the second equation are defined similarly.

Given ζ and χ, the bijective correspondence (1.54) is defined by

$$\rho(\alpha) := \mathcal{F}_{or}(\alpha) \circ \zeta_A \text{ and } \xi(\beta) := \chi_{\mathcal{P}} \circ \Psi(\beta)$$

for $\alpha \in Hom_{\Psi\mathtt{Op}}(\Psi(A), \mathcal{P})$ and $\beta \in Hom_{\Sigma\text{-}\mathtt{Mod}}(A, \mathcal{F}_{or}(\mathcal{P}))$.

After describing ζ and χ, rather than proving (1.56), we establish directly the bijection (1.54) by showing that ρ and ξ are inverse to each other. The map $\zeta_A \in Hom_{\Sigma\text{-}\mathtt{Mod}}(A, \mathcal{F}_{or}\Psi(A))$ is defined by morphisms

$$\zeta_A(X) : A(X) \to (\mathcal{F}_{or}\Psi(A))(X)$$

which were, for each $X \in \mathtt{Set}_f$, introduced in Remark 1.86. Thus $\rho(\alpha)(X)$. $A(X) \to (\mathcal{F}_{or}\mathcal{P})(X)$ is the composition of

$$(\mathcal{F}_{or}\alpha)(X) : (\mathcal{F}_{or}\Psi(A))(X) \to (\mathcal{F}_{or}\mathcal{P})(X)$$

with $\zeta_A(X)$.

To define χ, note that $\mathrm{c}(X)$ is the terminal object of $\mathtt{Tree}_X$, i.e. there is a unique tree morphism $\pi : T \to \mathrm{c}(X)$ for any tree $T \in \mathtt{Tree}_X$, which is given by contracting all internal edges of T.

Then, for any pseudo-operad $\mathcal{P}$, there is, by Corollary 1.74, a morphism $\mathcal{P}(\pi)$: $\mathcal{P}(T) \to \mathcal{P}(\mathrm{c}(X)) \cong \mathcal{P}(X)$. This extends to the colimit (1.47) over $T \in \mathtt{Tree}_X$ and determines a collection of $\mathcal{C}$-morphisms, one for each finite set $X \in \mathtt{Set}_f$,

$$(1.57) \qquad \chi_{\mathcal{P}}(X) : \Psi(\mathcal{F}_{or}\mathcal{P})(X) \to \mathcal{P}(X).$$

Together these form a morphism $\chi_{\mathcal{P}} \in Hom_{\Psi\mathtt{Op}}(\Psi\mathcal{F}_{or}(\mathcal{P}), \mathcal{P})$ for each $\mathcal{P} \in \Psi\mathtt{Op}$ and define the natural transformation χ. Thus $\xi(\beta)(X) : \Psi(A)(X) \to \mathcal{P}(X)$ is the composition of $\Psi(\beta) : \Psi(A)(X) \to \Psi(\mathcal{F}_{or}\mathcal{P})(X)$ with the morphism $\Psi(\mathcal{F}_{or}\mathcal{P})(X) \to \mathcal{P}(X)$ given by the pseudo-operad structure on $\mathcal{P}$.

The equation $\rho(\xi(\beta)) = \beta$ is immediate, since ρ composes $\xi(\beta) : \Psi(A) \to \mathcal{P}$ with $\zeta_A(X) : A(X) \cong A(\mathrm{c}(X)) \to \Psi(A)(X)$ and this composition is just β since $\mathrm{c}(X)$ has no edges to collapse.

To prove $\xi(\rho(\alpha)) = \alpha$, we need to use the compatibility condition on $\alpha \in Hom_{\Psi\mathsf{Op}}(\Psi(A), \mathcal{P})$ which says that the following diagram commutes, for each pair of finite sets Y, Z and $x \in Y$:

$$\begin{tikzcd}
\Psi(A)(Y) \odot \Psi(A)(Z) \arrow[r, "\alpha(Y) \odot \alpha(Z)"] \arrow[d, "\circ_x"] & \mathcal{P}(Y) \odot \mathcal{P}(Z) \arrow[d, "\circ_x"'] \\
\Psi(A)(Y \sqcup_x Z) \arrow[r, "\alpha(Y \sqcup_x Z)"] & \mathcal{P}(Y \sqcup_x Z).
\end{tikzcd}$$

The vertical arrow on the left is the colimit of the isomorphisms $A(T) \odot A(S) \xrightarrow{\sim} A(T \circ_x S)$ over trees $T \in \mathtt{Tree}_Y$, $S \in \mathtt{Tree}_Z$. Since $\Psi(A)(X)$ is a coproduct of $A(T)$ over the trees in $\mathtt{Tree}_X$, the morphism $\alpha(X)$ is determined by its compositions with $\iota_T : A(T) \to \Psi(A)(X)$. Any tree can be represented as a sequence of $\circ_x$-products of corollae as x varies over the internal edges of the tree. For each tree T, repeated use of the diagram shows that $\alpha(X)\iota_T$ is the $\odot$-product of α's acting on corollae followed by the composition in $\mathcal{P}$. But this is the ξ extension of $\rho(\alpha)$, proving that $\xi(\rho(\alpha)) = \alpha$. ∎

We defined the free pseudo-operad first because the associativity and equivariance axioms for a pseudo-operad (as opposed to the unit axiom for an operad) are naturally expressed in terms of trees. To define the free operad, we simply adjoin a unit to the free pseudo-operad. This leads us to define a new functor

$$\Gamma(A) := \mathbf{1} \coprod \Psi(A). \tag{1.58}$$

PROPOSITION 1.92. *The functor* $\Gamma(A)$ *defines the* free operad *generated by an object* $A \in \Sigma\text{-}\mathsf{Mod}$.

PROOF. If we define $\eta_x : \mathbf{1} \to \Gamma(A)(\{x\})$ by the obvious inclusion, then we have the commutative diagrams (1.35) for the unit:

$$\begin{tikzcd}
\Gamma(A)(X) \odot \mathbf{1} \arrow[r, "r"] \arrow[d, "\mathbb{1} \odot \eta_x"'] & \Gamma(A)(X) \\
\Gamma(A)(X) \odot \Gamma(A)(\{x\}) \arrow[ur, "\circ_x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \odot \Gamma(A)(X) \arrow[r, "l"] \arrow[d, "\eta_x \odot \mathbb{1}"'] & \Gamma(A)(X) \\
\Gamma(A)(\{x\}) \odot \Gamma(A)(X) \arrow[ur, "\circ_x"'] &
\end{tikzcd} \tag{1.59}$$

The adjointness relation

$$Hom_{\Sigma\text{-}\mathsf{Mod}}(A, \mathcal{F}_{or}(\mathcal{P})) \leftrightarrow Hom_{\mathsf{Op}}(\Gamma(A), \mathcal{P}) \tag{1.60}$$

follows from Theorem 1.91. ∎

Note that $\Psi(A)(1)$ is the free nonunital associative monoid on $A(1)$ while $\Gamma(A)(1)$ is the free unital associative monoid on $A(1)$. Another description of $\Gamma(A)$

uses the trivial tree T_0, which is a unit relative to grafting,

$$T_0 \circ_x S = S = S \circ_x T_0.$$

The $\circ_x$-operation for the free pseudo-operad, defined using the natural identification (1.48), when applied to $A(T_0)$ on the right or left has the property of a unit:

$$\begin{tikzcd} A(S)\odot A(T_0) \arrow[r, "\circ_x"] \arrow[d, "="'] & A(S\circ_x T_0) \arrow[d, "="] & A(T_0)\odot A(S) \arrow[r, "\circ_x"] \arrow[d, "="'] & A(T_0\circ_x S) \arrow[d, "="] \\ A(S)\odot \mathbf{1} \arrow[r, "\cong"] & A(S) & \mathbf{1}\odot A(S) \arrow[r, "\cong"] & A(S) \end{tikzcd}$$

Thus, we could simply define $A(T_0) := \mathbf{1}$ and allow the trivial tree in formula (1.52):

$$\Gamma(A) = A(T_0)\coprod \Psi(A).$$

1.9.1. Appendix. In this appendix we assume, for simplicity, that the underlying monoidal category $\mathcal{C}$ is the category $\mathtt{Mod}_{\mathbf{k}}$ of $\mathbf{k}$-modules. In some examples, in particular the discussion of quadratic operads and Koszul operads (see Section 3.2), it is convenient to pick representatives of the equivalence classes in $\mathcal{T}ree$ by using planar n-trees. For planar trees the orientation of the plane orders the edges at each vertex and the labels $A(In(v))$ can be replaced with the simpler object $A(\mathsf{a}(v))$, where $\mathsf{a}(v) := \#In(v)$ is the arity of v. Moreover, the vertices of an up-rooted planar tree have a natural order which we call the *planar order*. In the planar order the root vertex is first, followed by the vertices at height one (connected by one edge to the root) in order from left to right, next in order are vertices at height two, listed from left to right, and so on. In this way all tensor products over the set of vertices have a natural order. The drawback is that the indexing set for the coequalizer expressed in terms of planar trees in Definition 1.80 involves isomorphisms as well as automorphisms, since distinct planar trees may be isomorphic as abstract trees.

For example, there is only one (isomorphism class of a) binary rooted tree with three leaves, but there are two (isomorphism classes of) planar binary trees with three leaves. For each of the planar trees there are six labelings of the leaves given by the six permutations (i,j,k) of $(1,2,3)$. For a Σ-module A with basis $\{\alpha_l\}$ in arity two (and assuming $A(3)=0$), $\Gamma(A)(3)$ is defined as a quotient space of the vector space of dimension $12(\dim A(2))^2$ with a basis indexed by the labeled planar trees shown in Figure 13 as α_l and α_m run over a basis for $A(2)$.

REMARK 1.93. Using planar trees can be viewed as a kind of 'blowing up' of the formulas of the previous sections by replacing each abstract tree by the set of planar trees that are equivalent to it.

If we use planar trees to describe $A[T,\ell]$, then

$$A[T,\ell] := \left(\bigoplus_{\text{equivalent planar }(T,\lambda)} A(T,\lambda)\right)/W,$$

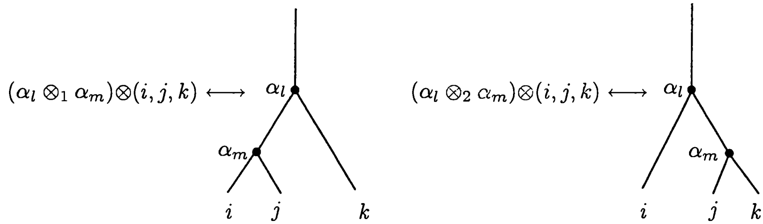


FIGURE 13. Generators of the space $\Gamma(A)(3)$.

where the form of the relation W follows from Definition 1.80. In the example given above, the subspace W is spanned by the terms

$$
\begin{array}{rrcl}
 & (i) \quad (\alpha_l \otimes_1 \alpha_m)\otimes(i,j,k) & - & (\alpha_l \cdot \tau \otimes_2 \alpha_m)\otimes(k,i,j), \\
(1.61) & (ii) \quad (\alpha_l \otimes_1 \alpha_m)\otimes(i,j,k) & - & (\alpha_l \otimes_1 \alpha_m \cdot \tau)\otimes(j,i,k), \\
 & (iii) \quad (\alpha_l \otimes_2 \alpha_m)\otimes(i,j,k) & - & (\alpha_l \otimes_2 \alpha_m \cdot \tau)\otimes(i,k,j).
\end{array}
$$

By a standard isomorphism theorem, one can define the quotient space in two steps, first taking the quotient by the subspace generated by the terms in line (i), and then taking the quotient by the subspace generated by the cosets of the elements in lines (ii) and (iii).

Equivalently, $A[T,\ell]$ could be defined by using only $(\alpha_l \otimes_1 \alpha_m)\otimes(i,j,k)$ (the terms pictured on the left) or alternatively, $(\alpha_l \otimes_2 \alpha_m)\otimes(i,j,k)$ (the terms pictured on the right). In general, we can always choose a set of planar trees representing the distinct equivalence classes of abstract trees. If we pick the set on the left, then $A[T,\ell]$ is defined as the quotient of the space with basis $\{(\alpha_l \otimes_1 \alpha_m)\otimes(i,j,k)\}$ modulo the subspace spanned by

$$
(\alpha_l \otimes_1 \alpha_m)\otimes(i,j,k) - (\alpha_l \otimes_1 \alpha_m \cdot \tau)\otimes(j,i,k). \tag{1.62}
$$

There are occasions where this is not the most convenient form (cf., the example of the associative operad given in Section 3.2 below).

1.10. Collections, K-collections and K-operads

In their paper on Koszul operads [**GK94**], Ginzburg and Kapranov work with K-collections for K an algebra over a field $\mathbf{k}$ and give a construction of the free operad using a modified form of $\Psi(A)$ with tensor products over K. The idea behind the definition of a K-collection is expressed in the next proposition, which is a list of useful, if obvious, observations. For the next two sections, we will restrict our attention to the category $\mathcal{C} = \mathbf{k}\text{-}\mathtt{Mod}$ although most of what we do can be done in an arbitrary symmetric monoidal category.

PROPOSITION 1.94. *If a Σ-module A has a 'partial operad structure' involving only the structure morphisms $\gamma_{1,1}$, $\gamma_{1,n}$ and $\gamma_{n;1,\,.\,,1}$, $n \geq 2$, which are assumed to satisfy the relevant operad axioms, then setting $K := A(1)$,*

$$
\begin{array}{rll}
\gamma_{1,1} : K \otimes K \to K & & \textit{defines a monoid structure on } K, \\
\gamma_{1;n} : K \otimes A(n) \to A(n) & & \textit{defines a left } K\textit{-module structure on } A(n) \textit{ and} \\
\gamma_{n;1,\ ,1} : A(n) \otimes K^{\otimes n} \to A(n) & & \textit{defines a right } K^{\otimes n}\textit{-module structure on } A(n).
\end{array}
$$

Moreover, the morphism $\gamma_{n;1,\ \ ,1}$ descends to $A(n) \otimes_{\Sigma_n} K^{\otimes n}$. The product $\otimes_{\Sigma_n}$ is defined using the right Σ_n-action on $A(n)$ and the left Σ_n-action on $K^{\otimes n}$ given by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_n) := x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}, \ \textit{for } x_1, \ldots, x_n \in K \textit{ and } \sigma \in \Sigma_n.$$

The last part of the proposition follows from equivariance of the structure morphisms and the existence of a right action of $A[n,n]$ (see the definition in equation (1.28)) on $A(n)$ which extends the right action of $K^{\bigcirc n}$:

$$A(n) \otimes A[n,n] \to A(n).$$

EXAMPLE 1.95. If $\mathcal{C} = \mathtt{Mod}_{\mathbf{k}}$, then K is a $\mathbf{k}$-algebra and $A[n,n]$ is the 'twisted group algebra,' $K_\tau[\Sigma_n]$, defined as the $\mathbf{k}$-module spanned by the expressions

$$(y_1 \otimes \cdots \otimes y_n)\sigma \ \text{ for } \ y_i \in K, \ \sigma \in \Sigma_n,$$

where $\otimes = \otimes_{\mathbf{k}}$, with multiplication given by:

$$(x_1 \otimes \cdots \otimes x_n)\,\sigma \cdot (y_1 \otimes \cdots \otimes y_n)\,\tau = \left(x_1 y_{\sigma^{-1}(1)} \otimes \cdots \otimes x_n y_{\sigma^{-1}(n)}\right) \sigma\tau.$$

DEFINITION 1.96. A *collection* in $\mathcal{C}$ is a Σ-module such that $A(1)$ is a $\mathcal{C}$-monoid and each $A(n)$ is a left $A(1)$- and right $A[n,n]$-module. For a fixed $\mathcal{C}$-monoid K, a K-*collection* in $\mathcal{C}$ is a collection such that $A(1) = K$. The category of collections will be denoted by $\mathtt{Col}$ and the category of K-collections by $\mathtt{Col}_K$.

Thus a K-collection is a Σ-module with the properties listed in Proposition 1.94.

REMARK 1.97. Introducing K-collections allows us to remove the restriction that $A(1) = \mathbf{k}$ in various constructions. Furthermore, if we assume that K is a semisimple $\mathbf{k}$-algebra, then every K-module decomposes into a direct sum of irreducible submodules and such a decomposition can be applied to each component $A(n)$, leading to the concept of a colored operad introduced, in the topological setting, by Boardman and Vogt [**BV73**].

For example, if $K = \mathbf{k} \oplus \mathbf{k}$, we obtain bicolored operads describing algebras and their maps; see also Section 2.9 or [**Mar99c, KS00**].

A morphism $\alpha \in \mathit{Hom}_{\mathtt{Col}}(A,B)$ consists of a morphism of $\mathcal{C}$ monoids $\alpha(1) : A(1) \to B(1)$ and morphisms $\alpha(n) : A(n) \to B(n)$, $n \geq 2$, compatible (in the obvious sense) with the $A(1)$-$A[n,n]$ bimodule structure on $A(n)$ and the $B(1)$-$B[n,n]$ bimodule structure on $B(n)$. In the category $\mathtt{Col}_K$, all morphisms satisfy, by definition, $\alpha(1) = \mathbb{1}_K$ so we can start at degree 2 and require that $\alpha(n)$ be a morphism of K-$K[n,n]$ bimodules.

In [**GK94**, paragraph 1.2.11], the definition of a K-collection in the category $\mathtt{Mod}_{\mathbf{k}}$ is a sequence $A = \{A(n)\}_{n\geq 2}$ such that $A(n)$ is a left K- and right $K_\tau[\Sigma_n]$-module; see Example 1.95. Thus it is a special case of our definitions.

We will define, by modifying Smirnov's construction, a new monoidal structure for the category $\mathtt{Col}_K$. First define $\mathbf{1}_K \in \mathtt{Col}_K$ by

$$\mathbf{1}_K(n) = \begin{cases} K, & \text{for } n = 1, \text{ and} \\ \mathbf{0}, & \text{for } n \geq 2, \end{cases} \tag{1.63}$$

and let $K[n,n]$ denote $\mathbf{1}_K[n,n] = K_\tau[\Sigma_n]$.

In the new monoidal structure $\square_K$ on $\mathtt{Col}_K$, we use the multiplication $\otimes_{K[n,n]}$, defined for X a right and Y a left $K[n,n]$-module, by letting $X \otimes_{K[n,n]} Y$ be the coequalizer of the two morphisms

$$X \otimes K[n,n] \otimes Y \underset{1\!\!1 \otimes \lambda}{\overset{\rho \otimes 1\!\!1}{\rightrightarrows}} X \otimes Y,$$

where ρ is the right $K[n,n]$-module structure on X and λ the left $K[n,n]$-module structure on Y.

Suppose that A and B are K-collections; for $n \geq 2$ we modify the formula in Definition 1.63, defining

$$(1.64) \qquad (A \Box_K B)(n) := \coprod_{j=1}^{n} A(j) \otimes_{K[j,j]} B[j,n].$$

Note that $B[j,n]$ is defined as before using $\otimes$ so that it is a left $K[j,j]$-, right $K[n,n]$-module. The unit object is defined in (1.63). The associativity of $\Box_K$ is proved just as for $\Box$.

The following definition should be compared to Definition 1.67.

DEFINITION 1.98. A *K-operad* is a monoid in the category of $\mathtt{Col}_K$ relative to the $\Box_K$-product.

We have *defined* a K-operad as a monoid in $\mathtt{Col}_K$, but it is easy to see that K-operads are precisely operads in the sense of May (Definition 1.4) with $\mathcal{P}(1) = K$.

1.11. The GK-construction

We now describe the Ginzburg and Kapranov (which we abbreviate GK) construction of the free K-operad generated by a K-collection.

The definition of a K-operad implies that the underlying Σ-module is a K-collection. The Ginzburg-Kapranov construction involves tensor products reduced using the K-$K[n,n]$ bimodule structure on a K-collection. There is a further simplification in considering only reduced trees, that is, with no vertices having one incoming edge. Furthermore, since the identity axiom is part of the K-$K[n,n]$ bimodule structure on a K-collection, $A(T_0) = A(1) = K$, where T_0 is the trivial tree, provides a unit for the free operad.

To begin with, they define an object $A_K(T)$ involving tensor products over K. It is convenient to describe the reduction of a tensor product over the field $\mathbf{k}$ to a tensor product over K in terms of collapsing edges adjacent to vertices of arity one. Just as for a pseudo-operad, where there is a morphism $A(T) \to A(T/e)$ corresponding to contracting an internal edge e, so for a K-collection, there is a morphism corresponding to contracting an internal edge which has a vertex of arity one. A vertex of arity one may be adjacent to a leaf or the root edge, as in the case of the vertices a and c in Figure 14.

In either of the latter two cases, the vertex is adjacent to only one internal edge and there is only one choice of an internal edge to collapse. But if the vertex of arity one is adjacent to two internal edges as in the case of the vertex marked b in Figure 14, then there are two possible edges, e, e', to collapse. Both possibilities leave us with the same tree, $T/e = T/e'$. Thus we have two morphisms,

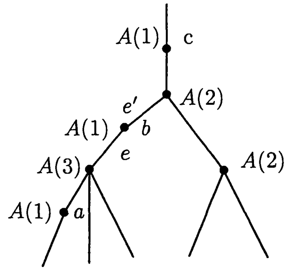


FIGURE 14. An unreduced tree with vertices of arity one.

$$A(T) \underset{\pi_{e'}}{\overset{\pi_e}{\rightrightarrows}} A(T/e).$$

The GK reduction procedure involves taking the coequalizer of all these morphisms. More precisely, for any reduced tree T, we can construct an unreduced tree $\hat{T}$, having one new vertex of arity one on each of the internal edges. Let n be the number of internal edges and $\kappa(T)$ the set of 2^n tree morphisms $\hat{T} \to T$ corresponding to all the possibilities of collapsing one of the adjacent edges at each of the new vertices of arity one. Define

$$A_K(T) := \operatorname*{coequalizer}_{\pi\,\in\,\kappa(T)} \{A(\pi) : A(\hat{T}) \to A(T)\}. \tag{1.65}$$

Less formally, $A_K(T)$ is defined by replacing the tensor products $\otimes_{\mathbf{k}}$ in $A(T)$ with $\otimes_K$.

Let $\mathtt{Rtree}_X$ be the full subcategory of $\mathtt{Tree}_X$ consisting of reduced trees and let $\mathit{Iso}(\mathtt{Rtree}_X)$ be the category of isomorphisms of $\mathtt{Rtree}_X$. It is possible to show, similarly as in Proposition 1.72, that the correspondence $T \mapsto A_K(T)$ determines a covariant functor $\mathit{Iso}(\mathtt{Rtree}_X) \to \mathcal{C}$.

DEFINITION 1.99. (GK construction of a free operad) For $A \in \mathtt{Col}_K$ define

$$\Gamma_K(A)(X) := \operatorname*{colim}_{T\,\in\,\mathtt{Rtree}_X} A_K(T).$$

The operad structure is given by grafting, as in the proof of Proposition 1.78. The unit is provided by

$$\Gamma_K(A)(\{x\}) := A(T_0) = K,$$

where T_0 is the trivial tree with one leaf.

For any tree T, there is an associated reduced form T^{red} given by deleting all vertices of arity one. By choosing a sequence of edge collapsings, one for each vertex of arity one, we can define a morphism $\psi : A(T) \to A(T^{\mathrm{red}})$. By composing with the coequalizer epimorphism $\pi_{T^{\mathrm{red}}} : A(T^{\mathrm{red}}) \to A_K(T^{\mathrm{red}})$, that is, reduction to $\otimes_K$, we get a morphism

$$\pi_{T^{\mathrm{red}}} \circ \psi : A(T) \to A_K(T^{\mathrm{red}}). \tag{1.66}$$

Passing to the colimit over $\mathtt{Tree}_X$ gives a morphism

$$\pi_K : \Psi(A) \longrightarrow \Gamma_K(A). \tag{1.67}$$

REMARK 1.100. If $A \in \mathtt{Col}_K$, then $\Psi(A)$ is nearly a K-operad, except that

$$\Psi(A)(1) = K \oplus K^{\otimes 2} \oplus K^{\otimes 3} \cdots,$$

with the right module (or left module) structure given by right (left) multiplication of the last factor on the right (left). On the other hand

$$\Gamma_K(A)(1) = K.$$

THEOREM 1.101. *The functor* $\Gamma_K : \mathtt{Col}_K \longrightarrow \mathtt{Op}_K$ *is left adjoint to the forgetful functor* $\mathcal{F}_{or} : \mathtt{Op}_K \longrightarrow \mathtt{Col}_K$, *hence defines the* free operad functor on $\mathtt{Col}_K$.

PROOF. One can give a direct proof similar to the proof of Theorem 1.91 or one can prove this as a corollary of that theorem.

1.12. Triples

In this section, we review the definition of a triple (monad) and give a description of pseudo-operads and operads in terms of triples and algebras over a triple. The relevant triples come from the endofunctors Ψ, Γ and Γ_K.

Let $End(\mathcal{C})$ be the strict symmetric monoidal category of endofunctors on $\mathcal{C}$ where multiplication is composition of functors, $\mathbf{1}$ is the identity functor and all of r, l, a in Section 1.1 are identity morphisms.

DEFINITION 1.102. A *triple* (also called a *monad*) T on a category $\mathcal{C}$ is an associative and unital monoid (T, μ, v) in $End(\mathcal{C})$. The multiplication μ and unit morphism v satisfy the axioms given by commutativity of the diagrams in Figure 15.

Triples arise naturally from pairs of adjoint functors. Given an adjoint pair

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "G"] & \mathcal{B}, \arrow[l, bend left, "F"] \end{tikzcd}$$

with associated bijection

$$Hom_{\mathcal{A}}(F(X), Y) \leftrightarrow Hom_{\mathcal{B}}(X, G(Y)),$$

there is a triple in $\mathcal{B}$ defined by $T = GF$. The unit of the adjunction $\mathbb{1} \to GF$ defines the unit v of the triple and the counit of the adjunction $FG \to \mathbb{1}$ induces a natural transformation $GFGF \to GF$ which defines the multiplication μ. In fact, it is a theorem of Eilenberg and Moore (see [**EM65**]) that all triples arise in this way from adjoint pairs. This is exactly the situation with the free pseudo-operad functor that was described in Section 1.9. We will show how operads and pseudo-operads can actually be defined using the concept, introduced in the next definition, of an algebra over a triple.

DEFINITION 1.103. A T-*algebra* (A, α) or *algebra over the triple* T is an object A of $\mathcal{C}$ together with a structure morphism $\alpha : T(A) \to A$ satisfying

$$\alpha(T(\alpha)) = \alpha(\mu_A) \text{ and } \alpha v_A = \mathbb{1}_A.$$

See Figure 16.

Morphisms of T-algebras must commute with the structure maps. The category of T-algebras in $\mathcal{C}$ will be denoted $\mathtt{Alg}_T(\mathcal{C})$. Since the free pseudo-operad functor Ψ and the free operad functor Γ described in Section 1.9 are left adjoints to $\mathcal{F}_{or} : \Psi\mathtt{Op} \to \Sigma\text{-}\mathtt{Mod}$ and $\mathcal{F}_{or} : \mathtt{Op} \to \Sigma\text{-}\mathtt{Mod}$, respectively, the functors $\mathcal{F}_{or}\Psi$ (denoted simply Ψ) and $\mathcal{F}_{or}\Gamma$ (denoted Γ) define triples on $\Sigma\text{-}\mathtt{Mod}$. Similarly, Γ_K is left

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\upsilon"] \arrow[dr, "\mathbb{1}"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\upsilon T"] \arrow[dr, "\mathbb{1}"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd}$$

FIGURE 15. Associativity and unit axioms for a triple.

$$\begin{tikzcd} T(T(A)) \arrow[r, "T(\alpha)"] \arrow[d, "\mu"'] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\upsilon_A"] \arrow[dr, "\mathbb{1}"'] & T(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

FIGURE 16. T-algebra structure.

adjoint to $\mathcal{F}_{or} : \mathtt{Op}_K \to \mathtt{Col}_K$ and there is a corresponding triple $\mathcal{F}_{or}\Gamma_K$ (denoted Γ_K) on $\mathtt{Col}_K$.

DEFINITION 1.104. The multiplication μ_Ψ is defined by setting $B = \Psi(A)$ in equation (1.57),

$$\mu_\Psi : \Psi(\Psi(A)) \xrightarrow{\chi_{\Psi(A)}} \Psi(A). \tag{1.68}$$

We can use the right and left unit morphisms $B \odot \mathbf{1} \to B$ and $\mathbf{1} \odot B \to B$, respectively, to define

$$\iota : \Psi(\mathbf{1} \coprod \Psi(A)) \to \Psi(\Psi(A))$$

and use ι to define the multiplication μ_Γ for the triple Γ as the composite

(1.69)

$$\Gamma(\Gamma(A)) = \mathbf{1} \coprod \Psi\left(\mathbf{1} \coprod \Psi(A)\right) \xrightarrow{(\mathbb{1},\iota)} \mathbf{1} \coprod \Psi(\Psi(A)) \xrightarrow{(\mathbb{1},\mu_\Psi)} \mathbf{1} \coprod \Psi(A) = \Gamma(A).$$

For the triple Γ_K, the multiplication μ_{Γ_K} is the reduction of μ_Γ to $\Gamma_K(\Gamma_K(A))$ using the fact that operad composition for a K-operad (in this case $\Gamma_K(A)$) is defined over $\otimes_K$.

We describe μ_Ψ in a little more detail. If we write (see the 'simplified formula' (1.52))

$$\begin{aligned} \Psi(\Psi(A))(X) &= \coprod_{T \in \mathit{Tree}(X)} \Psi(A)(T) = \coprod_{T \in \mathit{Tree}(X)} \bigodot_{v \in \mathit{Vert}(T)} \Psi(A)(\mathit{In}(v)) \\ &= \coprod_{T \in \mathit{Tree}(X)} \bigodot_{v \in \mathit{Vert}(T)} \left(\coprod_{S_v \in \mathit{Tree}(\mathit{In}(v))} A(S_v) \right) \end{aligned}$$

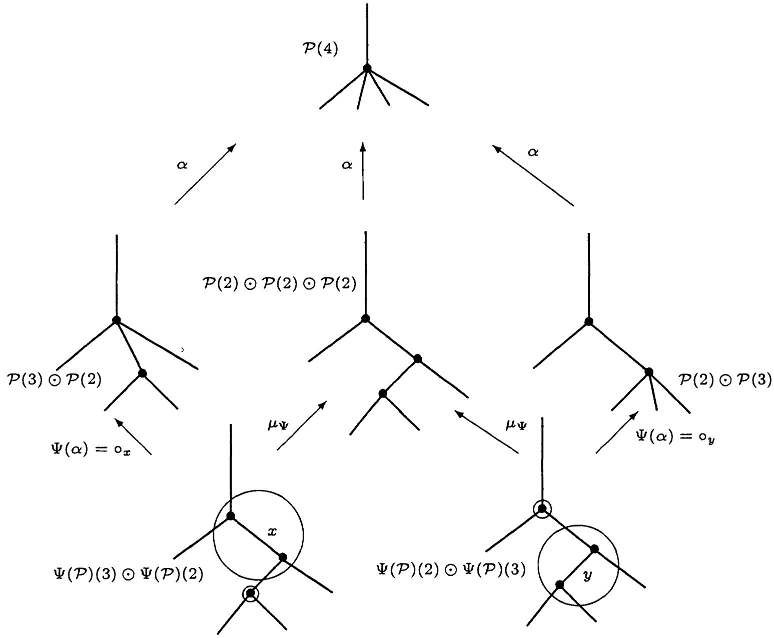


FIGURE 17. This diagram shows how the associativity axiom for $\circ_x$-compositions follows from the triple axiom $\alpha\mu_\Psi = \alpha\Psi(\alpha)$ for a Ψ-algebra, which implies that the subdiagrams on the right and left sides of the figure commute. Both subdiagrams have the same binary tree as a starting point if we forget the circles and the two distinct compositions of arrows on the extreme right and extreme left of the figure correspond to the two sides of the equation $\circ_x(\mathbb{1} \odot \circ_y) = \circ_y(\circ_x \odot \mathbb{1})$ in the particular case of $\mathcal{P}(2) \odot \mathcal{P}(2) \odot \mathcal{P}(2)$. Note that the multiplication μ_Ψ can be represented in terms of grafting (inserting) subtrees into the vertices of a given tree. On the other hand for a Ψ-algebra $\mathcal{P}$, the multiplication $\alpha : \mu_\Psi(\mathcal{P}) \to \mathcal{P}$ is represented in terms of collapsing all the internal edges of a tree.

and then move the $\odot$-product past the coproduct, the result is the coproduct

$$\coprod_{\{(T,S_v)|T \in \mathit{Tree}(X),\, S_v \in \mathit{Tree}(\mathit{In}(v))\}} \left(\bigodot_{v \in \mathit{Vert}(T)} A(S_v) \right).$$

Let $T \circ \{S_v | v \in \mathit{Vert}(T)\}$ be the new tree given by replacing the vertex $v \in \mathit{Vert}(T)$ (more precisely, replacing the corolla at the vertex v) with a tree S_v having $\mathit{In}(v)$ leaves (compare the insertion operads in Section I.1.20). Then the component of $\Psi(\Psi(A))$ corresponding to the index (T, S_v),

$$\bigodot_{\substack{v \in \mathit{Vert}(T) \\ S_v \in \mathit{Tree}(\mathit{In}(v))}} A(S_v) \cong \bigodot_{w \in \mathit{Vert}(T \circ \{S_v | v \in \mathit{Vert}(T)\})} A(\mathit{In}(w)) = A(T \circ \{S_v | v \in \mathit{Vert}(T)\}),$$

is also a component of $\Psi(A)(X)$ and we get for each finite set X and each component $(\Psi(A))(T)$ of $\Psi(\Psi(A))(X)$ a morphism of that component into $\Psi(A)(X)$. These morphisms combine to form the natural transformation $\mu_\Psi : \Psi(\Psi(A)) \to \Psi(A)$. A very similar argument will come up again in Theorem 3.24 of Section 3.1.

THEOREM 1.105. *A Σ-module $\mathcal{P}$ is a Ψ-algebra if and only if it is a pseudo-operad and it is a Γ-algebra if and only if it is an operad. An object $\mathcal{P} \in \mathtt{Col}_K$ is a Γ_K-algebra if and only if it is a K-operad. In shorthand notation, we can write*

$$\begin{aligned} \mathtt{Alg}_\Psi(\Sigma\text{-}\mathtt{Mod}) &\cong \Psi\mathtt{Op}, \\ \mathtt{Alg}_\Gamma(\Sigma\text{-}\mathtt{Mod}) &\cong \mathtt{Op} \textit{ and} \\ \mathtt{Alg}_{\Gamma_K}(\mathtt{Col}_K) &\cong \Psi\mathtt{Op}_K. \end{aligned}$$

PROOF. We outline the proof of the implication in one direction, from algebra to pseudo-operad or operad. The converse is left as an exercise for the reader. Let $\mathcal{P}$ be a Ψ-algebra. Restriction of the structure morphism $\alpha : \Psi(\mathcal{P}) \longrightarrow \mathcal{P}$ to the components of $\Psi(\mathcal{P})$ given by trees with one internal edge defines the pseudo-operad composition maps, $\circ_x$. See Figure 10.

The associativity axioms for the $\circ_x$-operations follow from the Ψ-algebra axiom $\alpha\Psi(\alpha) = \alpha\mu_\Psi$. Figure 17 gives a pictorial proof in a special case. The right and left subdiagrams in the figure commute because of the axiom $\alpha\mu_\Psi = \alpha\Psi(\alpha)$ for a Ψ-algebra. The two subdiagrams actually have the same binary tree as a starting point if we forget the circles. The two distinct compositions of arrows on the extreme right and extreme left of the figure correspond to the two sides of the equation $\circ_x(1\!\!1 \odot \circ_y) = \circ_y(\circ_x \odot 1\!\!1)$ for the associativity axiom in the particular case of $\mathcal{P}(2) \odot \mathcal{P}(2) \odot \mathcal{P}(2)$. From this example the general argument is clear, but writing out the details would be tedious.

Shifting our attention to μ_Γ, the associativity can be proved as was the associativity of μ_Ψ. The unit morphism can be discussed as follows. The structure morphism $\alpha : \Gamma(\mathcal{P})(\{x\}) = \mathbf{1} \coprod \mathcal{P}(\{x\}) \to \mathcal{P}(\{x\})$ restricted to the component $\mathbf{1}$ is the unit morphism $\eta_x : \mathbf{1} \to \mathcal{P}(\{x\})$. We will prove the unit axiom for an operad as expressed in the commutative diagrams (1.35). Consider a tree T with two vertices, one of arity one and one of arity n, and the component of $\Gamma(\mathcal{P})(T)$ of $\Gamma(\Gamma(\mathcal{P}))$. Label the vertex of arity one by $\mathbf{1}$ and the vertex of arity n by $\mathcal{P}(\mathrm{c}(X))$ (a component of $\Gamma(\mathcal{P})(X)$) where X is the set of incoming edges. Consider for the moment the ordered $\odot$-product with the label on the root vertex appearing first. In this case the condition $\alpha\Gamma(\alpha) = \alpha\mu_\Gamma$ is expressed by the diagrams below.

$$\begin{tikzcd} \mathcal{P}(\mathrm{c}(X)) \odot \mathbf{1} \arrow[r, "\mu_\Gamma"] \arrow[d, "\Gamma(\alpha)"'] & \mathcal{P}(\mathrm{c}(X)) \arrow[d, "\alpha"] & \mathbf{1} \odot \mathcal{P}(\mathrm{c}(X)) \arrow[r, "\mu_\Gamma"] \arrow[d, "\Gamma(\alpha)"'] & \mathcal{P}(\mathrm{c}(X)) \arrow[d, "\alpha"] \\ \mathcal{P}(X) \odot \mathcal{P}(\{x\}) \arrow[r, "\alpha"] & \mathcal{P}(X) & \mathcal{P}(\{x\}) \odot \mathcal{P}(X) \arrow[r, "\alpha"] & \mathcal{P}(X) \end{tikzcd}$$

The arrow $\Gamma(\alpha)$ applies the unit morphism $\eta_x : \mathbf{1} \to \mathcal{P}(\{x\})$ at the vertex of arity one and the identification $\mathcal{P}(\mathrm{c}(X)) = \mathcal{P}(X)$ at the other vertex. According to Definition 1.104, the restriction of μ_Γ to the component $\mathbf{1} \odot \mathcal{P}(\mathrm{c}(X))$ or $\mathcal{P}(\mathrm{c}(X)) \odot \mathbf{1}$ applies the unit axiom for the $\odot$ product to remove the factor $\mathbf{1}$. The vertical arrows on the right are both the identification $\mathcal{P}(c(X)) = \mathcal{P}(X)$ and the bottom arrow is $\circ_x$. Thus these two diagrams are equivalent to the diagrams (1.35).

The implication from Γ_K-algebra to K-operad is essentially the same as the implication from Γ-algebra to operad. ∎

CHAPTER 2

Topology – Review of Classical Results

2.1. Iterated loop spaces

May introduced the concept and terminology of operads to study iterated loop spaces. Here we provide an overview of the main techniques of several authors, referring to the original papers for some of the more technical details.

By an *iterated loop space* we mean a k-fold loop space for $1 < k < \infty$ or an infinite loop space. Let us remark that the standard terminology is an n-fold loop space, but since in our book the letter 'n' almost exclusively denotes the arity, we decided, to avoid confusion, to change the terminology a bit.

DEFINITION 2.1. A *k-fold loop space* $\Omega^k X$ is the space of based maps of the sphere S^k to a space X, $1 \leq k < \infty$. Equivalently, $\Omega^k X = \{\lambda : (I^k, \partial I^k) \to (X, *)\}$.

It is helpful to interpret 'k-fold loop space' as the sequence $\{Y_i = \Omega Y_{i+1} |\ 1 \leq i < k\}$ with

$$Y_k = X, Y_{k-1} = \Omega X, \ldots, Y_1 = \Omega^k X.$$

An *infinite loop space* ($k = \infty$) is then a sequence $\{Y_i = \Omega Y_{i+1} |\ 1 \leq i\}$.

May emphasized three uses of operads in this study: a recognition principle, a 'geometric' approximation construction and a theory of homology operations. His recognition principle carried further the work of Stasheff [**Sta63a**] recognizing loop spaces, Beck [**Bec69**] for k-fold loop spaces and Boardman-Vogt for k-fold and infinite loop spaces [**BV73**].

By a *recognition principle* (in the strong sense) for k-fold loop spaces is meant a specification of appropriate structure on a space Y such that Y possesses this structure if and only if Y has the homotopy type of a k-fold loop space. In the cases we consider, the structure is specified as that of an algebra over a certain operad. There are operads (cf. the little k-cubes operad below (Definition 2.2)) for which the algebras are of the homotopy type of a k-fold loop space, but for which the reverse implication may fail. For that inverse implication, one must replace the operad by an equivalent cofibrant one [**Vog**]. One may use for example a functorial cofibrant replacement provided by the W-construction of Boardman and Vogt [**BV73**, pages 72–75].

For $k = 1$, recognition is provided by the non-Σ operad $\underline{\mathcal{K}}$ of Stasheff (Section I.1.6), which happens to be cofibrant (see discussions in Section I.1.18) and which is in fact in a sense a minimal cofibrant replacement for the little intervals operad which we recall in Section 2.2.

For the two extreme cases, $k = 1$ and $k = \infty$, we can characterize the appropriate operad whose algebras have the homotopy type of an n-fold loop space quite simply as a non-Σ operad with each component contractible, respectively as a

Σ-operad with each component contractible and with a free Σ-action [**May72**]. For $k = 2$, Fiedorowicz [**BFSV**] has recently given a characterization of appropriate operads in terms of a braided structure. For $2 < k < \infty$, the question is open.

Geometric *approximation* constructs a space of the homotopy type of $\Omega^k S^k X$ for $1 \le k < \infty$ or

$$\Omega^\infty S^\infty X := \varinjlim \Omega^s S^s X \quad \text{for } k = \infty,$$

built from products of X and certain standard spaces depending only on k. For $k = 1$, this was done by James [**Jam55**], for $k < \infty$ by Milgram [**Mil66**], Boardman and Vogt [**BV68**], May [**May72**], Segal [**Seg73**] and for $k = \infty$ first by Dyer and Lashof [**DL**] and later by Barratt [**Bar71**] and Segal [**Seg74**]. The standard spaces are related to operads and include classical configuration spaces, the permutahedra and Milgram's polyhedra built from permutahedra as well as May's construction in terms of the little cubes operads.

Prior to his introduction of operads, May in [**May70**] reformulated the algebra behind Steenrod's work so that it applied also to the homology of iterated loop spaces. There the operations were first introduced, mod 2, by Kudo and Araki [**KA56**] and, mod $p > 2$, by Dyer and Lashof [**DL62**]. The notions of operad and especially E_∞-operad and E_∞-space simplify the construction, analysis and understanding of these operations and those of Steenrod as well as their relation. May in [**KM95**, page 33] wrote: 'Actually, [**May70**] should be read as a paper about operad actions. Unfortunately, it was written shortly before operads were invented.'

Although the simplification is somewhat less substantial, the use of operads is also relevant to our understanding of Massey operations in Section 2.6, originally defined in cohomology [**MU57, Mas58**] and their generalizations [**Sta63b, May69, Ret93, Pol98**].

2.2. Recognition

For $k = 1$, recognition is provided by the nonsymmetric operad $\underline{\mathcal{K}}$ of Stasheff (cf. Section I.1.6) while for $1 < k < \infty$, an algebra over the little k-cubes operad invented by Boardman and Vogt is of the homotopy type of a k-fold loop space. The little k-cubes operad is defined as follows.

Let I^k denote the standard k-dimensional unit cube. *A little k-cube* is a linear embedding $\mathrm{c} : I^k \hookrightarrow I^k$ with parallel axes, that is, the components of

$$\mathrm{c}(t_1, \dots, t_n) = (\mathrm{c}^1(t_1), \mathrm{c}^2(t_2), \dots, \mathrm{c}^n(t_n)),\ t_i \in I \text{ for } 1 \le i \le n,$$

are linear functions $\mathrm{c}^i : I \to I$ of the form $\mathrm{c}^i(t) = (y_i - x_i)t + x_i$ with $0 \le x_i < y_i \le 1$, $1 \le i \le n$. This means that each function c^i consists of a shrinking by the factor $y_i - x_i$ and translation by x_i.

DEFINITION 2.2. The *little k-cubes operad* $\mathcal{C}_k = \{\mathcal{C}_k(n)\}_{n \ge 0}$ consists of the spaces $\mathcal{C}_k(n)$ of n-tuples $(\mathrm{c}_n, \dots, c_n)$ of little k-cubes such that the images $\mathrm{c}_1(\overset{\circ}{I^k}), \dots$, $c_n(\overset{\circ}{I^k})$ of the interior $\overset{\circ}{I^k}$ of I^k are mutually disjoint. For $j = 0$, $\mathcal{C}_k(0)$ is defined to be the one-point space.

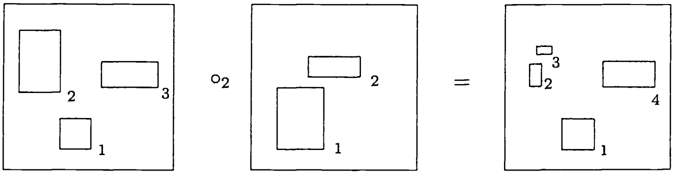


FIGURE 1. The $\circ_2$-product of an element of $\mathcal{C}_2(3)$ with an element of $\mathcal{C}_2(2)$. The result is an element of $\mathcal{C}_2(4)$.

As topological space, $\mathcal{C}_k(n)$ can be identified with an open subspace of $\mathbb{R}^{2nk}$ using the $\{y_i, x_i\}_{i=1}^k$ as coordinates. The operad structure is obvious· The symmetric group Σ_n acts on $\mathcal{C}_k(n)$ by permuting the labels of little cubes and the structure map γ is given by composition of embeddings. To be more precise, let $c : \bigsqcup_{s=1}^n I_s^k \to I^k$ and $c_s : \bigsqcup_{j=1}^{m_s} I_{j,s}^k \to I^k$ (where I_s^k and $I_{j,s}^k$ are identical copies of the standard k-cube) be elements of $\mathcal{C}_k(n)$, respectively $\mathcal{C}_k(m_s)$, $1 \le s \le n$. Then $\gamma(c; c_1, \dots, c_n) \in \mathcal{C}_k(m_1 + \cdots + m_n)$ is the map

$$\gamma(c; c_1, \dots, c_n) : \bigsqcup_{s=1}^{n} \bigsqcup_{j=1}^{m_s} I_{j,s}^k \to I^k$$

given by

$$\gamma(c; c_1, \dots, c_n)|_{I_{j,s}^k} := c \circ c_s,$$

where we interpret c_s as a map $c_s : \bigsqcup_{j=1}^{m_s} I_{j,s}^k \to I_s^k$. The composition law is illustrated in Figure 1. The unit $1 \in \mathcal{C}_k(1)$ is the identity map $\mathbb{1}_{I^k} : I^k \to I^k$

The operad $\mathcal{C}_1$ is also called the *little intervals operad*, while $\mathcal{C}_2$ is called simply the *little squares operad* and denoted $\mathcal{C}$. The 'infinite' version of $\mathcal{C}_k$ is introduced as follows.

DEFINITION 2.3. The operad $\mathcal{C}_\infty = \{\mathcal{C}_\infty(n)\}_{n \ge 0}$ is defined by $\mathcal{C}_\infty = \varinjlim \mathcal{C}_k$ with respect to the morphisms of operads $\sigma_k : \mathcal{C}_k \to \mathcal{C}_{k+1}$ given by

$$\sigma(n)(c_1, \dots, c_n) := (c_1 \times \mathbb{1}_I, \dots, c_n \times \mathbb{1}_I), \ n \ge 1.$$

Boardman and Vogt constructed an action $A_{BV} : \mathcal{C}_k \to \mathcal{E}nd_X$ of the little k-cubes operad $\mathcal{C}_k$ on a k-fold loop space $X = \Omega^k Y$ as follows. Given an n-tuple $\lambda_i : (I^k, \partial I^k) \to (Y, *) \in \Omega^k Y$, $1 \le i \le n$, and a little k-cube $c = (c_1, \dots, c_n) \in \mathcal{C}_k(n)$, the action $A_{BV}(c)(\lambda_1, \dots, \lambda_n)$ is the map defined to be λ_i (suitably rescaled) on the image of c_i and to be $*$ on the complement of all the images of the maps c_i.

Therefore each k-fold loop space is a $\mathcal{C}_k$-space. On the other hand, the following theorem, whose proof we indicate in Section 2.9, is true:

THEOREM 2.4. *(Boardman-Vogt* **[BV73]**, *May* **[May72]***) A connected $\mathcal{C}_k$-space X is of the homotopy type of a k-fold loop space.*

As for Stasheff's result recognizing a space of the homotopy type of a loop space by an action of the operad $\underline{\mathcal{K}}$ of associahedra, an iterated 'de-looping' is constructed explicitly. For this purpose, we use further generalizations of bar constructions.

2.3. The bar construction: theme and variations

The bar construction occurred first in the work of Eilenberg and Mac Lane (see [**EML53**]) in their extension of Hopf's work [**Hop42**], one of the earliest works on the cohomology of groups. Originally invented for the group ring of a discrete group, it has been reincarnated time and again in a variety of categories: differential graded algebraic [**Sem**], topological [**Mil56, DL59, Mil66**], categorical [**May72, BV73**]. As far as operads are concerned, there is a bar construction for algebras over a given operad $\mathcal{P}$ and also a bar construction for operads themselves.

To be more precise, the term 'bar construction' is ambiguous, even in a specified category. Originally there was an acyclic bar construction BA for an algebra A over a ground ring R which was free as an A-module generated by an R-module $\overline{B}A$, the reduced bar construction. Over time, with increased attention paid to the reduced bar construction (cf. a classifying space versus the corresponding universal principal bundle), the notation BA was commonly used for the reduced construction; we will follow this convention.

These two variants and more are covered by the general *'two-sided bar construction'* $B(L, A, R)$ which we now describe in a way that works equally well if

(i) A is an algebra (ordinary or dg) with right module R and left module L,
(ii) A is a monoid (abstract or topological) with right A-space R and left A-space L,
(iii) L, A, R are categories with functors $A \to R$ and $A \to L$,
(iv) A is itself an operad and R and L are either modules over A or algebras over A or one of each or
(v) A is a monad (= triple), L is an A-functor and R is an A-space.

We assume we are working in a category with finite products. Given the data L, A, R, form the facial (presimplicial) object $B_\bullet(L, A, R)$ which in degree n is $L \times A^{\times n} \times R$, with face maps

$$d_i : B_n(L, A, R) \to B_{n-1}(L, A, R)$$

given by

- the structure map on $L \times A$ if $i = 0$,
- the structure map on $A \times A$ if $1 \leq i < n$ and
- the structure map on $A \times R$ if $i = n$.

Then the two-sided bar construction $B(L, A, R)$ is the realization of $B_\bullet(L, A, R)$ in the original category.

Three special cases are of particular importance, corresponding to the classical universal bundle for a topological group or monoid G and a point $*$:

$$B(G, G, G) \simeq G \to B(G, G, *) = EG \to B(*, G, *) = BG.$$

In the general setting, $*$ is the terminal object in the category, so the A-action is trivial. The contractibility of the universal total space EG is a special case of the (weak) equivalence

$$X \leftarrow B(A, A, X) \tag{2.1}$$

which holds for any A-object X.

Now we are ready to sketch a proof of Theorem 2.4 according to May. Boardman and Vogt in their book [**BV73**] used a somewhat different iterative method.

As we already explained in Section 1.12, each operad $\mathcal{P}$ defines a triple P : Top $\to$ Top with the property that $\mathcal{P}$-algebras are precisely algebras over the triple P. In particular, the little k-cubes operad $\mathcal{C}_k$ induces the triple C_k. Given a triple P, May introduced P-functors as functors on which the triple P acts in an appropriate sense; see [**May72**, page 86]. The triple P is a functor over itself.

May considered two important C_k-functors, the functor S^k : Top $\to$ Top assigning to each topological space X its k-fold suspension S^k and the functor $\Omega^k S^k$: Top $\to$ Top assigning to each topological space X the k-fold iterated loop space of its k-fold suspension $\Omega^k S^k X$. He then proved [**May72**, Theorem 13.1] that there is, for a connected space X, a sequence of (weak) homotopy equivalences

$$X \leftarrow B(C_k, C_k, X) \to B(\Omega^k S^k, C_k, X) \to \Omega^k B(S^k, C_k, X). \tag{2.2}$$

The map on the left is a homotopy equivalence because of the general property (2.1). The middle map is a homotopy equivalence because the functor $\Omega^k S^k$ is, in a suitable sense, homotopy equivalent to C_k, as follows from the approximation theorem (Theorem 2.7). Finally, the right map is a homotopy equivalence as a consequence of a certain interchange rule between the geometric realization and the functor Ω. Sequence (2.2) then *explicitly* represents X as the k-fold loop space of $B(S^k, C_k, X)$ up to homotopy type.

2.4. Approximation

The first approximation theorem of the type mentioned in Section 2.1 is *James' reduced product* construction JX, the free topological monoid generated by X, as a model for ΩSX for a well-pointed space X:

$$JX := \bigsqcup_{n\geq 0} X^n/\sim .$$

Here $X^0 = *$ and the equivalence relation is defined by

$$X^n \ni (x_1, \dots, x_i = *, \dots, x_n) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X^{n-1},$$

for any $n \geq 1$ and $1 \leq i \leq n$.

We are going to describe approximations similar to that of James for iterated loop spaces. In proving his cobar construction models the chains on a loop space, Adams [**Ada56**] gave a combinatorial map from the cube I^n to the space of paths in the simplex Δ^{n+1} from the first to the last vertex. Milgram generalizes this by giving a combinatorial map from the polytope known as the permutahedron P_n to the space of paths in the cube I^n from $(0, \dots, 0)$ to $(1, \dots, 1)$ The permutahedron P_n can be described as an $(n-1)$-dimensional convex polytope with $n!$ vertices, one for each permutation of n variables.

DEFINITION 2.5. As a subspace of $\mathbb{R}^n$, the *permutahedron* P_n is the convex hull of the permutation group Σ_n itself embedded into $\mathbb{R}^n$ by its action on the point $(1, 2, \dots, n) \in \mathbb{R}^n$.

As a cellular complex, the permutahedron P_n is the realization of the poset $\mathfrak{P}_n$ of ordered partitions of the set $\{1, 2, \dots, n\}$. To be precise, the poset $\mathfrak{P}_n$ consists

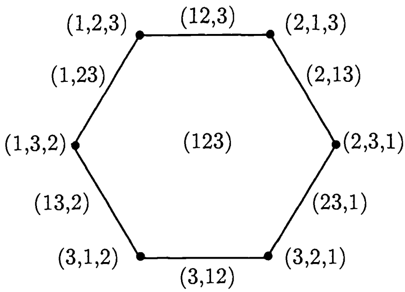


FIGURE 2. The poset structure of $\mathfrak{P}_3$.

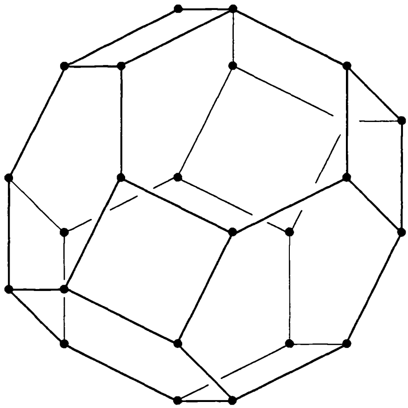


FIGURE 3. The polyhedron P_4.

of ordered partitions $(j_1, \ldots, j_w)$, $j_i \subset \{1, 2, \ldots, n\}$, $1 \leq w \leq n$. The partial order is generated by

$$(j_1, \ldots, j_w) < (j_1, \ldots, j_i \cup j_{i+1}, \ldots, j_w), \; 1 \leq i < w.$$

Geometrically, the cell corresponding to $(j_1, \ldots, j_w) \in \mathfrak{P}_n$ (and denoted again $(j_1, \ldots, j_w)$) is the convex hull of the subgroup leaving the decomposition $(j_1, \ldots, j_w)$ fixed. It is immediate to see from this description that the face $(j_1, \ldots, j_w)$ has dimension $n - w$. There is a unique maximal element $(j_1) \in \mathfrak{P}_n$, $j_1 = \{1, 2, \ldots, n\}$ and $n!$ minimal elements (vertices) indexed by permutations

$$\Sigma_n \ni \sigma \longleftrightarrow (\sigma(1), \ldots, \sigma(n)) \in \mathfrak{P}_n. \tag{2.3}$$

The poset structure of $\mathfrak{P}_3$ is illustrated in Figure 2. The polyhedron P_4 is portrayed in Figure 3.

It is clear from the definition of the permutahedron that each P_n is acted on by the symmetric group Σ_n. The natural *left* action is given on vertices by

$$\tau(\sigma(1), \ldots, \sigma(n)) := (\tau\sigma(1), \ldots, \tau\sigma(n)),$$

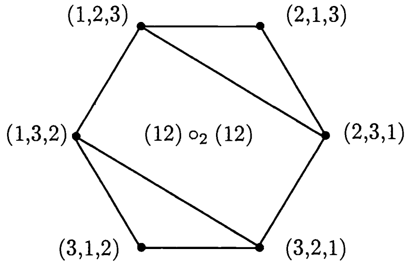


FIGURE 4. The noncellularity of the operad structure on $P = \{P(n)\}_{n\geq 1}$.

for $\sigma, \tau \in \Sigma_n$. Since we would like to consider the sequence $P = \{P(n)\}_{n\geq 1}$ as a collection, we must, to comply with the conventions accepted in this book, flip the above action to the *right* action setting

$$(\sigma(1), \dots, \sigma(n))\tau := (\tau^{-1}\sigma(1), \dots, \tau^{-1}\sigma(n)).$$

It is clear that this action respects the cellular structure, giving rise to a cellular right Σ_n-action on P_n for each $n \geq 1$.

Recall that the collection $\Sigma := \{\Sigma_n\}_{n\geq 0}$ has an operad structure given by its identification with the operad $\mathcal{M}on$ for topological monoids. This induces a corresponding operad structure on $\mathcal{P} = \{P_n\}_{n\geq 1}$ by taking convex hulls (affine extensions in Berger's terminology [**Ber97**]), but this structure is not cellular with respect to the above cells. For example, consider the 1-cell (12) of P_2 and describe the image $(12) \circ_2 (12)$ in P_3. To this end, we must understand how $\circ_2$ acts on the vertices $(1, 2)$ and $(2, 1)$ of (12). It can be easily calculated that

$$(1,2) \circ_2 (1,2) = (1,2,3), \quad (1,2) \circ_2 (2,1) = (1,3,2),$$
$$(2,1) \circ_2 (1,2) = (2,3,1), \quad (2,1) \circ_2 (2,1) = (3,2,1),$$

therefore the image $(12) \circ_2 (12)$ is the rectangle depicted in Figure 4.

Milgram [**Mil66**] builds an approximation to $\Omega^k S^k X$ using the permutahedra, though not in an obviously operadic way. He defines, for $k \geq 1$ and $n \geq 1$, spaces

$$J_n^{(k)} =: ((P_n)^{k-1} \times \Sigma_n)/\sim, \tag{2.4}$$

where the equivalence relation identifies certain boundary cells. For $k = 1$, $J_n^{(1)}$ reduces to a point. Let us explicitly describe the space $J_n^{(2)}$. Recall that the faces of the permutahedron P_n are indexed by decompositions $(j_1, \dots, j_w)$ of the set $\{1, \dots, n\}$. Denote $a_i := \mathrm{card}(j_i)$. Then the decompositions $(j_1, \dots, j_w)$ are in one-to-one correspondence with elements $\tau \in \Sigma_n$ with the property that, for $1 \leq s \leq w$,

$$\tau(a_1 + \dots + a_{s-1} + 1) < \tau(a_1 + \dots + a_{s-1} + 2) < \dots < \tau(a_1 + \dots + a_s), \tag{2.5}$$

that is, τ is increasing on each interval $[a_1 + \dots + a_{s-1} + 1, a_1 + \dots + a_s]$. Such elements are called $(a_1, \dots, a_w)$-*unshuffles*; let $unsh(a_1, \dots, a_w) \subset \Sigma_n$ denote the set of all these unshuffles.

Now let

$$J_n^{(2)} := (P_n \times \Sigma_n)/\sim,$$

with the relation $\sim$ defined as follows: For each face $(j_1,\dots,j_w)$ of P_n and for each $p \in (j_1,\dots,j_w)$ we identify

$$p\tau \times \sigma \sim p \times \tau\sigma,$$

where $\tau \in unsh(a_1,\dots,a_w)$ is the unshuffle corresponding to $(j_1,\dots,j_w)$.

For higher $k > 2$, the description of the relation in (2.4) is more complicated, so we will not give it here and refer to [**Ber97**] or [**Mil74**, page 24] instead.

The collection $J^{(k)} = \{J_n^{(k)}\}_{n\geq 1}$ does not form an operad nor do the configuration spaces $\mathsf{Con}(\mathbb{R}^k) = \{\mathsf{Con}(\mathbb{R}^k, n)\}_{n\geq 1}$, where by $\mathsf{Con}(\mathbb{R}^k, n)$ we denote the configuration space of n distinct labeled points in the affine space $\mathbb{R}^k$ with the group Σ_n acting by permuting the labels. But it turns out that for the purpose of approximation the relevant structure is that of 'degeneracies' (compare the operation of 'adding neutral arguments' mentioned in Section I.1.1). In the case of algebras, these correspond to the reductions achieved by replacing one variable by a unit. This structure is formalized in the definition of a preoperad [**Ber96**] (or coefficient system; cf. [**CMT78**]).

DEFINITION 2.6. A (topological) *preoperad* is a functor $\mathcal{P} : \Lambda^{op} \to \mathsf{Top}$, where Λ is the category of (nonempty) finite sets and injective maps.

It is clear that morphisms of the category Λ are generated by permutations and maps $d_i^n : \{1,\dots,n\} \to \{1,\dots,n+1\}$ defined, for $1 \leq i \leq n+1$, by

$$d_i^n(t) := \begin{cases} t, & \text{for } 1 \leq t < i, \text{ and} \\ t+1, & \text{for } i \leq t \leq n. \end{cases} \tag{2.6}$$

It is clear from this description that a preoperad $\mathcal{P}$ is uniquely determined by a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ together with degeneracy maps $D_i^* := \mathcal{P}(d_i^n) : \mathcal{P}(n+1) \to \mathcal{P}(n)$, $i = 1,\dots,n+1$, satisfying the appropriate relations. For a *pointed operad* $\mathcal{P}$, these degeneracy maps are given by *evaluation*:

$$D_i^*(t) := \gamma_{\mathcal{P}}(t; 1,\dots,1,*,1,\dots,1) \quad (* \in \mathcal{P}(0) \text{ at the } i\text{th position})$$

where $t \in \mathcal{P}(n+1)$, $1 \in \mathcal{P}(1)$ is the unit of the operad, $* \in \mathcal{P}(0)$ is the constant and $1 \leq i \leq n+1$.

In the particular case of the operad $\mathcal{M}on$ for monoids with $\mathcal{M}on(n) = \Sigma_n$, $n \geq 1$, the degeneracies $D_i^* : \Sigma_{n+1} \to \Sigma_n$ are given by deleting i and its image $\sigma(i)$ for each permutation $\sigma \in \Sigma_{n+1}$. To be more precise,

$$D_i^*(\sigma) = e_{\sigma(i)}^n \circ \sigma \circ d_i^n,$$

where d_i was defined in (2.6) and

$$e_j^n : \{1,\dots,j-1\} \cup \{j+1,\dots,n+1\} \to \{1,\dots,n\}$$

is, for $1 \leq j \leq n+1$, the unique order-preserving isomorphism. Degeneracies for the permutahedra are then induced by taking convex hulls of the maps D_i^* defined above. They can be shown to commute with the defining relation of (2.4), thus giving rise to a preoperad structure on the collection $J^{(k)} = \{J^{(k)}(n)\}_{n\geq 1}$.

Similarly, for the little cubes operad, we have degeneracies $D_i^* : \mathcal{C}_k(n+1) \to \mathcal{C}_k(n)$ given by omitting the ith little cube, and for the configuration spaces, degeneracies $D_i^* : F(\mathbb{R}^k, n+1) \to F(\mathbb{R}^k, n)$ given by omitting the ith point of the configuration.

In general, for any preoperad $\mathcal{P}$ and space $(X,*)$, we can form the space

$$\mathcal{P}X := (\bigsqcup_{n\geq 1} \mathcal{P} \times_{\Sigma_n} X^n)/\sim,$$

where the equivalence relation is given by

$$\mathcal{P}(n) \times X^n \ni (t, x_1, \ldots, x_i = *, \ldots, x_n)$$
$$\sim (D_i^*(t), x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathcal{P}(n-1) \times X^{n-1},$$

$1 \leq i \leq n$.

Now apply this construction using $J^{(k)}$ to define the *Milgram model* $J^{(k)}X$ of $\Omega^k S^k X$ for any space $(X,*)$. For $n=1$, $J^{(1)}X$ is just the free monoid generated by the pointed space $(X,*)$, that is, the James model for ΩSX. Using the preoperads $\mathsf{Con}(\mathbb{R}^k, n)$ or $\mathcal{C}_k X$ instead, we get the *May models* $\mathsf{Con}(\mathbb{R}^k)X$ and $\mathcal{C}_k X$ of $\Omega^k S^k X$.

THEOREM 2.7. *(Berger* [**Ber96**]*) The configuration preoperad* $\mathsf{Con}(\mathbb{R}^k)$*, Milgram's preoperad* $J^{(k)}$ *and the little cubes operad* $\mathcal{C}_k$ *are homotopy equivalent as preoperads. For a well-pointed space* $(X,*)$*, the May models and the Milgram model are all (weakly) homotopy equivalent.*

Recall that May proved in [**May72**, Theorem 6.1, page 50] that $\mathcal{C}_k X$ is weakly homotopy equivalent to $\Omega^k S^k X$. Thus all three models in Theorem 2.7 have the homotopy type of the k-loop space $\Omega^k S^k X$.

2.5. Γ-spaces

Segal's recognition principle and construction is similar but with several significant differences. In [**Seg74**], he treats only infinite loop spaces (loop spectra in his terminology.) In an unpublished manuscript, Bousfield [**Bou92**] adapts Segal's approach to finitely iterated loop spaces. Instead of finding an operad which is sufficiently elastic to keep track of all the homotopies necessary, Segal uses a special discrete category Γ to define 'special' simplicial spaces. In the following definition, 2^T denotes the set of all subsets of T.

DEFINITION 2.8. Γ is the category whose objects are all finite sets and whose morphisms from S to T are maps $\theta : S \to 2^T$ such that $\theta(a)$ and $\theta(b)$ are disjoint if a and b are distinct.

Let $\mathbf{n}$ denote the set $\{1, 2, \ldots, n\}$. The following definition of Γ spaces should be compared to that of a *finitary algebraic theory* of Lawvere [**Law63**]

DEFINITION 2.9. A Γ-*space* is a contravariant functor $\mathbf{A}$ from Γ to topological spaces such that

(i) $\mathbf{A}(\mathbf{0})$ is contractible and
(ii) for any n, the map $p_n : \mathbf{A}(\mathbf{n}) \to \mathbf{A}(\mathbf{1}) \times \cdots \times \mathbf{A}(\mathbf{1})$ induced by the n obvious maps $\mathbf{1} \to \mathbf{n}$ is a homotopy equivalence.

Any Γ-space can be regarded as a simplicial space (hence the terminology 'special' simplicial spaces) via the functor from the category Δ to Γ which takes the finite ordered set $[n] = \{0, 1, \cdots, n\}$ to $\mathbf{n}$ and a nondecreasing map $f : [p] \to [q]$ to $\theta : \mathbf{p} \to \mathbf{q}$ with $\theta(i) = \{j \in \mathbf{q} : f(i-1) < j \leq f(i)\}$. Any simplicial space $\mathbf{A}$ has a realization as a topological space (in fact, there are several, all of the same homotopy type for reasonable spaces). Using an appropriate realization, Segal constructs a 'classifying space' $B\mathbf{A}$ for any Γ-space $\mathbf{A}$ which is again a Γ-space. Also, any Γ-space has an H-space structure on $\mathbf{A}(1)$ given by the map

$$\mathbf{A}(\mathbf{1}) \times \mathbf{A}(\mathbf{1}) \xrightarrow{p_2^{-1}} \mathbf{A}(\mathbf{2}) \xrightarrow{\mathbf{A}(\nu)} \mathbf{A}(\mathbf{1}),$$

where p_2^{-1} is an arbitrary homotopy inverse of $p_2 : \mathbf{A}(\mathbf{2}) \to \mathbf{A}(\mathbf{1}) \times \mathbf{A}(\mathbf{1})$ and $\nu : \mathbf{1} \to \mathbf{2}$ is the map sending $1 \in \mathbf{1}$ to $\{1,2\} \subset 2^{\mathbf{2}}$.

PROPOSITION 2.10. *$\mathbf{A}(1)$ has the homotopy type of $\Omega B\mathbf{A}(1)$ if and only if the H-space structure on $\mathbf{A}(1)$ has a homotopy inverse.*

Segal observes that $B\mathbf{A}(1)$ in turn always has a homotopy inverse for its H-space structure and so $\mathbf{A}(1)$ is then an infinite loop space.

As for approximation, Segal considers $\coprod B\Sigma_n$ (where $B\Sigma_n$ is the classifying space of the symmetric group) as $B\Sigma(\mathbf{1})$ of a Γ-space $B\Sigma$ induced from the category of finite sets with disjoint union as monoidal structure. Similarly, for a space X, he considers a Γ-space $B\Sigma_X$ with $B\Sigma_X(\mathbf{1}) = \coprod (E\Sigma_n \times X^n)/\Sigma_n$ (where $E\Sigma_n$ is the total space of a universal principal Σ_n-bundle). The following two theorems are proved in [**BP72, Pri71**].

THEOREM 2.11. *(Barratt-Priddy-Quillen) As an infinite loop space, $B(B\Sigma)$ is homotopy equivalent to $\Omega^\infty S^\infty$, the 'sphere spectrum.'*

THEOREM 2.12. *As an infinite loop space, $B(B\Sigma_X)$ is homotopy equivalent to $\Omega^\infty S^\infty X_+$, where X_+ denotes the disjoint union of X with a point.*

2.6. Homology operations

In this section we present a unifying operadic approach to primary (co)homology operations, due to May [**May70**], and a brief survey of multi-variable higher order operations. We will respect the original sign and degree conventions, though in some cases they might be different from conventions used today.

We first review the Steenrod algebra in terms of higher homotopy commutativity. Let $K := C^*(X)$ be the complex of singular cochains on a topological space X. The classical *cup product* $\smile: K \otimes K \to K$ is a (strictly) associative multiplication. It is commutative up to a homotopy, which is traditionally denoted by $\smile_1$:

$$u \smile v - (-1)^{|u||v|} v \smile u = \delta(u \smile_1 v) + \delta u \smile_1 v + u \smile_1 \delta v. \tag{2.7}$$

The homotopy $\smile_1: K \otimes K \to K$ is homotopy anticommutative and there exists a hierarchy $\{\smile_i\}_{i \geq 1}$ of homotopies such that

$$\begin{aligned}(-1)^i u \smile_i v &- (-1)^{|u||v|} v \smile_i u \\ &= \delta(u \smile_{i+1} v) + (-1)^i \delta u \smile_{i+1} v + (-1)^{i+|u|} u \smile_{i+1} \delta v.\end{aligned}$$

Using the concept of *graded* commutativity, we may read the above equation as saying that the degree $-i$ map $\smile_i$ is graded commutative up to a homotopy $\smile_{i+1}$. *Steenrod squares* $\mathrm{Sq}^i : H^p(X;\mathbb{Z}_2) \to H^{p+i}(X;\mathbb{Z}_2)$ in mod 2 cohomology are then obtained as $\mathrm{Sq}^i([u]) := [u \smile_{p-i} u]$.

The above can be rewritten as follows. Let $\mathbb{Z}[\mathbb{Z}_2]$ be the group ring of the abelian group $\mathbb{Z}_2$ and let $W = (W, d)$ be the 'standard' $\mathbb{Z}[\mathbb{Z}_2]$-free resolution of $\mathbb{Z}$

taken with the opposite grading, i.e. $W_i :=$ the $\mathbb{Z}[\mathbb{Z}_2]$-free module with one generator e_i of degree $i \leq 0$, and the differential d acting as

$$de_i := e_{i+1} + (-1)^i T e_{i+1}, \quad T \text{ the generator of } \mathbb{Z}_2.$$

The reason why we take W with the opposite grading is so that the differential of the resolution has the same degree as the differential in the cochain complex $C^*(X)$, which is $+1$. Another possibility would be to flip the grading of $C^*(X)$, the convention used in [**May70**, par. 5]. It is easy to verify that the map $\theta : W \otimes K^{\otimes 2} \to K$ defined by

$$\begin{aligned} \theta(e_{-i} \otimes u \otimes v) &= (-1)^{i(i-1)/2} u \smile_i v \text{ and} \\ \theta(Te_{-i} \otimes u \otimes v) &= (-1)^{i(i-1)/2 + |u||v|} v \smile_i u \end{aligned} \tag{2.8}$$

is $\mathbb{Z}_2$-equivariant with $\theta(e_0 \otimes u \otimes v) = u \smile v$. An elementary exposition of these facts can be found in [**MT68**, Section 2].

More generally, it follows from an acyclic models argument that, for an arbitrary prime number $r > 2$ and a $\mathbb{Z}[\mathbb{Z}_r]$-free resolution W of $\mathbb{Z}$ such that W_0 is generated by e_0, there exists a $\mathbb{Z}_r$-equivariant map $\theta : W \otimes K^{\otimes r} \to K$ of chain complexes such that $\theta|_{\{e_0\} \otimes K^{\otimes r}}$ is the iterated cup product. This map gives rise to mod r reduced powers $P^i : H^p(X; \mathbb{Z}_r) \to H^{p+2i(r-1)}(X; \mathbb{Z}_r)$; see [**SE62**, VII] for details.

A similar situation arises when we take $K := C_*(Y)$, the singular chain complex of an $(n+1)$-fold loop space. Then, as we know from Section 2.2, the space Y admits an action of the little $(n+1)$-cubes operad $\mathcal{C}_{n+1}$ which has the property that $\mathcal{C}_{n+1}(r)$ is a Σ_r-free, $(n-1)$-connected space.

Let W be the same $\mathbb{Z}[\mathbb{Z}_r]$-free resolution as above, but this time with the standard, positive grading. The resolution W will not act on the chains $C_*(Y)$, but $W^{(n)}$, the n-skeleton of W, will. The $\mathbb{Z}[\mathbb{Z}_r]$-freeness of $W^{(n)}$ together with the $(n-1)$-acyclicity of $C_*(\mathcal{C}_{n+1}(r))$ give a $\mathbb{Z}_r$-equivariant map $W^{(n)}_* \to C_*(\mathcal{C}_{n+1}(r))$. This map, combined with the action $\mathcal{C}_{n+1}(r) \times Y^{\times r} \to Y$, induces a $\mathbb{Z}_r$-equivariant map $\theta : W^{(n)}_* \otimes K^{\otimes r} \to K$ with the property that $\theta|_{\{e_0\} \otimes K^{\otimes r}}$ is chain homotopic to the loop space multiplication. By the same mechanism as above, the map θ gives rise to *Kudo-Araki operations*, introduced in [**KA56**]

$$Q^i_{(2)} : H_p(Y; \mathbb{Z}_2) \to H_{p+i}(Y; \mathbb{Z}_2), \text{ for } p \leq i \leq p+n,$$

and, if $r > 2$ is an odd prime, *Dyer-Lashof operations*, introduced in [**DL62**]:

$$Q^i_{(r)} : H_p(Y; \mathbb{Z}_r) \to H_{p+2i(r-1)}(Y; \mathbb{Z}_r), \text{ for } p(r-1) \leq 2i(r-1) < p(r-1) + n.$$

Let Y be an $(n+1)$-fold loop space as above and consider again the action $\theta : W^{(n)} \otimes K^{\otimes 2} \to K$. As in (2.8), the formula $u \smile_i v := (-1)^{i(i-1)/2} \theta(e_i \otimes u \otimes v)$ defines, for $0 \leq i \leq n$, a family of bilinear degree i maps such that $\smile_i$ is, for $0 \leq i \leq n-1$, graded commutative up to the homotopy $\smile_{i+1}$. The obstruction cycle to the graded homotopy commutativity of $\smile_n$ (taken with appropriate signs) induces, for any coefficient ring Λ, the so-called *Browder operation* [**Bro60**]:

$$\lambda_n : H_p(Y; \Lambda) \otimes H_q(Y; \Lambda) \to H_{p+q+n}(Y; \Lambda).$$

The importance of the operation λ_n is related to the property that $\lambda_n = 0$ if Y is an $(n+2)$-fold loop space, thus it forms an obstruction for Y to be a $(n+2)$-fold loop space. The map λ_n is a degree n map which satisfies the following form of the *graded Jacobi identity*

$$(-1)^{(a+n)(c+n)}\lambda_n(x,\lambda_n(y,z)) + (-1)^{(b+n)(c+n)}\lambda_n(y,\lambda_n(z,x)) + \\ +(-1)^{(c+n)(b+n)}\lambda_n(z,\lambda_n(x,y)) = 0,$$

for $x \in H_a(Y;\Lambda)$, $y \in H_b(Y;\Lambda)$ and $z \in H_c(Y;\Lambda)$, and which is symmetric in the sense that

$$\lambda_n(x,y) = (-1)^{ab+1+n(a+b+1)}\lambda_n(y,x), \text{ for } x \in H_a(Y;\Lambda) \text{ and } y \in H_b(Y;\Lambda).$$

The sign convention follows the original work of Browder. Today, one would replace $\lambda_n(x,y)$ by $(-1)^{|x|}\lambda_n(x,y)$; this new λ_n would then be a graded antisymmetric degree n-Lie bracket; see Section I.1.12.

Let us remark that W. Browder introduced his operations for a quite large class of topological spaces, which he called H_n-spaces. An *H_n-space* is, by definition, a topological space Y equipped with an '$\mathbb{S}^n$-action,' which is a $\mathbb{Z}_2$-equivariant map

$$\phi : \mathbb{S}^n \times Y^{\times 2} \longrightarrow Y,$$

with $\mathbb{Z}_2$ acting on the sphere $\mathbb{S}^n$ via the antipodal map. It is easy to see that any $(n+1)$-fold loop space is an H_n-space.

Common features of the above constructions are summarized in the following definition due to May [**May70**]. For an integer r, a commutative ring Λ, a subgroup $\pi \subset \Sigma_r$ of the symmetric group Σ_r and for a $\Lambda[\pi]$-free resolution W of Λ with $W_0 = \Lambda e_0$, he introduced a category $\mathcal{C}(\pi,n,\Lambda)$, whose objects are pairs (K,θ) consisting of a homotopy associative (but not *strongly* homotopy associative in the sense of Section I.1.8) differential Λ-algebra K and a $\Lambda[\pi]$-equivariant morphism $\theta : W^{(n)} \otimes K^{\otimes r} \to K$ of differential complexes, where $W^{(n)}$ denotes the n-skeleton of W. The map θ has to satisfy several conditions, the most important being the requirement that the restriction of θ to $\{e_0\} \otimes K^{\otimes r}$ is Λ-homotopic to the iterated product $K^{\otimes r} \to K$.

If π is the cyclic group of prime order r, we write simply $\mathcal{C}(r,n)$ instead of $\mathcal{C}(\pi,n,\mathbb{Z}_r)$. Suppose that $(K,\theta) \in \mathcal{C}(r,n)$. If $r=2$, then θ induces operations

$$P_i : H_p(K) \to H_{p+i}(K), \text{ for } i \le p+n.$$

Similarly, for an odd prime $r > 2$, there are the operations

$$P_i : H_p(K) \to H_{p+2i(r-1)}(K),$$

for $2i(r-1) \le p(r-1)+n$, and

$$\beta P_i : H_p(K) \to H_{p+2i(r-1)-1}(K),$$

for $2i(r-1) \le p(r-1)+n+1$. Let us remark that, in general, $P_0 \neq 1\!\!1$.

The operation βP_i need not be the composition of the Bockstein with P_i as suggested by the notation. This is, however, true for so-called r-reduced objects $(K,\theta) \in \mathcal{C}(r,n)$. By definition, (K,θ) is r-reduced if it is obtained as the mod r reduction of some $(\widetilde{K},\widetilde{\theta}) \in \mathcal{C}(\pi,n,\mathbb{Z})$, where $\widetilde{K}$ is a flat $\mathbb{Z}$-module. Observe that both examples discussed above are r-reduced.

Let us give another, purely algebraic, example. For an associative Λ-algebra A, let $CH^p(A,A) := Hom_\Lambda(A^{\otimes p},A)$ be the complex of Hochschild cochains of A with coefficients in itself, with Hochschild differential δ [**Mac63a**, X.3]. In the classical paper [**Ger63**], M. Gerstenhaber considered the (obviously associative) *cup product* $\smile: CH^p(A;A)\otimes CH^q(A;A) \to CH^{p+q}(A;A)$:

$$(f \smile g)(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q) := f(a_1 \otimes \cdots \otimes a_p)g(b_1 \otimes \cdots \otimes b_q).$$

The cup product induces an associative product (which we denote by the same symbol) in the Hochschild cohomology $HH^*(A;A)$. Some indications based on the relation to deformation theory of associative algebras suggested that this product must be commutative, though the definition did not look commutative at all!

To prove this, M. Gerstenhaber introduced, for $f \in CH^p(A;A)$ and $g \in CH^q(A;A)$, the *composition product* $f \circ g \in CH^{p+q-1}(A;A)$ by

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{p+q-1}) :=$$
$$:= \sum_{1\leq i\leq p} (-1)^{(i-1)q} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+q-1}) \otimes a_{q+i+1} \otimes \cdots \otimes a_{p+q-1}).$$

The above formula is clearly the sum of $\circ_i$-compositions $f \circ_i g$ (obvious linear versions of $f \circ_i g$'s recalled in Section I.1.3) taken with appropriate signs. Gerstenhaber then proved the formula

$$f \circ \delta g - \delta(f \circ g) + (-1)^{q-1}\delta f \circ g = (-1)^{q-1}[g \smile f - (-1)^{pq} f \smile g]$$

which means that the chain-level cup product is homotopy commutative. A harmless modification of signs, $f \smile_1 g := (-1)^{pq+q} f \circ g$, gives a $\smile_1$-product satisfying (2.7). The corresponding Browder operation λ_1 defines the bracket product $[x,y] := (-1)^{|x|}\lambda_1(x,y)$ in Hochschild cohomology. The triple

$$(HH^*(A;A), \smile, [-,-])$$

(again with a small sign modification) is an example of algebraic structure which today is called a *Gerstenhaber* or *G-algebra*; see Section I.1.17.

We saw that Hochschild complex exhibits some properties of 2-fold loop spaces. This lead P. Deligne [**Del93**] to conjecture that the complex $CH^*(A;A)$ admits an action of some operad, chain homotopy equivalent to the operad of singular chains on the little 2-cubes operad $\mathcal{C}_2$; see Section I.1.19.

Heuristically, the existence of a 'W-action' $\theta : W \otimes K^{\otimes r} \to K$ means that the multiplication in K is homotopy commutative (or, if $r > 2$, that its iterates are homotopy symmetric), with a hierarchy of higher coherent homotopies. We saw that the 'W-action' was in some concrete examples induced from an action of an acyclic or at least highly connected operad, which meant that the multiplication in K was homotopically very close to a commutative and associative one, again with a coherent system of higher homotopies.

It is thus a very important fact that the first example of this section, the example of the reduced chain complex of a topological space, also admits an action of an acyclic operad.

One such action was constructed by V.A. Smirnov [**Smi85**] (and independently also by V.A. Hinich and V.V. Schechtman in [**HS87**]) who proved that the '*Eilenberg-Zilber operad*' $\mathcal{EZ}$ of endomorphisms of the reduced chain complex of the simplex acts on $C_*(X)$. Notice, however, that the Eilenberg-Zilber operad is not concentrated in positive degrees and it is not Σ-free.

Another, closely related, construction belongs to Justin Smith [**Smi94**]. He constructed an operad $R = \{R(n)\}_{n\geq 0}$ with the property that $R(n)$ is the bar resolution of $\mathbb{Z}$ over the symmetric group ring $\mathbb{Z}[\Sigma_n]$ and proved that also this operad acts naturally on the singular chain complex $C_*(X)$. Both actions contain very strong information about the space X itself, for instance, under some mild assumptions they fix the weak homotopy type of X. For a p-adic version, see M.A. Mandell [**Man01**].

The operations considered by May for which operads are so useful are operations derived from the higher homotopy commutativity embodied in the Eilenberg-Zilber theorem, so the action of the symmetric groups plays a key role. On the other hand, the nonsymmetric A_∞-operad $\underline{\mathcal{A}ss}_\infty$ is relevant to the higher order operations of several variables introduced by Uehara and Massey [**MU57, Mas58**], even though the Alexander-Whitney map provides a strictly associative multiplication at the cochain level.

The definition of *Massey products* in the homology of a loop space [**Sta70**] makes essential use of the A_∞-structure on the chains of an A_∞-space. For example, the definition of the Massey triple product $<u,v,w>$ uses the associating homotopy m_3. For $u, v, w \in H(A)$ such that $uv = 0 = vw$, we can choose representatives $\bar{u} \in u, \bar{v} \in v, \bar{w} \in w$ and chains $a, b \in A$ such that $\bar{u}\bar{v} = da$ and $\bar{v}\bar{w} = db$. Since the multiplication is not associative, $(-1)^{|u|}\bar{u}b - a\bar{w}$ need not be a cycle, but $(-1)^{|u|}\bar{u}b - a\bar{w} + m_3(\bar{u} \otimes \bar{v} \otimes \bar{w})$ is and represents $<u,v,w>$, which is defined as the coset of $H_*(A)$ modulo the indeterminacy $uH_*(A) + H_*(A)w$ to accommodate the choices made.

A_∞-algebras can also be used to define Massey products in the homology of associative cochain algebras as follows. As proved in [**Kad85**], for a given associative dg algebra A and a linear map $A \longrightarrow H(A)$ inducing an isomorphism of homology, $H(A)$ admits an A_∞-structure for which the linear map extends to an A_∞-map. Since $d = 0$, the algebra $H(A)$ is strictly associative, but the induced A_∞-structure maps $m_i : H(A)^{\otimes i} \longrightarrow H(A)$, $i \geq 3$, need not be trivial. For example, $m_3 : H(A)^{\otimes 3} \longrightarrow H(A)$ may be a nontrivial homotopy between the unique triple product and itself. The operation m_3 is related to the Massey triple product as follows: If $uv = vw = 0$, then $m_3(u,v,w)$ represents $<u,v,w>$ in $H(A)$ modulo $uH(A) + H(A)w$.

The usefulness of the Massey product is well illustrated in two important examples: the Massey triple product which detects the linking of the Borromean rings [**Mas69, Mas98**] and the Massey triple products in $H^*(Sp(5)/SU(5))$ which distinguish the homotopy type of $Sp(5)/SU(5)$ from that of the connected sum $(S^6 \times S^{25})\#(S^{10} \times S^{21})$ [**Bor53, Sta83**] which has the same cohomology algebra.

If the A_∞-structure on $H(A)$ induced from that on A is trivial, A is called *formal*, a term first introduced in rational homotopy theory. The concept of *formality* plays an important role in many recent developments, especially in relation to deformation quantization [**Kon97**] which uses the L_∞-operad (see Section I.1.10 and Example 3.133), providing the Lie analog of A_∞-structure (Section I.1.8 and Example 3.132). The Lie analogs of Massey products are known as Lie-Massey brackets or just Massey brackets [**Ret93**].

2.7. The linear isometries operad and infinite loop spaces

Many infinite loop spaces are recognized in terms of an operad, called the *linear isometries operad* and denoted $\mathcal{L}i$, which more recently has played a major role in a streamlined approach to the stable homotopy category [**KM95, May98**]. Following Boardman and Vogt [**BV73**], we describe this operad as follows.

Let LI denote the symmetric monoidal category of real inner-product spaces of countable dimension and linear isometric maps. Let us point out that by a linear isometric map we mean just a linear isometric *embedding*, not necessarily an isomorphism.

Topologize objects of LI by the direct limit topology of all the finite dimensional subspaces, which are given the usual metric topology. Topologize the morphism sets with the compactly generated function space topology. The monoidal structure is that of direct sum with the canonical identifications of vector spaces for associativity, commutativity and $\mathbb{R}$ as unit for tensor products. There is a special object in the category LI, the direct limit

$$\mathbb{R}^\infty := \varinjlim \mathbb{R}^n.$$

The basic property of $\mathbb{R}^\infty$ is formulated in the following lemma, whose proof we take almost literally from [**BV73**, pages 207–208].

LEMMA 2.13. *The space* $\mathrm{LI}(V, \mathbb{R}^\infty)$ *is contractible for any vector space* $V \in Ob(\mathrm{LI})$.

PROOF. A contracting homotopy is easy to construct, its construction being somewhat analogous to the 'Rota swindle.' Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for $\mathbb{R}^\infty$. Let $h : \mathbb{R}^\infty \to \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ be the isometry

$$h(e_{2n}) = (e_n, 0) \text{ and } h(e_{2n-1}) = (0, e_n), \ n \geq 1.$$

By applying the Gram-Schmidt orthogonalization process to the homotopy given by

$$g_t(e_n) := (1-t)e_n + te_{2n}, \ n \geq 1, \ t \in [0,1],$$

we obtain a homotopy through isometries from $1\!\!1_{\mathbb{R}^\infty}$ to $g := g_1$, given by $g(e_n) := e_{2n}$.

Similarly, construct a homotopy between the two axial inclusions $i_1, i_2 : V \to V \oplus V$ by applying the Gram-Schmidt process to the homotopy

$$(1-t)i_1 + ti_2, \ t \in [0,1].$$

Let $k : V \to \mathbb{R}^\infty$ be a fixed isometry. Let us compose the above homotopies to define a contraction of $\mathcal{L}i(V, \mathbb{R}^\infty)$ into the point $h^{-1} \circ (k \oplus k) \circ i_2 \in \mathcal{L}i(V, \mathbb{R}^\infty)$.

For any isometry $f : V \to \mathbb{R}^\infty$, the contraction runs from $f = h^{-1} \circ h \circ f$ to

$$h^{-1} \circ h \circ g \circ f = h^{-1} \circ i_1 \circ f = h^{-1} \circ (f \oplus k) \circ i_1$$

(using the homotopy between $1\!\!1_{\mathbb{R}^\infty}$ and g) and thence to $h^{-1} \circ (f \oplus k) \circ i_2 = h^{-1} \circ (k \oplus k) \circ i_2$ (using the homotopy between i_1 and i_2). ■

The 'classical' infinite loop spaces can be described in terms of symmetric monoidal functors $\mathrm{LI} \to \mathrm{Top}$. The category LI determines the *linear isometries operad*, denoted $\mathcal{L}i$

DEFINITION 2.14. The operad $\mathcal{L}i = \{\mathcal{L}i(n)\}_{n \geq 1}$ is given by

$$\mathcal{L}i(n) := \mathrm{LI}(\oplus^n \mathbb{R}^\infty, \mathbb{R}^\infty), \ n \geq 1.$$

Strictly speaking, the n-fold direct sum is given by a fixed choice of iteration. The symmetric groups act by permuting the summands in $\oplus^n\mathbb{R}^\infty$; since the maps in $\mathcal{L}i(n)$ are imbeddings, the induced action is free.

THEOREM 2.15. *The operad $\mathcal{L}i$ is an E_∞-operad. Any continuous symmetric monoidal functor* $\mathtt{F} : \mathtt{LI} \to \mathtt{Top}$ *induces, up to homotopy equivalence, an infinite loop space structure on* $\mathtt{F}(\mathbb{R}^\infty)$.

PROOF. The first part of the theorem follows immediately from Lemma 2.13. For the second part, let $X := \mathtt{F}(\mathbb{R}^\infty)$. Application of a continuous symmetric monoidal functor $\mathtt{F} : \mathtt{LI} \to \mathtt{Top}$ induces, for $n \geq 1$, maps

$$\mathcal{L}i(n) = \mathtt{LI}(\oplus^n\mathbb{R}^\infty, \mathbb{R}^\infty) \longrightarrow \mathcal{E}nd(X)(n) = Map(X^n, X)$$

which form a map of operads: $\mathcal{L}i \to \mathcal{E}nd(X)$. Thus $\mathtt{F}(\mathbb{R}^\infty)$ is an $\mathcal{L}i$-space and hence it has the homotopy type of an infinite loop space, by Proposition 2.33. ∎

Applications of Theorem 2.15 abound. The classical ones are given in the following examples.

EXAMPLE 2.16. Let $\mathtt{F} : \mathtt{LI} \to \mathtt{Top}$ be given by $\mathtt{F}(V) := O(V)$, the orthogonal group of V, so $\mathtt{F}(\mathbb{R}^\infty) = O(\mathbb{R}^\infty)$ is the stable orthogonal group, which is thus an infinite loop space. We may think of the multiplication as being induced by

$$A \in O(p), B \in O(q) \longmapsto A \oplus B \in O(p+q)$$

where, in terms of matrices,

$$A \oplus B := \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right).$$

Observe that the infinite loop space structure above does not use the group structure of the orthogonal group.

We can apply the ordinary classifying space functor B so that $\mathtt{F}(V) := BO(V)$. This choice also gives a monoidal functor, as follows from the homeomorphism

$$B(G \times H) \cong BG \times BH$$

for any two compact groups G and H. Thus the classifying space $BO := BO(\mathbb{R}^\infty)$ for real K-theory is again an infinite loop space with respect to the multiplication given by the classifying map for the Whitney sum of bundles.

The analogous results for U and BU (respectively Sp and BSp) follow immediately by considering $\mathtt{F}(V) = U(V \otimes \mathbb{C})$ (respectively $\mathtt{F}(V) = U(V \otimes \mathbb{H})$ where $\mathbb{H}$ denotes the quaternions).

EXAMPLE 2.17. Only slightly more subtle is $\mathtt{F}(V) := \mathcal{H}(V \cup \infty)$, the space of homotopy self-equivalences of the sphere obtained as the one-point compactification of V. For $\mathtt{F}(V) := Homeo(V)$, the space of homeomorphisms of V to itself, the topology is considerably more subtle; Boardman and Vogt resort to replacing $Homeo(V)$ by the weakly equivalent realization of its singular complex.

The maps $V \to V \otimes \mathbb{C} \to V \otimes \mathbb{H}$ induce infinite loop space maps as do the forgetful maps $V \otimes \mathbb{C} \to V \otimes \mathbb{R}^2$ and $V \otimes \mathbb{H} \to V \otimes \mathbb{R}^4$ as well as the maps $O(V) \to Homeo(V) \to \mathcal{H}(V \cup \infty)$ and all the respective maps of classifying spaces.

EXAMPLE 2.18. The tensor product of matrices

$$\otimes : O(p) \times O(q) \to O(pq), \quad A, B \mapsto A \otimes B,$$

induces a different structure on the infinite orthogonal group $O(\mathbb{R}^\infty)$. This, and the corresponding structure on the classifying space BO, can again be shown to be an infinite loop space structure, but this time using Segal's approach (Section 2.5) and not that of Boardman and Vogt, as they point out. The analogous results are true also for U, Sp, etc.

2.8. W-construction

Another major contribution of Boardman and Vogt [**BV73, BV68**] is their W-construction, which was their main tool in the analysis of homotopy invariant structures. Here we will present the essential ideas of the W-construction, its properties and applications. A detailed modern sophisticated treatment has recently been given by Vogt [**Vog**].

Essential ideas of the construction $W\mathcal{P}$ can be demonstrated on the operad they call the *operad of trees*, but which we prefer to call the *operad of metric trees*. To simplify our exposition, we will in fact ignore (homotopy) units and work with *reduced* (that is, with no vertices of arity one) trees only.

To simplify our exposition, we will make no distinction between isomorphism classes of trees and trees representing these classes; see also remarks in Section 1.9 Recall that $edge(T)$ denotes the set of internal edges of a tree T.

DEFINITION 2.19. A *metric tree* is a tree in the sense of Section 1.5 (i.e. with one root and a finite number of leaves) together with a *length function* (also called a *metric*) $h : edge(T) \to [0, 1]$.

For a given tree T, the set of metrics $Met(T)$ on T can be realized as a cube I^k, where the dimension k is the number of internal edges of the tree.

Recall that for an edge $e \in edge(T)$, T/e denotes the tree obtained from T by shrinking e to a point. For any length function $h : edge(T/e) \to [0, 1]$, let $s_e(h) : edge(T) \to [0, 1]$ be the obvious function which is 0 on e and agrees with h on the other edges. So we have defined, for any $e \in edge(T)$, a 'degeneracy' $s_e : Met(T/e) \to Met(T)$.

Now define, for a given $n \geq 2$, the space of *reduced metric trees* $\mathcal{R}mtree(n)$ of arity n to be the polytope formed from the union of all (isomorphism classes of) reduced metric trees of arity n by identifying $Met(T/e)$ as a cube with the corresponding face of the cube $Met(T)$ or, formally,

$$Met(T) \ni s_e(h) \sim h \in Met(T/e),$$

for any $T \in \mathcal{R}tree(n)$ and $e \in edge(T)$. Here $\mathcal{R}tree(n)$ denotes the set of isomorphism classes of reduced rooted n-trees (recall that it is in fact the nth component of the operad of reduced rooted trees recalled in Section 1.5). Since $\mathcal{R}tree(n)$ is finite, the polytope $\mathcal{R}mtree(n)$ is finite as well.

For metric trees, the grafting operation should be considered as identifying the root edge with the appropriate leaf edge and assigning the length of this new internal edge to be 1. This grafting induces (in fact, piecewise cubical) embeddings

$$\circ_i : \mathcal{R}mtree(r) \times \mathcal{R}mtree(s) \to \mathcal{R}mtree(r+s-1), \ r, s \geq 2,$$

giving $\mathcal{R}mtree = \{\mathcal{R}mtree(n)\}_{n\geq 2}$ the structure of a topological pseudo-operad.

The W-construction on an operad $\mathcal{P}$ is then a generalization of the operad $\mathcal{R}mtree$ in the sense that we consider metric trees with vertices colored by elements of the operad $\mathcal{P}$. A precise definition is given below.

The original construction [**BV73**] described $W\mathcal{P}$ as a quotient of the space of $\mathcal{P}$-colored metric (not necessarily reduced) trees by two types of relations, the first type used (in our terminology) the pseudo-operad structure of $\mathcal{P}$ only; the second, more complicated one involved the unit of $\mathcal{P}$. We will avoid introducing the second type of relation by assuming that

$$\mathcal{P}(1) = * \text{ (the one-point space);} \tag{2.9}$$

these operads are general enough for our purposes. The functor $(-)^+ : \mathsf{Op} \to \Psi\mathsf{Op}$ defined by

$$\mathcal{P}^+(n) := \begin{cases} \emptyset, & \text{for } n = 1, \text{ and} \\ \mathcal{P}(n), & \text{for } n \geq 2, \end{cases} \tag{2.10}$$

induces an equivalence of the category of (topological) operads with $\mathcal{P}(1) = *$ and the category of pseudo-operads $\mathcal{Q}$ with $\mathcal{Q}(1) = \emptyset$, the inverse functor given by formally adjoining the unit.

So we start with an operad $\mathcal{P}$ satisfying (2.9), then consider the pseudo-operad $\mathcal{P}^+$, construct a pseudo-operad $W\mathcal{P}^+$, and finally define the operad $W\mathcal{P}$ by adjoining the unit to $W\mathcal{P}^+$. Recall (Section 1.9) that

$$\mathcal{P}^+(T) = \bigtimes_{v \in Vert(T)} \mathcal{P}^+(In(v))$$

denotes the set of all $\mathcal{P}^+$-colorings of the tree T. Clearly $\mathcal{P}^+(T)$ may be nonempty only when T is reduced. Recall also that the pseudo-operad structure on $\mathcal{P}^+$ induces, for each $e \in edge(T)$, a map

$$\gamma_e : \mathcal{P}^+(T) \to \mathcal{P}^+(T/e). \tag{2.11}$$

Now let $W\mathcal{P}^+(1) := \emptyset$ and, for $n \geq 2$, define $W\mathcal{P}^+(n)$ as the quotient of

$$\coprod_{T \in \mathcal{R}tree(n)} Met(T) \times \mathcal{P}^+(T) \tag{2.12}$$

by the relation:

$$Met(T) \times \mathcal{P}^+(T) \ni (s_e(h), f) \sim (h, \gamma_e(f)) \in Met(T/e) \times \mathcal{P}^+(T/e),$$

where $T \in \mathcal{R}tree(n)$, $e \in edge(T)$ and γ_e is the contraction as in (2.11). The above definition is of the same informal type as that of (1.52); the doubting reader may rewrite it in terms of colimits.

Intuitively, the above relation means that we remove edges of length 0 by composing in $\mathcal{P}^+$, as indicated in Figure 5. The quotient is topologized using the topology on $\mathcal{P}$ together with that of the cubes.

The pseudo-operad structure on $W\mathcal{P}^+$ is given as follows. Observe that (2.12) is the union of products of ingredients making up the pseudo-operad $\mathcal{R}mtree$ (the $Met(-)$-factor) and those making up the free pseudo-operad (the $\mathcal{P}^+(-)$-factor). The pseudo-operad structure on $W\mathcal{P}^+$ is then induced diagonally by these two

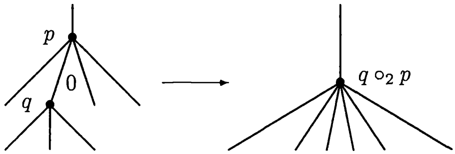


FIGURE 5. Removing an edge of length 0.

pseudo-operad structures. Finally, the operad $W\mathcal{P}$ is obtained by adjoining the unit $1 \in W\mathcal{P}(1) := *$ to $W\mathcal{P}^+$.

The augmentation $\varepsilon : W\mathcal{P} \to \mathcal{P}$ has components $\varepsilon(n) : W\mathcal{P}(n) \to \mathcal{P}(n)$ induced by the composition

$$Met(T) \times \mathcal{P}^+(T) \xrightarrow{\text{proj}} \mathcal{P}^+(T) \xrightarrow{\gamma_T} \mathcal{P}^+(n) = \mathcal{P}(n),\ T \in \mathcal{R}tree(n),\ n \geq 2,$$

where γ_T denotes the composition along $T \in \mathcal{R}tree(n)$. Informally, this means that ε is defined by shrinking every edge (length) to 0 and correspondingly composing the operad elements associated to the vertices at the ends of the edge.

As a matter of fact, the collection $\mathcal{P}$ is a subcollection (but <u>not</u> a suboperad) of $W\mathcal{P}$ under the natural inclusion $\iota : \mathcal{P} \to W\mathcal{P}$ given by

$$\iota(n)(p) := h_{c(n)} \times p \in Met(c(n)) \times \mathcal{P}^+(n),$$

where $h_{c(n)}$ is the unique trivial metric on the corolla $c(n)$ with n input leaves, $n \geq 2$: Metric contraction of the edges induces a deformation contraction of $W\mathcal{P}$ to $\mathcal{P}$.

THEOREM 2.20. *The* W-construction *is a functor* $W : \mathsf{Op} \to \mathsf{Op}$ *which, from any (topological) operad* $\mathcal{P}$*, produces an operad* $W\mathcal{P}$ *and an augmentation* $\varepsilon : W\mathcal{P} \to \mathcal{P}$ *which is a cofibrant resolution of* $\mathcal{P}$.

For a detailed proof, see [**BV73**, Proposition II.3.6, Theorem II.3.17]. 'Cofibrant' refers to an appropriate Closed Model Category structure, which we briefly discussed in Section I.1.18. One of the implications of the cofibrancy is that, given a diagram of operads

$$\begin{tikzcd} & \mathcal{B} \arrow[d, "g"] \\ W\mathcal{P} \arrow[ur, dotted, "\bar{f}"] \arrow[r, "f"'] & \mathcal{C} \end{tikzcd}$$

with g a homotopy equivalence of operads, there exists a lift $\bar{f} : W\mathcal{P} \to \mathcal{P}$ such that $g \circ \bar{f}$ and f are homotopic *as maps of operads.*

EXAMPLE 2.21. The operad $\mathcal{C}mon = \{\mathcal{C}mon(n)\}_{n \geq 1}$ for commutative associative topological monoids (not to be mistaken with the operad $\mathcal{C}om$ for commutative associative algebras) is given by $\mathcal{C}mon(n) = *$, for $n \geq 1$. The pseudo-operad $\mathcal{R}mtree$ of metric trees is then exactly the W-construction on $\mathcal{C}mon$, that is, $\mathcal{R}mtree = W\mathcal{C}mon^+$.

There is an obvious non-Σ version $\underline{W}$ of the W-construction defined for non-Σ operads in exactly the same way as W was for ordinary operads, except that

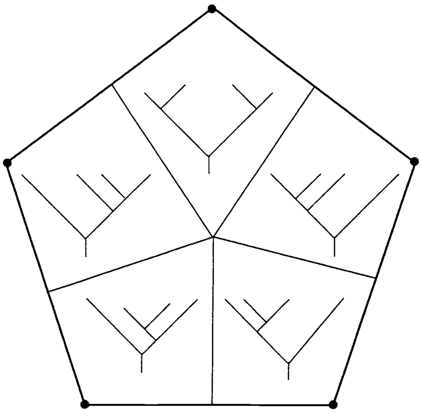


FIGURE 6. A portrait of CK_4 as the pentagon decomposed into five cubes. The cubes are indexed by binary trees $T \in \mathcal{R}tree(4)$.

we use planar trees instead of abstract ones. If $\underline{\mathcal{P}}$ is a non-Σ operad, these two constructions are related by

$$W(\underline{\mathcal{P}}(n)) \cong \underline{W}(\underline{\mathcal{P}})(n) \times \Sigma_n, \ n \geq 1.$$

EXAMPLE 2.22. The $\underline{W}$-construction applied to the non-Σ operad $\underline{\mathcal{M}on}$ for associative monoids defined by $\underline{\mathcal{M}on}(n) := *$ for each $n \geq 1$ coincides with the operad associated to the non-Σ pseudo-operad $\mathcal{R}m\underline{tree} := \{\mathcal{R}m\underline{tree}(n)\}_{n \geq 2}$ of planar metric reduced trees, an obvious analog of the pseudo-operad $\mathcal{R}mtree$ defined above. The space $\mathcal{R}m\underline{tree}(n)$ can also be described as a certain cubical decomposition CK_n of the associahedron K_n. A portrait of CK_4 is given in Figure 6.

2.9. Algebraic structures up to strong homotopy

As mentioned in Section I.1.18, the W-construction plays a key role in Boardman and Vogt's study of homotopy invariant algebraic structures. To investigate that concept fully, we first need to address the issue of maps of algebras over an operad $\mathcal{P}$. We will work in the category of (compactly generated) topological spaces, though with some subtlety the analogous discussion makes sense in any monoidal closed model category.

MAPS OF A_∞-SPACES. Suppose we are given two $\mathcal{P}$-spaces X and Y. A (strict) *$\mathcal{P}$-homomorphism* $f : X \to Y$ respects the structure maps precisely:

$$f(p(x_1, \dots, x_n)) = p(f(x_1), \dots, f(x_n))$$

for each $n \geq 1$, $x_1, \dots, x_n \in X$ and each n-ary operation $p \in \mathcal{P}(n)$.

However, from a homotopy point of view this is too much to ask; it is important to consider maps that respect the structure *up to (strong) homotopy.* The earliest example of this phenomenon is perhaps that of the structures of well-pointed H-spaces (X, e) and H-maps. Recall that an *H-space* is a space X with a map $m : X \times X \to X$ such that $m(x, e) = x = m(e, x)$ and an *H-map* $f : X \to Y$ between two H-spaces is a map f such that fm is homotopic to $m(f \times f)$ (preserving the base point e).

Higher homotopies are of interest even for strict algebraic structures If X and Y are topological monoids (associative H-spaces), Sugawara [**Sug61**] defined $f : X \to Y$ to be *strongly homotopy multiplicative* if there exists a family of maps

$$f_i : I^{i-1} \times X^i \to Y$$

such that $f_1 = f$ and for $1 \leq j \leq i-1$,

$$(2.13) \quad f_i(t_1, ..., t_{i-1}; a_1, ..., a_i) = \begin{cases} f_{i-1}(t_1, ..., \hat{t}_j, ..., t_{i-1}; a_1, ..., a_j a_{j+1}, ..., a_i), & \text{if } t_j = 0, \\ f_j(t_1, ..., t_{j-1}; a_1, ..., a_j) f_{i-j}(t_{j+1}, ..., t_i; a_{j+1}, ..., a_i), & \text{if } t_j = 1, \end{cases}$$

where $\hat{t}_j$ indicates t_j is omitted.

A very important example of such a map which helps explain some of the subtlety of transferring structure via a homotopy equivalence is the example comparing $X = Map(W, W)$ and $Y = Map(Z, Z)$ via the map $f : X \to Y$ induced by homotopy inverses $\phi : W \leftrightarrows Z : \psi$.

Let $h : I \to Map(Z, Z)$ be a homotopy with $h(0) = Id_Z$ and $h(1) = \phi\psi$. Define $f : Map(W, W) \to Map(Z, Z)$ by $f(a) = \phi a \psi$ for $a \in Map(W, W)$ and further

$$f_i(t_1, \ldots, t_{i-1}; a_1, \ldots, a_i) = \phi a_1 h(t_1) a_2 h(t_2) \cdots h(t_{i-1}) a_i \psi.$$

The boundary conditions (2.13) are easy to verify.

If X is an A_∞-space with structure maps

$$(2.14) \qquad m_s : K_s \times X^s \to X, \ s \geq 1$$

and Y is a topological monoid, we then say $f : X \to Y$ is an A_∞*-map* if there exists a family of maps

$$(2.15) \qquad f_i : K_{i+1} \times X^i \to Y, \ i \geq 1,$$

such that $f_1 = f$ and f_i restricted to a face of K_{i+1} is of the appropriate form, which means the following. Recall that the associahedron K_{i+1} is of dimension $i-1$ and has codimension one faces $K_r \circ_j K_s$, where $r + s = i + 2$ and $1 \leq j \leq r$. The restriction of f_i to $K_r \circ_j K_s$ is

$$f_i(\rho \circ_j \sigma; a_1, ..., a_i) = \begin{cases} f_{r-1}(\rho; a_1, ..., a_{j-1}, m_s(\sigma, a_j, ..., a_{j+s-1}), a_{j+s}, ..., a_i), & \text{for } j < r, \text{ and} \\ f_{r-1}(\rho; a_1, ..., a_{r-1}) f_{s-1}(\sigma; a_r, ..., a_i), & \text{for } j = r, \end{cases}$$

see [**Sta70**, page 54]. Observe that the associahedra appear both in definition (2.14) of an A_∞-space (as they should) and in definition (2.15) of an A_∞-map with the arity shifted by one. We have no conceptual explanation for this latter phenomenon.

If both X and Y are A_∞-spaces, still more complicated polyhedra, occasionally called *multiplihedra*, are used in defining A_∞-maps. For example, we begin with an ordinary homotopy, i.e. $f(xy) \simeq f(x)f(y)$, but for three variables, the polygon has six edges; see Figure 7. For four variables [**Sta70**, page 53] the polyhedron is depicted in Figure 8.

The following characterization of A_∞-maps between A_∞-spaces in terms of the B-construction (see Section I.1.6) was proved in [**Sta70**, Theorem 8.12].

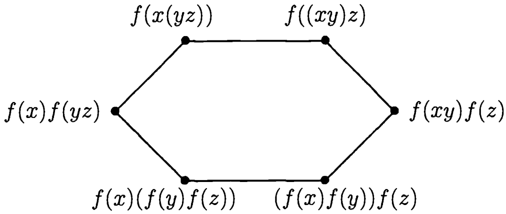


FIGURE 7. The multiplihedron for 3 variables.

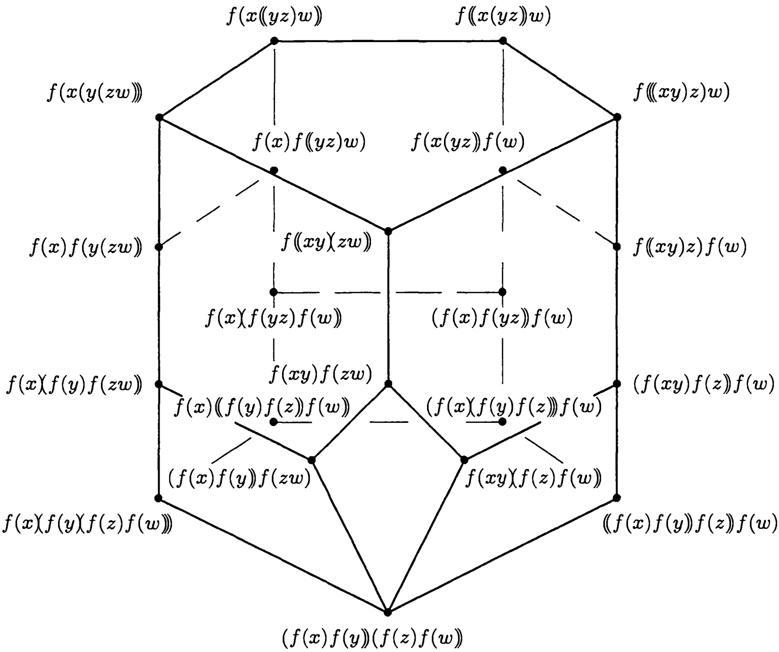


FIGURE 8. The multiplihedron for four variables.

PROPOSITION 2.23. *Suppose that X and Y are A_∞-spaces. There exists a one-to-one correspondence between homotopy classes of A_∞-maps $X \to Y$ and homotopy classes of continuous maps $BX \to BY$.*

A_∞-maps play a key role in the homotopy characterization of associative structures. Assuming modest topological restrictions, e.g. being of the homotopy type of a CW-complex, a space is of the homotopy type of a topological monoid if and only if it is an A_∞-space; moreover, given the A_∞-space, the homotopy equivalence with a topological monoid will be via A_∞-maps. Thus A_∞-structures provide homotopy invariant remains of monoidal structure. Being homotopy invariant, A_∞-structures provide a model for a more general theory of homotopy invariant algebraic structures – the leitmotif of Boardman and Vogt's work [**BV73**].

THE MEANING OF HOMOTOPY INVARIANT ALGEBRAIC STRUCTURE. With this background, we can look at Boardman and Vogt's study of homotopy invariant algebraic structures. We take algebraic structures on a space X to mean the structure of a $\mathcal{Q}$-space for some operad $\mathcal{Q}$. To be called a *homotopy invariant algebraic structure*, the structure should be given by an operad $\mathcal{Q}$ satisfying the three criteria of Boardman and Vogt [**BV73**, page 1], where $\mathcal{Q}$-map refers to the analog of A_∞-map for $\mathcal{Q}$-spaces, explicated below:

(i) If X is a $\mathcal{Q}$-space and $f : X \to Y$ is a homotopy equivalence, then Y admits the structure of a $\mathcal{Q}$-space and f that of a $\mathcal{Q}$-map.
(ii) If $f : X \to Y$ is a $\mathcal{Q}$-map of $\mathcal{Q}$-spaces and g is homotopic to f, then g admits the structure of a $\mathcal{Q}$-map.
(iii) If $f : X \to Y$ is a $\mathcal{Q}$-map of $\mathcal{Q}$-spaces and a homotopy equivalence, then any homotopy inverse to f admits the structure of a $\mathcal{Q}$-map.

Notice this leads to a further issue: One would like the structure of a $\mathcal{Q}$-map on any homotopy inverse to f to make it a homotopy inverse *as $\mathcal{Q}$-maps*, but that requires defining 'homotopy of $\mathcal{Q}$-maps,' beginning an infinite regression [**Mar99c**].

MAPS OF $\mathcal{P}$-SPACES. To address these issues of homotopy invariance, we need the analog of A_∞-maps for algebras over general operads and, as Boardman and Vogt point out [**BV73**, page 68], to study such maps of spaces over more general operads, it serves us well to have a subtle generalization of operad known as a bicolored operad. Still more colorful operads can be defined, but they are currently not of great importance.

We are not going to give here a formal definition of colored operads (which can be found in [**BV73**] or [**Mar99c**]); we only indicate how to construct a $\{\mathtt{B},\mathtt{W}\}$-colored operad $\mathcal{P}_{\mathtt{B}\to\mathtt{W}}$ so that an algebra over $\mathcal{P}_{\mathtt{B}\to\mathtt{W}}$ consists of two $\mathcal{P}$-spaces A and B and a continuous map $f : A \to B$ which is also a $\mathcal{P}$-homomorphism; see again [**Mar99c**] for details.

DEFINITION 2.24. For an ordinary operad $\mathcal{P}$, define the $\{\mathtt{B},\mathtt{W}\}$-colored operad $\mathcal{P}_{\mathtt{B}\to\mathtt{W}}$ as the quotient

$$\mathcal{P}_{\mathtt{B}\to\mathtt{W}} := \frac{\mathcal{P}_{\mathtt{B}} * \mathcal{P}_{\mathtt{W}} * \Gamma(f)}{(fa_{\mathtt{B}} = a_{\mathtt{W}} f^{\times n},\ \forall a \in \mathcal{P}(n),\ n \geq 1)}. \tag{2.16}$$

In this formula, $\mathcal{P}_{\mathtt{B}}$ (respectively $\mathcal{P}_{\mathtt{W}}$) denotes the copy of $\mathcal{P}$ 'concentrated' in the color $\mathtt{B}$ (respectively $\mathtt{W}$). The symbol f, interpreted as $f : \mathtt{B} \to \mathtt{W}$, is a new generator of the free $\{\mathtt{B},\mathtt{W}\}$-colored operad, $\Gamma(f)$, which consists only of f, since there is no way to compose f with itself. The asterix $*$ denotes the free product of colored operads, so $\mathcal{P}_{\mathtt{B}\to\mathtt{W}}$ consists of all manifestly meaningful compositions of f with operations of $\mathcal{P}_{\mathtt{B}}$ (respectively $\mathcal{P}_{\mathtt{W}}$). The equation $fa_{\mathtt{B}} = a_{\mathtt{W}} f^{\times n}$ generating the ideal in the denominator of (2.16) expresses the fact that f commutes with all operations of the operad $\mathcal{P}$. The operad $\mathcal{P}_{\mathtt{B}\to\mathtt{W}}$ is the same as $\mathcal{P} \otimes \mathcal{L}_1$ of [**BV73**]

HOMOTOPY INVARIANT ALGEBRAIC STRUCTURES. As the operad $\mathcal{K}$ for A_∞-spaces can be regarded as $W\mathcal{M}on$ (or $\underline{\mathcal{K}}$ as the nonsymmetric version $\underline{W}(\underline{\mathcal{M}on})$ (see Example 2.22)), so for any operad $\mathcal{P}$, the operad $W\mathcal{P}$ provides the proper notion of a homotopy invariant structure. The more common parlance is to speak of a $W\mathcal{P}$ structure as a $\mathcal{P}$-structure *up to (strong) homotopy*.

DEFINITION 2.25. A *strongly homotopy $\mathcal{P}$-space* is a topological space with an action of the operad $W\mathcal{P}$.

We adopt the terminology that a *$W\mathcal{P}$-homomorphism* of $W\mathcal{P}$-spaces means a single map that respects the $W\mathcal{P}$-structure precisely, while a strongly homotopy $\mathcal{P}$-map refers to the higher homotopy notion exemplified by an A_∞-map of A_∞-spaces. We use $W(\mathcal{P}_{\mathtt{B}\to\mathtt{W}})$ to make that higher homotopy notion precise. Notice that our terminology differs slightly from the one used in [**BV73**].

DEFINITION 2.26. A *strongly homotopy $\mathcal{P}$-map* between $W\mathcal{P}$-spaces A and B is an algebra over $W(\mathcal{P}_{\mathtt{B}\to\mathtt{W}})$.

The importance of $W\mathcal{P}$-structures is expressed by the following theorem proved in [**BV73**, Theorem 4.37].

THEOREM 2.27. *A topological space X admits a $W\mathcal{P}$-structure if and only if X is of the homotopy type of a $\mathcal{P}$-space. Moreover, given X of the homotopy type of a $\mathcal{P}$-space Y, X is a $W\mathcal{P}$-space in such a way that the homotopy equivalences between X and Y are strongly homotopy $\mathcal{P}$-maps.*

REMARK 2.28. Here the cofibrancy of $W\mathcal{P}$ is crucial for the 'if' statement. For example, although a space over the little n-cubes operad $\mathcal{C}_n$ has the homotopy type of an n-fold loop space, the converse is, for $n \geq 2$, not true in general, because the operad $\mathcal{C}_n$ is not cofibrant.

For the A_∞-case, the proof of Theorem 2.27 was first accomplished by constructing a 'classifying space' BX and then taking Y to be the Moore space of loops on BX. Later [**Ada, BV73, Lad76**], other constructions were provided. Here we sketch a more straightforward approach.

THE M-CONSTRUCTION. We construct the space Y from the given data on X via the *M-construction*, a strict $\mathcal{P}$-space MX, sometimes called the *rectification* of the $W\mathcal{P}$-space X.

THEOREM 2.29. *Under mild topological assumptions, there exists, for a $W\mathcal{P}$-space X, a strict $\mathcal{P}$-space MX such that X is a strong deformation retract of MX,*

$$\iota : X \hookrightarrow MX,$$

with ι a strongly homotopy $\mathcal{P}$-map.

A construction of MX for a general operad $\mathcal{P}$ was given by Boardman and Vogt in [**BV73**, Theorem 4.49]. We illustrate basic ideas of the construction on A_∞-spaces (i.e. with $\mathcal{P} = \mathcal{A}ss$), in which case the construction is actually due to Adams [**Ada**].

In Example 2.22 we recalled the non-Σ pseudo-operad of reduced metric planar trees $\mathcal{R}m\underline{tree} := \{\mathcal{R}m\underline{tree}(n)\}_{n\geq 2}$ and observed that the space $\mathcal{R}m\underline{tree}(n)$ provides a certain cubical decomposition CK_n of the associahedron K_n. Let $\mathcal{R}m\underline{tree}^e(n)$ (e for *extended*) denote, for $n \geq 2$, the space of (isomorphism classes of) metric planar n-trees such that a length in $[0,1]$ is assigned also to the root edge, so, in fact

$$\mathcal{R}m\underline{tree}^e(n) \cong \mathcal{R}m\underline{tree}(n) \times [0,1].$$

For $n = 1$ we put $\mathcal{R}m\underline{tree}^e(1) := \{T_0\}$, where T_0 is the trivial 1-tree interpreted as a metric tree with the unique edge (which is both the root and leaf) of length 1.

For $S \in \mathcal{R}m\underline{tree}^e(k)$, $T \in \mathcal{R}m\underline{tree}(l)$, $1 \leq i \leq k$ and $l \geq 2$, let $S \circ_i T \in \mathcal{R}m\underline{tree}^e(k+l-1)$ be the tree obtained by grafting T at the ith leaf of S, with the metric induced by the metric of S and T and the length of the internal edge created

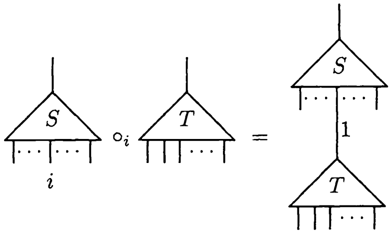


FIGURE 9. The right $\mathcal{R}m\underline{tree}$-module structure of $\mathcal{R}m\underline{tree}^e$.

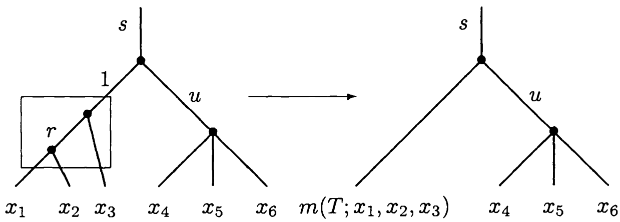


FIGURE 10. Representing elements of MX.

by grafting set equal to 1, as indicated in Figure 9 These operations induce on $\{\mathcal{R}m\underline{tree}^e(n)\}_{n\geq 1}$ a right $\mathcal{R}m\underline{tree}$-module structure in the sense of Definition 3.26, but we will not need this observation. If $k = 1$, $T_0 \in \mathcal{R}m\underline{tree}^e(1)$ the unique element and $T \in \mathcal{R}m\underline{tree}(l)$, then $T_0 \circ_1 T$ is T interpreted as an element of $\mathcal{R}m\underline{tree}^e(l)$ with root of length 1.

DEFINITION 2.30. Let X be an A_∞-space with an action

$$\{m_n : \mathcal{R}m\underline{tree}(n) \times X^{\times n} \to X,\ n \geq 2\}.$$

Define MX to be the quotient of the disjoint union

$$\bigsqcup_{n\geq 1} \mathcal{R}m\underline{tree}^e(n) \times X^{\times n}$$

modulo the relation

$$\begin{aligned}\mathcal{R}m\underline{tree}^e(n) \times X^{\times n} &\ni S \circ_i T \times (x_1, ..., x_n)\\ &\sim S \times (x_1, ..., x_{i-1}, m_l(T; x_i, ..., x_{i+l-1}), x_{i+l}, ..., x_n) \in \mathcal{R}m\underline{tree}^e(k) \times X^{\times k},\end{aligned}$$

for $k \geq 1$, $l \geq 2$, $n = k + l - 1$ and $1 \leq i \leq k$.

An element of MX can be represented by a metric tree with vertices labeled by elements of X. There are two types of relations. The first type comes from $\mathcal{R}m\underline{tree}$ and says that internal edges of length 0 are collapsed. The second relation says that if an internal edge or root has length 1, the tree "below" (in the up-rooted convention) this edge or root is interpreted as an element of $\mathcal{R}m\underline{tree}$ and used to compose the X-labels of this tree; see Figure 10.

Let us indicate how to prove that the space MX in Definition 2.30 indeed has the properties stated in Theorem 2.29. For two metric trees $U \in \mathcal{R}m\underline{tree}^e(a)$, $V \in \mathcal{R}m\underline{tree}^e(b)$ and $a + b \geq 1$, let $U \cdot V \in \mathcal{R}m\underline{tree}^e(a + b)$ be the tree obtained

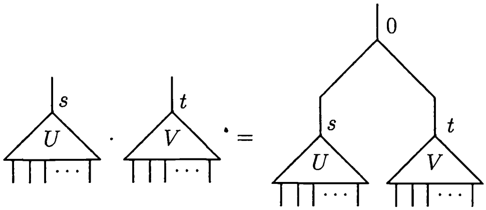


FIGURE 11. The product $U \cdot V$.

by grafting U (respectively V) at the left (respectively right) leaf of the planar 2-corolla, with the metric induced by the metric of U and V and the length of the root equal 0. This operation is symbolized in Figure 11. Define a multiplication on MX by

$$(U \times (x_1, \dots, x_a)) \cdot (V \times (y_1, \dots, y_b)) := U \cdot V \times (x_1, \dots, x_a, y_1, \dots, y_b),$$

for $U \in \mathcal{R}m\underline{tree}^e(a)$ and $V \in \mathcal{R}m\underline{tree}^e(b)$. It is easy to verify that this formula defines an associative multiplication on MX. There is a natural inclusion $\iota : X \hookrightarrow MX$ given by

$$X \ni x \mapsto T_0 \times x \in \mathcal{R}m\underline{tree}^e(1) \times X,$$

where T_0 is the trivial tree with the unique edge being the root of length 1. As follows from [**BV73**], this map is an A_∞-homotopy equivalence.

The proof for a general operad $\mathcal{P}$ is similar. The rectification MX is again constructed by 'extending' more general $\mathcal{P}$-colored trees forming $W\mathcal{P}$ instead of planar trees above.

E_∞-STRUCTURES. As recalled in Section 2.3, given a space X over the little n-cubes operad $\mathcal{C}_n$, May produced a space Y so that X had the homotopy type of $\Omega^n Y$ by building Y as a two-sided bar construction. In contrast, Boardman and Vogt [**BV73**, Chapter VI] approach the problem of identifying iterated loop spaces by investigating how (if at all) structure on a space X can be transferred to its classifying space BX. Here X must be at least an A_∞-space and they take the classifying space BX to be the classifying space of Dold and Lashof [**DL59**] or of Milgram [**Mil67**] applied to the monoid MX, though one could as well use the direct construction in Section I.1.6. There are mild topological restrictions assumed for the rest of this section.

In the following Proposition 2.32, which gives the induction step for the approach of Boardman and Vogt, we need the *$\otimes$-product of operads*.

DEFINITION 2.31. Let $\mathcal{P}$ and $\mathcal{S}$ be two topological operads. Then $\mathcal{P}\otimes\mathcal{S}$ is defined to be the operad with the property that a topological space X is an algebra for this operad if and only if X is both a $\mathcal{P}$- and an $\mathcal{S}$-algebra and these two structures commute with each other.

The product $\mathcal{P}\otimes\mathcal{S}$ is constructed explicitly in [**BV73**]. Observe that this product is essentially different from the tensor product of modules, though we denote both by the same symbol. The following proposition is [**BV73**, Propositon 6.21].

PROPOSITION 2.32. *For an operad $\mathcal{P}$ and for a $W(\mathcal{K} \otimes \mathcal{P})$-space X, the classifying space BX is a $W\mathcal{P}$-space.*

Boardman and Vogt apply this result iteratively to show that a space X over the little n-cubes operad $\mathcal{C}_n$ admits a classifying space BX (so X has the homotopy type of ΩBX), which admits a classifying space B^2X (so X has the homotopy type of $\Omega^2 B^2 X$), which admits ..., so that ultimately X has the homotopy type of $\Omega^n B^n X$.

To do this, they observe there is a map of operads from $\mathcal{K}$ to the little 1-cubes or *little intervals* operad $\mathcal{C}_1$ which is a homotopy equivalence since both have contractible components indexed by Σ_n. Thus, given an operad $\mathcal{P}$, they can show that a $W(\mathcal{C}_1 \otimes \mathcal{P})$-action on X induces a $W(\mathcal{K} \otimes \mathcal{P})$-action on X and hence a $W\mathcal{P}$-action on BX. Finally there are fairly manifest operad maps $\mathcal{C}_{n-1} \longrightarrow \mathcal{C}_n$ from which more subtle operad maps $\mathcal{C}_1 \otimes \mathcal{C}_{n-1} \longrightarrow \mathcal{C}_n$ can be constructed. This then permits the iteration: X being a $\mathcal{C}_n$-space implies BX is a $\mathcal{C}_{n-1}$-space so BBX exists and is a $\mathcal{C}_{n-2}$-space ...., until B^nX is just a plain, ordinary space, but its existence is what we are after.

The operad maps $\mathcal{C}_{n-1} \longrightarrow \mathcal{C}_n$ allow the definition of $\mathcal{C}_\infty$ as the direct limit, $\mathcal{C}_\infty = \varinjlim \mathcal{C}_n$, so that if X is a $\mathcal{C}_\infty$-space, it has the homotopy type of $\Omega^n B^n X$ for all $n \geq 0$. Thus Theorem 2.4 is established.

Similar results hold for more general E_∞-operads $\mathcal{E}$ by comparison with $W\mathcal{C}_\infty$. Recall that an operad $\mathcal{E}$ is an E_∞-operad if each $\mathcal{E}(n)$, $n \geq 1$, is contractible and the right Σ_n-action is free.

This means that there exists a morphism of operads $\mathcal{E} \longrightarrow \mathcal{M}on$ which is a 'free topological' resolution, though not necessarily a cofibrant one From the cofibrancy of $W\mathcal{C}_\infty$ we know that there is a morphism of operads $W\mathcal{C}_\infty \longrightarrow \mathcal{E}$; see [**Vog**]. For application to identifying infinite loop spaces, we use the *linear isometries operad* $\mathcal{L}i$ presented in Section 2.7. This is an E_∞-operad (Theorem 2.15), thus there exists a map of operads $W\mathcal{C}_\infty \longrightarrow \mathcal{L}i$, therefore each $\mathcal{L}i$-space is also a $W\mathcal{C}_\infty$-space. We conclude:

PROPOSITION 2.33. *Each $\mathcal{L}i$-space is an infinite loop space.*

CHAPTER 3

Algebra

3.1. The cobar complex of an operad

In order to study the homological algebra of operads, in particular, homotopy algebra structures, it is necessary to extend the base category from modules over a commutative ring $\mathbf{k}$, which was the main example in Chapter 1, to the category of differential graded $\mathbf{k}$-modules. To simplify the presentation, we will assume that $\mathbf{k}$ is a field of characteristic zero and restrict our attention to the categories $\mathtt{Vec}$ of $\mathbf{k}$-vector spaces, $\mathtt{gVec}$ of graded $\mathbf{k}$-vector spaces and $\mathtt{dgVec}$ of differential graded $\mathbf{k}$-vector spaces.

In the following sections we define several complexes. The cobar complex $\mathbf{C}(\mathcal{P})$ and the dual dg operad $\mathbf{D}(\mathcal{P})$ are more or less the same, up to operadic suspension (defined below). The Koszul complex $\mathbf{K}(\mathcal{P})$ is important, in part, because the cochain complex of the free $\mathcal{P}$-algebra generated by a vector space V factors into a tensor product of $\mathbf{K}(\mathcal{P})$ and tensor powers of V. The categorical cobar complex $\mathbf{N}(\mathcal{P})$ is the middle link in a quasi-isomorphism between $\mathbf{C}(\mathcal{P})$ and $\mathbf{K}(\mathcal{P})$ in the sense that there are two spectral sequences on $\mathbf{N}(\mathcal{P})$, one reducing at stage $\mathbf{E}_1$ to $\mathbf{C}(\mathcal{P})$, the second one reducing at stage $\mathbf{E}_1$ to $\mathbf{K}(\mathcal{P})$.

The objects (V^*, d) of $\mathtt{dgVec}$ are cochain complexes of vector spaces with $\mathbf{k}$-linear differentials raising degree by 1. The symmetric monoidal structure was defined in (1.2). We use $\otimes$ to denote both $\otimes_{\mathbf{k}}$ and the graded tensor product. The meaning will be clear from the context.

For the sign factor in the symmetry $\sigma(v \otimes w) := (-1)^{vw} w \otimes v$, we will use the notation $(-1)^v := (-1)^{deg(v)}$, $(-1)^{vw} := (-1)^{deg(v)deg(w)}$, etc. There is a potential ambiguity in the symbol $(-1)^{vw}$ since the exponent could be either $deg(v)deg(w)$ or $deg(vw)$. In our convention it will always have the meaning $deg(v)deg(w)$.

The *dual complex* $V^{\#} = (V^{\#}, d^{\#})$ of $V = (V, d)$ is defined by

$$
\begin{aligned}
(V^{\#})^i &:= Hom(V^{-i}, \mathbf{k}), \\
d^{\#}(\alpha) &:= (-1)^{\alpha} \alpha \circ d,
\end{aligned}
\tag{3.1}
$$

where $\alpha \in V^{\#}$ and Hom denotes $Hom_{\mathbf{k}}$. We use the superscript $\#$ to denote the linear dual complex to avoid confusion with $*$ for grading and $\bullet$ for simplicial degree. In general, we will use the symbols $\mathcal{P}$, $\mathcal{Q}$, etc., for operads and the symbols A, B, etc., for Σ-modules.

DEFINITION 3.1. A *differential graded* Σ*-module* (dg Σ-module) is a Σ-module $A := \{A(n)\}_{n \geq 1}$ such that $A(n) \in \mathtt{dgVec}$ for all $n \geq 1$ and the differential $d(n)^i : A(n)^i \longrightarrow A(n)^{i+1}$ is $\mathbf{k}$-linear and Σ_n-equivariant. The component $A(n)^i$ is said to have *internal degree* i. Differential graded Σ-modules form a category $\mathtt{dg}$-Σ-$\mathtt{Mod}$.

The morphisms of dg-Σ-Mod $\varphi : A \to B$ satisfy the condition that for each n, $\varphi(n) : A(n) \to B(n)$ is a Σ_n-equivariant morphism of dgVec.

DEFINITION 3.2 A *differential graded operad* (or *dg operad*) is a differential graded Σ-module with an operad structure for which the operad structure maps are dgVec morphisms. A dg pseudo-operad is a differential graded Σ-module for which the pseudo-operad structure maps $\circ_i$ are dgVec morphisms.

Using the graded tensor product in dgVec, one can extend the monoidal structure on Σ-Mod defined in (1.64) to dg-Σ-Mod by

$$(A \Box B)(n) := \bigoplus_{j=1}^{n} A(j) \otimes_{\Sigma_j} B[j,n].$$

One can also define a dg operad as a monoid in dg-Σ-Mod relative to the $\Box$-monoidal structure.

The remainder of this section uses the definition of the cobar complex in [**GK94**, Section 3.2], but our point of view is slightly different. As a first step in defining the cobar complex of an operad, we need to dualize the $\circ_i$-operations. In general, we will not need the unit axiom so it will be enough to consider pseudo-operads; however, to simplify the statements we will use the generic term "operad" unless it is necessary to make a distinction. Moreover, to make sure that all dual operations are well defined, we consider only operads $\mathcal{P}$ such that $\mathcal{P}(n)$ is of finite type for all n, that is, all the graded components of $\mathcal{P}(n)$ are finite dimensional.

DEFINITION 3.3. Let $\mathcal{P}$ be a dg operad. Dualizing the $\circ_i$-operations defines a family of dg maps on the dual dg Σ-module $\{\mathcal{P}^{\#}(n)\}_{n\geq 1}$. If we define

$$\Delta_i^{n,m} : \mathcal{P}^{\#}(m+n-1) \to \mathcal{P}^{\#}(n) \otimes \mathcal{P}^{\#}(m)$$

by

$$\Delta_i^{n,m}(\lambda)(\alpha \otimes \beta) := \lambda(\alpha \circ_i \beta),$$

then

$$\Delta_i^{n,m}(\lambda) := \sum \lambda_{(i,1)} \otimes \lambda_{(i,2)},$$

where

$$\lambda(\alpha \circ_i \beta) := \sum (-1)^{\alpha\lambda_{(i,2)}} \lambda_{(i,1)}(\alpha)\lambda_{(i,2)}(\beta),$$

for $\lambda \in \mathcal{P}^{\#}(m+n-1)$. The above two equations use Sweedler notation, i.e. the summation index is suppressed.

REMARK 3.4. The assumption that $\mathcal{P}(n)$ is of finite type implies that the (graded) dual of the tensor product is the (graded) tensor product of the (graded) duals $(\mathcal{P}(n) \otimes \mathcal{P}(m))^{\#} \cong \mathcal{P}^{\#}(n) \otimes \mathcal{P}^{\#}(m)$. The operations $\Delta_i^{m,n}$ define the structure of *pseudo-cooperad* with axioms dual to those of a pseudo-operad. We leave the precise formulation of the axioms to the enthusiastic reader.

When the pseudo-operad structure is described in terms of $\mathcal{P}(T)$ as in Theorem 1.73 and Corollary 1.74, the basic operations are the $\circ_x$-operations, one for each tree-morphism $\pi_x : T' \to T'/e_x = T$ such as that appearing in Figure 1.

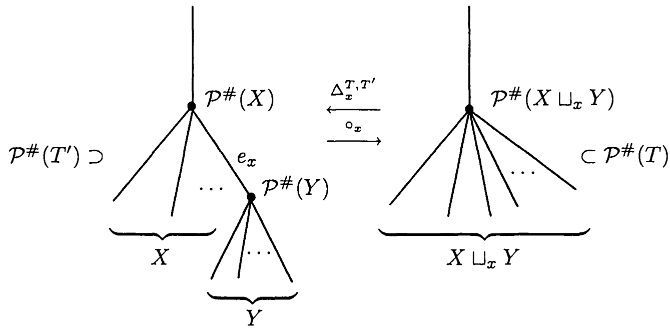


FIGURE 1. The operation $\Delta_x^{T,T'}$ dualizing $\circ_x$.

Recall that $X \sqcup_x Y := (X \sqcup Y) - \{x\}$ (see (1.15)) is the set resulting from replacing the element x of X with the set Y and the pseudo-operad composition $\circ_x$ is a morphism

$$\circ_x : \mathcal{P}(X) \otimes \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \sqcup_x Y).$$

Dualizing this operation in the same way the morphism $\Delta_i^{n,m}$ was defined in Definition 3.3, we have a morphism $\Delta_x^{X,Y} : \mathcal{P}^{\#}(X \sqcup_x Y) \to \mathcal{P}^{\#}(X) \otimes \mathcal{P}^{\#}(Y)$ for arbitrary sets X, Y. Let T' be a tree with an edge e_x such that the input vertex (the one farthest from the root) has incident edges labeled by Y and the output vertex (the one closest to the root) has incident edges labeled by X, $(x \in X)$, i.e. the situation as in Figure 1. Let $T := T'/e_x$. Applying $\Delta_x^{X,Y}$ at vertex $\pi(e_x)$ in the tensor product $\mathcal{P}^{\#}(T)$ (where $\pi(x)$ is the vertex which is created by collapsing the edge e_x) defines an operation:

$$\Delta_x^{T,T'} : \mathcal{P}^{\#}(T) \to \mathcal{P}^{\#}(T'). \tag{3.2}$$

The following result on extending the dg structure on $A(n)$ or $A^{\#}(n)$ follows immediately from the fact that we defined $A(T)$ for any monoidal category, in particular, for the monoidal category of differential graded vector spaces, so $A(T)$ is an object of the same category. The only subtlety here is that $A(T)$ involves the unordered tensor product of $A(In(v))$ over the vertices of T (a colimit over the different simple orderings (see Definition 1.58)). The differential commutes with the symmetry; therefore, it commutes with the symmetric group action permuting factors in the ordered tensor products and hence defines a differential on the colimit $A(T)$.

PROPOSITION 3.5. *If A is a dg Σ-module, then for any tree T the dg structure on A (or $A^{\#}$) extends to $A(T)$ (or $A^{\#}(T)$) and the $\Delta_x^{T,T'}$ in equation (3.2) are dg maps.*

Recall (Section 1.5) that for any tree T, *edge*(T) denotes the set of internal edges of T. We also denoted the number of internal edges of T by $|T| := card(edge(T))$. For any $A \in \mathtt{dgVec}$, the suspension is defined by

$$(\uparrow A)^i := A^{i-1}.$$

DEFINITION 3.6. For any finite set S, let $\uparrow \mathbf{k}^S$ be the free $\mathbf{k}$-module with basis S considered as a dg vector space concentrated in degree 1 and define the one-dimensional dg vector space concentrated in degree $|S|$:

$$det(S) = det(\mathbf{k}^S) := \wedge^{|S|}(\uparrow \mathbf{k}^S).$$

We may interpret $\mathbf{k}^S$ also as the space of $\mathbf{k}$-valued functions on S, therefore $det(\mathbf{k}^S)$ is 'contravariant in S.' Define

$$det(T) := \uparrow det(\mathbf{k}^{edge(T)}).$$

Thus $det(T)$ is concentrated in degree $|T|+1$. An ordering $(e_1, e_2, \dots, e_{|T|})$ of the edges of T determines a basis element $e_1 \wedge e_2 \wedge \cdots \wedge e_{|T|} \in det(T)$. Since the correspondence $T \mapsto edge(T)$, $f \mapsto f^*$, is a contravariant functor on the category $\mathtt{Tree}$ of rooted labeled trees (see Section 1.5), the correspondence $T \mapsto det(T)$ extends to a covariant functor on the category $Iso(\mathtt{Tree})$ of isomorphisms of $\mathtt{Tree}$.

REMARK 3.7. The degree shift in the definition of $det(T)$ can be explained as follows. Let $eedge(T)$ (extended set of edges) be the set of internal edges of T plus the root edge. What we really want in definitions below is $det(T)$ defined as the determinant of the set $eedge(T)$, that is, $det(T) = det(eedge(T))$. But, since the root of a rooted tree is always 'marked,' there are canonical isomorphisms

$$\begin{aligned} det(eedge(T)) &\cong det(edge(T) \sqcup \{\text{the root edge of } T\}) \\ &\cong det(edge(T)) \otimes det(\{\text{the root edge of } T\}) \cong \uparrow det(edge(T)), \end{aligned}$$

therefore the definition of $det(T)$ based on the extended set of edges would be the same as the former one.

The *cobar bicomplex* of an operad $\mathcal{P}$ in arity n (operad-degree n) is a double complex

$$\mathbf{C}(\mathcal{P})(n)^{*,*} = \bigoplus_{i \geq 1, j \in \mathbb{Z}} \mathbf{C}(\mathcal{P})(n)^{i,j}$$

with

$$\mathbf{C}(\mathcal{P})(n)^{i,j} = \operatorname*{colim}_{\substack{T \in Iso(\mathtt{Rtree}_n) \\ |T|+1=i}} \mathcal{P}^{\#}(T)^j \otimes det(T),$$

where $\mathtt{Rtree}_n$ is the category of reduced rooted labeled n-trees; compare the definition of the free pseudo-operad (1.47). We call i the *tree degree* and j the *internal degree*.

The cobar differential will consist of two pieces: the tree differential and the internal differential.

The *tree differential* is a sum $\delta := \sum \delta^i$, where the ith degree component

$$\begin{aligned} \delta^i : \mathbf{C}(\mathcal{P})^{i,*} = &\operatorname*{colim}_{|T|+1=i} \mathcal{P}^{\#}(T) \otimes det(T) \\ \longrightarrow &\operatorname*{colim}_{|T'|+1=i+1} \mathcal{P}^{\#}(T') \otimes det(T') = \mathbf{C}(\mathcal{P})(n)^{i+1,*} \end{aligned} \tag{3.3}$$

is the colimit of its matrix components

$$\delta_{T,T'} := \begin{cases} \Delta_e^{T,T'} \otimes e \wedge (-), & \text{if } T = T'/e, \text{ and} \\ 0, & \text{if } T \not\cong T'/e, \end{cases} \tag{3.4}$$

where $e \wedge (-)$ is the map

$$e \wedge (-) : det(T) \to e \wedge det(T) \cong det(T')$$

and $\Delta_e^{T,T'}$ was introduced in (3.2). The condition $\delta^2 = 0$ follows immediately from dualizing the associativity identities for $\circ_x$. The tree differential gives rise to what we will call the *δ-complex*

$$\mathcal{P}^{\#}(n)^* \xrightarrow{\delta} \operatorname*{colim}_{|T|=1} \mathcal{P}^{\#}(T)^* \otimes det(T) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \operatorname*{colim}_{|T|=n-2} \mathcal{P}^{\#}(T)^* \otimes det(T), \tag{3.5}$$

where the component corresponding to trees T with $|T| = i$ has tree degree $i+1$ and the condition $T \in Iso(\mathtt{Rtree}_n)$ is implicit in all the colimits.

The *internal differential* $\delta_{\mathcal{P}}$ has components

$$\delta_{\mathcal{P}}^j = d_{\mathcal{P}^{\#}}^j \otimes 1\!\!1 : \mathcal{P}^{\#}(T)^j \otimes det(T) \longrightarrow \mathcal{P}^{\#}(T)^{j+1} \otimes det(T), \tag{3.6}$$

where $d_{\mathcal{P}^{\#}}^j : \mathcal{P}^{\#}(T)^j \to \mathcal{P}^{\#}(T)^{j+1}$ is the induced differential; see Proposition 3.5.

REMARK 3.8. The precise meaning of the 'matrix components' (3.4) is the following. Suppose that there is some $T' \in \mathtt{Rtree}_n$ and $e \in edge(T)$ such that $f : S \xrightarrow{\cong} T'/e = T$ and $g : S' \xrightarrow{\cong} T'$ for some isomorphisms f and g in $\mathtt{Rtree}_n$. Then we define

$$\delta_{S,S'} := [\mathcal{P}^{\#}(g^{-1})\Delta_e^{T,T'}\mathcal{P}^{\#}(f)] \otimes [det(g^{-1})(e \wedge (-))det(f)]. \tag{3.7}$$

Otherwise we put $\delta_{S,S'} := 0$. The first line of (3.4) is (3.7) for $S = T'/e$ and $S' = T'$; (3.7) is then the unique extension of (3.4) compatible with the colimits. Equation (3.6) has the similar obvious meaning.

DEFINITION 3.9. The *cobar complex* of an operad $\mathcal{P}$ is the differential graded Σ-module $\mathbf{C}(\mathcal{P}) := \{\mathbf{C}(\mathcal{P})(n)\}_{n\geq 1}$ with arity one component set equal to 0,

$$\mathbf{C}(\mathcal{P})(1) := 0$$

and the arity n component for $n \geq 2$ given by the total complex of the cobar bicomplex in arity n

$$(\mathbf{C}(\mathcal{P})(n)^{*,*}, d := \delta + (-1)^{|T|}\delta_{\mathcal{P}}),$$

where δ is the tree differential and $\delta_{\mathcal{P}}$ is the internal differential defined above.

In sections 3.4, 3.5 and 3.6 we use another representation of $\mathbf{C}(\mathcal{P})$, as the direct sum over representatives of isomorphism classes of labeled trees,

$$\mathbf{C}(\mathcal{P}) = \bigoplus_{T \in Tree(n)} \mathcal{P}^{\#}[T] \otimes det(T); \tag{3.8}$$

compare Proposition 1.82 and formula (1.52).

DEFINITION 3.10. Given $A \in$ dg-Σ-Mod, the *reduced suspension* $\uparrow A$ is the dg Σ-module defined by

$$(\uparrow A)(n) := \begin{cases} 0, & \text{if } n = 1, \text{ and} \\ \uparrow(A(n)), & \text{if } n \geq 2. \end{cases}$$

This definition implies that the free pseudo-operad $\Psi(\uparrow A)$ on the Σ-module $\uparrow A$ is a colimit over *reduced* trees. We use the language of pseudo-operads here to emphasize that we have set the arity one components equal to zero in the construction of $\mathbf{C}(\mathcal{P})$.

THEOREM 3.11. *For each arity $n \geq 2$, there is an isomorphism of dg vector spaces between the cobar complex graded by total degree with differential $\delta_{\mathcal{P}}$ (ignoring the tree differential) and the free pseudo-operad $\Psi(\uparrow \mathcal{P}^{\#})$ on the dg Σ-module $\uparrow \mathcal{P}^{\#}$ with differential $d_{\mathcal{P}^{\#}}$ induced from the differential on $\mathcal{P}$:*

$$\mathbf{C}(\mathcal{P})(n) \cong \Psi(\uparrow \mathcal{P}^{\#})(n). \tag{3.9}$$

PROOF. It is clear that if we ignore the factors $det(T)$ and degrees, then $(\mathbf{C}(\mathcal{P})(n), \delta_{\mathcal{P}})$ is isomorphic to $(\Psi(\mathcal{P}^{\#})(n), d_{\mathcal{P}^{\#}})$. The following lemma ([**GK94**, Lemma 3.2.9]) describes what happens when we include $det(T)$ and the grading in $\mathcal{P}$.

LEMMA 3.12. *For any finite set I of m elements and $W_i \in$ `dgVec`, $i \in I$, there is a canonical isomorphism*

$$\varphi : \bigotimes_{i \in I} (\uparrow W_i) \to \bigotimes_{i \in I} W_i \otimes det(\mathbf{k}^I).$$

For $A \in$ `dg-`Σ`-Mod`, there is a canonical isomorphism

$$(\uparrow A)(T) \cong A(T) \otimes det(T) \tag{3.10}$$

of covariant functors on Iso(`Tree`).

PROOF. See the appendix to this section. ■

COROLLARY 3.13. *The cobar complex $\mathbf{C}(\mathcal{P})$, with the pseudo-operad structure defined by isomorphism (3.9) in Theorem 3.11 and differential $d = \delta + (-1)^{|T|}\delta_{\mathcal{P}}$, is a dg operad. It defines a contravariant functor $\mathbf{C}$: `dg`Ψ`Op` $\longrightarrow$ `dg`Ψ`Op` which preserves quasi-isomorphisms.*

PROOF. The operad composition on $\mathbf{C}(\mathcal{P})$ comes from identification (3.9) with $\Psi(\uparrow \mathcal{P}^{\#})$. It follows immediately from the definition, which uses the grafting of trees (cf. the proof of Proposition 1.78), that both the tree differential and the internal differential are derivations relative to the $\circ_i$-operations. The functor $\mathbf{C}$ preserves quasi-isomorphisms because the same is true of the functor $\mathcal{P} \longrightarrow \mathcal{P}^{\#}(T)$ for any tree T. ■

REMARK 3.14. We constructed $\mathbf{C}(\mathcal{P})$ in such a way that it 'ignores' the component $\mathcal{P}(1)$. More precisely, recall that, given $\mathcal{P}$, the pseudo-operad $\mathcal{P}^{+}$ was defined in (2.10) by

$$\mathcal{P}^{+}(n) := \begin{cases} \mathcal{P}(n), & \text{if } n \geq 2, \text{ and} \\ 0, & \text{for } n = 1. \end{cases}$$

Then $\mathbf{C}(-)$ factors through the functor $\mathcal{P} \mapsto \mathcal{P}^{+}$ defined above. Our approach differs slightly from the one of [**GK94**] based on K-collections

The complex $\mathbf{D}(\mathcal{P})$ which we define below is a form of the cobar complex with cochain degree shifted by $-n+1$ so that the maximal cochain degree is normalized to be 0. Both complexes $\mathbf{C}(\mathcal{P})$ and $\mathbf{D}(\mathcal{P})$ are necessary for the study of quadratic operads and the Koszul property (the topic of Section 3.3). The construction of $\mathbf{D}(\mathcal{P})$ uses the notions of operadic suspension and desuspension, which will be important in a number of other contexts as well.

DEFINITION 3.15. For any dg Σ-module A, the *operadic suspension* $\mathfrak{s}A$ is the dg Σ-module whose arity n component is defined by

$$\mathfrak{s}A(n) := \uparrow^{n-1} A(n) \otimes \mathrm{sgn}_n.$$

The *operadic desuspension* $\mathfrak{s}^{-1}A$ is defined analogously:

$$\mathfrak{s}^{-1}A(n) := \downarrow^{n-1} A(n) \otimes \mathrm{sgn}_n.$$

It is easy to see that if $\mathcal{P}$ is an operad, then the Σ-modules $\mathfrak{s}\mathcal{P}$ and $\mathfrak{s}^{-1}\mathcal{P}$ have the operadic composition naturally induced from the operadic composition on $\mathcal{P}$.

LEMMA 3.16. *The operadic desuspension of the endomorphism operad is isomorphic to the endomorphism operad of the suspension:*

$$\mathfrak{s}^{-1}\mathcal{E}nd_V \cong \mathcal{E}nd_{\uparrow V}.$$

PROOF. The isomorphism of the lemma is a consequence of the following sequence of isomorphisms and identifications

$$\begin{aligned}
\mathcal{E}nd^i_{\uparrow V}(n) &= Hom^i\left((\uparrow V)^{\otimes n}, \uparrow V\right) = \bigoplus_{j\in\mathbb{Z}} Hom\left(\left((\uparrow V)^{\otimes n}\right)^j, (\uparrow V)^{i+j}\right) \\
&\cong \bigoplus_{j\in\mathbb{Z}} Hom((V^{\otimes n})^{j-n}, V^{i+j-1}) \otimes \mathrm{sgn}_n = Hom^{i+n-1}(V^{\otimes n}, V) \otimes \mathrm{sgn}_n \\
&= \mathcal{E}nd_V^{i+n-1}(n) \otimes \mathrm{sgn}_n \cong (\mathfrak{s}^{-1}\mathcal{E}nd_V(n))^i.
\end{aligned}$$

■

REMARK 3.17. It is not surprising that $\mathfrak{s}^{-1}\mathcal{E}nd_V(n)$ and $\mathcal{E}nd_{\uparrow V}(n)$ are isomorphic as vector spaces. What is surprising is that the signum representation arises from the commutation relations between the suspension and $\otimes$. This calculation explains why we need sgn in the definition of the operadic suspension.

DEFINITION 3.18. The *dual dg pseudo-operad* $\mathbf{D}(\mathcal{P})$ is the operadic desuspension of $\mathbf{C}(\mathcal{P})$:

$$\mathbf{D}(\mathcal{P}) := \mathfrak{s}^{-1}\mathbf{C}(\mathcal{P}).$$

REMARK 3.19. Theorem 3.24 below shows that by iterating the functor $\mathcal{P} \longmapsto \mathbf{D}(\mathcal{P})$ we get a 'standard resolution' of $\mathcal{P}$. Moreover, we will see in Theorem 3.39 that $\mathbf{D}(\mathcal{P})$ is also related to the 'quadratic dual operad,' hence the name dual dg pseudo-operad. The next proposition and the remarks that follow describe the structure of $\mathbf{D}(\mathcal{P})$ in a little more detail.

PROPOSITION 3.20. *The suspension functor and the free pseudo-operad functor commute up to a natural transformation, that is, there is a natural transformation of functors from* $\mathtt{dg}\text{-}\Sigma\text{-}\mathtt{Mod}$ *to* $\mathtt{dg}\Psi\mathtt{Op}$*:*

$$\Phi : \mathfrak{s} \circ \Psi \longrightarrow \Psi \circ \mathfrak{s}.$$

The proof is given in the appendix to this section. Putting Definition 3.18, Proposition 3.20 and Theorem 3.11 together, we conclude

$$\mathbf{D}(\mathcal{P}) := \mathfrak{s}^{-1}\mathbf{C}(\mathcal{P}) \cong \mathfrak{s}^{-1}\Psi(\uparrow\mathcal{P}^{\#}) \cong \Psi(\mathfrak{s}^{-1}\uparrow\mathcal{P}^{\#}). \tag{3.11}$$

REMARK 3.21. Ginzburg and Kapranov **[GK94]** construct $\mathbf{D}(\mathcal{P})$ using what they call the determinant operad, which is defined as follows.

DEFINITION 3.22. The *determinant operad* $\bigwedge$ is the desuspension of the commutative operad,

$$\bigwedge := \mathfrak{s}^{-1}\mathcal{C}om.$$

In other words, the arity n component $\bigwedge(n)$ is a one-dimensional graded vector space concentrated in internal degree $1-n$ carrying the signum representation of Σ_n.

The tensor product of operads $\mathbf{C}(\mathcal{P})\otimes\mathcal{C}om$ defined by $(\mathbf{C}(\mathcal{P})\otimes\mathcal{C}om)(n) := \mathbf{C}(\mathcal{P})(n)\otimes\mathcal{C}om(n)$ is naturally isomorphic to $\mathbf{C}(\mathcal{P})$. Therefore, since tensoring with $\mathcal{C}om$ followed by $\mathfrak{s}^{-1}$ is equivalent to tensoring with $\mathfrak{s}^{-1}\mathcal{C}om$,

$$\mathbf{D}(\mathcal{P}) = \mathfrak{s}^{-1}\mathbf{C}(\mathcal{P}) \cong \mathfrak{s}^{-1}(\mathbf{C}(\mathcal{P})\otimes\mathcal{C}om) \cong \mathbf{C}(\mathcal{P})\otimes\mathfrak{s}^{-1}\mathcal{C}om \cong \mathbf{C}(\mathcal{P})\otimes\bigwedge.$$

The last term is the definition of $\mathbf{D}(\mathcal{P})$ in **[GK94]**.

DEFINITION 3.23. Recall that $|T| = card(edge(T))$, the number of internal edges of the tree T. Recall also that $Edg(T)$ denotes the set of all edges of T excluding the root edge. Let $L(T) = card(Leaf(T))$ and $E(T) := card(Edg(T)) = L(T)+|T|$. Define

$$Det(T) := \downarrow^{2L(T)-2} \wedge^{E(T)}(\uparrow\mathbf{k}^{Edg(T)}).$$

Thus $Det(T)$ is a graded vector space of dimension 1 concentrated in (non-positive) degree $|T|+2-L(T)$, which is the degree of the 'operadic desuspension' of $det(T)$. With this definition, the complex $\mathbf{D}(\mathcal{P})$ relative to the tree differential δ is

$$\mathcal{P}^{\#}(n) \xrightarrow{\delta} \operatorname*{colim}_{|T|=1} \mathcal{P}^{\#}(T)\otimes Det(T) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \operatorname*{colim}_{|T|=n-2} \mathcal{P}^{\#}(T)\otimes Det(T).$$

The component $\mathcal{P}^{\#}(T)\otimes Det(T)$ corresponding to a tree T has *tree degree* $|T|+2-L(T)$, so the complex is concentrated in tree $(2-L(T),\dots,0)$. The *internal degree* is induced by the grading on the $\mathcal{P}^{\#}(n)$ and the usual rule for the tensor products appearing in $\mathcal{P}^{\#}(T)$. The Σ_n-action is described in more detail in the appendix to this section.

THEOREM 3.24. *If $\mathcal{P}$ is a dg pseudo-operad such that each $\mathcal{P}(n)$ is of finite type, then there is a canonical quasi-isomorphism $\mathbf{D}(\mathbf{D}(\mathcal{P})) \longrightarrow \mathcal{P}^{+}$, where $\mathcal{P}^{+}$ was defined in (2.10).*

PROOF. See the appendix to this section.

It follows from this theorem and Theorem 3.11 that $\mathbf{D}(\mathbf{D}(\mathcal{P}))$ is a free resolution of $\mathcal{P}^{+}$. It is an algebraic equivalent of the W-construction of Boardman and Vogt **[BV73]** which we recalled in Section 2.8. This becomes clear in the course of the proof given in the appendix, where it is shown that $\mathbf{D}(\mathbf{D}(\mathcal{P}))$ is indexed by trees with vertices labeled by elements of $\mathcal{P}$, and with edges of two types: one type (which we will call primary edges) arising from the first application of the functor

$\mathbf{D}$ and the other type (which we will call secondary edges) arising from the second application of the functor $\mathbf{D}$. Note that in Figure 3 in the appendix, the secondary edges are those which appear inside the circles.

The differential in $\mathbf{D}(\mathbf{D}(\mathcal{P}))$ has three components. The first component comes from the differential in the dg operad $\mathcal{P}$ (internal differential), the second component is a sum of terms each of which involves collapsing a secondary edge and composing the labels on the adjacent vertices using the composition law for the operad $\mathcal{P}$ and the last component of the differential is a sum of terms each of which involves relabeling a secondary edge as a primary edge.

In the W-construction recalled in Section 2.8 the metric edges are considered as cells of dimension 1 and trees with m metric edges label the m-cells (an m-fold product of 1-cells). If an edge is labeled with $t = 0$ or $t = 1$, the corresponding product represents two $(m-1)$-dimensional faces, but the relations in the W-construction imply that the edges with $t = 0$ are collapsed and the labels on the adjacent vertices are combined by operad composition, giving rise to a tree with $m-1$ metric edges.

Given a cellular (topological) operad $\mathcal{Q}$, let $C_*(\mathcal{Q})$ be its operad of cellular chains. Let us see what happens in the preceding paragraph if $\mathcal{P} = C_*(\mathcal{Q})$. For any tree indexing a component of $\mathbf{D}(\mathbf{D}(C_*(\mathcal{Q})))$, define a corresponding metric tree (cell) in the W-construction on $\mathcal{Q}$ with a metric edge for each secondary edge and an edge with label $t = 1$ for each primary edge. Using this correspondence one can define a map from $\mathbf{D}(\mathbf{D}(C_*(\mathcal{Q})))$ to $C_*(W(\mathcal{Q}))$. The internal differential on the complex $C_*(W(\mathcal{Q}))$, that is, the one induced by the differential on $C_*(\mathcal{Q})$, is the same as the internal differential on $\mathbf{D}(\mathbf{D}(C_*(\mathcal{Q})))$. The differential given by the sum of the second and third components of the differential in $\mathbf{D}(\mathbf{D}(C_*(\mathcal{Q})))$, as described in the proof, corresponds to the differential on the cellular chains arising from the cells given by metric trees in the W-construction because the relations in the W-construction imply that assigning the value $t = 0$ to a metric edge collapses that edge and composes the labels on the adjacent vertices by the operad law.

3.1.1. Appendix. The technical lemmas proved in this appendix show that the symmetries in dgVec (in particular, the signs) are compatible with the various maps. They are then used to prove Theorem 3.11.

Proof of Lemma 3.12. If we choose an ordering $\{i_1, \dots, i_m\}$ of I and represent $\bigotimes_{i\in I} W_i$ as $W_{i_1} \otimes \cdots \otimes W_{i_m}$, then we can define

$$\varphi : \bigotimes_{1\leq j\leq m} \uparrow W_{i_j} \to \bigotimes_{1\leq j\leq m} W_{i_j} \otimes det(\mathbf{k}^I)$$

by

$$\varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m) = (-1)^{\sum_{1\leq i\leq m} i w_i}(w_1 \otimes \cdots \otimes w_m) \otimes (i_1 \wedge \cdots \wedge i_m), \tag{3.12}$$

where $\uparrow w_j \in \uparrow W_{i_j}$ and $w_j \in W_{i_j}$ are corresponding elements and, in our notation, $(-1)^w := (-1)^{deg(w)}$. By the definition of $det(\mathbf{k}^I)$, φ preserves degree. The following equations show that φ commutes with the differentials:

$$\begin{aligned}
&\varphi(d(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m)) \\
&\quad = \sum_{1\le j\le m} \epsilon_1 \varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow dw_j \otimes \cdots \otimes \uparrow w_m) \\
&\quad = \sum_{1\le j\le m} \epsilon_2 (w_1 \otimes \cdots \otimes dw_j \otimes \cdots \otimes w_m) \otimes (i_1 \wedge \cdots \wedge i_m) = d\varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m).
\end{aligned}$$

The sign factors above are

$$\begin{aligned}
\epsilon_1 &= (-1)^{(\sum_{i<j}(w_i+1)+1)} \text{ and} \\
\epsilon_2 &= (-1)^{(\sum_{i\neq j} iw_i + j(w_j+1) + \sum_{i<j} w_i + j)} = (-1)^{(\sum iw_i + \sum_{i<j} w_i)}.
\end{aligned}$$

Next we show that φ commutes with $\sigma \in \Sigma_m$ and hence is well defined independent of the choice of ordering of I. It is enough to consider σ_j transposing factors w_j, w_{j+1},

$$\begin{aligned}
&\varphi(\sigma_j(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m)) \\
&\quad = \epsilon_1 \varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_{j+1} \otimes \uparrow w_j \otimes \cdots \otimes \uparrow w_m) \\
&\quad = \epsilon_2 (w_1 \otimes \cdots \otimes w_{j+1} \otimes w_j \otimes \cdots \otimes w_m) \otimes (i_1 \wedge \cdots \wedge i_{j+1} \wedge i_j \wedge \cdots \wedge i_m) \\
&\quad = \sigma_j \left(\epsilon_3 (w_1 \otimes \cdots \otimes w_j \otimes w_{j+1} \otimes \cdots \otimes w_m)\right) \otimes (i_1 \wedge \cdots \wedge i_j \wedge i_{j+1} \wedge \cdots \wedge i_m) \\
&\quad = (\sigma_j \otimes id)\varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m).
\end{aligned}$$

The signs are given as follows

$$\begin{aligned}
\epsilon_1 &= (-1)^{(w_j+1)(w_{j+1}+1)}, \\
\epsilon_2 &= (-1)^{\sum iw_i - w_{j+1} + w_j + (w_j+1)(w_{j+1}+1)} = (-1)^{(\sum iw_i + w_j w_{j+1} + 1)} \text{ and} \\
\epsilon_3 &= (-1)^{\sum iw_i}.
\end{aligned}$$

The extra term $-w_{j+1}+w_j$ in the exponent in the third line comes from the reversed order of w_j and w_{j+1}. Therefore φ commutes with all permutations and generates a morphism of the colimit defining the unordered tensor product.

To prove isomorphism (3.10), note that there is a canonical identification of $\bigwedge^{|T|+1}(\mathbf{k}^{Vert(T)})$ and $\bigwedge^{|T|}(\mathbf{k}^{edge(T)})$ coming from the ordered splitting

$$\mathbf{k}^{Vert(T)} \cong \mathbf{k}^{edge(T)} \oplus \mathbf{k}\iota_{\text{root}},$$

where a vertex corresponds to its outgoing edge and ι_{root} is the basis element of $\mathbf{k}^{Vert(T)}$ corresponding to the root vertex. Then there is a dg isomorphism

$$(\uparrow A)(T) \cong A(T) \otimes det(\mathbf{k}^{Vert(T)}) \cong A(T) \otimes \uparrow det(\mathbf{k}^{edge(T)}) = A(T) \otimes det(T). \qquad \blacksquare$$

The proof of Proposition 3.20 uses a lemma, for which we need some preliminary definitions and reminders. For the rest of the appendix, we return to the notation (T, ℓ) for a labeled tree. Recall that in Proposition 1.82 we showed that the arity n component of the free pseudo-operad generated by a Σ-module A can be represented as a direct sum over isomorphism classes of (unlabeled) trees with n-leaves:

$$\Psi(A)(n) = \bigoplus \mathcal{A}[T],$$

where $\mathcal{A}[T]$ was defined in Definition 1.81. In Proposition 1.87 we then proved the isomorphism

$$\mathcal{A}[T] \cong A(T,\ell) \otimes_{\Sigma(T,\ell)} \mathbf{k}[\Sigma_n]$$

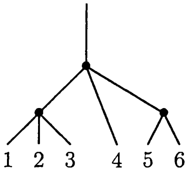


FIGURE 2. Labeling leaves of a planar tree.

of Σ_n-modules. Define

$$\mathfrak{s}(\mathcal{A}[T]) := \uparrow^{n-1} \mathcal{A}[T] \otimes \mathrm{sgn}_n,$$

and

$$(\mathfrak{s}\mathcal{A})[T] := (\mathfrak{s}\mathcal{A})(T,\ell) \otimes_{\Sigma(T,\ell)} \mathbf{k}[\Sigma_n].$$

In other words, $(\mathfrak{s}\mathcal{A})[T]$ is $\mathcal{A}[T]$ derived from the Σ-module $\mathfrak{s}A$ instead of A.

LEMMA 3.25. *For each Σ-module A, there is a Σ_n-equivariant isomorphism*

$$\Phi_{A,T} : \mathfrak{s}(\mathcal{A}[T]) \longrightarrow (\mathfrak{s}\mathcal{A})[T].$$

The correspondence $A \mapsto \Phi_{A,T}$ is a natural transformation of functors $A \mapsto \mathfrak{s}(\mathcal{A}[T])$ and $A \mapsto (\mathfrak{s}\mathcal{A})[T]$ on dg-Σ-Mod.

PROOF. Let $\mathfrak{s}(A(T,\ell)) := \uparrow^{n-1} A(T,\ell) \otimes \mathrm{sgn}_n$, where sgn_n in fact means here the restriction of the signum representation to the subgroup $\Sigma(T,\ell) \subset \Sigma_n$. Initially, we define

$$\Psi_{A,T,\ell} : \mathfrak{s}A(T,\ell) \to (\mathfrak{s}A)(T,\ell) \tag{3.13}$$

by induction on the height of the tree T. Then we will show that $\Psi_{A,T,\ell}$ is equivariant relative to $\Sigma(T,\ell)$, and therefore, extends to a well-defined map $\mathfrak{s}(\mathcal{A}[T]) \to (\mathfrak{s}\mathcal{A})[T]$. The most convenient way to present $A(T,\ell)$ is to choose as a representative of the equivalence class $[T,\ell]$ a planar tree with leaves labeled from left to right in the standard counterclockwise planar orientation, as indicated in Figure 2.

For a tree of height one, that is, a corolla, the assertions of the lemma simply restate the definition of the functor $\mathfrak{s}$. Suppose that we have constructed the map of (3.13) for trees of height m and we want to construct $\Psi_{A,T,\ell}$ for trees of height $m+1$. Any labeled tree (T,ℓ) of height $m+1$ is the result of grafting labeled trees $(T_1,\ell_1),\dots,(T_k,\ell_k)$ of height at most m to a corolla $t(k)$. We represent the labeled tree (T,ℓ) by

$$(T,\ell) \cong (\cdots(t(k) \circ_k (T_k,\ell_k)) \circ_{k-1} \cdots) \circ_1 (T_1,\ell_1),$$

where the $\circ_i$-operation on trees is grafting along the ith leaf. Then $A(T,\ell)$ can be represented

$$A(T,\ell) \cong A(k) \otimes_G (A(T_1,\ell_1) \otimes \cdots \otimes A(T_k,\ell_k)), \tag{3.14}$$

where G is the subgroup of Σ_k which permutes isomorphic trees in the sequence $(T_1,\dots,T_k)$; see (1.14). Under this identification, the image $\Sigma(T,\ell)$ of the tree automorphism group in Σ_n is the normalizer of $\Sigma(T_1,\ell_1) \times \cdots \times \Sigma(T_k,\ell_k)$ in Σ_n.

The map $\Psi_{A,T,\ell}$ is defined inductively as the composition of two maps. If $\wedge_k$ is a basis element for the one-dimensional signum representation of Σ_k, then

$$\Psi_{A,T,\ell} := (\mathbb{1} \otimes_G \Psi_{A,T_1,\ell_1} \otimes \cdots \otimes \Psi_{A,T_k,\ell_k}) \circ \Psi'_{A,T}$$

where

$$\Psi'_{A,T} : \mathfrak{s}(A(T)) \to (\mathfrak{s}A)(k) \otimes_G \mathfrak{s}(A(T_1)) \otimes \cdots \otimes \mathfrak{s}(A(T_k))$$

is defined by

$$\begin{aligned}\Psi'_{A,T}(\uparrow^{n-1} (\alpha_0 \otimes \cdots \otimes \alpha_k \otimes \wedge_n)) \\ .= \quad (-1)^{\xi(\uparrow^{n-1}\overline{\alpha})}(\uparrow^{k-1} \alpha_0 \tau \otimes \wedge_k) \otimes \cdots \otimes (\uparrow^{m_k-1} \alpha_k \otimes \wedge_{m_k}),\end{aligned}$$

for $\overline{\alpha} .= \alpha_0 \otimes \cdots \otimes \alpha_k \otimes \wedge_n$ and the exponent of the sign factor (coming from moving $\uparrow$ across the terms of the tensor product) is:

$$\xi(\uparrow^{n-1}\overline{\alpha}):=(m_1-1)|\alpha_0| + (m_2-1)(|\alpha_0| + |\alpha_1|) + \cdots + (m_k-1)(|\alpha_0| + \cdots + |\alpha_{k-1}|).$$

The equivariance of the Ψ_{A,T_i,ℓ_i} follows from the induction hypothesis; therefore, the $\Sigma(T,\ell)$-equivariance of the composition $\Psi_{A,T,\ell}$ is proved once we have proved the equivariance of $\Psi'_{A,T}$.

We will prove the equivariance of $\Psi'_{A,T}$ when T has height 2. The proof for the general inductive step from height m to height $m+1$ is completely analogous.

If T has height 2 and is the result of grafting k corollae with $m_1, \ldots, m_k$ leaves respectively to a corolla with k leaves, the group $\Sigma(T,\ell)$ is the normalizer of the subgroup $\Sigma_{m_1} \times \cdots \times \Sigma_{m_k} \subset \Sigma_n$ and is generated by two types of elements. The first type of element is in the image of the imbeddings $\iota_i : \Sigma_{m_i} \hookrightarrow \Sigma_n$. The second type of element transposes two adjacent intervals of integers of the same length m_i.

The elements $\hat{\sigma} := \iota_i(\sigma)$ of the first type act on the tensor product $A(k) \otimes A(m_1) \otimes \cdots \otimes A(m_k)$ according to the action of Σ_{m_i} in the ith factor (counting $A(k)$ as the 0th factor). The elements ρ of the second type act by transposing adjacent factors $A(m_i) = A(m_{i+1})$ while at the same time acting on $A(k)$ as the transposition of i and $i+1$. Clearly $\Psi'_{A,T}$ is equivariant with respect to the action of the first type of elements, since

$$\mathrm{sgn}_n|_{\Sigma_{m_i}} = \mathrm{sgn}_{m_i}.$$

We need only verify that it is equivariant with respect to the action of the second type. If $m_i = m_{i+1}$, let ρ be the transposition of intervals i and $i+1$ and $\tau = (i, i+1) \in \Sigma_k$ be the transposition of $i, i+1$. Then

$$\begin{aligned}&\Psi'_{A,T}\left(\uparrow^{n-1} (\alpha_0 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_k \otimes \wedge_n)\rho\right) \\ &= \Psi'_{A,T}\left(\mathrm{sgn}_n(\rho)(-1)^{\alpha_i\alpha_{i+1}} \uparrow^{n-1} (\alpha_0\tau \otimes \cdots \otimes \alpha_{i+1} \otimes \alpha_i \otimes \cdots \otimes \alpha_k \otimes \wedge_n)\right) \\ &= (-1)^{\xi(\uparrow^{n-1}\overline{\alpha}')+\alpha_i\alpha_{i+1}+m_i^2}(\uparrow^{k-1} \alpha_0\tau \otimes \wedge_k) \otimes \cdots \\ &\quad \cdots \otimes (\uparrow^{m_i-1} \alpha_{i+1} \otimes \wedge_{m_i}) \otimes (\uparrow^{m_i-1} \alpha_i \otimes \wedge_{m_i}) \otimes \cdots \otimes (\uparrow^{m_k-1} \alpha_k \otimes \wedge_{m_k}).\end{aligned}$$

The new terms in the exponent of the coefficient come from the substitution $\mathrm{sgn}_n(\rho) = (-1)^{m_i^2}$ and the transposition of α_i and α_{i+1}. The expression $\overline{\alpha}'$ is the tensor product with factors α_i and α_{i+1} transposed. On the other hand

$$\begin{aligned}
&\Psi'_{A,T}(\uparrow^{n-1} (\alpha_0 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_k \otimes \wedge_n))\rho \\
&= \epsilon_1 \left((\uparrow^{k-1} \alpha_0 \otimes \wedge_k) \otimes \cdots \otimes (\uparrow^{m_i-1} \alpha_{i+1} \otimes \wedge_{m_i}) \otimes (\uparrow^{m_i-1} \alpha_i \otimes \wedge_{m_i}) \otimes \right.\\
&\qquad\qquad \left. \cdots \otimes (\uparrow^{m_k-1} \alpha_k \otimes \wedge_{m_k})\right) \rho \\
&= \epsilon_2 (\uparrow^{k-1} \alpha_0 \otimes \wedge_k)\tau \otimes \cdots \otimes (\uparrow^{m_i-1} \alpha_{i+1} \otimes \wedge_{m_i}) \otimes (\uparrow^{m_i-1} \alpha_i \otimes \wedge_{m_i}) \otimes \\
&\qquad\qquad \cdots \otimes (\uparrow^{m_k-1} \alpha_k \otimes \wedge_{m_k}) \\
&= \epsilon_3 (\uparrow^{k-1} \alpha_0 \tau \otimes \wedge_k) \otimes \cdots \otimes (\uparrow^{m_i-1} \alpha_{i+1} \otimes \wedge_{m_i}) \otimes (\uparrow^{m_i-1} \alpha_i \otimes \wedge_{m_i}) \otimes \\
&\qquad\qquad \cdots \otimes (\uparrow^{m_k-1} \alpha_k \otimes \wedge_{m_k}).
\end{aligned}$$

The sign factors are

$$\begin{aligned}
\epsilon_1 &= (-1)^{\xi(\uparrow^{n-1}\overline{\alpha})}, \\
\epsilon_2 &= (-1)^{\xi(\uparrow^{n-1}\overline{\alpha})+(\alpha_i+m_i-1)(\alpha_{i+1}+m_i-1)} \quad \text{and} \\
\epsilon_3 &= (-1)^{\xi(\uparrow^{n-1}\overline{\alpha})+(\alpha_i+m_i-1)(\alpha_{i+1}+m_i-1)+1}
\end{aligned}$$

One checks immediately that the exponents $\xi(\uparrow^{n-1} \overline{\alpha}) + (|\alpha_i| + m_i - 1)(|\alpha_{i+1}| + m_i - 1) + 1$ and $\xi(\uparrow^{n-1} \overline{\alpha}') + |\alpha_i||\alpha_{i+1}| + m_i^2$ have the same parity so $\Psi_{A,T}$ and ρ commute.

Since $\Psi_{A,T,\ell}$ is $\Sigma(T,\ell)$-equivariant, the morphism $\Psi_{A,T,\ell} \otimes_{\Sigma(T,\ell)} 1\!\!1_{\mathbf{k}[\Sigma_n]}$ has a well-defined extension to $\mathfrak{s}(\mathcal{A}[T])$. Then define

$$\Phi_{A,T} := \Psi_{A,T,\ell} \otimes_{\Sigma(T,\ell)} 1\!\!1_{\mathbf{k}[\Sigma_n]}.$$

All constructions above are clearly functorial in A, therefore $\Phi_{-,T} : \mathfrak{s}(-(T)) \longrightarrow (\mathfrak{s}-)(T)$ defines a natural transformation of functors on dg-Σ-Mod. ■

Proof of Proposition 3.20. Define

$$\Phi_A := \bigoplus \Phi_{A,T} : \bigoplus \mathfrak{s}(\mathcal{A}[T]) = \mathfrak{s}\Psi(A) \longrightarrow \bigoplus (\mathfrak{s}\mathcal{A})[T] = \Psi(\mathfrak{s}A),$$

where the summation is over isomorphism classes of unlabeled reduced trees.

Lemma 3.25 shows that Φ_A is a natural transformation. The verification that it is a morphism of operads (i.e. is compatible with operad composition) and hence that Φ is a natural transformation of functors from dg-Σ-Mod to dgOp will be left to the industrious reader. ■

Proof of Theorem 3.24. The complex $\mathbf{D}(\mathbf{D}(\mathcal{P}))$ is tri-graded, with multi-index (i, j, k). The index i is the tree-degree coming from the last application of the functor $\mathbf{D}$. The corresponding tree differential will be denoted by δ_1. The index j is the *negative* of the tree-degree coming from the first application of $\mathbf{D}$, negative because the dual $\mathbf{D}(\mathcal{P})^\#$ enters in the definition of $\mathbf{D}(\mathbf{D}(\mathcal{P}))$. The internal differential on $\mathbf{D}(\mathcal{P})^\#$ has two components δ_2 and $d_\mathcal{P}$. The differential δ_2 is dual to the tree differential on $\mathbf{D}(\mathcal{P})$ from the first application of the functor $\mathbf{D}$ and therefore expresses the composition law on the operad $\mathcal{P}$. The differential $d_\mathcal{P}$ is induced by the internal differential on $\mathcal{P}$ and raises the last index k by one. If

$\mathsf{a}(v) = card(In(v))$, then by definition,

$$\begin{aligned}
\mathbf{D}(\mathbf{D}(\mathcal{P}))^{i,j,k}(n) &= \operatorname*{colim}_I \mathbf{D}(\mathcal{P})^{\#}(T)^{j,k} \otimes Det(T) \\
&\cong \operatorname*{colim}_I \left(\bigotimes_{v \in Vert(T)} \mathbf{D}(\mathcal{P})^{\#}(In(v)) \right)^{j,k} \otimes Det(T) \\
&\cong \operatorname*{colim}_I \left(\bigotimes_K \left(\operatorname*{colim}_J \ (\mathcal{P}(S_v)^{k_v})^{\#} \otimes Det(S_v) \right)^{\#} \right) \otimes Det(T) \\
&\cong \operatorname*{colim}_I \left(\bigotimes_K \left(\operatorname*{colim}_J \mathcal{P}(S_v)^{k_v} \otimes Det(S_v)^{\#} \right) \right) \otimes Det(T).
\end{aligned}$$

The indexing categories for the colimits are

$$\begin{aligned}
I &= \{T \in Iso(\mathtt{Rtree}_n),\ |T| + 2 - n = i\} \text{ and} \\
J &= \{S_v \in Iso(\mathtt{Rtree}(In(v))),\ |S_v| + 2 - \mathsf{a}(v) = -j_v\}
\end{aligned}$$

and the indexing set for the tensor product is

$$K = \{(v, j_v, k_v) \mid v \in Vert(T), \textstyle\sum j_v = j, \sum k_v = k\}.$$

We believe that the meaning of the above definitions is clear, for example, I denotes the full subcategory of $Iso(\mathtt{Rtree}_n)$ of trees with $|T| + 2 - n = i$.

These isomorphisms are explained as follows. Since $\mathcal{P}(n)^k$ is finite dimensional for all n, k and $\#$ indicates the graded dual, we have $((\mathcal{P}(S_v)^k)^{\#})^{\#} \cong \mathcal{P}(S_v)^k$. The index i is nonpositive and equals $|T| + 2 - n$, reflecting the fact that s^{-1} shifts tree degree $|T| + 1$ by $1 - n$. Thus $i = 0$ corresponds to binary trees in $\mathtt{Rtree}_n$ ($n - 1$ vertices). The dg vector space $\mathbf{D}(\mathcal{P})(T)$ is the colimit of tensor products of $\mathbf{D}(\mathcal{P})(\mathsf{a}(v))$, one for each vertex v of T and we use j_v to denote cochain degree of the element associated to vertex v. The factors $\mathbf{D}(\mathcal{P})(In(v))$ are a colimit over labeled trees with $card(In(v))$ leaves such that there is a tree $S_v \in \mathtt{Rtree}(In(v))$ assigned to each vertex of T, that is, purely on a tree level, the vertices of T are labeled by trees S_v. The relation between the number of internal edges of S_v and the degree j_v is $j_v = -(|S_v| + 2 - \mathsf{a}(v))$.

In order to describe the differential

$$\delta_1 : \mathbf{D}(\mathbf{D}(\mathcal{P}))^{i,j,k} \longrightarrow \mathbf{D}(\mathbf{D}(\mathcal{P}))^{i+1,j,k}$$

it is useful to construct a 'big tree' S by grafting each tree S_v into T at the corresponding vertex v. There is a morphism of trees $\pi : S \longrightarrow T$ such that $S_v = \pi^{-1}(v)$ and π contracts each subtree S_v onto the corresponding vertex v in T. The vertices of the big tree S are partitioned into the sets $Vert(S_v)$ indexed by the vertices of T, thus the vertices of T can be identified with subsets which constitute the partition. As an example, consider Figure 3 where the same big tree S (a binary tree with three vertices) is associated to four trees. Each of the trees T_i corresponds to a partition of the vertices of S into subsets consisting of the vertices of a subtree.

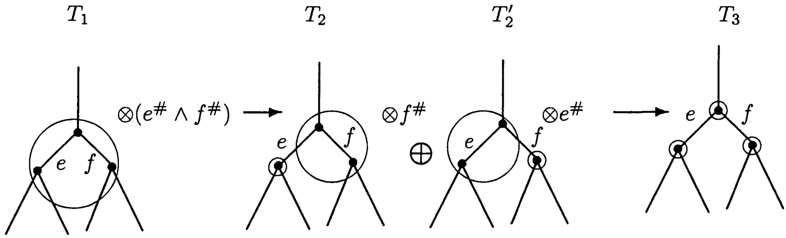


FIGURE 3. One small piece of the complex $\mathbf{D}(\mathbf{D}(\mathcal{P}))$ with differential δ_1 in the case of a binary tree S with three vertices. The tree T_1 is a corolla with the vertex labeled by a binary tree. The trees T_2 and T_2' are trees with one binary and one tertiary vertex and the tertiary vertex is labeled by a binary tree. The tree T_3 is a binary tree. The labels are the encircled subtrees collapsed to vertices.

Coming back to the equation above, we can re-express the last line:

$$
\begin{aligned}
&\operatorname*{colim}_{I}\left(\bigotimes_{K}\left(\operatorname*{colim}_{J}\mathcal{P}(S_v)^{k_v}\otimes Det(S_v)^{\#}\right)\right)\otimes Det(T)\\
=\;&\operatorname*{colim}_{L}\left(\bigotimes_{M}\mathcal{P}(In(w))^{k_{v,w}}\otimes Det(S_v)^{\#}\right)\otimes Det(T)\\
\cong\;&\operatorname*{colim}_{L}\;\mathcal{P}(S)^k\otimes\left(\bigotimes_{v\in Vert(T),\,S_v=\pi^{-1}(v)} Det(S_v)^{\#}\right)\otimes Det(T)\\
\cong\;&\operatorname*{colim}_{L}\;\mathcal{P}(S)^k\otimes\left(\bigotimes_{v\in Vert(T),\,S_v=\pi^{-1}(v)} det(S_v)^{\#}\right).
\end{aligned}
$$

The new indexing category for the colimit is the category of isomorphisms of objects

$$L=\{\pi:S\to T\mid |T|+2-n=i,|S|+2-n=-j\} \tag{3.15}$$

The indexing set for the unordered tensor product in the second row is

$$M=\{(v,w)\mid v\in Vert(T),\; S_v=\pi^{-1}(v),\; w\in Vert(S_v),\textstyle\sum k_{v,w}=k\}.$$

The substitution of $\bigotimes det(S_v)^{\#}$ in the last line is a consequence of the following isomorphisms (compare the definition of a 'coefficient system' for modular operads in Section 5.3):

$$Det(T)\otimes\left(\bigotimes_{v\in Vert(T)} det(S_v)\right)\cong Det(S)\cong\bigotimes_{v\in Vert(T)} Det(S_v).$$

By duality we get

$$\left(\bigotimes_{v\in Vert(T)} Det(S_v)^{\#}\right)\otimes Det(T)\cong\bigotimes_{v\in Vert(T)} det(S_v)^{\#}$$

The new indexing $|S|+2-n=-j$ in the last three lines is explained by the following sequence of equations:

$$\begin{aligned}
-j &= \sum_{v\in \mathit{Vert}(T)} -j_v = \sum_{v\in \mathit{Vert}(T)} (|S_v|+2-\mathsf{a}(v)) \\
&= \sum_{v\in \mathit{Vert}(T)} (|\mathit{Vert}(S_v)|+1-\mathsf{a}(v)) = |\mathit{Vert}(S)|+|\mathit{Vert}(T)| - \sum_{v\in \mathit{Vert}(T)} \mathsf{a}(v) \\
&= |\mathit{Vert}(S)|+|\mathit{Vert}(T)|-(|T|+n) = |\mathit{Vert}(S)|+1-n = |S|+2-n,
\end{aligned}$$

that is, $-j=|S|+2-n$.

The tree differential δ_1 is dual to pseudo-operad composition in $\mathbf{D}(\mathcal{P})$,

$$\circ_i : \mathcal{P}(T)\otimes\mathcal{P}(T') \xrightarrow{\cong} \mathcal{P}(T\circ_i T'),$$

in which $T\circ_i T'$ is the tree given by grafting T' to T along the leaf labeled i; see Definition 1.37. Applying the differential at a vertex v of T consists of replacing v of T by an edge e_v and expressing S_v as the grafting of two trees along the edge e_v. This edge e_v can be identified with one of the internal edges of S_v. The operation does not change the big tree S and does not affect the labels from the operad $\mathcal{P}$; it merely repartitions the vertices of S – see Figure 3.

From this description it becomes clear that the entire δ_1 complex can be decomposed into a direct sum over subcomplexes $\mathbf{D}(\mathbf{D}(\mathcal{P}))_S$ indexed by (isomorphism classes of) 'big trees' S. For a given S the subcomplex is itself a colimit over the category of isomorphisms of objects $\pi : S\to T$ where $S_v=\pi^{-1}(v)$ as v runs over the vertices of T,

$$\mathbf{D}(\mathbf{D}(\mathcal{P}))_S^{(i,n-2-|S|,k)} := \operatorname*{colim}_N \mathcal{P}(S)^k \otimes \bigotimes_{v\in \mathit{Vert}(T)} \mathit{det}(\pi^{-1}(v))^{\#},$$

for N the category of isomorphisms of objects $\{(T,\pi)\mid \pi : S\to T,\ |T|+2-n=i\}$. The trees T which appear in this subcomplex arise from all possible images of S under a tree-morphism. Figure 3 shows a typical example. For any subset of internal edges $J\subset \mathit{edge}(S)$, there is a tree-morphism $\pi : S\to T_J = S/J$, with $|T_J|=|S|-|J|=i+n-2$. In terms of the subsets J we have

$$\begin{aligned}
\mathbf{D}(\mathbf{D}(\mathcal{P}))_S^{(i,n-2-|S|,k)} &= \bigoplus_Q \mathcal{P}(S)^k \otimes \bigotimes_{v\in \mathit{Vert}(T_J)} \mathit{det}(\pi^{-1}(v))^{\#} \\
&\cong \bigoplus_R \mathcal{P}(S)^k \otimes (e^{\#}_{i_1}\wedge\cdots\wedge e^{\#}_{i_{|J|}}),
\end{aligned}$$

where

$$\begin{aligned}
Q &= \{J\subset \mathit{edge}(S),\ |S|-|J|=i+n-2\} \text{ and} \\
R &= \{J=\{e_{i_1},\ldots,e_{i_{|J|}}\}\subset \mathit{edge}(S),\ |J|=|S|-i-n+2\}.
\end{aligned}$$

Observe that the colimits in the above formulas were replaced by direct sums, because subsets $J\subset \mathit{edge}(S)$ index representatives of isomorphism classes of elements of the category L in (3.15), by $J\mapsto (S\to S/J)$. The term $e^{\#}_{i_1}\wedge\cdots\wedge e^{\#}_{i_{|J|}}$ is the wedge product of all the internal edges in all the subtrees S_v and δ_1 decreases $|J|$ by one.

For the corolla S with n leaves and no internal edges, $|T|=|S|=0$, $i=2-n$ and $j=n-2$. Since there are no internal edges in S, both δ_1 and δ_2 (corresponding to operad composition in $\mathcal{P}$, coming from dualizing the first application of $\mathbf{D}$) are zero. Thus for each $n > 0$,

$$\left(\mathbf{D}(\mathbf{D}(\mathcal{P}))(n)^{(2-n,n-2,*)}, \delta\right) \cong (\mathcal{P}(n)^*, d_{\mathcal{P}(n)}).$$

If we prove that for any tree S other than a corolla the corresponding subcomplex $(\mathbf{D}(\mathbf{D}(\mathcal{P}))_S, \delta_1)$ is acyclic, then, since the total differential on $\mathbf{D}(\mathbf{D}(\mathcal{P}))_S$ can be expressed as the differential of a double complex,

$$d_{\mathbf{D}(\mathbf{D}(\mathcal{P}))} = \sum \delta_1^i + (-1)^i \delta_{\mathbf{D}(\mathcal{P})},$$

the acyclicity follows from a standard argument in the theory of double complexes (see for example [**Mac63a**, II.6]).

Then the cohomology is concentrated in the complementary subcomplex given by the direct sum over corollae S and the quasi-isomorphism referred to in the statement of Theorem 3.24 is just projection on that subcomplex.

To prove δ_1 acyclicity for $|S| > 0$, we will show that in this case the subcomplex $(\mathbf{D}(\mathbf{D}(\mathcal{P}))_S, \delta_1)$ is isomorphic to the tensor product of $\mathcal{P}(S)$ with the augmented chain complex of an $(|S|-1)$-simplex.

For each tree S with internal edges $\{e_1, \ldots, e_{|S|}\}$, define $T_J := S/\{e_j | j \in J\}$,

$$\bigwedge(S) .= \bigwedge(\mathbf{k}\{e_i^\# \mid 1 \leq i \leq |S|\}),$$

and

$$\xi_S : \mathbf{D}(\mathbf{D}(\mathcal{P}))_S \longrightarrow \mathcal{P}(S) \otimes \bigwedge(S),$$

as a sum $\xi_S = \sum \xi_{S,J}$, where for $J = \{i_1, \ldots, i_{|J|}\}$,

$$\xi_{S,J} : \mathcal{P}(S) \otimes \Big(\bigotimes_{v \in \mathit{Vert}(T_J)} \mathit{det}(S_v)^\# \Big) \longrightarrow \mathcal{P}(S) \otimes (e_{i_1}^\# \wedge \cdots \wedge e_{i_J}^\#).$$

It is clear from the discussion above that

$$\delta_1 : \mathcal{P}(S) \otimes \bigotimes_{v \in \mathit{Vert}(T_J)} \mathit{det}(S_v)^\# \longrightarrow \bigoplus_{\substack{1 \leq k \leq |J| \\ J_k = J - e_{i_k}}} \mathcal{P}(S) \otimes \bigotimes_{v \in \mathit{Vert}(T_{J_k})} \mathit{det}(S_v)^\#$$

is a sum of terms each one given by deleting one edge from the set J. It composes with ξ_S as follows

$$\xi_S \circ \delta_1 = \left(\sum_{1 \leq i \leq |S|} e_i \rfloor \right) \circ \xi_S,$$

where the operation $\rfloor$ contracts e_i with $e_i^\#$. The chain complex $(\bigwedge\{e_i^\#\}, d := (\sum e_i \rfloor))$ is clearly isomorphic to the augmented simplicial chain complex of the $(|S|-1)$-simplex, which is acyclic. ■

3.2. Quadratic operads

In this section we work with operads $\mathcal{P}$ such that $\mathcal{P}(1) = \mathbf{k}$ The definition of a quadratic operad is based on a fairly straightforward analogy with the definition of a quadratic algebra as the quotient of a free associative algebra by an ideal of quadratic relations. We have already defined free operads so it remains only to define ideals in an operad. First we introduce the concept of a module over an operad (see [**Mar96c**]) and then that of an ideal. We illustrate the fundamental examples using Young tableaus.

DEFINITION 3.26. By a *left module over an operad* $\mathcal{P}$ we mean a Σ-module $M = \{M(n)\}_{n\geq 1}$ together with maps

$$\lambda_{l,\mathbf{m}} : \bigoplus_{1\leq i\leq l} \left\{ \mathcal{P}(l)\otimes\mathcal{P}(m_1)\otimes\cdots\otimes M(m_i)\otimes\cdots\otimes\mathcal{P}(m_l) \right\} \longrightarrow M(m_1+\cdots+m_l)$$

for any $l \geq 1$, $m_1,\ldots,m_l \geq 1$, satisfying the operadic form of the standard axioms for a left module over an algebra.

Similarly, a *right module over an operad* $\mathcal{P}$ is a Σ-module $M = \{M(n)\}_{n\geq 1}$ together with maps

$$\rho_{l;\mathbf{m}} : M(l) \otimes \mathcal{P}(m_1)\otimes\cdots\otimes\mathcal{P}(m_l) \longrightarrow M(m_1+\cdots+m_l)$$

for any $l \geq 1$, $m_1,\ldots,m_l \geq 1$, satisfying the corresponding operadic form of the identities for a right module.

The terms 'left module' or 'right module' obviously reflect the position in which M appears relative to the analogs $\lambda_{l,\mathbf{m}}$ and $\rho_{l,\mathbf{m}}$ of operad composition morphisms $\gamma_{l,\mathbf{m}}$, but since the factor M appears both on the right and on the left of $\mathcal{P}$ for $\lambda_{l,m}$, there is a certain asymmetry.

We can also combine the maps appearing in these definitions using the $\square$-product introduced in Section 1.1. For example, a right $\mathcal{P}$-module structure is given by a morphism

$$\rho : M\square\mathcal{P} \longrightarrow M$$

satisfying the standard identity between the composite maps

$$\rho\circ(\rho\otimes 1\!\!1) = \rho\circ(1\!\!1\otimes\mu_{\mathcal{P}})\circ a_{M,\mathcal{P},\mathcal{P}} : (M\square\mathcal{P})\square\mathcal{P} \longrightarrow M, \tag{3.16}$$

where $\mu_{\mathcal{P}} : \mathcal{P}\square\mathcal{P} \to \mathcal{P}$ is the multiplication on $\mathcal{P}$ as a monoid for the $\square$-product of Σ-modules and $a_{M,\mathcal{P},\mathcal{P}}$ is the associativity constraint for the $\square$-product. The unit axiom can be expressed by saying that the composition

$$M \cong M\square\mathbf{1} \xrightarrow{\rho} M \tag{3.17}$$

is the identity or, in 'coordinates,'

$$\rho_{l;1,\ ,1}(m\otimes 1\otimes\cdots\otimes 1) = m,$$

for each $l \geq 1$, $m \in M(l)$ and the unit $1 \in \mathcal{P}(1)$. The analogous description of a left $\mathcal{P}$-module uses the following definition.

DEFINITION 3.27. Given three Σ-modules A, B, C, define their *relative $\square$-product* $A\square(B,C)$ to be the sub-Σ-module of the $\square$-product of A with the Σ-module $B\oplus C$ consisting of terms with just one $\otimes$ factor from C. The component of arity m is

$$\bigoplus_{\substack{1\leq i\leq q\leq m,\\ m_1+\ \ +m_q=m}} A(q)\otimes_{\Sigma_{q-1}}\left(C(m_1)\otimes B(m_2)\otimes\cdots\otimes B(m_q)\right),$$

where Σ_{q-1} acts by permuting the factors of $B(m_2)\otimes\cdots\otimes B(m_q)$.

The associativity constraint for the $\square$-product of Σ-modules defines an associativity constraint for the relative $\square$-product,

$$a'_{A,B,(C,D)} : A\square(B\square C, B\square(C,D)) \xrightarrow{\cong} (A\square B)\square(C,D). \tag{3.18}$$

Using the relative $\square$-product, we can define a left $\mathcal{P}$-module structure on a Σ-module M as a morphism

$$\lambda : \mathcal{P}\square(\mathcal{P}, M) \longrightarrow M,$$

satisfying identities analogous to (3.16) and (3.17).

DEFINITION 3.28. A *$\mathcal{P}$-module M* is a Σ-module with right and left $\mathcal{P}$-module structures which are compatible in the obvious sense.

Equivalently, left and right $\mathcal{P}$-modules can be described in terms of the left and right $\circ_i$-operations:

$$\begin{aligned} \circ_i^\lambda &= \circ_i : \mathcal{P}(n) \otimes M(m) \to M(m+n-1) \text{ and} \\ \circ_i^\rho &= \circ_i : M(n) \otimes \mathcal{P}(m) \to M(n+m-1). \end{aligned}$$

The operad $\mathcal{P}$ itself is a $\mathcal{P}$-module. If $\gamma : \mathcal{P} \to \mathcal{Q}$ is an operad map, then γ induces a $\mathcal{P}$-module structure on $\mathcal{Q}$. The definition of an ideal in an operad is derived from the definition of a $\mathcal{P}$-module in the standard way.

DEFINITION 3.29. An *ideal in an operad $\mathcal{P}$* is a sub-Σ-module which is also a $\mathcal{P}$-submodule.

The forgetful functor $U : \mathtt{Mod}_{\mathcal{P}} \to \Sigma\text{-}\mathtt{Mod}$ from the category of $\mathcal{P}$-modules to the category of Σ-modules has a left adjoint $\mathcal{P}\langle - \rangle : \Sigma\text{-}\mathtt{Mod} \to \mathtt{Mod}_{\mathcal{P}}$ and $\mathcal{P}\langle B\rangle$ is called the *free $\mathcal{P}$-module* on the Σ-module B. It is defined by

$$\mathcal{P}\langle B\rangle := \mathcal{P}\square(\mathbf{1}, B\square\mathcal{P}),$$

which in 'coordinates' means

$$\mathcal{P}\langle B\rangle(n) \cong \bigoplus_I \mathcal{P}(q)\otimes(B(r)\otimes_{\Sigma_r}(\mathcal{P}(s_1)\otimes\cdots\otimes\mathcal{P}(s_r))), \tag{3.19}$$

where $\mathbf{1}$ is the unit Σ-module defined in (1.63) and the direct sum is over the index set

$$I := \{(q, r, s_1, \ldots, s_r) \mid 1 \le q \le n,\ 1 \le r \le n,\ q + s_1 + \cdots + s_r = n+1\}.$$

The structure maps are defined by applying the associativity constraint to regroup terms followed by operad composition in $\mathcal{P}$. Our assumption $\mathcal{P}(1) = \mathbf{k}$ is important here. Without this assumption, the free $\mathcal{P}$-module would be the quotient of (3.19) by relations for the unit.

DEFINITION 3.30. Let $\mathcal{P}$ be an operad and $R \subset \mathcal{P}$ a sub-Σ-module of $\mathcal{P}$. The *operadic ideal generated by R* is the image in $\mathcal{P}\langle R\rangle$ under the $\mathcal{P}$-module morphism $\mathcal{P}\langle R\rangle \longrightarrow \mathcal{P}$ induced by the inclusion $R \hookrightarrow \mathcal{P}$ of Σ-modules.

Any operad $\mathcal{P}$ can be represented as a quotient $\Psi(E)/(R)$, where E and R are Σ-modules and (R) is the operadic ideal generated by R in $\Psi(E)$; we write

$$\mathcal{P} \cong \langle E; R\rangle := \Psi(E)/(R). \tag{3.20}$$

DEFINITION 3.31. A Σ-module E such that $E(n) = 0$ if $n \neq 2$ is called a *quadratic Σ-module.* An operad is called a *quadratic operad* if it is a quotient of the free operad on a quadratic Σ-module modulo an ideal generated by a subspace $R \subset \Psi(E)(3)$ of *defining relations.*

REMARK 3.32. Identities such as $\alpha = \alpha \cdot \sigma$ or $\alpha = -\alpha \cdot \sigma$ for a basis element $\alpha \in E(2)$ and $\sigma \in \Sigma_2$ are not considered as defining relations, but rather as a description of $E(2)$ as a sum of copies of the trivial representation of Σ_2 and the sign representation. These identities in turn determine the decomposition of the space R as a representation of Σ_3, as explained below. We want to point out that the free operad generated by a quadratic Σ-module E has the special feature that the nonzero $E(T)$ appearing in the definition of $\Psi(E)$ (Definition 1.77) are indexed by binary trees.

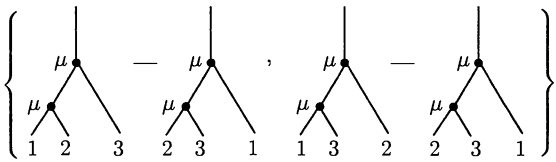
In a similar way we could define operads of type $(k, 2k-1)$ which are generated by a Σ-module E satisfying $E(n) = 0$ for $n \neq k$ and have defining relations $R \subset \Psi(E)(2k-1)$. In this case the free operad $\Psi(E)$ would be described by trees having all vertices with arity k.

EXAMPLE 3.33. The commutative associative operad *Com* is generated by the Σ-module

$$E_{Com}(n) = \begin{cases} \mathbf{k} \cdot \mu & \text{if } n = 2 \text{ and} \\ 0 & \text{if } n \neq 2, \end{cases}$$

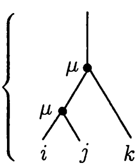
where $\mathbf{k} \cdot \mu$ is the trivial representation of Σ_2. The ideal of relations is generated by:

R_{Com} = Span



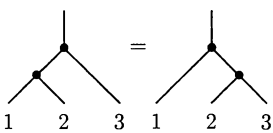
which is the Σ_3-invariant subspace of

$\Psi(E_{Com})(3) =$ Span



$\mid (i,j,k) \in \{(1,2,3),(1,3,2),(2,3,1)\}$.

We could also say that R_{Com} is the subspace generated over Σ_3 by the 'associativity'



which is perhaps the most natural way to introduce this operad. We could also give a more formal definition such as $\Psi(\mu)/(\mu \circ_1 \mu - \mu \circ_2 \mu)$

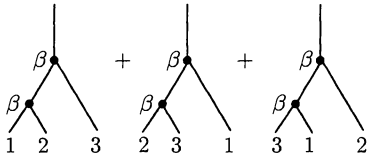
In terms of the representation theory of the symmetric groups, E_{Com} has Young diagram $\ydiagram{2}$ and $\Psi(E_{Com})(3)$ is the corresponding induced representation of Σ_3. It is three-dimensional and decomposes as the sum of the one-dimensional trivial representation isomorphic to $Com(3)$ and the two-dimensional irreducible representation given by the relations R_{Com}. Expressed in Young diagrams:

$$\Psi(E_{Com})(3) \cong \Psi(\ydiagram{2})(3) \cong \ydiagram{3} \bigoplus \ydiagram{2,1} \cong Com(3) \bigoplus R_{Com}.$$

EXAMPLE 3.34. The Lie operad $\mathcal{L}ie$ is generated by the $\mathbf{k}$-Σ-module

$$E_{\mathcal{L}ie}(n) = \begin{cases} \mathbf{k} \cdot \beta & \text{if } n = 2 \\ 0 & \text{if } n \neq 2, \end{cases}$$

where $\mathbf{k} \cdot \beta$ is the signum representation of Σ_2. The ideal of relations is generated by the Jacobi identity:

$R_{\mathcal{L}ie} := \mathrm{Span}\Bigg\{$  $\Bigg\}.$

In this case the generator $E_{\mathcal{L}ie}$ is the signum representation of Σ_2 with Young diagram $\ydiagram{1,1}$. The corresponding induced representation of Σ_3 is again three-dimensional, but this time it decomposes into the sum of the one-dimensional signum representation given by the subspace of relations and the complementary two-dimensional irreducible representation isomorphic to $\mathcal{L}ie(3)$:

$$\Psi(E_{\mathcal{L}ie})(3) \cong \Psi(\ydiagram{1,1})(3) \cong \ydiagram{1,1,1} \bigoplus \ydiagram{2,1} \cong R_{\mathcal{L}ie} \bigoplus \mathcal{L}ie(3).$$

Another definition of $\mathcal{L}ie(n)$ is given in Definition 1.28.

EXAMPLE 3.35. For the associative operad $\mathcal{A}ss$, $E(2)$ is the regular representation of Σ_2 with basis $\{\alpha, \alpha \cdot \tau\}$. Then $\Psi(E_{\mathcal{A}ss})(3)$ is the direct sum of two copies of the regular representation

$$\Psi(E_{\mathcal{A}ss})(3) = \mathrm{Span}\{(\alpha \otimes_1 \alpha) \otimes (i,j,k),\ (\alpha \otimes_2 \alpha) \otimes (i,j,k)\}.$$

The defining relations are

$$R_{\mathcal{A}ss} := \mathrm{Span}\{(\alpha \otimes_1 \alpha - \alpha \otimes_2 \alpha) \otimes (i,j,k)\}.$$

For the notation $\otimes_1$ and $\otimes_2$ see Figure 13 in Appendix 1.9.1.

In the next definition we introduce a duality which is central to the concept of Koszul operads.

DEFINITION 3.36. Let E be a Σ-module. Then the *dual Σ-module* $E^{\#} = \{E^{\#}(n)\}_{n \geq 1}$ is defined by $E^{\#}(n) := Hom_{\mathbf{k}}(E(n), \mathbf{k})$. The Σ_n-representation on $E(n)$ determines a dual representation on $E^{\#}(n)$ by

$$(\lambda \cdot \sigma, \alpha) := (\lambda, \alpha \cdot \sigma^{-1}), \text{ for } \lambda \in E^{\#}(n),\ \sigma \in \Sigma_n \text{ and } \alpha \in E(n).$$

The *Czech dual* is the Σ-module $E^\vee = \{E^\vee(n)\}_{n\geq 1}$ with $E^\vee(n) := E^\#(n) \otimes \mathrm{sgn}_n$. This is equivalent to defining $E^\vee(n)$ as the dual space with Σ_n-representation

$$(\lambda \cdot \sigma, \alpha) := \mathrm{sgn}_n(\sigma)(\lambda, \alpha \cdot \sigma^{-1}). \tag{3.21}$$

There is a very important construction in the theory of operads called the *quadratic dual operad*, defined as a quotient of the free operad $\Psi(E^\vee)$ by relations 'orthogonal' to the relations defining the original operad $\mathcal{P}$. Moreover the quadratic dual operad is naturally a quotient of $\mathbf{D}(\mathcal{P})$.

The full definition is given below. In order to describe these relations, we need to extend the pairing between $E(n)$ and $E^\vee(n)$ to a pairing between $\Psi(E)(n)$ and $\Psi(E^\vee)(n)$. Recall (Definition 1.77) that $\Psi(E)(n)$ and $\Psi(E^\vee)(n)$ are the colimits

$$\Psi(E)(n) = \operatorname*{colim}_{T\, \in\, \mathtt{Tree}_n} E(T),\quad \Psi(E^\vee)(n) = \operatorname*{colim}_{T\, \in\, \mathtt{Tree}_n} E^\vee(T)$$

over labeled n-trees, where $E(T) = E(T,\ell)$ and $E^\vee(T) = E^\vee(T,\ell)$ were defined in Definition 1.80.

The first condition defining the extension is that $E(T,\ell)$ and $E^\vee(T',\ell')$ are orthogonal, $E(T,\ell) \perp E^\vee(T',\ell')$, when $(T,\ell) \not\cong (T',\ell')$ (not isomorphic as labeled trees).

If $(T,\ell) = (T',\ell')$, we order vertices $(v_1,\ldots,v_p)$, fix a labeling $(i_1,\ldots,i_n)$ and represent elements of $E(T,\ell)$ and $E^\vee(T,\ell)$ as products $\alpha_1 \otimes\cdots\otimes \alpha_p$ and $\lambda_1 \otimes\cdots\otimes\lambda_p$ respectively. Then we define

$$\Big((\lambda_1 \otimes \cdots \otimes \lambda_p) \otimes (i_1,\ldots,i_n), (\alpha_1 \otimes\cdots\otimes \alpha_p) \otimes (i_1,\ldots,i_n)\Big) := \epsilon(\lambda_1,\alpha_1)\cdots(\lambda_p,\alpha_p),$$

where the outer parentheses represent the dual pairing and the sign is

$$\epsilon = (-1)^{\alpha_1(\lambda_2+\cdots+\lambda_p)+\alpha_2(\lambda_3+\cdots+\lambda_p)+\cdots+\alpha_{p-1}\lambda_p}\mathrm{sgn}_n(i_1,\ldots,i_n).$$

The above data are enough to define a pairing between colimits; compare Remark 3.8. For example, the pairing between $\Psi(E^\vee)(3)$ and $\Psi(E)(3)$ is defined by

$$\begin{aligned}&((\lambda \otimes_1 \lambda') \otimes (i,j,k), (\alpha \otimes_1 \alpha') \otimes (i',j',k'))\\ &:= \begin{cases} (-1)^{\lambda'\alpha}\mathrm{sgn}_3(i,j,k)(\lambda,\alpha)(\lambda',\alpha'), & \text{if } (i,j,k) = (i',j',k'),\\ (-1)^{\lambda'\alpha}\mathrm{sgn}_3(i,j,k)(\lambda,\alpha)(\lambda',\alpha'\cdot\tau), & \text{if } (j,i,k) = (i',j',k'),\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{3.22}$$

In this example we use the product $\otimes_1$ defined in Figure 13 of Appendix 1.9.1.

DEFINITION 3.37. The *quadratic dual* of a quadratic operad $\mathcal{P} \cong \langle E; R\rangle$ is $\mathcal{P}^! := \langle E^\vee; R^\perp\rangle$, where $R^\perp \subseteq \Psi(E^\vee)(3)$ is the annihilator with respect to pairing (3.22) of the relations $R \subseteq \Psi(E)(3)$ defining $\mathcal{P}$.

It is an instructive exercise to show that $(-)^!$ is an involution, $(\mathcal{P}^!)^! \cong \mathcal{P}$.

EXAMPLE 3.38. The quadratic dual of the commutative operad is generated by

$$E^\vee_{\mathcal{C}om} = \square\!\square^{\,\vee} = \begin{array}{c}\square\\[-4pt]\square\end{array} = E_{\mathcal{L}ie}.$$

The space of defining relations for $\mathcal{C}om^{!}$ is the annihilator in $\Psi(E_{\mathcal{L}ie})(3)$ (relative to the pairing (3.22)) of the defining relations for the commutative operad. Since the annihilator is a subrepresentation of complementary dimension, it is the one-dimensional subrepresentation given by $R_{\mathcal{L}ie}$:

$$R_{\mathcal{C}om^{!}} = R_{\mathcal{C}om}^{\perp} = \ydiagram{2,1} \overset{\perp}{=} \ydiagram{1,1,1} = R_{\mathcal{L}ie}.$$

Therefore $\mathcal{L}ie$ is the quadratic dual of $\mathcal{C}om$ and conversely:

$$\mathcal{C}om^{!} = \mathcal{L}ie, \quad \mathcal{L}ie^{!} = \mathcal{C}om.$$

The duality between commutative coalgebras and Lie algebras, observed independently by J. Moore and D. Quillen in the late 1960's, can be considered as the prehistory of quadratic duality in operad theory. The general theory was developed in [**GK94**].

In describing the associative operad, we use a basis for $\Psi(E)(3)$ involving both terms with $\otimes_1$ and $\otimes_2$. In this case we need to use the identities (1.61) to reduce to terms of type $\otimes_1$ and apply equations (3.22). For example, the pairing between two expressions involving $\otimes_2$ is given by:

$$\begin{aligned}
&((\lambda \otimes_2 \lambda') \otimes (i,j,k), (\alpha \otimes_2 \alpha') \otimes (i',j',k')) \\
&\qquad = ((\lambda \cdot \tau \otimes_1 \lambda') \otimes (j,k,i), (\alpha \cdot \tau \otimes_1 \alpha') \otimes (j',k',i')) \\
&\qquad = -((\lambda \otimes_1 \lambda') \otimes (j,k,i), (\alpha \otimes_1 \alpha') \otimes (j',k',i')).
\end{aligned} \tag{3.23}$$

The minus sign appears because of the identity

$$(\lambda \cdot \tau, \alpha \cdot \tau) = \operatorname{sgn}(\tau)(\lambda, \alpha) = -(\lambda, \alpha)$$

in the definition of the Czech dual.

In the associative operad, $E_{\mathcal{A}ss}(2)$ is the regular representation of Σ_2 and so is $E^{\vee}_{\mathcal{A}ss}(2)$. We denote the basis of $E^{\vee}_{\mathcal{A}ss}(2)$ by $\{\lambda, \lambda \cdot \tau\}$. From the definition of the pairing in equation (3.23), it is clear that

$$R_{\mathcal{A}ss}^{\perp} = \operatorname{Span}\{(\lambda \otimes_1 \lambda - \lambda \otimes_2 \lambda) \otimes (i,j,k)\}$$

and thus

$$\mathcal{A}ss^{!} = \mathcal{A}ss.$$

THEOREM 3.39. *For a quadratic operad $\mathcal{P}$, there is a natural transformation of functors*

$$\Theta_{\mathcal{P}} : \mathbf{D}(\mathcal{P}) \to \mathcal{P}^{!},$$

inducing an isomorphism

$$H^0(\mathbf{D}(\mathcal{P})(n), \delta) \cong \mathcal{P}^{!}(n), \ \textit{for each } n \geq 2.$$

Note that the '0' in H^0 refers to the 'tree-degree' not the total degree and that $\mathbf{D}(\mathcal{P})$ is negatively graded.

PROOF. By definition

$$
\begin{aligned}
\mathbf{D}(\mathcal{P})(n)^0 &= \operatorname*{colim}_{\text{binary } T \in \mathtt{Rtree}_n} \mathfrak{s}^{-1}(\mathcal{P}^{\#}(T) \otimes det(T)) \\
&\cong \operatorname*{colim}_{\text{binary } T \in \mathtt{Rtree}_n} \mathfrak{s}^{-1}(\uparrow \mathcal{P}^{\#}(T)) \text{ by Lemma 3.12} \\
&\cong \operatorname*{colim}_{\text{binary } T \in \mathtt{Rtree}_n} (\mathfrak{s}^{-1} \uparrow \mathcal{P}^{\#})(T) \text{ by Proposition 3.20} \\
&\cong \operatorname*{colim}_{\text{binary } T \in \mathtt{Rtree}_n} \mathcal{P}^{\vee}(T).
\end{aligned}
$$

The last isomorphism comes from the fact that the trees are binary and

$$(\mathfrak{s}^{-1} \uparrow \mathcal{P}^{\#})(2) \cong \downarrow (\uparrow \mathcal{P}^{\#})(2) \otimes \mathrm{sgn}_2 \cong \mathcal{P}^{\#}(2) \otimes \mathrm{sgn}_2 = \mathcal{P}^{\vee}(2).$$

Thus $\mathbf{D}(\mathcal{P})^0(n) \cong \Psi(\mathcal{P}^{\vee}(2))(n)$. The morphism $\Theta_{\mathcal{P}}$ is defined by composing with the projection onto $\mathcal{P}^!(n)$:

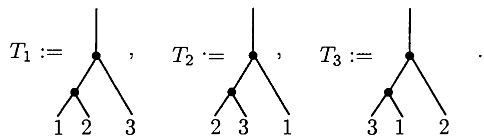
$$\Theta_{\mathcal{P}} : \mathbf{D}(\mathcal{P})^0(n) \xrightarrow{\cong} \Psi(\mathcal{P}^{\vee}(2))(n) \xrightarrow{\pi} \big(\Psi(\mathcal{P}^{\vee}(2))(n)/(R_{\mathcal{P}}^{\perp})(n)\big) = \mathcal{P}^!(n),$$

and requiring that $\Theta|_{\mathbf{D}(\mathcal{P})(n)^{<0}} = 0$.

Finally we show that $\delta(\mathbf{D}(\mathcal{P})(n)^{-1}) = Ker(\Theta_{\mathcal{P}})$, proving that there is an isomorphism:

$$\overline{\Theta}_{\mathcal{P}} : H^0(\mathbf{D}(\mathcal{P})(n), \delta) = \mathbf{D}(\mathcal{P})(n)^0/\delta(\mathbf{D}(\mathcal{P})(n)^{-1}) \xrightarrow{\cong} \mathcal{P}^!(n).$$

To simplify the exposition, represent $\mathbf{D}(\mathcal{P})(n)^{-1}$ as the direct sum over isomorphism classes of 'almost binary' rooted trees with one distinguished vertex having three incoming edges and all other vertices binary. For any such tree, denote the distinguished vertex by v_T. The differential δ acts nontrivially only on the coefficient from $\mathcal{P}^{\#}(3)$ at v_T. The image $\delta(\mathcal{P}^{\#}(T) \otimes Det(T))$ is contained in the direct sum of three terms labeled by the new binary trees created by grafting one of the following subtrees at the vertex v_T:



There is an exact sequence for the defining relations $R_{\mathcal{P}}$ of a quadratic operad $\mathcal{P}$:

$$0 \longrightarrow R_{\mathcal{P}} \xrightarrow{i} \bigoplus_{i=1,2,3} \mathcal{P}(T_i) \xrightarrow{\circ} \mathcal{P}(3) \longrightarrow 0. \tag{3.24}$$

The middle term is $\Psi(\mathcal{P}(2))(3)$, i is the inclusion and $\circ$ contracts the unique internal edge of the tree T_i. Dualizing, we get the exact sequence

$$0 \longleftarrow R_{\mathcal{P}}^{\#} \xleftarrow{i^{\#}} \bigoplus_{i=1,2,3} \mathcal{P}^{\#}(T_i) \xleftarrow{\delta} \mathcal{P}^{\#}(3) \longleftarrow 0. \tag{3.25}$$

Therefore $\delta(\mathcal{P}^{\#}(3)) = Ker(i^{\#}) = R_{\mathcal{P}}^{\perp}$. For each tree T in the indexing set of $\mathbf{D}(\mathcal{P})(n)^{-1}$, tensor the sequence (3.25) with the tensor product of $\mathcal{P}^{\#}(2)$ over the binary vertices of T. Take the sum of all these sequences inside the complex $\mathbf{D}(\mathcal{P})(n)$ to get

$$\delta(\mathbf{D}(\mathcal{P})(n)^{-1}) = \varphi^{-1}(R_{\mathcal{P}}^{\perp}) = Ker(\Theta_{\mathcal{P}}). \qquad \blacksquare$$

3.3. Koszul operads

In this section we define and describe Koszul operads. The characteristic property, which makes this class of operads particularly well suited to the study of homotopy algebras, is that the dual cobar complex $\mathbf{D}(\mathcal{P})$ together with the map $\Theta_{\mathcal{P}}$ in Theorem 3.39 provides a minimal model for $\mathcal{P}^!$ in the sense of Definition 3.124.

DEFINITION 3.40. A quadratic operad $\mathcal{P}$ is called a *Koszul operad* if the map $\Theta_{\mathcal{P}} : \mathbf{D}(\mathcal{P}) \to \mathcal{P}^!$ of Theorem 3.39 is a quasi-isomorphism.

That is, for each $n \geq 2$, the complexes $\mathbf{D}(\mathcal{P})(n)^*$ are exact everywhere except in degree 0; therefore,

$$H^*(\mathbf{D}(\mathcal{P})(n), \delta) = H^0(\mathbf{D}(\mathcal{P})(n), \delta) \cong \mathcal{P}^!(n).$$

The Koszul property for an operad can be described in terms of the Koszul complex, which is based on an analogy with the theory of quadratic algebras. The concept of a Koszul algebra was introduced by Priddy [**Pri70**]. In the case of a Koszul quadratic algebra, the Koszul complex is defined as the tensor product of the quadratic dual algebra and the vector space dual to the original algebra. For example, in the case of the symmetric algebra, where the quadratic dual algebra equals the exterior algebra, the Koszul complex is the tensor product of the symmetric and exterior algebras, and as a complex is isomorphic to the de Rham complex of a vector space with polynomial coefficients (as opposed to smooth functions).

DEFINITION 3.41. Let $\mathcal{P}$ be a quadratic operad. The *Koszul complex*

$$\mathbf{K}(\mathcal{P}^!) = \bigoplus_{n\geq 2,\ p\geq 1} \mathbf{K}(\mathcal{P}^!)(n)^p$$

of a quadratic operad $\mathcal{P}$ has as component in degree p and arity n:

$$\mathbf{K}(\mathcal{P}^!)(n)^p := \mathcal{P}^!(p) \otimes_{\Sigma_p} (\mathrm{sgn}_p \otimes \mathcal{P}^{\#}[p,n]).$$

The differential $\delta_{\mathbf{K}}^p : \mathbf{K}(\mathcal{P}^!)(n)^p \to \mathbf{K}(\mathcal{P}^!)(n)^{p+1}$ is the composition (deleting subscripts involving the symmetric group action in the tensor products for ease in reading):

$$\begin{aligned}
\mathcal{P}^!(p) \otimes (\mathrm{sgn}_p \otimes \mathcal{P}^{\#}[p,n]) \quad &\longrightarrow \quad \mathcal{P}^!(p) \otimes (\mathrm{sgn}_p \otimes (\mathcal{P}^{\#}[p,p+1] \otimes \mathcal{P}^{\#}[p+1,n])) \\
&\cong \mathcal{P}^!(p) \otimes (\mathcal{P}^![p,p+1] \otimes \mathrm{sgn}_{p+1}) \otimes (\mathcal{P}^{\#}[p+1,n]) \\
&\cong (\mathcal{P}^!(p) \otimes \mathcal{P}^![p,p+1]) \otimes (\mathrm{sgn}_{p+1} \otimes \mathcal{P}^{\#}[p+1,n]) \\
&\quad \longrightarrow \mathcal{P}^!(p+1) \otimes (\mathrm{sgn}_{p+1} \otimes \mathcal{P}^{\#}[p+1,n]).
\end{aligned}$$

The first arrow is $1\!\!1_{\mathcal{P}^!(p)} \otimes (1\!\!1_{\mathrm{sgn}_p} \otimes \widetilde{\Delta})$, where

$$\widetilde{\Delta} : \mathcal{P}^{\#}[p,n] \to \mathcal{P}^{\#}[p,p+1] \otimes \mathcal{P}^{\#}[p+1,n]$$

is the component with $r = p+1$ of the dual to operadic multiplication

$$\Delta : \mathcal{P}^{\#}[p,n] \to \bigoplus_{p<r<n} \mathcal{P}^{\#}[p,r] \otimes \mathcal{P}^{\#}[r,n].$$

The isomorphism between the second and the third terms is explained in Lemma 3.42 below and the last arrow is composition in the quadratic dual operad $\mathcal{P}^!$.

Observe that $\mathbf{K}(\mathcal{P}^!)(n)^1 = \mathcal{P}^\#(n)$ and $\mathbf{K}(\mathcal{P}^!)(n)^n = \mathcal{P}^!(n) \otimes \mathrm{sgn}_n$. We defer the proof that $\delta_{\mathbf{K}}^2 = 0$ as well as the proof of the next lemma to the end of this section.

LEMMA 3.42 *There is a left Σ_p-, right Σ_{p+1}-isomorphism*

$$\mathrm{sgn}_p \otimes \mathcal{P}^\#[p,p+1] \xrightarrow{\cong} \mathcal{P}^![p,p+1] \otimes \mathrm{sgn}_{p+1}.$$

The following theorem giving an alternative characterization of the Koszul property using the Koszul complex is the key to proving that an operad is Koszul if and only if the operadic homology of free algebras vanishes in all but one degree. Using the latter result we will prove that the operads $\mathcal{C}om$, $\mathcal{L}ie$ and $\mathcal{A}ss$ are Koszul.

THEOREM 3.43. *A quadratic operad $\mathcal{P}$ is Koszul if and only if for all arities $n \geq 2$ the Koszul complexes $\mathbf{K}(\mathcal{P}^!)(n)$ are exact, that is, the cohomology is zero in all degrees.*

In the proof of this theorem, which will be given in Section 3.6, we actually use a dual complex.

DEFINITION 3.44. The *dual Koszul complex* $\mathbf{K}(\mathcal{P}^!)^\#$ of a quadratic operad $\mathcal{P}$ has as component in degree p and arity n:

$$(\mathbf{K}(\mathcal{P}^!)^\#)(n)^p := \mathcal{P}^!(p)^\# \otimes_{\Sigma_p} (\mathrm{sgn}_p \otimes \mathcal{P}[p,n]) \cong \mathcal{P}^!(p)^\vee \otimes_{\Sigma_p} \mathcal{P}[p,n].$$

The differential is defined by the standard dualization (3.1).

Now we will prove Lemma 3.42 and show that $\delta_{\mathbf{K}}$ is indeed a differential, that is, $\delta_{\mathbf{K}}^2 = 0$.

The structure of $\mathcal{P}^\#[p,p+1]$ was described in Lemma 1.69. If we define $V := \mathcal{P}^\#(2) \otimes \mathcal{P}^\#(1)^{\otimes p-1} \cong \mathcal{P}^\#(2)$ (we are assuming $\mathcal{P}^\#(1) = \mathbf{k}$), then, as a right Σ_2-, left Σ_{p-1}-module,

$$\mathcal{P}^\#[p,p+1] \cong \mathbf{k}[\Sigma_p] \otimes_{\Sigma_{p-1}} (V \otimes_{\Sigma_2} \mathbf{k}[\Sigma_{p+1}]),$$

where Σ_{p-1} is identified with the subgroup of Σ_p leaving 1 fixed and the subgroup of Σ_{p+1} leaving 1 and 2 fixed. Lemma 3.42 asserts the existence of an isomorphism

$$\mathrm{sgn}_p \otimes (\mathbf{k}[\Sigma_p] \otimes_{\Sigma_{p-1}} (V \otimes_{\Sigma_2} \mathbf{k}[\Sigma_{p+1}])) \longrightarrow (\mathbf{k}[\Sigma_p] \otimes_{\Sigma_{p-1}} [(V \otimes \mathrm{sgn}_2) \otimes_{\Sigma_2} \mathbf{k}[\Sigma_{p+1}]]) \otimes \mathrm{sgn}_{p+1}.$$

In the second term, $\mathbf{k}[\Sigma_2]$ acts diagonally on $V \otimes \mathrm{sgn}_2$. The isomorphism is defined by

$$\varphi : \wedge_p \otimes \sigma \otimes [v \otimes \tau] \longmapsto (\mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau))\sigma \otimes [(v \otimes \wedge_2) \otimes \tau] \otimes \wedge_{p+1},$$

for $\sigma \in \Sigma_p$, $\tau \in \Sigma_{p+1}$, where $\wedge_i$ is, for $i \geq 1$, the generator of the signum representation sgn_i. Equivariance with respect to the left Σ_p-action follows from:

$$\begin{aligned}
\varphi(\omega \cdot (\wedge_p \otimes \sigma \otimes [v \otimes \tau])) &= \varphi(\mathrm{sgn}_p(\omega) \wedge_p \otimes \omega\sigma \otimes [v \otimes \tau]) \\
&= (\mathrm{sgn}_p(\omega)\mathrm{sgn}_p(\omega\sigma)\mathrm{sgn}_{p+1}(\tau))\ \omega\sigma \otimes [(v \otimes \wedge_2) \otimes \tau] \otimes \wedge_{p+1} \\
&= (\mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau))\ \omega\sigma \otimes [(v \otimes \wedge_2) \otimes \tau] \otimes \wedge_{p+1} \\
&= \omega\left((\mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau))\ \sigma \otimes [(v \otimes \wedge_2) \otimes \tau] \otimes \wedge_{p+1}\right) \\
&= \omega \cdot \varphi((\wedge_p \otimes \sigma \otimes [v \otimes \tau])),
\end{aligned}$$

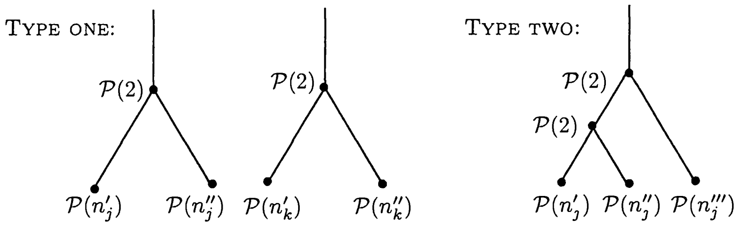


FIGURE 4. Situation before dualizing. We have four new terms of type one, two in position j and two in position k created in either order. We also have three new terms of type two created in any cyclic order of n_j, n_j'', n_j'''.

for $\omega \in \Sigma_p$. Equivariance relative to the right Σ_{p+1}-action follows from a similar calculation.

Proof of the fact that this definition is compatible with the tensor product over $\mathbf{k}[\Sigma_{p-1}]$ between σ and $v \otimes \tau$ and the tensor product over Σ_2 is left to the reader. ∎

LEMMA 3.45. *The Koszul operator $\delta_{\mathbf{K}}$ satisfies the condition for a differential:* $\delta_{\mathbf{K}}^2 = 0$.

PROOF. By Definition 3.41, $\delta_{\mathbf{K}}^2$ involves the maps

$$\tilde{\Delta}^2 := (\mathbb{1} \otimes \tilde{\Delta}) \circ \tilde{\Delta} : \mathcal{P}^{\#}[p,n] \longrightarrow \mathcal{P}^{\#}[p,p+1] \otimes \mathcal{P}^{\#}[p+1,p+2] \otimes \mathcal{P}^{\#}[p+2,n].$$

The term $\mathcal{P}^{\#}[p,p+1] \otimes \mathcal{P}^{\#}[p+1,p+2]$ is a direct sum of components $\mathcal{P}^{\#}[g] \otimes \mathcal{P}^{\#}[f]$ indexed by pairs of surjections $[p+2] \overset{f}{\twoheadrightarrow} [p+1] \overset{g}{\twoheadrightarrow} [p]$. It is necessary to distinguish between two types of terms.

Terms of the first type arise from the dual map to the tensor product of a pair of disjoint binary compositions $\mathcal{P}(2) \otimes \mathcal{P}(n_j') \otimes \mathcal{P}(n_j'') \to \mathcal{P}(n_j' + n_j'')$ and $\mathcal{P}(2) \otimes \mathcal{P}(n_k') \otimes \mathcal{P}(n_k'') \to \mathcal{P}(n_k' + n_k'')$. Terms of the second type arise from the dual map to a sequence of two compositions, where the first composition can involve any pair from the three terms $\mathcal{P}(n_j'), \mathcal{P}(n_j''), \mathcal{P}(n_j''')$ and the second composition involves the remaining term. The situation before dualizing is schematically shown in Figure 4. Looking at one component of $\mathcal{P}^{\#}[p,n]$ we get the two types of terms in the image of $\tilde{\Delta}^2$,

TYPE ONE:

$$\mathcal{P}^{\#}[p,n] \supset \bigotimes_{i=1}^{p} \mathcal{P}^{\#}(n_i) \longrightarrow$$

$$\left(\bigoplus_I \mathcal{P}^{\#}(2) \otimes \left(\mathcal{P}^{\#}(n_j') \otimes \mathcal{P}^{\#}(n_j'')\right) \otimes \left(\mathcal{P}^{\#}(2) \otimes \mathcal{P}^{\#}(n_k') \otimes \mathcal{P}^{\#}(n_k'')\right)\right) \otimes \left(\bigotimes_{i \neq j,k} \mathcal{P}^{\#}(n_i)\right)$$

with the direct sum in the last row indexed by

$$I = \{(j,k,n_j',n_k',n_j'',n_k'') \mid 1 \le j \neq k \le p,\ n_j' + n_j'' = n_j, n_k' + n_k'' = n_k\}$$

and TYPE TWO:

$$\bigotimes_{i=1}^{p} \mathcal{P}^{\#}(n_i) \xrightarrow{\tilde{\Delta}^2}$$

$$\left(\mathcal{P}^{\#}(T_1)\oplus\mathcal{P}^{\#}(T_2)\oplus\mathcal{P}^{\#}(T_3)\right)\otimes\left(\bigoplus_{J}\mathcal{P}^{\#}(n_j')\otimes\mathcal{P}^{\#}(n_j'')\otimes\mathcal{P}^{\#}(n_j''')\right)\otimes\left(\bigotimes_{i\neq j}\mathcal{P}^{\#}(n_i)\right),$$

with the direct sums in the last line indexed by

$$J = \{(j, n_j', n_j'', n_j''') \mid 1 \le j \le p, n_j' + n_j'' + n_j''' = n_j\}.$$

The notation involving the binary trees T_i in the first factor of the last term is the same as in the proof of Theorem 3.39 and just as in that proof this factor is just $\Psi(\mathcal{P}^{\#}(2))(3)$. Modulo the tensor product with $\bigotimes_{i\neq j}\mathcal{P}^{\#}(n_i)$, the component of type two is dual to the compositions described by

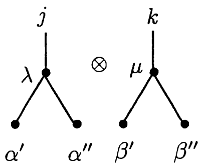
$$\begin{aligned}
&\bigoplus_{n_j'+n_j''+n_j'''=n_j} (\mathcal{P}(T_1)\oplus\mathcal{P}(T_2)\oplus\mathcal{P}(T_3))\otimes\left(\mathcal{P}(n_j')\otimes\mathcal{P}(n_j'')\otimes\mathcal{P}(n_j''')\right) \longrightarrow \\
&\bigoplus_{n_j'+n_j''+n_j'''=n_j} \mathcal{P}(2)\otimes\Big(\left(\mathcal{P}(n_j'+n_j'')\otimes\mathcal{P}(n_j''')\right)\oplus\left(\mathcal{P}(n_j'+n_j''')\otimes\mathcal{P}(n_j'')\right)\oplus\left(\mathcal{P}(n_j''+n_j''')\otimes\mathcal{P}(n_j')\right)\Big) \\
&\qquad \longrightarrow \mathcal{P}(n_j'+n_j''+n_j''') = \mathcal{P}(n_j).
\end{aligned}$$

By associativity, the composite arrow above is the same as the composite

$$\begin{aligned}
&\bigoplus_{n_j'+n_j''+n_j'''=n_j} (\mathcal{P}(T_1)\oplus\mathcal{P}(T_2)\oplus\mathcal{P}(T_3))\otimes\left(\mathcal{P}(n_j')\otimes\mathcal{P}(n_j'')\otimes\mathcal{P}(n_j''')\right) \\
&\qquad \longrightarrow \mathcal{P}(3)\otimes\left(\mathcal{P}(n_j')\otimes\mathcal{P}(n_j'')\otimes\mathcal{P}(n_j''')\right) \longrightarrow \mathcal{P}(n_j'+n_j''+n_j''') = \mathcal{P}(n_j).
\end{aligned}$$

Using the exact sequence (3.25) from the last part of the proof of Theorem 3.39, we see that the map dual to the compositions has image $R_{\mathcal{P}}^{\perp}\otimes(\mathcal{P}^{\#}(n_j')\otimes\mathcal{P}^{\#}(n_j'')\otimes\mathcal{P}^{\#}(n_j'''))$. To calculate $\delta_{\mathbf{K}}^2$ we need to take the image of $R_{\mathcal{P}}^{\perp}$ in $\Gamma(\mathcal{P}^{\vee}(2))(3)$ and project to the operad $\mathcal{P}^{!}$, but $R_{\mathcal{P}}^{\perp} \to 0 \in \mathcal{P}^{!}(3)$, so this part of $\delta_{\mathbf{K}}^2$ is zero.

We can represent the component of type one in the image of $\tilde{\Delta}^2$ more precisely as a sum of terms:



There are two expressions containing $\lambda\otimes\mu$ pictured here, one with $\lambda\in\mathcal{P}^{\#}(2)$ created by the first application of $\tilde{\Delta}$ and then $\mu\in\mathcal{P}^{\#}(2)$ created by the second application of $\tilde{\Delta}$, and the other with μ created first and then λ. The isomorphism defined in Lemma 1.69 introduces opposite signs for the two different orders of creating the terms; therefore, the two expressions cancel. This completes the proof. ■

3.4. A complex relating the two conditions for a Koszul operad

In this section we construct a complex $\mathbf{N}(\mathcal{P})$ which we will use to prove the equivalence of the condition in Theorem 3.43 to the defining property of a Koszul operad as given in Definition 3.40. The proof, which is given in the next section, is based on the existence of two spectral sequences for the cohomology of $\mathbf{N}(\mathcal{P})$, one has an $\mathbf{E}_1$ term which is isomorphic to the cobar complex $\mathbf{C}(\mathcal{P})$ and the other has an $\mathbf{E}_1$ term isomorphic to the Koszul complex $\mathbf{K}(\mathcal{P})$.

The complex $\mathbf{N}(\mathcal{P})$ is constructed by analogy to the theory of algebras where the graded components of the cobar complex are iterated $\otimes$-products. The following isomorphism for iterated $\square$-products is an extension of the isomorphism in Proposition 1.65:

$$(3.26)\quad (\cdots((A\square A)\cdots A)\square A)(n) \cong \bigoplus_{1\le r_1\le r_2\le \cdots \le r_{p-1}\le n} A(r_1)\otimes_{\Sigma_{r_1}} A[r_1,r_2]\otimes\cdots\otimes_{\Sigma_{r_{p-1}}} A[r_{p-1},n].$$

For convenience in displaying the formula, we have used a Σ-module A, although we are actually going to apply the constructions to $\mathcal{P}^{\#}$; moreover, in contrast to the weak inequalities in (3.26), the complex will be further normalized by requiring strict inequalities. Define

$$(3.27)\quad \mathbf{N}(\mathcal{P})(n)^p := \begin{cases} \mathcal{P}^{\#}(n), & \text{for } p=1, \text{ and} \\ \bigoplus \mathcal{P}^{\#}(r_1)\otimes_{\Sigma_{r_1}}\mathcal{P}^{\#}[r_1,r_2]\otimes_{\Sigma_{r_2}}\cdots\otimes_{\Sigma_{r_{p-1}}}\mathcal{P}^{\#}[r_{p-1},n], & \text{for } 2\le p\le n-1, \end{cases}$$

where the summation runs over $1 < r_1 < r_2 < \cdots < r_{p-1} < n$.

The degree p will be called the *$\square$-degree* or *box degree* (the number of $\square$-product factors). The *$\square$-differential* $\delta_{\mathbf{N}}$ is defined by extending the map

$$(3.28)\qquad \delta_{r,s,t} : \mathcal{P}^{\#}[r,t] \longrightarrow \mathcal{P}^{\#}[r,s]\otimes_{\Sigma_s}\mathcal{P}^{\#}[s,t],$$

dual to the composition law $\gamma : \mathcal{P}\square\mathcal{P} \to \mathcal{P}$ for the operad $\mathcal{P}$,

$$\gamma_{r,s,t} : \mathcal{P}[r,s]\otimes_{\Sigma_s}\mathcal{P}[s,t] \hookrightarrow (\mathcal{P}\square\mathcal{P})[r,t] \longrightarrow \mathcal{P}[r,t].$$

Then $\delta_{\mathbf{N}}(n)^p : \mathbf{N}(\mathcal{P})(n)^p \to \mathbf{N}(\mathcal{P})(n)^{p+1}$ is defined on the component

$$\mathbf{N}(\mathcal{P})(r_1,\ldots,n) := \mathcal{P}^{\#}[r_1]\otimes_{\Sigma_{r_1}}\mathcal{P}^{\#}[r_1,r_2]\otimes_{\Sigma_{r_2}}\cdots\otimes_{\Sigma_{r_{p-1}}}\mathcal{P}^{\#}[r_{p-1},n]$$

of the direct sum in (3.27) by

$$\delta_{\mathbf{N}}(n)^p|_{\mathbf{N}(\mathcal{P})(r_1,\ldots,n)} := \sum_I (-1)^{i-1}\delta_{r_{i-1},s_i,r_i},$$

where the summation is over $I = \{(i,s_i) \mid 1\le i\le p,\ r_{i-1}<s_i<r_i\}$ with $r_0=1$ and $r_p=n$.

DEFINITION 3 46. Let $\mathbf{N}(\mathcal{P}) := \{\mathbf{N}(\mathcal{P})(n)\}_{n\ge1}$ be the dg Σ-module

$$\mathbf{N}(\mathcal{P})(n) := \bigoplus_{1\le p\le n-1}\mathbf{N}(\mathcal{P})(n)^p,$$

for $n\ge 2$, where $\mathbf{N}(\mathcal{P})^p$ is defined in (3.27). Set $\mathbf{N}(\mathcal{P})(1)=0$. Define a second grading by *internal degree*

$$\mathbf{N}(\mathcal{P})(n)^p := \bigoplus_{q\in\mathbb{Z}}\mathbf{N}(\mathcal{P})^{p,q},$$

induced by the dg structure on the operad $\mathcal{P}^{\#}$ with *internal differential*

$$d^{\#} : \mathbf{N}(\mathcal{P})(n)^{p,q} \longrightarrow \mathbf{N}(\mathcal{P})(n)^{p,q+1}.$$

The *categorical cobar complex* is the dg Σ-module whose arity n component is the total complex

$$\mathbf{N}(\mathcal{P})(n)^{*,*} := \bigoplus_{p\geq 1, q\in\mathbb{Z}} \mathbf{N}(\mathcal{P})(n)^{p,q}$$

with the differential defined on the component $\mathbf{N}(\mathcal{P})(n)^{p,q}$ by

$$\delta := \sum_{p} (\delta_{\mathbf{N}}(n)^p + (-1)^p d^{\#}).$$

It is possible to define an operad structure on $\mathbf{N}(\mathcal{P})$ using the $\square$-product, but since we will not need it, we leave the details to the interested reader. The condition $\delta_{\mathbf{N}}(n)^{p+1} \circ \delta_{\mathbf{N}}(n)^p = 0$ follows from associativity of the operad composition. The compatibility of $\delta_{\mathbf{N}}$ and $d^{\#}$ follows from the assumption that $\mathcal{P} \in \mathtt{dgOp}$.

At this point it might be helpful to give a table of the various constructions which have appeared in the last few sections:

$$\begin{array}{rcl}
(\mathbf{C}(\mathcal{P}), d) & = & \text{the operadic bar construction on } \mathcal{P},\\
(\mathbf{D}(\mathcal{P}), \delta) & = & \text{the operadic desuspension of } \mathbf{C}(\mathcal{P}),\\
(\mathbf{K}(\mathcal{P}), \delta_{\mathbf{K}}) & = & \text{the Koszul complex of } \mathcal{P} \text{ and}\\
(\mathbf{N}(\mathcal{P}), \delta_{\mathbf{N}}) & = & \text{the categorical cobar complex on } \mathcal{P}.
\end{array}$$

REMARK 3.47. In [**GK94**], V. Ginzburg and M.M. Kapranov introduce a category $Cat(\mathcal{P})$ (actually a PROP (see Section I.1.2)) associated to an operad $\mathcal{P}$. The objects are the natural numbers and the 'Hom-sets' are $Hom_{Cat(\mathcal{P})}(n,m) := \mathcal{P}[m,n]$. The categorical cobar complex is a subcomplex of the simplicial cochain complex of the classifying space of this category. They use this construction and results from an earlier paper of Beilinson, Ginzburg and Schechtman on Koszul categories, [**BGS88**], to compare alternative characterizations of a Koszul operad. We present here the purely operadic proof given in [**SVO99**].

To understand the structure of $\mathbf{N}(\mathcal{P})$ and its relation to the cobar complex $\mathbf{C}(\mathcal{P})$, we consider first the component of $\square$-degree 2:

$$(\mathcal{P}^{\#} \square \mathcal{P}^{\#})(n) \supset \mathbf{N}(\mathcal{P})(n)^2 = \bigoplus_{1<r<n} \mathcal{P}^{\#}(r) \otimes_{\Sigma_r} \mathcal{P}^{\#}[r,n]. \tag{3.29}$$

It will be useful to introduce the following equivalence relation on the set $Surj[j,n]$ of surjections $f : [n] \twoheadrightarrow [j]$;

$$f \sim g \text{ if and only if there exists } \sigma \in \Sigma_j,\ \sigma \circ f = g. \tag{3.30}$$

Let $[f]$ stand for the equivalence class of f and $\widetilde{Surj}[j,n]$ be the set of equivalence classes (which can be identified with the set of unordered partitions of n into j sets). Defining

$$\mathcal{P}[[f]] := \bigoplus_{g\in[f]} \mathcal{P}[g] \tag{3.31}$$

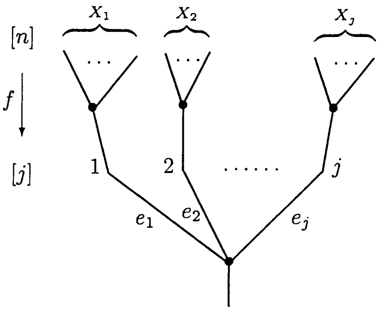


FIGURE 5. Constructing the tree $T_{[f]}$ for $f : [n] \twoheadrightarrow [j]$. There is no particular significance to the fact that the internal edges of $T_{[f]}$ appear in this figure as broken lines.

where $\mathcal{P}[g]$ was introduced in (1.22), we have

$$\bigoplus_{1<r<n} \mathcal{P}^{\#}(r) \otimes_{\Sigma_r} \mathcal{P}^{\#}[r,n] \cong \bigoplus_{\substack{1<r<n \\ [f]\in \widetilde{Surj}[r,n]}} \mathcal{P}^{\#}(r) \otimes_{\Sigma_r} \mathcal{P}^{\#}[[f]].$$

In order to compare the complex $\mathrm{N}(\mathcal{P})$ and the cobar complex we describe a surjection by a corresponding tree. Define a *strict surjection* to be a surjection which is not a bijection, that is $card(f^{-1}(i)) \geq 2$ for at least one i. Let $t(X)$ be the corolla with leaves labeled by X. If $\{x\}$ is a singleton set, define $t(\{x\})$ to be the trivial tree with one edge labeled by x.

DEFINITION 3.48. For a strict surjection $f : [n] \twoheadrightarrow [j]$ such that $f^{-1}(i) = X_i$, define the *tree* $T_{[f]}$ *corresponding to a surjection* f as the tree with j internal edges given by grafting $t(X_i)$, onto $t([j])$ along the ith leaf e_i,

$$T_{[f]} = (\cdots(t([j]) \circ_{e_1} t(X_1)) \circ_{e_2} \cdots) \circ_{e_j} t(X_j);$$

see Figure 5.

Note that the isomorphism class of the unlabeled tree $T_{[f]}$ depends only on the equivalence class $[f]$, as suggested by the notation. It is also clear that

$$\mathcal{P}^{\#}(r) \otimes_{\Sigma_r} \mathcal{P}^{\#}[[f]] \cong \mathcal{P}^{\#}[T_{[f]}]; \tag{3.32}$$

see Definition 1.80 and (3.31) for the notation.

This isomorphism together with the isomorphism in (3.29) presents $\mathrm{N}(\mathcal{P})^2$ in terms of a direct sum of $\mathcal{P}^{\#}(T)$'s:

$$\mathrm{N}(\mathcal{P})(n)^2 \cong \bigoplus_{\substack{1<r<n \\ [f]\in \widetilde{Surj}[r,n]}} \mathcal{P}^{\#}[T_{[f]}]. \tag{3.33}$$

See Figure 6. There is no twisting by $det(T_{[f]})$ but otherwise the summands are the same as those in the description (3.8) of $\mathbf{C}(\mathcal{P})$. However, the $\Box$-degree of all the summands is 2 and the tree-degree of the summand $\mathcal{P}^{\#}[T_{[f]}]$ is the number of vertices in $T_{[f]}$ which, for trees of the type associated to $\mathrm{N}(\mathcal{P})^2$, could be as high as $[\frac{n}{2}]+1$. For $\Box$-degree $p > 2$, $\mathrm{N}(\mathcal{P})^p$ is not described as a direct sum indexed by trees as is $\mathbf{C}(\mathcal{P})$. It is necessary to add an additional refinement in the indexing set,

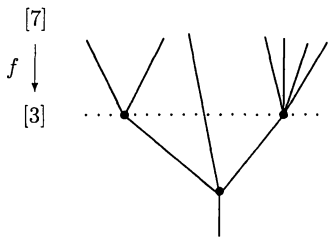


FIGURE 6. The tree representing the particular surjection $f : [7] \twoheadrightarrow [3]$ which indexes $\mathcal{P}^{\#}(3) \otimes [\mathcal{P}^{\#}(2) \otimes \mathcal{P}^{\#}(1) \otimes \mathcal{P}^{\#}(4)] \subset \mathcal{P}^{\#}(3) \otimes \mathcal{P}^{\#}[3,7] \subset \mathrm{N}(\mathcal{P}^{\#})(7)^2$.

separating the vertices of a tree into levels. The trees corresponding to $\mathrm{N}(\mathcal{P})(n)^2$ have two levels, the root at level 0 and all other vertices at level 1. This refinement turns out to be the key to comparing the different forms of the Koszul condition. The direct sum in (3.27) can be refined into a direct sum indexed by equivalence classes of what we shall call surjection sequences, defined as follows.

DEFINITION 3.49. A *surjection sequence (surse)* of length p on the set $[n]$ is a sequence of p strict surjections (not bijections)

$$[n] \xrightarrow{f_p} [r_{p-1}] \xrightarrow{f_{p-1}} \cdots \xrightarrow{f_2} [r_1] \xrightarrow{f_1} [1]. \tag{3.34}$$

Such a surjection sequence is said to be of *type* $(r_1, \ldots, r_{p-1}, n)$. We reverse the order to fit with the order in the tensor product in (3.26) and denote a surjection sequence by $\mathbf{f} = (f_1, \ldots, f_p)$. Let SS^p_n be the set of all such sequences and $SS(r_1, \ldots, r_{p-1}, n)$ the subset of surses of type $(r_1, \ldots, r_{p-1}, n)$.

We can describe $\mathrm{N}(\mathcal{P})$ as a direct sum over equivalence classes of strict surjection sequences, where the equivalence relation is defined by the action of the direct product of symmetric groups $\Sigma_{r_1} \times \cdots \times \Sigma_n$ on $SS(r_1, \ldots, r_{p-1}, n)$ given by the formula:

$$\hat{\sigma}\mathbf{f} = (\sigma_1, \ldots, \sigma_p)(f_1, \ldots, f_p) := (f_1, \sigma_1 \circ f_2 \circ \sigma_2^{-1}, \ldots, \sigma_{p-1} \circ f_p \circ \sigma_p^{-1}), \tag{3.35}$$

where $\hat{\sigma} = (\sigma_1, \ldots, \sigma_p) \in \Sigma_{r_1} \times \cdots \times \Sigma_n$ and $\mathbf{f} = (f_1, \ldots, f_p) \in SS(r_1, \ldots, r_{p-1}, n)$. The restricted action of $\Sigma_{r_1} \times \cdots \times \Sigma_{r_{p-1}}$ (without the factor Σ_n) defines an equivalence relation on SS^p_n with the orbits as equivalence classes. Let $[\mathbf{f}]$ denote the equivalence class of $\mathbf{f}$ and $\widetilde{SS}^p_n$ the set of equivalence classes.

REMARK 3.50. In defining the equivalence relation, we do not use the full action of $\Sigma_{r_1} \times \cdots \times \Sigma_n$ but only of $\Sigma_{r_1} \times \cdots \times \Sigma_{r_{p-1}}$ so that there remains a residual action of Σ_n on the set of equivalence classes. A surjection sequence defines a sequence of nested ordered partitions of $[n]$. An equivalence class can be identified with a sequence of nested unordered partitions of $[n]$.

DEFINITION 3.51. For any surjection sequence $\mathbf{f} \in SS(r_1, \ldots, n)$, there are contravariant functors $\mathtt{dg}\text{-}\Sigma\text{-}\mathtt{Mod} \longrightarrow \mathtt{dgVec}$ defined by

$$\begin{aligned} A \mapsto A^{\#}[\mathbf{f}] &:= A^{\#}[f_1] \otimes A^{\#}[f_2] \otimes \cdots \otimes A^{\#}[f_p] \text{ and} \\ A \mapsto A[\mathbf{f}] &:= A[f_1] \otimes A[f_2] \otimes \cdots \otimes A[f_p]. \end{aligned}$$

For convenience in notation we define *reduced forms* $A^{\#}[\mathrm{f}]'$, respectively $A[\mathrm{f}]'$, by deleting the factors $A^{\#}[f_i^{-1}(j)]$, respectively $A[f_i^{-1}(j)]$ for which $|f_i^{-1}(j)| = 1$

The next proposition brings in the symmetric group action which does not appear in the definition of $A[\mathrm{f}]$ or $A^{\#}[\mathrm{f}]$. We state the proposition only for $A^{\#}[\mathrm{f}]$ because that is the case we need, but obviously there is a completely parallel proposition for $A[\mathrm{f}]$.

PROPOSITION 3.52. *Given a Σ-module A, define an equivalence relation on*

$$\bigoplus_{\mathrm{f}\in SS(r_1, \ldots, n)} A^{\#}[\mathrm{f}] \tag{3.36}$$

by

$$A^{\#}[\mathrm{f}] \ni \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_p \sim \alpha_1\sigma_1 \otimes \sigma_1^{-1}\alpha_2\sigma_2 \otimes \cdots \otimes \sigma_{p-1}\alpha_p \in A^{\#}[\hat{\sigma}\mathrm{f}] \tag{3.37}$$

for all $(\sigma_1, \ldots, \sigma_{p-1}) \in \Sigma_{r_1} \times \cdots \times \Sigma_{r_{p-1}}$. Then there is an isomorphism

$$\Big(\bigoplus_{\mathrm{f}\in SS(r_1, \ldots, n)} A^{\#}[\mathrm{f}]\Big)/\sim\, \cong A^{\#}[r_1]\otimes_{\Sigma_{r_1}} A^{\#}[r_1, r_2]\otimes_{\Sigma_{r_2}} \cdots \otimes_{\Sigma_{r_{p-1}}} A^{\#}[r_{p-1}, n]. \tag{3.38}$$

The right action of Σ_{r_i} and the left action of $\Sigma_{r_{i-1}}$ on $A[r_{i-1}, r_i]$ were defined in equation (1.29) and Definition 1.55.

PROOF. Taking the quotient of $A^{\#}[r_1] \otimes A[r_1, r_2] \otimes \cdots \otimes A^{\#}[r_{p-1}, n]$ by equivalence (3.37) reduces the tensor products over $\mathbf{k}$ to the tensor products $\otimes_{\Sigma_{r_i}}$. ∎

Proposition 3.52 allows us to define a functor $A^{\#}[[\mathrm{f}]]$ for each equivalence class $[\mathrm{f}] \in \widetilde{SS}_n^p$

DEFINITION 3.53. For any $\mathrm{f} \in SS(r_1, \ldots, n)$, the equivalence relation (3.37) restricts to $\bigoplus_{\mathrm{g}\in[\mathrm{f}]} A^{\#}[\mathrm{g}]$ and defines a functor from $\mathtt{dg}$-Σ-$\mathtt{Mod}$ to $\mathtt{dgVec}$:

$$A \mapsto A^{\#}[[\mathrm{f}]] := \Big(\bigoplus_{\mathrm{g}\in[\mathrm{f}]} A^{\#}[\mathrm{g}]\Big)/\sim$$

Using Proposition 3.52 and Definition 3.53 we can describe $\mathrm{N}(\mathcal{P})$ as a direct sum over equivalence classes of strict surjection sequences:

$$\mathrm{N}(\mathcal{P})(n)^p \cong \bigoplus_{[\mathrm{f}]\in\widetilde{SS}_n^p} \mathcal{P}^{\#}[[\mathrm{f}]]. \tag{3.39}$$

REMARK 3.54. Each $A[[\mathrm{f}]]$ is a partially ordered tensor product with the factors in $A[r_i, r_{i+1}]$ forming an unordered tensor product, but if $i < j$, then all the factors in $A[r_i, r_{i+1}]$ appear before any of the factors in $A[r_j, r_{j+1}]$. This partial ordering of the factors in $A[[\mathrm{f}]]$ is basically equivalent to the creation of a level structure on the vertices of a tree, as we will explain in the next section.

REMARK 3.55. If we allowed for one of the surjections in the sequence to be a bijection, we would have factors $A^{\#}[r, r]$. This is like having tensor factors $\mathbf{k}$ in the cobar complex of the dual coalgebra of a finite dimensional $\mathbf{k}$-algebra. Removing this redundancy leads us to the normalized cochain complex $\mathrm{N}(A)(n)$ which is defined by strict inequalities.

3.5. Trees with levels

In this section, we generalize Definition 3.48 associating to any equivalence class $[\mathbf{f}] \in \widetilde{SS}_n^p$ a tree $T_{[\mathbf{f}]}$ with leaves labeled by $[n] = \{1, \dots, n\}$ and vertices separated into p levels. Then $\mathbf{N}(\mathcal{P})(n)^p$ for $p > 2$ can be presented as a direct sum of $\mathcal{P}^{\#}[T_{[\mathbf{f}]}]$:

$$\mathbf{N}(\mathcal{P})(n)^p \cong \bigoplus_{[\mathbf{f}] \in \widetilde{SS}_n^p} \mathcal{P}^{\#}[T_{[\mathbf{f}]}]. \tag{3.40}$$

This isomorphism is a direct consequence of a natural isomorphism of functors

$$\mathcal{P}^{\#}[[\mathbf{f}]] \xrightarrow{\cong} \mathcal{P}^{\#}[T_{[\mathbf{f}]}]$$

given in Proposition 3.57 below.

The direct sum in (3.40) looks very much like the direct sum representation (3.8) of $\mathbf{C}(\mathcal{P})$ (without the twisting by $det(T_{[\mathbf{f}]})$), with the exception that the *gradings in* $\mathbf{N}(\mathcal{P})$ *and* $\mathbf{C}(\mathcal{P})$ *are quite different;* $\mathbf{N}(\mathcal{P})$ is graded by the length of the surjection sequence indexing a given summand or, equivalently, the number of levels in the tree indexing the given summand and $\mathbf{C}(\mathcal{P})$ is graded by the number of vertices in the tree indexing the summand.

In Definition 3.48, a surjection $f : [r] \twoheadrightarrow [s]$ was represented by an s-tuple of corollae $(t(X_1), \dots, t(X_s))$ where $X_j = f^{-1}(j)$, $1 \leq j \leq s$. In the present case, we do this for each surjection in the surjection sequence $\mathbf{f} = (f_1, \dots, f_p)$ as in (3.34), representing the surjection f_i by the ordered set of corollae $(t(X_{i,1}), \dots, t(X_{i,r_{i-1}}))$ where

$$X_{i,j} := f_i^{-1}(j) \tag{3.41}$$

for $1 \leq i \leq p$, $1 \leq j \leq r_{i-1}$. The vertices of the corollae $t(X_{i,j})$ are said to be at level i. Then we create a composite tree by grafting the corollae at level i to the leaves of the corollae at level $i-1$. The root vertex is considered as being at level 0 and there are no vertices corresponding to values onto which a surjection is one-to-one, which means an edge may pass through a level without introducing a vertex. More precisely, we have the following definition.

DEFINITION 3.56. Given a surjection sequence $\mathbf{f} = (f_1, \dots, f_p)$ as in (3.34), let $\mathbf{f}^i := (f_1, \dots, f_i)$, for $1 \leq i \leq p$. Define inductively an 'increasing sequence' of trees $T_{[\mathbf{f}^1]} \subset T_{[\mathbf{f}^2]} \subset \cdots$ with $T_{[\mathbf{f}^i]}$ containing $i+1$ levels of vertices $(0, 1, \dots, i)$ as follows.

For $i = 0$, $\mathbf{f}^1 = f_1$ is an ordinary surjection, so the tree $T_{[\mathbf{f}^1]} := T_{[f_1]}$ was defined in Definition 3.48. The tree $T_{[\mathbf{f}^i]}$ is, for $i > 1$, constructed by grafting to $T_{[\mathbf{f}^{i-1}]}$ along the leaf e_j the corolla $t(X_{i,j})$ with $X_{i,j}$ defined in (3.41), but only for those $1 \leq j \leq r_{i-1}$ such that $card(f_i^{-1}(j)) > 1$.

We define $T_{[\mathbf{f}]} := T_{[\mathbf{f}^p]}$. Vertices v of the corolla $t(X_{i,j})$ are said to be *vertices of level* i. We write $\ell_{\mathbf{f}}(v) := i$ and call $\ell_{\mathbf{f}} : Vert(T_{[\mathbf{f}]}) \to \mathbb{N}$ the *level function*.

For an example, see Figure 7. The construction is defined for a particular surjection sequence but as an abstract tree (with labeled leaves), $T_{[\mathbf{f}]}$ depends only on the equivalence class, as suggested by the notation. Two equivalent surjection sequences differ only by relabeling the elements of the domain and range of the component surjections which amounts to relabeling the vertices, which is irrelevant to the description of $T_{[\mathbf{f}]}$ as an abstract tree. Such 'trees with levels' also appears in the work of Loday [**Lod93**] and Ulyanov [**Uly99**].

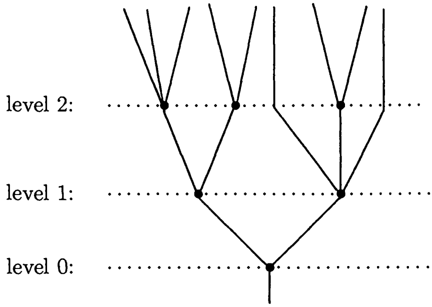


FIGURE 7. An example of the grafting of corollae at two adjacent levels of a surjection sequence. All vertices are indicated by a $\bullet$. Note that there are only three vertices created by the grafting at level 2. There are two edges (appearing as broken line segments) passing through level 2.

PROPOSITION 3.57. *The functors* $A \longmapsto A^{\#}[[\mathbf{f}]]$ *and* $A \longmapsto A^{\#}[T_{[\mathbf{f}]}]$ *from* dg-Σ-Mod *to* dgVec *are naturally isomorphic.*

PROOF. Any surse $\mathbf{f}$ defines an ordering of the vertices of $T_{[\mathbf{f}]}$. First, the vertices $\{v_1, \dots, v_n\}$ are ordered by levels. Then the vertices at a given level are simply ordered by the values of the surjection defining that level, with the vertex corresponding to $f_i^{-1}(j)$ coming before the vertex corresponding to $f_i^{-1}(j')$, whenever $j < j'$. Then we have an isomorphism

$$A^{\#}[\mathbf{f}] = A^{\#}[f_1] \otimes A^{\#}[f_2] \otimes \cdots \otimes A^{\#}[f_n] \cong \bigotimes_{i=1}^{n} A^{\#}(In(v_i)). \tag{3.42}$$

where the tensor products are ordered as above. By definition, $A^{\#}[T_{[\mathbf{f}]}]$ is the colimit of the $\bigotimes_{i=1}^{n} A^{\#}(In(v_i))$ with respect to the permutations of the vertices and for any particular ordering of the vertices

$$A^{\#}[T_{[\mathbf{f}]}] \cong \bigotimes_{i=1}^{n} A^{\#}(In(v_i)) \cong A^{\#}[\mathbf{f}].$$

These isomorphisms combine to give a morphism

$$\bigoplus_{\mathbf{g}\in[\mathbf{f}]} A^{\#}[\mathbf{g}] \longrightarrow A^{\#}[T_{[\mathbf{f}]}].$$

All the equivalence relations on $\bigoplus_{\mathbf{g}\in[\mathbf{f}]} A^{\#}[\mathbf{g}]$ defined by (3.37) are particular cases of the permutation relations defining the colimit $A^{\#}[T_{[\mathbf{f}]}]$. Hence there is an isomorphism

$$\psi_{[\mathbf{f}]} : A^{\#}[[\mathbf{f}]] \longrightarrow A^{\#}[T_{[\mathbf{f}]}] \tag{3.43}$$

and naturality of (3.43) relative to morphisms of dg-Σ-Mod follows from the naturality of the isomorphism in (3.42). ■

Separating the vertices of a tree into levels can be described in terms of order relations among the vertices.

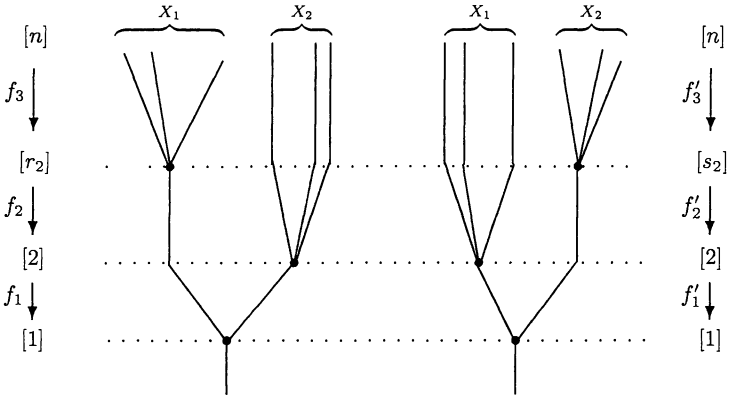


FIGURE 8. Two surjection sequences with the same tree, but different level functions. Each one indexes a distinct component of the image of $\delta_{\mathbf{N}}(\mathcal{P}^{\#}[T_{[\mathbf{f}]}])$, where $T_{[\mathbf{f}]}$ is the tree with the two nonroot vertices both at level one.

DEFINITION 3.58. The $\mathbf{f}$-*order* on the set $Vert(T_{[\mathbf{f}]})$ of the tree $T_{[\mathbf{f}]}$ is given by:

$$v \leq_{\mathbf{f}} v' \text{ if and only if } \ell_{\mathbf{f}}(v) \leq \ell_{\mathbf{f}}(v'),$$

where the level function $\ell_{\mathbf{f}} : Vert(T_{[\mathbf{f}]}) \to \mathbb{N}$ was introduced in Definition 3.56.

The ordering relates any two vertices, but it is not a simple ordering since $v \leq_{\mathbf{f}} v'$ and $v' \leq_{\mathbf{f}} v$ does not imply $v = v'$.

There is another order on the vertices which does not use a level function. Since a vertex of a tree T is connected to the root vertex by a unique path, the vertices along this path are simply ordered with the root as minimal vertex. In this way we get a partial ordering on the vertices of the tree.

DEFINITION 3.59 The *T-ordering* on the vertices of a tree T is given by:

$$v \leq_T v' \text{ if and only if } v \text{ lies on the path from } v' \text{ to the root.}$$

Order relations can be compared using the concept of *refinement*.

DEFINITION 3.60. Given two orders $\leq_1$ and $\leq_2$ on a set X, we say that $\leq_1$ *refines* $\leq_2$, or equivalently, $\leq_1$ is a *refinement* of $\leq_2$, if $x \leq_2 y$ implies $x \leq_1 y$ for all $x, y \in X$.

PROPOSITION 3.61. *Let $T \in Tree(n)$ and $\mathbf{f} \in \widetilde{SS}^p_n$. If $T = T_{[\mathbf{f}]}$, then the $\mathbf{f}$-order on the vertices of T is a refinement of the T-ordering. The map $[\mathbf{f}] \to T_{[\mathbf{f}]}$ from $\widetilde{SS}^p_n$ to n-labeled trees is many-to-one with inverse image of an n-labeled tree T corresponding bijectively to the refinements of the partial ordering of the vertices of T into an ordering into p levels.*

PROOF. The proof is clear. ∎

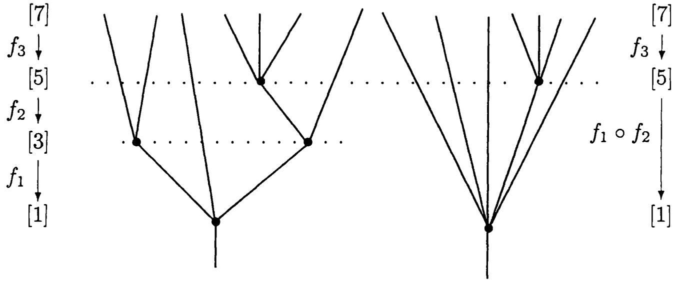


FIGURE 9. The three level tree for the surse

$$[7] \xrightarrow{f_3} [5] \xrightarrow{f_2} [3] \xrightarrow{f_1} [1]$$

maps to the two level tree for the surse

$$[7] \xrightarrow{f_3} [5] \xrightarrow{f_1 \circ f_2} [1],$$

collapsing two edges and decreasing the number of edges by two. Dualizing the corresponding component of the composition $\gamma_{1,3,5}$: $\mathcal{P}[1,3] \otimes_{\Sigma_3} \mathcal{P}[3,5] \to \mathcal{P}[1,5]$ describes one component of the differential $\delta_{\mathbf{N}}$ which creates a new edge in the tree.

We emphasize that if $\mathbf{N}(\mathcal{P})$ is described by the direct sum of $\mathcal{P}^{\#}[T_{[\mathbf{f}]}]$ as in equation (3.40), then the $\square$-degree (ignoring the internal degree from $\mathcal{P}^{\#}$) is not the number of vertices but rather the number of levels of vertices. Only the minimal cochain-degree components are isomorphic:

$$\mathbf{N}(\mathcal{P})(n)^1 = \mathcal{P}^{\#}(n) \cong \mathbf{C}(\mathcal{P})(n)^1.$$

For the maximal cochain degree, there is a surjection $\mathbf{N}(\mathcal{P})(n)^{n-1} \twoheadrightarrow \mathbf{C}(\mathcal{P})(n)^{n-1}$, which forgets the separation of vertices into levels:

$$\begin{aligned} \mathbf{N}(\mathcal{P})(n)^{n-1} &= \mathcal{P}^{\#}[2] \otimes_{\Sigma_2} \mathcal{P}^{\#}[2,3] \otimes_{\Sigma_3} \cdots \otimes_{\Sigma_{n-1}} \mathcal{P}^{\#}[n-1,n] \\ &\longrightarrow \bigoplus_{\substack{T \in \mathit{Tree}(n) \\ |T| = n-2}} \mathcal{P}^{\#}[T] \cong \mathbf{C}(\mathcal{P})(n)^{n-1}. \end{aligned}$$

The difference in the gradings of $\mathbf{C}(\mathcal{P})$ and $\mathbf{N}(\mathcal{P})$ makes the differentials quite different. The tree-differential in the cobar complex $\mathbf{C}(\mathcal{P})$ has 'matrix components' $\delta_{T,T'}$ (3.4), one for each new tree T' created from the tree T by expanding a vertex into an internal edge. The $\square$-differential in the $\mathbf{N}(\mathcal{P})$ creates one new level of vertices and may create many new internal edges or no new internal edges. Figures 8 and 9 show trees and the corresponding surjection sequences indexing components of $\mathbf{N}(\mathcal{P})(n)^3$. Figure 8 shows two surjection sequences and the corresponding trees with levels which arise from the $\square$-differential applied to $\mathcal{P}^{\#}[T_{[\mathbf{f}]}]$, where $T_{[\mathbf{f}]}$ is the tree with the two nonroot vertices both at level one. The part of the $\square$-differential on $\mathcal{P}^{\#}[T_{[\mathbf{f}]}]$ which lies in these components has not introduced a new edge in the indexing trees, only a new level. By contrast Figure 9 (read from right to left) shows a component of the $\square$-differential which does introduce a new edge.

3.6. The spectral sequences relating $\mathbf{N}(\mathcal{P})$ and $\mathbf{C}(\mathcal{P})$

The grading of $\mathbf{C}(\mathcal{P})$ defines a filtration of $\mathbf{N}(\mathcal{P})$ and generates a spectral sequence which relates the cohomologies of $\mathbf{N}(\mathcal{P})$ and $\mathbf{C}(\mathcal{P})$. In fact, we will show that the $\mathbf{E}_1$-term of the spectral sequence coming from the filtration by the number of vertices in the corresponding tree (see the next definition) is isomorphic to $\mathbf{C}(\mathcal{P})$.

DEFINITION 3.62. The *homogeneity* $\mathbf{h}(\mathbf{f})$ of a surjection sequence $\mathbf{f}$ is the number of vertices in the tree $T_{[\mathbf{f}]}$:

$$\mathbf{h}(\mathbf{f}) := |T_{[\mathbf{f}]}| + 1. \tag{3.44}$$

The homogeneity of an element $\alpha_1 \otimes \cdots \otimes \alpha_p \in \mathcal{P}^{\#}[\mathbf{f}]$ is the homogeneity of $\mathbf{f}$, which is the same as the number of $\otimes$-factors when the term is written in the reduced form, that is, the homogeneity of $\alpha_1 \otimes \cdots \otimes \alpha_p$ is p if $\alpha_1 \otimes \cdots \otimes \alpha_p \in \mathcal{P}^{\#}[\mathbf{f}]'$; see Definition 3.51.

Alternatively we could define the homogeneity of a single surjection as the number of values for which f is many-to-one,

$$\mathbf{h}(f) := card(\{j \mid card(f^{-1}(j)) > 1\}),$$

and the homogeneity of a surse $\mathbf{f} = (f_1, \ldots, f_p)$ as the sum of the homogeneities of its components

$$\mathbf{h}(\mathbf{f}) := \sum \mathbf{h}(f_i).$$

DEFINITION 3.63. Let $F^s(\mathbf{N}(\mathcal{P})(n))$ be the subspace of $\mathbf{N}(\mathcal{P})(n)$ spanned by the $A[[\mathbf{f}]]$ for $\mathbf{h}([\mathbf{f}]) \geq s$,

$$F^s(\mathbf{N}(\mathcal{P})(n)) := \bigoplus_{\{[\mathbf{f}] \in \widetilde{SS}^*_n \mid \mathbf{h}(\mathbf{f}) \geq s\}} \mathcal{P}^{\#}[T_{[\mathbf{f}]}].$$

These subspaces define a decreasing filtration of $\mathbf{N}(\mathcal{P})(n)$ called the *homogeneity filtration*.

The filtration is compatible with the differential on $\mathbf{N}(\mathcal{P})(n)$. The term

$$\mathbf{E}_0^{p,q} := \mathbf{E}_0^{p,q}(\mathbf{N}(\mathcal{P})) := \frac{F^p(\mathbf{N}^{p+q}(\mathcal{P}))}{F^{p+1}(\mathbf{N}^{p+q}(\mathcal{P}))}$$

in the initial stage of the spectral sequence is spanned by reduced trees with exactly p vertices (filtration degree) and $p+q$ levels (cochain degree). Since the number of levels is less than or equal to the number of vertices, $\mathbf{E}_0^{p,q} \neq 0$ only if $q \leq 0$, and the spectral sequence is in the fourth quadrant.

The differential $\delta_0 : \mathbf{E}_0^{p,q} \longrightarrow \mathbf{E}_0^{p,q+1}$ preserves homogeneity, introducing one new level but no new vertices in $T_{[\mathbf{f}]}$.

In order to compute the $\mathbf{E}_1$ term of the spectral sequence, we need to describe δ_0 in more detail. The differential $\delta_{\mathbf{N}}$ on $\mathbf{N}(\mathcal{P})$ is defined by dualizing the operad composition $\mathcal{P}[r,s] \otimes_{\Sigma_s} \mathcal{P}[s,t] \longrightarrow \mathcal{P}[r,t]$ (cf. equation (3.28)). For certain pairs of surjections, the morphisms $\mathcal{P}[f] \otimes \mathcal{P}[g] \to \mathcal{P}[f \circ g]$ preserve homogeneity ($\mathbf{h}(f \circ g) = \mathbf{h}(f) + \mathbf{h}(g)$). This occurs when the only compositions are of the type $\mathcal{P}(1) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(m)$ or $\mathcal{P}(m) \otimes \mathcal{P}(1) \xrightarrow{\circ_i} \mathcal{P}(m)$. Such compositions involve only the structure of $\mathcal{P}$ as a Σ-module, not the full operad structure. In such a composition, if the elements are expressed in reduced form, without the factors $\mathcal{P}(1)$, the only operation is a permutation of factors. Define a dual map to composition of this type

$$\delta'_{r,t} : \mathcal{P}^{\#}[f] \longrightarrow \bigoplus_{\substack{f=f_1\circ f_2 \\ \mathbf{h}(f)=\mathbf{h}(f_1)+\mathbf{h}(f_2)}} \mathcal{P}^{\#}[f_1] \otimes \mathcal{P}^{\#}[f_2].$$

Preserving homogeneity means that $[t] \xrightarrow{f} [r]$ and $[t] \xrightarrow{f_2} [s] \xrightarrow{f_1} [r]$ correspond to the same r-tuple of trees (as in Figure 8) which implies that the factors $\alpha_i \in \mathcal{P}^{\#}(f^{-1}(i))$, $i \in [r]$ are partitioned into two subsets, but no new elements are created:

$$\delta'_{r,t}(\alpha_1 \otimes \cdots \otimes \alpha_r) = \sum_{\sigma} \epsilon(\sigma)(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(j)}) \otimes (\alpha_{\sigma(j+1)} \otimes \cdots \otimes \alpha_{\sigma(r)}),$$

where $\epsilon(\sigma)$ is the Koszul sign factor (3.96) coming from the symmetry in **dgVec** and the summation is taken over all $(j, r-j)$-unshuffles σ; see (2.5). Therefore the differential $\delta_0 : \mathbf{E}_0^{p,q} \to \mathbf{E}_0^{p,q+1}$ acts on a component $\mathcal{P}^{\#}[[\mathbf{f}]]$ by mapping it to a sum of components $\mathcal{P}^{\#}[[\mathbf{g}]]$ where the trees $T_{[\mathbf{f}]}$ and $T_{[\mathbf{g}]}$ are the same but there is one more level of vertices in $T_{[\mathbf{g}]}$. This can be interpreted as refining the **f**-ordering of the vertices of the tree $T_{[\mathbf{f}]}$ into the **g**-ordering.

PROPOSITION 3.64. *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{P}^{\#}[[\mathbf{f}]] \arrow[r, "\psi_{[\mathbf{f}]}"] \arrow[d, "\delta_{0,\mathbf{f},\mathbf{g}}"'] & \mathcal{P}^{\#}[T_{[\mathbf{f}]}] \arrow[d, "1\!\!1"] \\ \mathcal{P}^{\#}[[\mathbf{g}]] \arrow[r, "\psi_{[\mathbf{g}]}"] & \mathcal{P}^{\#}[T_{[\mathbf{g}]}]. \end{tikzcd}$$

Here $\psi_{[\mathbf{f}]}$ and $\psi_{[\mathbf{g}]}$ are the isomorphisms defined in (3.43) and $\delta_{0,\mathbf{f},\mathbf{g}}$ is the 'matrix component' of δ_0. In terms of the ordered tensor products $\bigotimes \mathcal{P}^{\#}(In(v_i))$ defined in equation (3.42), the effect of δ_0 is a reordering of the tensor factors.

PROOF. Before passing to equivalence classes, for a particular pair of surjection sequences, the matrix component $\delta_{0,\mathbf{f},\mathbf{g}} : \mathcal{P}^{\#}[\mathbf{f}] \longrightarrow \mathcal{P}^{\#}[\mathbf{g}]$ of the differential is just a reordering of the factors $\mathcal{P}^{\#}(f_j^{-1}(i))$ of the reduced form It is clear that the equivalence classes $[\mathbf{g}]$ and $[\mathbf{f}]$ correspond to the same nested unordered partitions and

$$\mathcal{P}^{\#}[[\mathbf{g}]] \cong \mathcal{P}^{\#}[[\mathbf{f}]].$$

The composite morphism

$$\psi_{[\mathbf{f}]}\delta_{0,\mathbf{f},\mathbf{g}} : \mathcal{P}^{\#}[[\mathbf{g}]] \cong \mathcal{P}^{\#}[[\mathbf{f}]] \xrightarrow{\psi[\mathbf{f}]} \mathcal{P}^{\#}[T_{[\mathbf{f}]}] = \mathcal{P}^{\#}(T_{[\mathbf{g}]})$$

is the same as the isomorphism $\psi[\mathbf{g}]$; therefore, the diagram commutes. ∎

Let $(\mathbf{N}(\mathcal{P}), \delta_{\mathbf{N}})$ be the categorical cobar complex with the $\square$-differential. In the spectral sequence relative to the homogeneity filtration, the $\mathbf{E}_0^{p,*}$ term is a direct sum

$$\mathbf{E}_0^{p,*} \cong \bigoplus_{\{[\mathbf{f}]\,|\,\mathbf{h}(\mathbf{f})=p\}} \mathcal{P}^{\#}[[\mathbf{f}]]. \tag{3.45}$$

For a tree T with p vertices, define

$$\mathbf{E}_0^{p,*}[T] := \bigoplus_{\{[\mathbf{f}]\,|\,T_{[\mathbf{f}]}\cong T\}} \mathcal{P}^{\#}[[\mathbf{f}]]. \tag{3.46}$$

Then Proposition 3.64 implies that $\mathbf{E}_0^{p,*}(T)$ is a δ_0-subcomplex and we have the decomposition:

$$\mathbf{E}_0^{p,*} \cong \bigoplus_{\{T\,|\,|T|+1=p\}} \mathbf{E}_0^{p,*}[T]. \tag{3.47}$$

Having determined the coboundary operator δ_0 at the first stage of the spectral sequence, we proceed to calculate the cohomology. The calculation is based on a new Koszul algebra, which we call the 'surjection algebra,' defined as follows.

DEFINITION 3.65. The *surjection algebra* B is the associative $\mathbf{k}$-algebra with basis $\{1\} \cup \{b_f\}$, where $\{b_f\}$ is indexed by the set of nondecreasing proper (not bijective) surjections f of the finite sets $[n]$, $n \geq 1$, that is the functions $\{h_{\langle m_1, \ \ ,m_n\rangle}\}$ from Lemma 1.69. The defining relations for multiplication of basis elements are

$$b_f * b_g = \begin{cases} 0, & \text{if } f \circ g \text{ is not defined,} & \text{(case 1)} \\ 0, & \text{if } f \circ g \text{ is defined, but } \mathbf{h}(f \circ g) < \mathbf{h}(f) + \mathbf{h}(g), & \text{(case 2)} \\ b_{f \circ g}, & \text{if } f \circ g \text{ is defined and } \mathbf{h}(f \circ g) = \mathbf{h}(f) + \mathbf{h}(g). & \text{(case 3)} \end{cases}$$

If we use the same definition of homogeneity for the b_f as for the f, it is clear that the surjection algebra is generated as an algebra by elements b_f of homogeneity one. (In terms of Lemma 1.69, $f = h_{\langle 1, \ \ ,1,i,1 \ \ 1\rangle}$.) Moreover, the defining relations are quadratic:

$$b_f * b_g = \begin{cases} 0 & \text{in cases 1 or 2 and} \\ b_{f'} * b_{g'} & \text{in case 3 if } f \circ g = f' \circ g' \end{cases}$$

and therefore B is graded. The relations in case 3 simply say that all products of generators representing the same surjection are equal in the surjection algebra B

We claim that B is a Koszul algebra in the classical sense defined by Priddy in his seminal paper [**Pri70**]. The main point is that the special form of the relations insures the existence of a Poincaré-Birkhoff-Witt (PBW) basis for B.

PROPOSITION 3.66. *The surjection algebra has a PBW basis.*

PROOF. Recall the definition of a PBW basis from [**Pri70**, Section 5.1]. Such a basis consists of 'admissible expressions' which are certain products of the generators (homogeneity one elements). Assume that a simple order has been defined on the generators. Then a lexicographic order is defined on the products of generators. Shorter words come first in the lexicographic ordering, and for two words of the same length, $w = b_1 * b_2 * \cdots * b_p$ is smaller than $w' = b'_1 * b'_2 * \cdots * b'_p$ if in the left-most position where the two words differ $b_i < b'_i$. The two conditions required for the admissible expressions are the following.

(i) If the product of admissible expressions $(b_1 * \cdots * b_p) * (b'_1 * \cdots * b'_p)$ is not itself an admissible expression, then the combined word is less than all the terms which appear in its representation as a linear combination of admissible expressions.
(ii) If $b_1 * b_2 * \cdots * b_p$ is an admissible expression, then so are all the products $b_1 * b_2 * \cdots * b_q$ and $b_{q+1} * b_{q+2} * \cdots * b_p$ for $q < p$.

To show that the surjection algebra has a PBW basis, define a simple order on the (homogeneity one) generators. Then for any surjection g of homogeneity p, consider the set of all its factorizations into a product $b_g = b_{f_1} * \cdots * b_{f_p}$ of p generators of homogeneity one and pick as the the admissible expression representing b_g the word $b_{f_1} * b_{f_2} * \cdots * b_{f_p}$ which is maximal in the lexicographic ordering. From this definition it follows immediately that if the product of admissible expressions $(b_{f_1} * \cdots * b_{f_p}) * (b_{f'_1} * \cdots * b_{f'_p})$ is not itself an admissible expression, then the admissible expression representing this product has a label which is greater. There is no need to consider linear combinations. This is condition (i) required of a PBW basis. Condition (ii) is also an obvious consequence of our definition of 'admissible expression' and the definition of lexicographic order. ∎

COROLLARY 3.67. *The surjection algebra B is a Koszul algebra. Therefore its cohomology algebra is isomorphic to the dual Koszul algebra $B^!$, which by definition has generators β_f corresponding bijectively to the generators b_f of B, and defining relations orthogonal to those of B:*

$$\beta_f * \beta_g = -\beta_{f'} * \beta_{g'},$$

if the composition $f \circ g = f' \circ g'$ is defined and preserves homogeneity.

PROOF. See [**Pri70**, Theorems 2.5 and 5.3].

In order to complete the calculation of the δ_0-cohomology as given in Theorem 3.69 below, we describe in more detail the structure of the cochain complex $C(B, \mathbf{k}) := \mathit{Hom}_{\mathbf{k}}(B^{\otimes p}, \mathbf{k})$ for the (classical) Hochschild cohomology of the surjection algebra B with coefficients in $\mathbf{k}$. The basis $\{b_{f_1} \otimes b_{f_2} \otimes \cdots \otimes b_{f_p}\}$ of $B^{\otimes p}$ splits into the sum of two complementary subspaces

$$B^{\otimes p} \cong U_p \oplus V_p,$$

where

$$U_p = \mathrm{Span}\{b_{f_1} \otimes b_{f_2} \otimes \cdots \otimes b_{f_p} \,|\, \mathrm{domain}(f_{i-1}) \neq \mathrm{codomain}(f_i) \text{ for some } i\}$$

and

$$V_p = \mathrm{Span}\{b_{f_1} \otimes b_{f_2} \otimes \cdots \otimes b_{f_p} \,|\, \mathrm{domain}(f_{i-1}) = \mathrm{codomain}(f_i) \text{ for all } i\}.$$

Letting $C^p_U :=$ annihilator (V_p) and $C^p_V :=$ annihilator (U_p), we have a vector space decomposition $C^p(B, \mathbf{k}) = C^p_U \oplus C^p_V$. The bar differential $\partial : B^{\otimes p} \to B^{\otimes(p-1)}$ is defined by

$$\partial(b_{f_1} \otimes \cdots \otimes b_{f_p}) = \sum (-1)^i b_{f_1} \otimes \cdots \otimes b_{f_i} * b_{f_{i+1}} \otimes \cdots \otimes b_{f_p},$$

therefore, $\partial(U_p) \subseteq U_{p-1}$ and $\partial(V_p) \subseteq V_{q-1}$. Passing to the duals gives $\delta C^p_U \subseteq C^{p+1}_U$ and $\delta C^p_V \subseteq C^{p+1}_V$, which shows that $C^*(B, \mathbf{k})$ splits as a complex,

$$C^*(B, \mathbf{k}) \cong C^*_U \oplus C^*_V. \tag{3.48}$$

Let $\{b^{\#}_f\}$ be the dual basis to $\{b_f\}$, then C^p_V has a basis

$$\{b_{\mathfrak{f}} := b^{\#}_{f_1} \otimes b^{\#}_{f_2} \otimes \cdots \otimes b^{\#}_{f_p} \,|\, \mathrm{domain}(f_{i-1}) = \mathrm{codomain}(f_i), 1 < i \leq p\}$$

and the differential is given by

$$\begin{aligned}\delta_B(b^{\#}_{f_1}\otimes b^{\#}_{f_2}\otimes\cdots\otimes b^{\#}_{f_p}) &= \sum(-1)^i b^{\#}_{f_1}\otimes\cdots\otimes\delta(b^{\#}_{f_i})\otimes\cdots\otimes b^{\#}_{f_p},\\ \delta_B(b^{\#}_{f}) &= \sum_{\{b_f=b_g*b_h\}} b^{\#}_{g}\otimes b^{\#}_{h}.\end{aligned}$$

There is a further direct sum decomposition

$$C^*_V = \bigoplus_{T\in\mathcal{T}ree} C^*_V[T], \text{ where } C^p_V[T] := \operatorname{Span}\{b^{\#}_{\mathbf{f}} \mid T_{[\mathbf{f}]}\cong T\}. \tag{3.49}$$

PROPOSITION 3.68. *Given $T\in\mathcal{T}ree(n)$, let $p := |T|+1$. The cohomologies of the subcomplexes $C^*_V[T]$ are as follows:*

$$H^q(C_V[T]) = \begin{cases} 0, & \text{for } 1\le q<p, \text{ and}\\ det(T), & \text{for } q=p.\end{cases}$$

PROOF. The direct sum decompositions (3.48) and (3.49) imply

$$H(B,\mathbf{k})\cong H(C^*_U)\oplus H(C^*_V) \text{ and } H(C_V)\cong\bigoplus_{T\in\mathcal{T}ree} H(C^*_V[T]).$$

The Koszul condition for B implies that the cohomology algebra is generated by $H^1(B)$ and therefore the homogeneity of any nonzero cohomology class equals its cochain degree. The homogeneity of any cochain $b^{\#}_{\mathbf{f}} = b^{\#}_{f_1}\otimes b^{\#}_{f_2}\otimes\cdots\otimes b^{\#}_{f_p}\in C^q_V[T]$ for the tree T in the statement of the proposition is p, the number of vertices in T. Therefore the cohomology $H(C^*_V[T])$ vanishes in dimensions $q<p$, proving the first assertion

Corollary 3.67 implies that the pth cohomology group of $C^*_V[T]$ is spanned by terms $\beta_{\mathbf{f}} := \beta_{f_1}*\beta_{f_2}*\cdots*\beta_{f_p}$ for surjection sequences $\mathbf{f}$ such that $T_{[\mathbf{f}]}\cong T$, subject to the relations

$$\beta_{f_1}*\cdots*\beta_{f_i}*\beta_{f_{i+1}}*\cdots\beta_{f_p} = -\beta_{f_1}*\cdots*\beta_{f'_i}*\beta_{f'_{i+1}}*\cdots\beta_{f_p} \tag{3.50}$$

for any two surjection sequences

$$\begin{aligned}\mathbf{f} &= [n]\xrightarrow{f_p}[r_{p-1}]\cdots[r_{i+1}]\xrightarrow{f_{i+1}}[r_i]\xrightarrow{f_i}[r_{i-1}]\cdots[r_1]\xrightarrow{f_1}[1] \text{ and}\\ \mathbf{f}' &= [n]\xrightarrow{f_p}[r_{p-1}]\cdots[r_{i+1}]\xrightarrow{f'_{i+1}}[r'_i]\xrightarrow{f'_i}[r_{i-1}]\cdots[r_1]\xrightarrow{f_1}[1],\end{aligned}$$

differing in two adjacent positions, but determining the same tree. See Figure 8. The definition of $T_{[f]}$ implies that to each surjection f_i there is a unique associated vertex of the tree $T\cong T_{[f]}$. This vertex has a unique outgoing edge which we will denote e_i. The root edge is e_1 and the internal edges of T are $e_2,\dots,e_p$. Define a map $\varphi: H(C^*_V[T])\to det(T)$ by $\varphi(c\beta_{f_1}*\beta_{f_2}*\cdots*\beta_{f_p}) = c\,e_2\wedge\cdots\wedge e_p\in det(T)$. We claim this is a $\Sigma(T)$-equivariant isomorphism. It is clearly onto, since $det(T)$ is one-dimensional and the map is a nonzero linear map. We will show that the domain $H(C^*_V[T])$ is one-dimensional, carries the sign representation of $\Sigma(T)$ and φ is equivariant. Let $o_{\mathbf{f}}$ be the $\mathbf{f}$-order of T introduced in Definition 3.58 and define

$$o(T) := \{o_{\mathbf{f}} \mid \mathbf{f}\in SS^p_n,\ T_{[\mathbf{f}]}\cong T\}. \tag{3.51}$$

Consider two orderings $o_{\mathbf{f}}$ and $o_{\mathbf{f}'}$ which are related by transposing the order of two vertices v,w in adjacent levels, that is, the orders agree except on the vertices v,w but

$$v\le_{\mathbf{f}} w, \text{ whereas } v\ge_{\mathbf{f}'} w.$$

Equation (3.50) then implies that the corresponding elements $\beta_{\mathfrak{f}}$ and $\beta_{\mathfrak{f}'}$ are equal up to a change of sign. Since any two orderings are related by a sequence of such transpositions, the space generated by the $\beta_{\mathfrak{f}}$ is one-dimensional and carries the signum representation of the subgroup of Σ_p (acting as permutations of the vertices of T) which stabilizes the set $o(T)$. This subgroup contains $\Sigma(T)$ and obviously the sign factors agree for a permutation acting on the vertices and the same permutation acting on the corresponding edges. ∎

THEOREM 3.69. *Given a quadratic operad $\mathcal{P}$ and $T \in \mathcal{R}tree(n)$, let $p = |T|+1$.*

(i) *There is a* dgVec *isomorphism*

$$(\mathbf{E}_0^{p,*}[T], \delta_0) \cong (\mathcal{P}^{\#}[T] \otimes C_V^*[T], \mathbb{1} \otimes \delta_B). \tag{3.52}$$

(ii) *At the next stage of the spectral sequence, the $\mathbf{E}_1^{p,*}[T]$ term is given by:*

$$\mathbf{E}_1^{p,q}[T] := \begin{cases} H^{p+q}(\mathbf{E}_0^{p,*}[T]) = 0, & \textit{for } q < 0, \textit{ and} \\ H^p(\mathbf{E}_0^{p,0}[T]) \cong \mathcal{P}^{\#}[T] \otimes \det(T), & \textit{for } q = 0. \end{cases} \tag{3.53}$$

Taking the direct sum over T one has:

$$\mathbf{E}_1^{p,q} \cong \bigoplus \mathbf{E}_1^{p,q}[T] = \begin{cases} 0, & \textit{for } q < 0, \textit{ and} \\ \bigoplus \mathcal{P}^{\#}[T] \otimes \det(T) \cong \mathbf{C}(\mathcal{P})^p, & \textit{for } q = 0, \end{cases} \tag{3.54}$$

where the direct sum runs over all $T \in \mathcal{R}tree$ such that $|T| + 1 = p$.

(iii) *The isomorphisms in (3.54) combine to give a* dgVec *isomorphism*

$$\mathbf{E}_1^{*,0} \cong \mathbf{C}(\mathcal{P}). \tag{3.55}$$

PROOF. The isomorphism (3.52) follows from formulas (3.46) and (3.49), Proposition 3.64 and the definition of the surjection algebra. The description of the $\mathbf{E}_1$-term as a graded $\mathbf{k}$-vector space follows immediately from Proposition 3.68.

The differential δ_1 increases the number of vertices in a tree by one and is dual to the operad composition law for the operad E. This is precisely the definition of the differential δ on the cobar-dual (tree complex), $\mathbf{C}(\mathcal{P})$. ∎

COROLLARY 3.70. *If $\mathcal{P}$ is a Koszul operad in* dgVec, *then*

$$H^p(\mathbf{N}(\mathcal{P})(n)) = \begin{cases} 0, & \text{if } p \neq n-1 \text{ and} \\ \mathcal{P}^!(n) \otimes \mathrm{sgn}_n, & \text{if } p = n-1. \end{cases}$$

We now use these results to prove the equivalence of two characterizations of the Koszul condition for a quadratic operad. For the convenience of the reader, we restate Theorem 3.43:

THEOREM 3.43. *A quadratic operad $\mathcal{P}$ is Koszul if and only if for all arities $n \geq 2$ the cohomology of the Koszul complexes $\mathbf{K}(\mathcal{P}^!)(n)$ (Definition 3.41) vanishes in all degrees.*

PROOF. The proof is based on another spectral sequence coming from a decreasing filtration of the complex $\mathbf{N}(\mathcal{P})$, which 'pinches' the tail factor $\mathcal{P}^{\#}[r_{p-1}, n]$ in $\mathbf{N}(\mathcal{P})(n)$ by requiring that $r_{p-1} \geq s$, that is,

$$\overline{F}^s(\mathbf{N}(\mathcal{P})(n)^p) := \begin{cases} \bigoplus \mathcal{P}^{\#}(r_1) \otimes_{\Sigma_{r_1}} \cdots \otimes_{\Sigma_{r_{p-1}}} \mathcal{P}^{\#}[r_{p-1}, n], & \text{for } p > 1 \text{ and } s \geq 1, \\ \mathcal{P}^{\#}(n), & \text{for } p = 1 \text{ and } s = 1, \\ 0, & \text{for } p = 1 \text{ and } s > 1, \end{cases}$$

where the direct sum for the case $p > 1$ and $s > 1$ runs over all $1 < r_1 < \cdots < r_{p-1} < n$ with $s \leq r_{p-1}$.

At arity n and $\square$-degree p, the filtration is constant for $s < p$ (since $p \leq r_{p-1}$) and strictly decreasing for $p \leq s \leq n-1$:

$$\mathbf{N}(\mathcal{P})(n)^p = \overline{F}^p(\mathbf{N}(\mathcal{P})) \supsetneq \cdots \supsetneq \overline{F}^{n-1}(\mathbf{N}(\mathcal{P})(n)) \supsetneq \overline{F}^n(\mathbf{N}(\mathcal{P})(n)) = 0.$$

For $s = 1$,

$$\overline{\mathbf{E}}_0^{1,t}(\mathbf{N}(\mathcal{P})(n)) = \begin{cases} \mathcal{P}^\#(n), & \text{for } t = 0, \text{ and} \\ 0, & \text{for } t \neq 0. \end{cases}$$

For $s > 1$,

$$\overline{\mathbf{E}}_0^{s,t}(\mathbf{N}(\mathcal{P})(n)) = \frac{\overline{F}^s(\mathbf{N}(\mathcal{P})(n)^{s+t})}{\overline{F}^{s+1}(\mathbf{N}(\mathcal{P})(n)^{s+t})} \cong \mathbf{N}(\mathcal{P})(s)^{s+t-1} \otimes_{\Sigma_s} \mathcal{P}^\#[s,n]. \tag{3.56}$$

The term $\mathbf{N}(\mathcal{P})(s)^{s+t-1}$ reflects the fact that the tail (last term) in $\mathbf{N}(\mathcal{P})(s)^{s+t}$ is held fixed at $\mathcal{P}[s,n]$ and the number of remaining $\square$-products is reduced by one. We observed above that the filtration is trivial when the filtration degree is less than the $\square$-degree, which implies that $\overline{\mathbf{E}}_0^{s,t} \neq 0$ only if $s+t \leq s$, that is, $t \leq 0$. Therefore, the spectral sequence for $\mathbf{N}(\mathcal{P})(n)$ is in the fourth quadrant and, furthermore, it is between the lines $s+t = 1$ and $s+t-n = 1$.

If $\delta_{\mathbf{N}}(n)$ denotes, for $n \geq 2$, the differential on $\mathbf{N}(\mathcal{P})(n)$, then restricting δ_0 to $\overline{\mathbf{E}}_0^{s,*}(\mathbf{N}(\mathcal{P})(n))$ gives $\delta_{\mathbf{N}}(s) \otimes_{\Sigma_s} 1\!\!1$. Therefore

$$\overline{\mathbf{E}}_1^{s,t}(\mathbf{N}(\mathcal{P})(n)) = \begin{cases} H^{s+t-1}(\mathbf{N}(\mathcal{P})(s)) \otimes_{\Sigma_s} (\mathcal{P}^\#[s,n]) & \text{for } 2 \leq s < n,\ s+t > 1, \\ \mathcal{P}^\#(n) & \text{for } s = 1,\ t = 0 \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{3.57}$$

To prove the theorem in one direction, assume that the operad $\mathcal{P}$ is Koszul. Then Corollary 3.70 and equation (3.57) imply that the $\overline{\mathbf{E}}_1$-stage of the spectral sequence has only one nonzero row, $t = 0$:

$$\overline{\mathbf{E}}_1^{s,0}(\mathbf{N}(\mathcal{P})(n)) \cong \begin{cases} \mathcal{P}^!(s) \otimes_{\Sigma_s} (\mathrm{sgn}_s \otimes \mathcal{P}^\#[s,n]) & \text{for } 2 \leq s \leq n-1 \text{ and} \\ \mathcal{P}^\#(n) & \text{for } s = 1. \end{cases}$$

This is just the Koszul complex $\mathbf{K}(\mathcal{P}^!)(n)$ truncated at the nth term $\mathcal{P}^!(n)$. The fact that $\overline{\mathbf{E}}_1$ is supported on a single row $t = 0$ means that $\overline{\mathbf{E}}_2 \cong \overline{\mathbf{E}}_\infty$. But the complex $\mathbf{N}(\mathcal{P})(n)$ has cohomology only in degree $n-1$ and by Corollary 3.70 $H^{n-1}(\mathbf{N}(\mathcal{P})(n)) = \mathcal{P}^!(n) \otimes \mathrm{sgn}_n$. Therefore,

$$H^s(\overline{\mathbf{E}}_1^{*,0}(\mathbf{N}(\mathcal{P})(n))) = \overline{\mathbf{E}}_2^{s,0}(\mathbf{N}(\mathcal{P})(n)) \cong \begin{cases} \mathcal{P}^!(n) \otimes \mathrm{sgn}_n, & \text{if } s = n-1, \\ 0, & \text{if } s \neq n-1. \end{cases}$$

Thus the augmented complex $\overline{\mathbf{E}}_1^{*,0}(\mathbf{N}(\mathcal{P})(n)) \longrightarrow \mathcal{P}^!(n) \otimes \mathrm{sgn}_n$ is exact. But this is the full Koszul complex, which proves the theorem in one direction: 'Koszul implies exactness of the Koszul complex.'

To prove the theorem in the other direction, we must prove that if the Koszul complexes are exact, then the augmented complexes

$$\mathbf{N}(\mathcal{P})(n)^1 \to \mathbf{N}(\mathcal{P})(n)^2 \to \cdots \to \mathbf{N}(\mathcal{P})(n)^{n-1} \to \mathcal{P}^!(n) \otimes \mathrm{sgn}_n \tag{3.58}$$

are exact. The proof is by induction on n. For $n = 2$, $\mathbf{N}(\mathcal{P})(2)^1 = \mathcal{P}^{\#}(2)$ and so the complex (3.58) expresses the isomorphism $\mathcal{P}^{\#}(2) \cong \mathcal{P}^{!}(2) \otimes \mathrm{sgn}_2$. For $n = 3$, the complex (3.58) is the Koszul complex and so is exact by assumption.

Assuming we have shown that this complex is exact for $r < n$, we want to prove that it is exact for n. The calculation in the first part of the proof for indices r which are strictly less than n shows that

$$\overline{\mathbf{E}}_1^{r,0}(\mathbf{N}(\mathcal{P})(n)) \cong \mathcal{P}^{!}(r) \otimes_{\Sigma_r} (\mathrm{sgn}_r \otimes \mathcal{P}^{\#}[r,n]).$$

Once again $\overline{\mathbf{E}}_1$ is supported on a single row, so $\overline{\mathbf{E}}_2 \cong \overline{\mathbf{E}}_\infty$ which is, as before, the Koszul complex truncated by the last term $\mathcal{P}^{!}(n)$. By assumption, the Koszul complexes are exact so $\overline{\mathbf{E}}_\infty^{n-1,0} \cong \mathcal{P}^{!}(n)$ and therefore the augmented complex $\mathbf{N}(\mathcal{P})(n)^* \longrightarrow \mathcal{P}^{!}(n) \otimes \mathrm{sgn}_n$ is exact as required in the definition of a Koszul operad. ■

3.7. Coalgebras and coderivations

Let $\mathcal{P} := \{\mathcal{P}(n)\}_{n \geq 1}$ be an operad in the category $\mathtt{dgVec}$ of differential graded vector spaces over a field $\mathbf{k}$ of characteristic zero such that each $\mathcal{P}(n)$, $n \geq 2$, is of *finite type* and $\mathcal{P}(1) = \mathbf{k}$. Recall that we denoted the graded $\mathbf{k}$-dual of $\mathcal{P}$ by $\mathcal{P}^{\#} = \{\mathcal{P}^{\#}(n)\}_{n \geq 1}$, where $\mathcal{P}^{\#}(n) = \mathcal{P}(n)^{\#} = \mathit{Hom}_{\mathbf{k}}(\mathcal{P}, \mathbf{k})$. All tensor products are $\otimes_{\mathbf{k}}$ unless otherwise indicated and $\otimes_{\Sigma_n} := \otimes_{\mathbf{k}[\Sigma_n]}$. The coendomorphism operad $\mathcal{C}o\mathcal{E}nd_X$ was introduced in Definition 1.9.

DEFINITION 3.71. A $\mathcal{P}$-*coalgebra* X is a dg vector space equipped with a dg operad map $\lambda : \mathcal{P} \to \mathcal{C}o\mathcal{E}nd_X$ with components

$$\lambda_X(n) : \mathcal{P}(n) \longrightarrow \mathcal{C}o\mathcal{E}nd_X(n) = \mathit{Hom}_{\mathbf{k}}(X, X^{\otimes n}),\ n \geq 1,$$

or, equivalently, a dg vector space X with a sequence of dg maps

$$\bar{\lambda}_X(n) : X \longrightarrow \mathit{Hom}_{\mathbf{k}[\Sigma_n]}(\mathcal{P}(n), X^{\otimes n}) \cong \left(\mathcal{P}^{\#}(n) \otimes X^{\otimes n}\right)^{\Sigma_n},\ n \geq 1. \tag{3.59}$$

The last isomorphism uses the assumption that $\mathcal{P}(n)$ is of finite type. Morphisms of $\mathcal{P}$-coalgebras are defined in the obvious way Since the field $\mathbf{k}$ has characteristic zero, given a Σ_n-module $V \in \mathtt{dgVec}$, there is a projection $p : V \to V^{\Sigma_n}$ onto the Σ_n-invariants in V,

$$p(v) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma(v).$$

This projection factors through the space of coinvariants V_{Σ_n} and defines an isomorphism $V_{\Sigma_n} \cong V^{\Sigma_n}$:

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d] & V^{\Sigma_n} \\ V_{\Sigma_n} \arrow[ur, "\cong"'] & \end{tikzcd} \tag{3.60}$$

Therefore, we can also express the coalgebra structure map in (3.59) in the form

$$\bar{\lambda}_X(n) : X \longrightarrow \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} X^{\otimes n}. \tag{3.61}$$

When there is no possibility of confusion, the subscript in $\lambda_X(n)$ and $\overline{\lambda}_X(n)$ will be deleted.

DEFINITION 3.72. Let $\mathcal{P}$ be an operad in $\mathtt{dgVec}$. A $\mathcal{P}$-coalgebra X is called *nilpotent* if, for any $x \in X$, $\overline{\lambda}(n)(x) = 0$ for all n sufficiently large.

Our attention will focus on the definition of a 'standard construction' of the chain complex for the homology and cohomology of an operad algebra which uses the cofree nilpotent coalgebra over the quadratic dual operad; see Section 3.2. This approach reproduces the standard constructions for classical algebras: the bar construction for associative algebras, the Chevalley-Eilenberg chain complex for Lie algebras and the Harrison chain complex for commutative associative algebras. It should come as no surprise that the analogous construction works for algebras over an arbitrary quadratic Koszul operad.

The quadratic dual operad also provides an approach to the theory of strong homotopy algebras which were originally defined in terms of coherent sets of multivariable maps. The latter approach made manifest the description of a strong homotopy algebra as an algebra over a suitable differential graded resolution of the original operad, which in this case coincides with $\mathbf{D}(\mathcal{P}^!)$, the minimal model of $\mathcal{P}$; see Section 3.10.

We start by introducing cofree nilpotent coalgebras. Given a symmetric monoidal category $\mathcal{C}$ and an operad $\mathcal{P}$ in $\mathcal{C}$, the cofree coalgebra functor is, by definition, a right adjoint of the obvious forgetful functor from the category of $\mathcal{P}$-coalgebras in $\mathcal{C}$ to the category $\mathcal{C}$. In this section, $\mathcal{C}$ can be either $\mathtt{dgVec}$ or $\mathtt{gVec}$. The former case allows for operads with nontrivial differentials and can thus accommodate objects such as cofree A_∞-coalgebras. The latter case, $\mathcal{C} = \mathtt{gVec}$, is more restrictive, but still general enough for our purposes, so we choose this framework. Most of our constructions can be easily modified to $\mathcal{C} = \mathtt{dgVec}$.

DEFINITION 3.73. Let $\mathcal{P}$ be an operad in $\mathtt{gVec}$. A *cofree nilpotent $\mathcal{P}$-coalgebra* functor $\mathcal{F}^{\mathtt{c}}_{\mathcal{P}}$ is a right adjoint to the forgetful functor $U^{\mathtt{c}}_{\mathcal{P}}$ from the category of nilpotent $\mathcal{P}$-coalgebras in $\mathtt{gVec}$ to the category $\mathtt{gVec}$, that is, $\mathcal{F}^{\mathtt{c}}_{\mathcal{P}}$ establishes a bijective correspondence:

$$Hom_{\mathcal{P}\text{-}\mathtt{coalg}}(C, \mathcal{F}^{\mathtt{c}}_{\mathcal{P}}(X)) \longleftrightarrow Hom_{\mathtt{gVec}}(U^{\mathtt{c}}_{\mathcal{P}}(C), X)$$

for any nilpotent $\mathcal{P}$-coalgebra C.

In the standard notation for adjoints, $U^{\mathtt{c}}_{\mathcal{P}} \dashv \mathcal{F}^{\mathtt{c}}_{\mathcal{P}}$. Such a functor, if it exists, is unique up to equivalence. We now construct an example of such a functor.

DEFINITION 3.74. Let $\mathcal{P}$ be an operad such that each $\mathcal{P}(n)$, $n \geq 2$, is of finite type and X a vector space . Define

$$\mathcal{S}^{\mathtt{c}}_{\mathcal{P}}(X) := \bigoplus_{n\geq 1} \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} X^{\otimes n} \cong \bigoplus_{n\geq 1} \left(\mathcal{P}(n)^{\#} \otimes X^{\otimes n}\right)^{\Sigma_n},$$

with the grading

$$\mathcal{S}^{\mathtt{c}}_{\mathcal{P}}(X)^n := \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} X^{\otimes n}. \tag{3.62}$$

The graded vector space $\mathcal{S}^{\mathtt{c}}_{\mathcal{P}}(X)$ is a $\mathcal{P}$-coalgebra with structure maps:

$$\overline{\lambda}(n) : \mathcal{S}^{\mathtt{c}}_{\mathcal{P}}(X) \longrightarrow \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} \mathcal{S}^{\mathtt{c}}_{\mathcal{P}}(X)^{\otimes n},\ n \geq 1, \tag{3.63}$$

given by

$$
\begin{aligned}
\mathcal{S}^{c}_{\mathcal{P}}(X) \supset \mathcal{S}^{c}_{\mathcal{P}}(X)^{m} &= \mathcal{P}(m)^{\#} \otimes_{\Sigma_m} X^{\otimes m} \xrightarrow{\chi^{\#}\otimes id} \\
&\longrightarrow \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} \left(\bigoplus \left(\bigotimes_{i=1}^{n} \mathcal{P}(m_i)^{\#} \right) \otimes_{\Sigma_{m_1,\dots,m_n}} \otimes X^{\otimes m} \right) \\
&\xrightarrow{\xi} \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} \left(\bigoplus \left(\bigotimes_{i=1}^{n} (\mathcal{P}(m_i)^{\#} \otimes_{\Sigma_{m_i}} X^{\otimes m_i}) \right) \right) \\
&\cong \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} \left(\bigoplus (\mathcal{F}^{c}_{\mathcal{P}}(X)^{m_1} \otimes \cdots \otimes \mathcal{F}^{c}_{\mathcal{P}}(X)^{m_n}) \right),
\end{aligned}
\tag{3.64}
$$

where the summation is over $\{(m_1, \dots, m_n) \mid m_i \geq 1, m_1 + \cdots + m_n = m\}$ and $\Sigma_{m_1,\dots,m_n} := \Sigma_{m_1} \times \cdots \times \Sigma_{m_n}$. The map $\chi^{\#}$ in the first line is dual to the operad composition.

$$\chi_{n;m_1,\dots,m_n} : \mathcal{P}(n) \otimes_{\Sigma_n} (\mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n)) \longrightarrow \mathcal{P}(m)$$

and therefore maps into the space of invariants. The map ξ in the third line is simply reordering using the symmetry in **gVec**.

The unit $X \to \mathcal{S}^{c}_{\mathcal{P}}(X)$ and the counit $\mathcal{S}^{c}_{\mathcal{P}}(X) \to X$ of the adjunction are, respectively, the standard inclusion

$$X \to \mathbf{k} \otimes X = \mathcal{P}(1) \otimes X \hookrightarrow \mathcal{S}^{c}_{\mathcal{P}}(X)$$

and projection

$$\pi : \mathcal{S}^{c}_{\mathcal{P}}(X) \twoheadrightarrow X \otimes \mathcal{P}(1) = X \otimes \mathbf{k} \cong X$$

THEOREM 3.75. *The functor $X \mapsto \mathcal{S}^{c}_{\mathcal{P}}(X)$ is a cofree nilpotent $\mathcal{P}$-coalgebra functor*

$$\mathcal{F}^{c}_{\mathcal{P}}(X) \cong \mathcal{S}^{c}_{\mathcal{P}}(X).$$

PROOF. The verification that the $\mathcal{P}$-coalgebra $\mathcal{S}^{c}_{\mathcal{P}}(X)$ is nilpotent follows immediately from the fact that the coalgebra structure map $\overline{\lambda}(n)$ applied to $\mathcal{S}^{c}_{\mathcal{P}}(X)^{m}$ partitions the tensor products $X^{\otimes m}$ into n nonempty factors. Such a partition cannot involve more than m components; therefore,

$$\overline{\lambda}(n)|_{\mathcal{S}^{c}_{\mathcal{P}}(X)^{m}} = 0 \quad \text{if} \quad n > m.$$

The assertion that $\mathcal{S}^{c}_{\mathcal{P}}(X)$ is cofree is equivalent to the fact that for any nilpotent $\mathcal{P}$-coalgebra C and for a degree zero $\mathbf{k}$-linear map $\psi : C \to X$, there exists exactly one coalgebra homomorphism $\tilde{\psi} : C \to \mathcal{S}^{c}_{\mathcal{P}}(X)$ making the following diagram commutative:

$$
\begin{tikzcd}
C \arrow[r, "\tilde{\psi}"] \arrow[dr, "\psi"'] & \mathcal{S}^{c}_{\mathcal{P}}(X) \arrow[d, "\pi"] \\
& X.
\end{tikzcd}
\tag{3.65}
$$

Suppose that $\tilde{\psi}$ is such a homomorphism and let $\tilde{\psi}^{n} : C \to \mathcal{S}^{c}_{\mathcal{P}}(X)^{n}$ be the nth graded component of $\tilde{\psi}$. Since $\tilde{\psi}$ is a homomorphism,

$$(\mathbb{1} \otimes_{\Sigma_n} \tilde{\psi}^{\otimes n}) \overline{\lambda}_C(n) = \overline{\lambda}_{\mathcal{S}}(n) \tilde{\psi},$$

for each $n \geq 2$. Applying the projection

$$\mathcal{P}^{\#}(n) \otimes_{\Sigma_n} \mathcal{S}^{c}_{\mathcal{P}}(X)^{\otimes n} \twoheadrightarrow \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} (\mathcal{S}^{c}_{\mathcal{P}}(X)^{1})^{\otimes n} \cong \mathcal{P}^{\#}(n) \otimes_{\Sigma_n} X^{\otimes n}$$

to the above equation, we obtain the following commutative diagram·

$$\begin{tikzcd} C \arrow[r, "\tilde{\psi}^n"] \arrow[d, "\overline{\lambda}_C(n)"'] & \mathcal{S}^c_{\mathcal{P}}(X)^n \arrow[d, "\overline{\lambda}_{\mathcal{S}}(n)"] \\ \mathcal{P}^{\#}(n) \otimes C^{\otimes n} \arrow[r, "\mathbb{1} \otimes \psi^{\otimes n}"] & \mathcal{P}^{\#}(n) \otimes X^{\otimes n}. \end{tikzcd}$$

It follows directly from this diagram that $\tilde{\psi}^n = (\mathbb{1} \otimes_{\Sigma_n} \psi^{\otimes n}) \circ \overline{\lambda}_C(n)$. Since C is nilpotent, the sum

$$\tilde{\psi} := \sum_{n \geq 1} (\mathbb{1} \otimes \psi^{\otimes n}) \circ \overline{\lambda}_C(n). \tag{3.66}$$

is 'locally' (that is, when evaluated on a specific element of C) finite. It is easy to see that this definition of $\tilde{\psi}$ satisfies the required condition and is uniquely determined. ■

Henceforth, we will use the symbol $\mathcal{F}^c_{\mathcal{P}}(X)$ only for the cofree nilpotent $\mathcal{P}$-coalgebra.

REMARK 3.76. We have to restrict to nilpotent coalgebras in order that the definition of $\tilde{\psi}$ in (3.66) involves a finite sum. This is why $\mathcal{S}^c_{\mathcal{P}}(X)$ is <u>not</u> the free coalgebra without the nilpotency condition. See [**Blo85**].

It is worth noting that the construction is dual to the construction of free algebras, so the linear dual of the free algebra is, under mild finite dimension assumptions, the cofree nilpotent coalgebra. If we drop the nilpotency assumption, cofree coalgebras will be much more complicated objects than just 'duals' of free algebras. See also [**Blo85, Fox93**].

REMARK 3.77. It is also worth pointing out that since all $\mathcal{P}(n)$ are of finite type, there is a (noncanonical) isomorphism $\mathcal{P}(n) \cong \mathcal{P}^{\#}(n)$. Moreover, since $\mathbf{k}$ has characteristic zero, the space of Σ_n-coinvariants is isomorphic to the space of Σ_n-invariants, which implies that $\mathcal{F}^c_{\mathcal{P}}(X)$ and $\mathcal{F}_{\mathcal{P}}(X)$ (free nilpotent coalgebra and free algebra on X) are isomorphic in $\mathtt{gVec}$, though not canonically.

EXAMPLE 3.78. Let X be a graded vector space and $\mathcal{A}ss = \{\mathcal{A}ss(n)\}_{n \geq 1}$ with $\mathcal{A}ss(n) = \mathbf{k}[\Sigma_n]$ the operad for associative algebras; see Definition 1.12. Then

$$\mathcal{F}^c_{\mathcal{A}ss}(X)^n := (\mathbf{k}[\Sigma_n] \otimes X^{\otimes n})^{\Sigma_n} \cong \mathbf{k}[\Sigma_n] \otimes_{\mathbf{k}[\Sigma_n]} \otimes X^{\otimes n} \cong X^{\otimes n},$$

therefore $\mathcal{F}^c_{\mathcal{A}ss}(X) = \bigoplus_{n \geq 1} \mathcal{F}^c_{\mathcal{A}ss}(X)^n$ is the standard tensor coalgebra, that is, $\mathcal{F}^c_{\mathcal{A}ss}(X)^n \cong \bigotimes^n X$, with coproduct component of arity n of $\overline{\lambda}(m)$ given by the sum over all ordered partitions of a tensor product into a sequence of m subfactors:

$$\overline{\lambda}(m)(x_1 \otimes \cdots \otimes x_n) := \sum_{1 < j_1 < \quad < j_{m-1} < n} e_m \otimes (x_1 \otimes \cdots \otimes x_{j_1}) \otimes \cdots \otimes (x_{j_{m-1}+1} \otimes \cdots \otimes x_n),$$

where $e_m \in \mathbf{k}[\Sigma_m]$ is the basis element corresponding to the identity of Σ_m.

EXAMPLE 3.79. Let X be a graded vector space and $\mathcal{C}om = \{\mathcal{C}om(n)\}_{n \geq 1}$ with $\mathcal{C}om(n) = \mathbf{k}$ the operad for commutative algebras; see Definition 1.12. Then

$$\mathcal{F}^c_{\mathcal{C}om}(X)^n := \mathbf{1} \otimes_{\mathbf{k}[\Sigma_n]} X^{\otimes n} \cong \mathcal{S}ym^n(X),$$

the nth symmetric tensor product. As a vector space, $\mathcal{F}^c_{\mathcal{C}om}(X)$ is the subspace of symmetric tensors $Sym(X)$ in the tensor coalgebra $\mathcal{F}^c_{\mathcal{A}ss}(X)$ from Example 3.78. The coalgebra structure on the tensor coalgebra preserves the symmetric tensors and $\mathcal{F}^c_{\mathcal{C}om}(X)$ has the structure of a subcoalgebra of $\mathcal{F}^c_{\mathcal{A}ss}(X)$. Since $\mathcal{C}om$ is a quadratic operad with a single generator $\mu \in \mathcal{C}om(2)$, the coalgebra structure is generated by the degree 2 coproduct $\overline{\lambda} : Sym(X) \to Sym(X) \otimes Sym(X)$ corresponding to μ, which can also be identified with the coproduct given by identifying $Sym(X)$ with the universal enveloping algebra $\mathcal{U}(X, [-,-] = 0)$ of X considered as a Lie algebra with zero Lie bracket [**Ser65**, Chapter III].

DEFINITION 3.80. A $\mathcal{P}$-coalgebra X is said to be *cogenerated* by a subspace Y if there is a $\mathbf{k}$-linear projection $p : X \to Y$ such that the corresponding morphism of $\mathcal{P}$-coalgebras $X \to \mathcal{F}^c_{\mathcal{P}}(Y)$ is injective as a map of modules.

Under the adjoint relation in Definition 3.73, $1\!\!1_{\mathcal{F}^c_{\mathcal{P}}(X)}$ corresponds to the projection $\pi : \mathcal{F}^c_{\mathcal{P}}(X) \to X$. Therefore, $\mathcal{F}^c_{\mathcal{P}}(X)$ is cogenerated by X in the sense of the previous definition. We call $\mathcal{F}^c_{\mathcal{P}}(X)$ the cofree nilpotent $\mathcal{P}$-coalgebra *cogenerated* by X.

DEFINITION 3.81. A *coderivation* of a $\mathcal{P}$-coalgebra X is a graded $\mathbf{k}$-linear endomorphism D of X such that

$$\lambda_X(n)(b) \circ D = (-1)^{|b||D|} \left(\sum_{k=0}^{n-1} 1\!\!1_X^{\otimes k} \otimes D \otimes 1\!\!1_X^{\otimes n-k-1} \right) \circ \lambda_X(n)(b)$$

in $Hom_{\mathbf{k}}(X, X^{\otimes n})$, for all $n \geq 2$ and $b \in \mathcal{P}(n)$.

REMARK 3.82. A standard way of dealing with coderivations (or derivations) is to extend the field $\mathbf{k}$ to the commutative ring $\mathbf{k}_\epsilon := \mathbf{k}[\epsilon]/(\epsilon^2)$ of *dual numbers* and extend all the previous structures from the category of vector spaces over $\mathbf{k}$ to the category of $\mathbf{k}_\epsilon$-modules. Then D is a coderivation of the coalgebra X if and only if $1\!\!1_X + \epsilon D$ is an automorphism of $X_\epsilon := X \otimes \mathbf{k}_\epsilon$ as a $\mathcal{P}_\epsilon := \mathcal{P} \otimes \mathbf{k}_\epsilon$-coalgebra. The adjoint relation in Definition 3.73 applied to $\mathcal{P}_\epsilon$-coalgebras implies the next proposition which can also be proved directly as was Theorem 3.75.

PROPOSITION 3.83. *There is a bijection*

$$Coder_{\mathcal{P}\text{-coalg}}(\mathcal{F}^c_{\mathcal{P}}(X)) \xleftrightarrow{\cong} Hom_{\mathsf{gVec}}(\mathcal{F}^c_{\mathcal{P}}(X), X). \tag{3.67}$$

The bijection maps a coderivation D to $\pi \circ D \in Hom_{\mathsf{gVec}}(\mathcal{F}^c_{\mathcal{P}}(X), X)$ and conversely, $\alpha \in Hom_{\mathsf{gVec}}(\mathcal{F}^c_{\mathcal{P}}(X), X)$ is sent to the coderivation D_α defined by the formula

$$D_\alpha := \sum_{n \geq 1} \left(\sum_{j=0}^{n-1} 1\!\!1_{\mathcal{P}^\#(n)} \otimes (\pi^{\otimes j} \otimes \alpha \otimes \pi^{\otimes n-j-1}) \right) \circ \overline{\lambda}(n). \tag{3.68}$$

The sum in (3.68) is infinite but nilpotency implies that it involves only finitely many terms when applied to an element of $\mathcal{F}^c_{\mathcal{P}}(X)$.

DEFINITION 3.84. Let $Coder^p(\mathcal{F}^c_{\mathcal{P}}(X))$ be the subspace of coderivations of $\mathcal{F}^c_{\mathcal{P}}(X)$ which increase by p the degree induced by the grading on X The grading on $\mathcal{F}^{c}_{\mathcal{P}}(X)$ by the number of X factors (see (3.62)) induces a second grading on $Coder^{p}(\mathcal{F}^{c}_{\mathcal{P}}(X))$, namely

$$Coder^{p}(\mathcal{F}^{c}_{\mathcal{P}}(X)) = \bigoplus_{n\geq 0} Coder^{p,n}(\mathcal{F}^{c}_{\mathcal{P}}(X)),$$

where

$$Coder^{p,n}(\mathcal{F}^{c}_{\mathcal{P}}(X)) = \{D \in Coder^{p}(\mathcal{F}^{c}_{\mathcal{P}}(X)) \,|\, (\pi\circ D)(\mathcal{F}^{c}_{\mathcal{P}}(X)^{q}) = 0, \text{ for } q \neq n+1\}.$$

LEMMA 3.85. *The map* $\omega : Coder^{p,n}(\mathcal{F}^{c}_{\mathcal{P}}(X)) \to Hom^{p}(\mathcal{F}^{c}_{\mathcal{P}}(X)^{n+1}, X)$, *given by* $\omega(D) := \pi \circ D$, *is an isomorphism for all* p, $n \geq 0$.

PROOF. The lemma is an immediate consequence of Proposition 3.83. ∎

In the rest of the chapter, the operad $\mathcal{P}$ will be assumed to be quadratic. The following theorem is taken from [**FM97**].

THEOREM 3.86. *Let* V *be a graded vector space and let* $X := \downarrow V$. *Then there is a natural one-to-one correspondence between* $\mathcal{P}$*-algebra structures* $a : \mathcal{P} \to End_V$ *on* V *and coderivations* $d \in Coder^{1,1}(\mathcal{F}^{c}_{\mathcal{P}^!}(X))$ *of the* $\mathcal{P}^!$*-coalgebra* $\mathcal{F}^{c}_{\mathcal{P}^!}(X)$ *with* $d^2 = 0$.

The next lemma, which will be used in the proof of Theorem 3.86, is based on observing that the definition of $\mathcal{F}^{c}_{\mathcal{P}}(X)$ makes sense even when $\mathcal{P}$ is only a Σ-module. Then of course $\mathcal{F}^{c}_{\mathcal{P}}(X)$ will not be a coalgebra, but only a graded vector space so, in this general setting, it makes no sense to talk about coderivations. The use of desuspension $\downarrow V$ in the lemma is a device for keeping track of degrees.

LEMMA 3.87. *Let* V *be a graded vector space and* $X := \downarrow V$. *If* E *is a* Σ*-module in* gVec *and* $E^{\vee}$ *the Czech dual (Definition 3.36), then for each* $n \geq 2$, *there is an isomorphism*

$$\Phi_n : Hom^{i}(\mathcal{F}^{c}_{E^{\vee}}(X)^{n}, X) \cong Hom^{i+1-n}_{\Sigma_n}(E(n), \mathcal{E}nd_V(n)).$$

PROOF. By definition

$$\begin{aligned} Hom^{i}(\mathcal{F}^{c}_{E^{\vee}}(X)^{n}, X) &\cong Hom^{i}\Big(E(n)^{\vee\#} \otimes_{\Sigma_n} X^{\otimes n}, X\Big) \\ &\cong Hom^{i}\Big((E(n)\otimes \mathrm{sgn}_n) \otimes_{\Sigma_n} X^{\otimes n}, X\Big). \end{aligned}$$

Let $\varphi : V^{\otimes n} \xrightarrow{\cong} \mathrm{sgn}_n \otimes (\downarrow V)^{\otimes n} = \mathrm{sgn}_n \otimes X^{\otimes n}$ be the isomorphism of degree $-n$ described in Lemma 3.12 and

$$\pi : E(n)\otimes \mathrm{sgn}_n \otimes X^{\otimes n} \twoheadrightarrow (E(n)\otimes \mathrm{sgn}_n) \otimes_{\Sigma_n} X^{\otimes n}$$

be the projection, then we define the desired isomorphism

$$\Phi_n : Hom^{i}\Big((E(n)\otimes \mathrm{sgn}_n) \otimes_{\Sigma_n} X^{\otimes n}, X\Big) \longrightarrow Hom^{i+1-n}_{\Sigma_n}(E(n), Hom(V^{\otimes n}, V))$$

by

$$\Phi_n(f)(e)(v_1\otimes\cdots\otimes v_n) := \uparrow f(\pi(e\otimes\varphi(v_1\otimes\cdots\otimes v_n))),$$

for $v_1, \ldots, v_n \in V$ and $e \in E(n)$. ∎

PROOF OF THEOREM 3.86. To fix the notation, let $\mathcal{P} = \langle E; R\rangle$ as in Section 3.2. Then $E^\vee(2) \cong \mathcal{P}^!(2)$ and $E^\vee(n) = 0$ for $n \neq 2$. By Lemma 3.85, we have

$$Coder^{1,1}(\mathcal{F}^c_{\mathcal{P}^!}(X)) \cong Hom^1(\mathcal{F}^c_{\mathcal{P}^!}(X)^2, X) \cong Hom^1(\mathcal{F}^c_{E^\vee}(X)^2, X).$$

Since $\Psi(E)$ is free and E is concentrated in arity two,

$$Hom_{\Psi\mathtt{Op}}(\Psi(E), \mathcal{E}nd_V) \cong Hom_{\Sigma\text{-}\mathtt{Mod}}(E, \mathcal{E}nd_V) = Hom_{\Sigma_2}(E(2), \mathcal{E}nd_V(2)).$$

We define a map Ω of vector spaces by the commutativity of the following diagram, where Φ_2 is the isomorphism from Lemma 3.87:

$$\begin{tikzcd}
Coder^{1,1}(\mathcal{F}^c_{\mathcal{P}^!}(X)) \arrow[d, "\cong"'] & Hom_{\Psi\mathtt{Op}}(\Psi(E), \mathcal{E}nd_V) \arrow[l, "\Omega"'] \arrow[d, "\cong"] \\
Hom^1\left(\mathcal{F}^c_{E^\vee}(X)^2, X\right) & Hom^0_{\Sigma_2}(E(2), \mathcal{E}nd_V(2)). \arrow[l, "\Phi_2^{-1}"']
\end{tikzcd}$$

One can verify directly that $a \in Hom_{\mathtt{Op}}(\Psi(E), End_V)$ defines an algebra structure on V, i.e. factors through $\Psi(E)/(R)$, if an only if the coderivation $d = \Omega(a)$ satisfies $d^2 = 0$. We will give an explicit proof that $d^2 = 0$ in Section 3.8. ■

We now consider an application to the homotopy theory of operad algebras. In Section 3.10 a strongly homotopy $\mathcal{P}$-algebra is defined as an operad algebra over a minimal model of $\mathcal{P}$. As we will show in Example 3.118, for a quadratic Koszul operad $\mathcal{P}$, $\mathbf{D}(\mathcal{P}^!)$ is just such a minimal model. Proposition 3.88 below describes these strong homotopy algebras in terms of coderivations. In fact, this description is equivalent to the classical form of axioms in terms of structure operations, as explained in Remark 3.89.

PROPOSITION 3.88. *Let V be a differential graded vector space, $X = \downarrow V$ and $\mathcal{P}$ a Koszul operad. A coderivation $D \in Coder^1(\mathcal{F}^c_{\mathcal{P}^!}(X))$ with $D^2 = 0$ is equivalent to a strongly homotopy $\mathcal{P}$-algebra structure on V.*

PROOF. It follows from the above remarks that we need to prove that a $\mathbf{D}(\mathcal{P}^!)$-algebra structure on V is equivalent to a coderivation $D \in Coder^1(\mathcal{F}^c_{\mathcal{P}^!}(X))$ with $D^2 = 0$. Since

$$\mathbf{D}(\mathcal{P}^!) \cong \Psi(\mathfrak{s}^{-1}\uparrow(\mathcal{P}^{!\#})),$$

such a structure is given by a dg pseudo-operad map from $\Psi(\mathfrak{s}^{-1}\uparrow(\mathcal{P}^{!\#}))$ to $\mathcal{E}nd_V$. It is crucial here that this map is a differential graded map, but we will initially consider the map on the operad level. Since $\Psi(\mathfrak{s}^{-1}\uparrow(\mathcal{P}^{!\#}))$ is free,

$$\begin{aligned}
(3.69)\qquad Hom_{\Psi\mathtt{Op}}(\Psi(\mathfrak{s}^{-1}(\uparrow \mathcal{P}^{!\#})), \mathcal{E}nd_V) &\cong Hom^0_{\Sigma\text{-}\mathtt{Mod}}(\mathfrak{s}^{-1}\uparrow\mathcal{P}^{!\#}, \mathcal{E}nd_V) \\
&\cong \prod_{n\geq 2} Hom^0_{\Sigma_n}(\downarrow^{n-2}\mathcal{P}^!(n)^{\#}\otimes \mathrm{sgn}_n, \mathcal{E}nd_V(n)) \\
&\cong \prod_{n\geq 2} Hom^{2-n}_{\Sigma_n}(\mathcal{P}^!(n)^\vee, \mathcal{E}nd_V(n)) \cong \prod_{n\geq 2} Hom^1(\mathcal{F}^c_{\mathcal{P}^!}(X)^n, X).
\end{aligned}$$

The isomorphism in the last line follows from Lemma 3.87 and $((\mathcal{P}'(n))^\vee)^\vee = \mathcal{P}'(n)$. On the other hand, by Proposition 3.83,

$$Coder^1(\mathcal{F}^c_{\mathcal{P}'}(X)) \cong \prod_{n\geq 1} Hom^1(\mathcal{F}^c_{\mathcal{P}'}(X)^n, X), \tag{3.70}$$

which differs from the last expression in (3.69) only by the term $Hom^1(\mathcal{F}^c_{\mathcal{P}'}(X)^1, X) \cong Hom^1(X, X)$. The discrepancy arises because the equivalences in (3.69) do not take into account the differential on V. For $V \in \mathbf{dgVec}$, the differential defines an element of $Hom^1(V, V) \cong Hom^1(X, X)$. Moreover, an element

$$\mu \in Hom_{\Psi\mathrm{Op}}(\Psi(\mathfrak{s}^{-1}\uparrow(\mathcal{P}'^{\#})), \mathcal{E}nd_V)$$

defines the structure of a *differential* graded algebra on V if and only if it is a differential graded map, that is,

$$\mu d_{\mathbf{D}(\mathcal{P}')} = d_{\mathcal{E}nd_V}\mu. \tag{3.71}$$

Equation (3.71) is equivalent to the condition on the infinite sequence

$$\mu = (\mu_2, \dots) \in \prod_{n\geq 2} Hom^{2-n}_{\Sigma_n}(\mathcal{P}'^\vee(n)\otimes_{\Sigma_n} V^{\otimes n}, V) \cong Hom_{\Psi\mathrm{Op}}(\Psi(\mathfrak{s}^{-1}\uparrow(\mathcal{P}'^{\#})), \mathcal{E}nd_V)$$

which says that the augmented sequence

$$(\mu_1, \mu_2, \dots) \in \prod_{n\geq 1} Hom^{2-n}(\mathcal{P}'^\vee(n)\otimes_{\Sigma_n} V^{\otimes n}, V) \cong Coder^1(\mathcal{F}^c_{\mathcal{P}'}(X))$$

with $\mu_1 = d_V$ defines a coderivation $D_\mu \in Coder^1(\mathcal{F}^c_{\mathcal{P}'}(X))$ with $D^2_\mu = 0$. This proves the proposition. ∎

REMARK 3.89. The requirement that $\mathcal{P}$ is a quadratic Koszul operad which appears in Proposition 3.88 is necessary to guarantee that the dual dg operad $\mathbf{D}(\mathcal{P}')$ is quasi-isomorphic to $\mathcal{P}$ and therefore provides a minimal model. When this proposition is compared with Theorem 3.86, it becomes clear that strong homotopy algebras are obtained by passing to the differential graded setting with differential $d_1 \in Coder^{1,0}(\mathcal{F}^c_{\mathcal{P}'}(X))$ and then adding higher terms to the coderivation $d_2 \in Coder^{1,1}(\mathcal{F}^c_{\mathcal{P}'}(X))$ (which would describe a strict $\mathcal{P}$-algebra), that is, perturbing $d_1 + d_2$ to $D = d_1 + d_2 + d_3 + d_4 + \cdots$ with $d_i \in Coder^{1,i-1}(\mathcal{F}^c_{\mathcal{P}'}(X))$, $i \geq 3$.

EXAMPLE 3.90. By Proposition 3.88, a *strongly homotopy associative* (also called A_∞) structure on V is equivalent to a coderivation $D \in Coder^1(\mathcal{F}^c_{\mathcal{A}ss}(\downarrow V))$ with $D^2 = 0$. Such a coderivation is given by a sequence of maps $m_n : V^{\otimes n} \to V$ of degree $2-n$, for $n \geq 1$, satisfying the sequence of A_∞-identities given in Example 3.132.

Similarly, a *strongly homotopy Lie algebra* (also called L_∞- or sh Lie algebra) structure on V is equivalent to a coderivation $D \in Coder^1(\mathcal{F}^c_{\mathcal{C}om}(\downarrow V))$ with $D^2 = 0$ given by a sequence of brackets $(\lambda_n)_{n\geq 1}$, where the map $\lambda_n : V^{\otimes n} \to V$ has degree $2-n$ and is skew-symmetric. The λ_n are required to satisfy a sequence of L_∞-identities given in Example 3.133.

Strongly homotopy associative *commutative* algebras (also called C_∞- or balanced A_∞-algebras) are discussed in Example 3.134.

3.8. The homology and cohomology of operad algebras

The results of Section 3.7 can be used to define homology or cohomology theories of $\mathcal{P}$-algebras for any quadratic operad. These theories coincide with the standard theories: Hochschild, Harrison and Chevalley-Eilenberg for $\mathcal{P} = \mathcal{A}ss$, $\mathcal{C}om$ and $\mathcal{L}ie$, respectively. Recall that we assume that the characteristic of the ground field $\mathbf{k}$ is zero.

Moreover, the homology (or cohomology) can be used to give a very convenient characterization of the Koszul property: A quadratic operad $\mathcal{P}$ is Koszul if and only if the homology of the free $\mathcal{P}$-algebra $\mathcal{F}_{\mathcal{P}}(W)$ vanishes in all degrees other than 1 and equals W in degree 1, for each graded vector space W. The chain complex $C_{\mathcal{P}}(V)$ defining the homology for a $\mathcal{P}$-algebra V is the free operad coalgebra of the quadratic dual operad $\mathcal{F}^{c}_{\mathcal{P}^!}(\downarrow V)$.

Since the construction of the chain complex $C_{\mathcal{P}}(V)$ involves the quadratic dual of $\mathcal{P}$, we must assume that $\mathcal{P}$ is a quadratic operad, in particular, an operad in **gVec**. On the other hand, the $\mathcal{P}$-algebra V may have, in principle, a nontrivial differential. The differential of the chain complex $C_{\mathcal{P}}(V)$ would then consists of two parts – one induced by the $\mathcal{P}$-algebra multiplication on V and one induced by the differential of V. We could obtain in this way, for example, the 'two-step' cohomology of dg associative algebras ([**Mac63a**, X.11]). As we will not need this generality, we will assume that V is an algebra in **gVec**, though our constructions easily generalize to **dgVec**-algebras. In order to be consistent with our conventions that the differentials have degree 1, the chain complex $C_{\mathcal{P}}(V)$ will be defined in negative degrees.

In general, we will express everything in terms of differentials of degree 1 although analogous theories can be described for differentials of degree -1, in which case we will adopt the convention that complexes with upper indices have d of degree 1 and with lower indices degree -1.

DEFINITION 3.91. Let $\mathcal{P}$ be a quadratic operad in **Vec**, V a $\mathcal{P}$-algebra and $X := \downarrow V$. The *$\mathcal{P}$-algebra chain complex* of V is defined in *$\mathcal{P}$-degree* $-n$ by

$$C^{-n}_{\mathcal{P}}(V) := \mathcal{F}^{c}_{\mathcal{P}^!}(X)^n, \text{ for } n \geq 1.$$

The *$\mathcal{P}$-differential* $d_{\mathcal{P}}$ is the coderivation of $\mathcal{F}^{c}_{\mathcal{P}^!}(X)$ corresponding to the $\mathcal{P}$-algebra structure on V as defined in Theorem 3.86.

Since, for $\mathcal{P} = \mathcal{A}ss$, the chain complex $C^{*}_{\mathcal{P}}(V)$ coincides (modulo grading) with the bar construction $B(V) = B(\mathbf{k}, V, \mathbf{k})$ on an associative algebra V, we may also call the dg $\mathcal{P}^!$-coalgebra $(C^{*}_{\mathcal{P}}(V), \partial_{\mathcal{P}})$ the *bar construction on the $\mathcal{P}$-algebra V*. The complex $C^{*}_{\mathcal{P}}(V)$ is actually bigraded,

$$C^{*,*}_{\mathcal{P}}(V) = \bigoplus_{n\geq 1,p} C^{-n,p}_{\mathcal{P}}(V).$$

The *internal degree* p is induced by the grading of V. Let us note that the complex $C_{\mathcal{P}}$ has an alternative presentation; for $n \geq 1$:

$$\begin{aligned} C^{-n}_{\mathcal{P}}(V) &= \mathcal{F}^{c}_{\mathcal{P}^!}(X)^n \\ &\cong \mathcal{P}^!(n)^{\#} \otimes_{\Sigma_n} (\downarrow V)^{\otimes n} \cong \mathcal{P}^!(n)^{\#} \otimes_{\Sigma_n} (\mathrm{sgn}_n \otimes \downarrow^n (V^{\otimes n})) \\ &\cong (\downarrow^n \mathcal{P}^!(n)^{\#} \otimes \mathrm{sgn}_n)_{\Sigma_n} \otimes V^{\otimes n} \cong \downarrow \mathfrak{s}^{-1}(\mathcal{P}^{!\#})(n) \otimes_{\Sigma_n} V^{\otimes n}, \end{aligned}$$

where we used (3.60) and the definition of $\mathfrak{s}^{-1}$. Then also the differential $d_{\mathcal{P}}$ has an alternative description. Let $B(n) := \mathfrak{s}^{-1}(\mathcal{P}^{!\#})(n)$, then

$$
\begin{aligned}
d_{\mathcal{P}} : C_{\mathcal{P}}^{-n}(V) &= \downarrow (B(n) \otimes_{\Sigma_n} V^{\otimes n}) \\
&\longrightarrow \downarrow (B(n-1) \otimes_{\Sigma_{n-1}} B[n-1,n] \otimes_{\Sigma_n} V^{\otimes n}) \\
&\cong \downarrow (B(n-1) \otimes_{\Sigma_{n-1}} (\downarrow \mathcal{P})[n-1,n] \otimes_{\Sigma_n} V^{\otimes n}) \\
&\longrightarrow \downarrow ((B(n-1)) \otimes_{\Sigma_{n-1}} V^{\otimes n-1}) = C_{\mathcal{P}}^{-n+1}(V).
\end{aligned}
\tag{3.72}
$$

The arrow in the second line is the desuspension of the dual to the $\mathcal{P}^!$-structure morphisms :

$$\mathcal{P}^!(n-1) \otimes_{\Sigma_{n-1}} \mathcal{P}^![n-1,n] \longrightarrow \mathcal{P}^!(n)$$

and is easily derived from the isomorphism in Lemma 3.42. This map has degree 0 relative to the $\mathcal{P}$-degree. The isomorphism in the third line comes from the isomorphism $B(2) = \mathfrak{s}^{-1}(\mathcal{P}^{!\#})(2) \cong \downarrow \mathcal{P}(2)$. The last arrow is the extension of the structure morphism $\mathcal{P}(2) \otimes V^{\otimes 2} \longrightarrow V$, but due to the presence of $\downarrow \mathcal{P}(2)$ instead of $\mathcal{P}(2)$, this map has degree 1 relative to the $\mathcal{P}$-degree, as required of the $\mathcal{P}$-differential.

REMARK 3.92. Since all the morphisms appearing in (3.72) commute with maps of $\mathcal{P}$-algebras and every $\mathcal{P}$-algebra is the image of a free $\mathcal{P}$-algebra, in order to prove that $d_{\mathcal{P}}^2 = 0$, it is sufficient to consider the special case of $V := \mathcal{F}_{\mathcal{P}}(W)$. In this case the desired property follows from the fact (proved in Proposition 3.94 below) that $d_{\mathcal{P}}$ is equivalent to the dual of the Koszul differential, which was shown to have square zero in Lemma 3.45. We will need this description in the proof of Theorem 3.95 which characterizes the Koszul property for an operad $\mathcal{P}$ in terms of the homology of the free $\mathcal{P}$-algebra.

DEFINITION 3.93. Let $\mathcal{P}$ be a quadratic operad in $\mathtt{Vec}$ and V a $\mathcal{P}$-algebra. The *$\mathcal{P}$-algebra homology* (also called the *operadic homology*) of V with trivial coefficients is defined as

$$H_n(V) := H^{-n}(C_{\mathcal{P}}(V), d_{\mathcal{P}}),\ n \geq 1.$$

The next proposition describing $C_{\mathcal{P}}^*(\mathcal{F}_{\mathcal{P}}(W))$ in terms of the Koszul complexes introduced in Definition 3.41 leads to a very convenient criterion for deciding if an operad is Koszul; see Theorem 3.95 below. Our proof avoids the lengthy argument in the proof of Proposition 4.2.12 in [**GK94**].

PROPOSITION 3.94. *For the free $\mathcal{P}$-algebra $V = \mathcal{F}_{\mathcal{P}}(W)$ generated by W, the complex $C_{\mathcal{P}}^*(\mathcal{F}_{\mathcal{P}}(W))$ decomposes as a direct sum*

$$C_{\mathcal{P}}^*(\mathcal{F}_{\mathcal{P}}(W)) \cong \bigoplus_{m \geq 1} C_{\mathcal{P}}^*\{W, m\},$$

where the subcomplexes $C_{\mathcal{P}}^*\{W,m\}$ are isomorphic to the dual Koszul complexes $(\mathbf{K}(\mathcal{P}^!)(m))^\#$ tensored with $W^{\otimes m}$:

$$C_{\mathcal{P}}^{-n}\{W,m\} := \begin{cases} (\mathbf{K}(\mathcal{P}^!)(m)^\#)^n \otimes W^{\otimes m}, & \textit{for } 2 \leq m,\ 1 \leq n \leq m, \\ \mathcal{P}(1) \otimes W \cong W, & \textit{for } m = 1,\ n = 1 \textit{ and} \\ 0, & \textit{for } m = 1,\ n \neq 1, \end{cases}$$

with the differential $\delta_{\{W,m\}} := \delta^{\#}_{\mathbf{K}} \otimes 1\!\!1$. *Therefore,*

$$H_i(C_{\mathcal{P}}\{W,m\}) = \begin{cases} H_i\left(\mathbf{K}(\mathcal{P}^!)(m)^{\#}\right) \otimes W^{\otimes m}, & \textit{for all } i \textit{ if } m > 1, \\ \mathcal{P}(1) \otimes W \cong W, & \textit{for } m = 1,\ i = 1 \textit{ and} \\ 0, & \textit{for } m = 1,\ i \neq 1. \end{cases}$$

PROOF. For $n \geq 1$ the term of $\mathcal{P}$-degree $-n$ term decomposes as

$$\begin{aligned} C^{-n}_{\mathcal{P}}(\mathcal{F}_{\mathcal{P}}(W)) &= \downarrow s^{-1}(\mathcal{P}^{!\#})(n) \otimes_{\Sigma_n} \mathcal{F}_{\mathcal{P}}(W)^{\otimes n} \\ &\cong \downarrow s^{-1}(\mathcal{P}^{!\#})(n) \otimes_{\Sigma_n} \left(\bigoplus_{1 \leq m_1, \quad, m_n} \bigotimes_{i=1}^{n} \left(\mathcal{P}(m_i) \otimes_{\Sigma_{m_i}} W^{\otimes m_i}\right) \right) \\ &\xrightarrow{\xi^{-1}} \downarrow^n (\mathcal{P}^{!\vee})(n) \otimes_{\Sigma_n} \left(\bigoplus_{\substack{\{m \mid n \leq m\} \\ m_1 + \quad + m_n = m}} (\mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n)) \otimes_{\Sigma_{m_1, \quad, m_n}} W^{\otimes(m_1 + \quad + m_n)} \right) \\ &\cong \quad \downarrow^n \bigoplus_{\{m \mid n \leq m\}} \left(\mathcal{P}^{!\vee}(n) \otimes_{\Sigma_n} \mathcal{P}[n,m]\right) \otimes_{\Sigma_m} W^{\otimes m} \\ &\cong \quad \downarrow^n \bigoplus_{\{m \mid n \leq m\}} (\mathbf{K}(\mathcal{P}^!)(m)^n)^{\#} \otimes_{\Sigma_m} W^{\otimes m}, \end{aligned}$$

where $\Sigma_{m_1, \quad, m_n} = \Sigma_{m_1} \times \cdots \times \Sigma_{m_n}$.

The first congruence follows from the definition of $\mathcal{F}^{c}_{\mathcal{P}}(W)$ and the distributivity of the tensor product over direct sums. The arrow in the next line is the inverse of the isomorphism ξ in (3.64). The next to the last congruence can be proved by a short argument using the isomorphism in Lemma 1.69 describing the structure of $\mathcal{P}[n,m]$. The last congruence follows from Definition 3.44 of the dual Koszul complex.

The differential $d_{\mathcal{P}} : C^{-n}_{\mathcal{P}}\{W,m\} \longrightarrow C^{-n+1}_{\mathcal{P}}\{W,m\}$ fixes the factor $W^{\otimes m}$. Bringing the differential inside the $\downarrow^n$, it acts only on $\mathcal{P}^{!\vee}(n) \otimes_{\Sigma_n} \mathcal{P}[n,m]$, and in the following way,

$$\begin{aligned} \mathcal{P}^{!\vee}(n) \otimes_{\Sigma_n} \mathcal{P}[n,m] &\longrightarrow (\mathcal{P}^!(n-1)^{\#} \otimes_{\Sigma_{n-1}} \mathcal{P}^![n-1,n]^{\#} \otimes \mathrm{sgn}_n) \otimes_{\Sigma_n} \mathcal{P}[n,m] \\ &\xrightarrow{\cong} (\mathcal{P}^!(n-1)^{\#} \otimes \mathrm{sgn}_{n-1}) \otimes_{\Sigma_{n-1}} (\mathcal{P}[n-1,n] \otimes_{\Sigma_n} \mathcal{P}[n,m]) \\ &\longrightarrow \mathcal{P}^{!\vee}(n-1) \otimes_{\Sigma_{n-1}} \mathcal{P}[n-1,m], \end{aligned}$$

which is the dual of the Koszul differential. The isomorphism in the second line uses $\mathcal{P}^!(2)^{\#} \otimes \mathrm{sgn}_2 \cong \mathcal{P}(2)$ and Lemma 3.42. This proves the first part of the theorem. The second part follows immediately. ∎

The next theorem is the main tool for proving that a quadratic operad is Koszul.

THEOREM 3.95. *A quadratic operad* $\mathcal{P}$ *is Koszul if and only if for any vector space* W, *the* $\mathcal{P}$*-algebra homology of the free* $\mathcal{P}$*-algebra* $\mathcal{F}_{\mathcal{P}}(W)$ *generated by* W *equals* W *in degree* 1 *and vanishes in all other degrees.*

PROOF. According to Proposition 3.94, the vanishing condition is equivalent to the exactness of all the Koszul complexes $\mathbf{K}(\mathcal{P}^!)(m)$ for $m \geq 2$. By Theorem 3.43, this is equivalent to the fact that $\mathcal{P}$ is a Koszul operad. ∎

In Example 3.38 we showed that $\mathcal{A}ss^! \cong \mathcal{A}ss$ and $\mathcal{L}ie^! \cong \mathcal{C}om$. In the following Examples 3.96 and 3.97, we will show that all these operads are Koszul.

EXAMPLE 3.96. Let V be an $\mathcal{A}ss$-algebra, then since $\mathcal{A}ss^!(n) = \mathcal{A}ss(n) = \mathbf{k}[\Sigma_n]$, we have

$$C^{-n}_{\mathcal{A}ss}(V) = \mathbf{k}[\Sigma_n] \otimes_{\mathbf{k}[\Sigma_n]} (\downarrow V)^{\otimes n} \cong (\downarrow^n \mathbf{k}[\Sigma_n] \otimes \mathrm{sgn}_n) \otimes_{\mathbf{k}[\Sigma_n]} \otimes V^{\otimes n} \cong \downarrow^n V^{\otimes n}.$$

Modulo the presence of $\downarrow V$ instead of V, the differential $d^n_{\mathcal{A}ss}$ is the same as the Hochschild *homology* differential

$$d^n_{\mathrm{Hoch}} := \sum_{i=1}^{n-1} (-1)^{i-1} (\mathbb{1}^{\otimes i-1} \otimes \mu \otimes \mathbb{1}^{n-i-1}) : V^{\otimes n} \to V^{n-1}$$

for an associative algebra V with multiplication μ and trivial coefficients. It is well known (see [**Wei94**]) that for the free associative algebra on a vector space V

$$HH_n(\mathcal{F}_{\mathcal{A}ss}(V), \mathbf{k}) = \begin{cases} 0, & \text{for } n \neq 1 \text{ and} \\ V, & \text{for } n = 1. \end{cases}$$

Therefore, by Theorem 3.95, $\mathcal{A}ss$ is a Koszul operad.

EXAMPLE 3.97 Let V be a Lie algebra. Since $\mathcal{L}ie^!(n) = \mathcal{C}om(n) = \mathbf{1}$, the trivial representation of Σ_n,

$$C^{-n}_{\mathcal{L}ie}(V) = \mathbf{1} \otimes_{\mathbf{k}[\Sigma_n]} (\downarrow V)^{\otimes n} \cong (\downarrow^n \mathrm{sgn}_n) \otimes_{\mathbf{k}[\Sigma_n]} V^{\otimes n} \cong \downarrow^n \wedge^n X.$$

As expected, $d_{\mathcal{L}ie}$ is the Chevalley-Eilenberg differential, the fact rigorously proved by Balavoine in his thesis [**Bal96**]. It is well known (see [**Wei94**]) that for the free Lie algebra on a vector space V

$$H^{\mathrm{CE}}_n(\mathcal{F}_{\mathcal{L}ie}(V), \mathbf{k}) = \begin{cases} 0, & \text{for } n \neq 1 \text{ and} \\ V, & \text{for } n = 1. \end{cases}$$

Again by Theorem 3.95 $\mathcal{L}ie$ is a Koszul operad. Since an operad $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^!$ is Koszul, this shows that $\mathcal{C}om$ is Koszul as well.

For commutative algebras, the operadic homology is the Harrison homology. More details about the homology and cohomology of operad algebras can be found in [**Bal98**].

REMARK 3.98. There are also other examples of Koszul operads, with two basic operations such as the operad $\mathcal{P}oiss$ for Poisson algebras or the operad e_2 for Gerstenhaber algebras. Their Koszulness follows from the presence of a distributive law; see [**FM97, Mar96b**]. The Koszulness of operads for Leibniz and Zinbiel algebras introduced by Loday [**LFCG01**] is discussed in [**Bal94**]. Still more exotic examples of Koszul operads can be found in [**Cha01, Cha00, CL00, LFCG01**].

We saw that to prove that a given operad is Koszul is a quite nontrivial task. In fact, a moment's reflection convinces us that each 'generic' sufficiently nontrivial operad is non-Koszul, but to prove that a *concrete* operad is non-Koszul is also very difficult.

Given a gVec-operad $\mathcal{P}$, one may consider its *generating function* $g_{\mathcal{P}} \in \mathbf{k}[[t]]$ defined by

$$g_{\mathcal{P}}(t) := \sum_{n=1}^{\infty} \dim_{\mathbf{k}}(\mathcal{P}(n)) \frac{t^n}{n!}.$$

As proved in [**GK94**, Theorem 3.3.2], if $\mathcal{P}$ is quadratic Koszul, then

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-t)) = t. \tag{3.73}$$

By calculating initial terms of generating functions, one can show that (3.73) is violated by the operad for associative anticommutative algebras, which means that this operad is not Koszul. Since $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^!$ is, dually, the operad for 'commutative Lie algebras,' that is, algebras with a commutative multiplication satisfying the Jacobi identity, is also not Koszul.

Algebras over non-Koszul operads would exhibit very strange properties. We think that this is why they almost never occur in 'real life' and all 'classical' algebras are algebras over well-behaved Koszul operads; see also Remark 3.131.

Next we sketch the cohomology theory of a $\mathcal{P}$-algebra V with coefficients in V, which is the cohomology theory 'controlling' deformations of $\mathcal{P}$-algebra structures.

DEFINITION 3.99. Let $\mathcal{P}$ be a quadratic operad and $V \in \mathtt{gVec}$ a $\mathcal{P}$-algebra. Let $X := \downarrow V$ and, for $p \in \mathbb{Z}$, $n \geq 1$,

$$C^{p,n}_{\mathcal{P}}(V;V) := Coder^{p,n}(\mathcal{F}^{c}_{\mathcal{P}^!}(X)),$$

where $Coder^{p,n}(\mathcal{F}^{c}_{\mathcal{P}^!}(X))$ is defined in Definition 3.84. Define the *$\mathcal{P}$-algebra cochain complex of V with coefficients in V* by

$$C^{*}_{\mathcal{P}}(V;V) := \bigoplus_{n\geq 1} C^{n}_{\mathcal{P}}(V;V),$$

where

$$C^{n}_{\mathcal{P}}(V;V) := \bigoplus_{p\in\mathbb{Z}} C^{p,n}_{\mathcal{P}}(V;V).$$

Let $d_{\mathcal{P}} \in Coder^{1,1}(\mathcal{F}^{c}_{\mathcal{P}^!}(X))$ be the coderivation corresponding to the $\mathcal{P}$-algebra structure on V described in Theorem 3.86. Define the coboundary operator on $C^{*}_{\mathcal{P}}(V;V)$ as the graded commutator with $d_{\mathcal{P}}$, that is, for $D \in C^{p,n}_{\mathcal{P}}(V;V)$,

$$\delta_{\mathcal{P}}(D) := d_{\mathcal{P}} \circ D - (-1)^{p} D \circ d_{\mathcal{P}}. \tag{3.74}$$

It is easy to verify that $\delta_{\mathcal{P}}$ maps $C^{p,n}_{\mathcal{P}}(V;V)$ to $C^{p+1,n+1}_{\mathcal{P}}(V;V)$ and that $\delta^{2}_{\mathcal{P}} = 0$.

DEFINITION 3.100. Let V be an algebra over a quadratic operad $\mathcal{P}$ Define the *$\mathcal{P}$-algebra* (also called *operadic*) *cohomology $H^{p,n}_{\mathcal{P}}(V;V)$ of V with coefficients in V* as

$$H^{p,n}_{\mathcal{P}}(V;V) := H^{p,n}(C^{**}_{\mathcal{P}}(V;V), \delta_{\mathcal{P}}).$$

We call p and n the *internal* and the *$\mathcal{P}$-degrees*, respectively.

REMARK 3.101. As in the homology examples above, the cohomology theories just described agree with the standard definitions. For an associative algebra, one gets the Hochschild cohomology of the algebra with coefficients in the algebra; for a Lie algebra, one gets the Chevalley-Eilenberg cohomology and for a commutative algebra, one gets the Harrison cohomology.

3.8.1. Derived functors. Let $\varphi : A \to V$ be a fixed homomorphism of $\mathcal{P}$-algebras. By definition, derivations $D \in Der_{\mathcal{P}\text{-Alg}}(A, V)$ are linear maps $D : A \to V$ satisfying the identity

$$
\begin{aligned}
(3.75)\quad & D(\alpha_A(n)(p)(a_1 \otimes \cdots \otimes a_n)) \\
&= \sum \epsilon\, \alpha_V(n)(p)(\varphi(a_1) \otimes \cdots \otimes \varphi(a_{i-1}) \otimes D(a_i) \otimes \varphi(a_{i+1}) \otimes \cdots \otimes \varphi(a_n)),
\end{aligned}
$$

where $\epsilon := (-1)^{|D|(|a_1|+\cdots+|a_{i-1}|+|p|)}$, $p \in \mathcal{P}(n)$ and $a_i \in A$ for $i = 1, \dots, n$; see Definition 1.20 for the notation.

We will show that if $\mathcal{P}$ is a quadratic Koszul operad, then the $\mathcal{P}$-algebra cohomology of V with coefficients in V is equivalent to the derived functor of $Der_{\mathcal{P}\text{-Alg}}(-, V)$ in the category of $\mathcal{P}$-algebras over V, that is, $\mathcal{P}$-algebras A with a homomorphism of $\mathcal{P}$-algebras $\varphi : A \to V$.

We establish this equivalence by showing that if $\mathcal{P}$ is a Koszul operad, then composing the free $\mathcal{P}$-algebra functor with the cofree nilpotent $\mathcal{P}$-coalgebra functor gives a canonical resolution $\rho : Can(V) \to V$ in this category,

$$
\begin{aligned}
Can(V) &:= \mathcal{F}_{\mathcal{P}}(\uparrow \mathcal{F}^c_{\mathcal{P}^!}(\downarrow V)) \\
&\cong \bigoplus_{n\geq 1} \mathcal{P}(n) \otimes_{\Sigma_n} \left(\bigoplus_{1\leq m_1, \dots, m_n} \bigotimes_{i=1}^{n} (\uparrow \mathcal{P}^{!\#}(m_i) \otimes_{\Sigma_{m_i}} (\downarrow V)^{\otimes m_i}) \right) \\
&\cong \bigoplus_{n\geq 1} \mathcal{P}(n) \otimes_{\Sigma_n} \left(\bigoplus_{\substack{\{m \mid n\leq m\} \\ m_1+\cdots+m_n=m}} (\uparrow \mathcal{P}^{!\#}(m_1) \otimes \cdots \otimes \uparrow \mathcal{P}^{!\#}(m_n)) \otimes_{\Sigma_{m_1,\dots,m_n}} (\downarrow V)^{\otimes m} \right) \\
&\cong \bigoplus_{\{m \mid n\leq m\}} \left(\uparrow^n \mathcal{P}(n) \otimes_{\Sigma_n} (\mathrm{sgn}_n \otimes \mathcal{P}^{!\#}[n, m]) \right) \otimes_{\Sigma_m} (\downarrow V)^{\otimes m} \\
&\cong \uparrow^n \bigoplus_{\{m \mid n\leq m\}} \mathbf{K}(\mathcal{P})(m)^n \otimes_{\Sigma_m} (\downarrow V)^{\otimes m}.
\end{aligned}
$$

The *resolution degree* of the term of bidegree (n, m),

$$Can^{n,m}(V) := \mathbf{K}(\mathcal{P})(m)^n \otimes_{\Sigma_m} (\downarrow V)^{\otimes m}$$

is, for $n \leq m$, defined to be $n - m$:

$$Can^k(V) := \bigoplus_{n-m=k} Can^{n,m}(V),$$

therefore, the complex equals zero in positive degrees. The total differential ∂_C (of degree 1) is the sum of two terms:

$$\partial_C|_{Can^{n,m}(V)} := \partial_{\mathbf{K}} + (-1)^{n-m} \partial_{\mathcal{F}}. \tag{3.76}$$

The first summand $\partial_{\mathbf{K}}$ is defined by:

$$\partial_{\mathbf{K}}|_{Can^{n,m}(V)} := \uparrow^{n+1} \delta_{\mathbf{K}(\mathcal{P})} \downarrow^n \otimes 1\!\!1, \tag{3.77}$$

where $\delta_{\mathbf{K}(\mathcal{P})}$ is the Koszul differential; see Definition 3.41. That definition was actually for the complex $\mathbf{K}(\mathcal{P}^!)$, whereas here we use the complex $\mathbf{K}(\mathcal{P})$ with the corresponding differential. The subscript $\mathbf{K}(\mathcal{P})$ is introduced to make the distinction clear. The second summand $\partial_{\mathcal{F}}$ is the (graded) derivation on $\mathcal{F}_{\mathcal{P}}(\uparrow \mathcal{F}^c_{\mathcal{P}^!}(\downarrow V))$ extending $\uparrow d_{\mathcal{P}} \downarrow$, where $d_{\mathcal{P}}$ is the differential $d_{\mathcal{P}}$ on $\mathcal{F}^c_{\mathcal{P}^!}(\downarrow V)$ described in Theorem 3.86. Observe that bidegree of $\partial_{\mathbf{K}}$ is $(1, 0)$ and bidegree of $\partial_{\mathcal{F}}$ is $(0, 1)$.

Let $\pi : \mathcal{F}^c_{\mathcal{P}'}(\downarrow V) \to \downarrow V$ be the projection. Define ρ as the composition

$$\rho : \mathcal{F}_{\mathcal{P}}(\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V)) \xrightarrow{\mathcal{F}_{\mathcal{P}}(\uparrow\pi\downarrow)} \mathcal{F}_{\mathcal{P}}(V) \xrightarrow{\mu_V} V, \tag{3.78}$$

where $\mu_V : \mathcal{F}_{\mathcal{P}}(V) \to V$ is given by the $\mathcal{P}$-algebra structure on V.

PROPOSITION 3.102. *Let V be a $\mathcal{P}$-algebra and $(V, 0)$ the differential graded $\mathcal{P}$-algebra concentrated in resolution degree* 0 *and with trivial differential, then*

$$\rho : (Can(V), \partial_C) \longrightarrow (V, 0) \tag{3.79}$$

is a morphism of differential graded $\mathcal{P}$-algebras. If $\mathcal{P}$ is a Koszul operad, then ρ is a quasi-isomorphism and (3.79) (or simply $Can(V)$) will be called the canonical resolution *of V.*

PROOF. By definition of a $\mathcal{P}$-algebra, the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{P}}(\mathcal{F}_{\mathcal{P}}(V)) & \xrightarrow{\mathcal{F}_{\mathcal{P}}(\mu_V)} & \mathcal{F}_{\mathcal{P}}(V) \\ {\scriptstyle \mu_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \mu_V} \\ \mathcal{F}_{\mathcal{P}}(V) & \xrightarrow{\mu_V} & V, \end{array}$$

which shows that μ_V is a $\mathcal{P}$-algebra morphism (in fact, the commutativity of the above diagram is equivalent to V being an algebra over the triple $\mathcal{F}_{\mathcal{P}}(-)$; see Definition 1.103). By functoriality $\mathcal{F}_{\mathcal{P}}(\uparrow \pi \downarrow)$ is also a $\mathcal{P}$-algebra morphism and, therefore, ρ is a $\mathcal{P}$-algebra morphism.

The definition of ρ immediately implies that it annihilates all the graded components of $Can^{n,m}(V)$ for $n < m$, that is

$$\rho(\bigoplus_{k<0} Can^k(V)) = \rho(\bigoplus_{n<m} Can^{n,m}(V)) = 0.$$

Since $(V, 0)$ is concentrated in resolution degree 0, it is enough to show that

$$\rho(\partial_C(Can^{-1}(V))) = 0$$

in order to prove that ρ is a dg morphism. We will prove that

$$\rho(\partial_C(Can^{n,n+1}(V))) = 0$$

for any $n \geq 1$. There is an isomorphism

$$\begin{aligned} Can^{n,n+1}(V) &\cong \mathcal{P}(n) \otimes_{\Sigma_{n-1}} \left((\uparrow \mathcal{P}'^{\#}(2) \otimes_{\Sigma_2} (\downarrow V)^{\otimes 2}) \otimes (\uparrow \mathcal{P}'^{\#}(1) \otimes \downarrow V)^{\otimes n-1}\right) \\ &\cong \downarrow \mathcal{P}(n) \otimes_{\Sigma_{n-1}} \left((\mathcal{P}(2) \otimes_{\Sigma_2} V^{\otimes 2}) \otimes V^{\otimes n-1}\right), \end{aligned}$$

where we use $\otimes_{\Sigma_{m-1}}$ to represent the tensor product $\otimes_{\Sigma_n}$ with a single distinguished term among the factors on the right-hand side, where the distinguished factor appears as the factor in the first position on the right. With this convention we can represent a typical element of $Can^{n,n+1}(V)$ as a linear combination of terms

$$\alpha \otimes_{\Sigma_{n-1}} (\beta \otimes_{\Sigma_2} w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1}$$

with $\alpha \in \downarrow \mathcal{P}(n)$, $\beta \in \mathcal{P}(2)$ and $v_i, w_j \in V$. Computing the two differentials, we have

$$\begin{aligned} &\partial_{\mathbf{K}}(\alpha \otimes_{\Sigma_{n-1}} (\beta \otimes_{\Sigma_2} w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1}) \\ &\quad = j_1\left((\alpha \circ_1 \beta) \otimes_{\Sigma_2 \times \Sigma_{n-1}} w_1 \otimes w_2 \otimes v_1 \otimes \cdots \otimes v_{n-1}\right) \end{aligned}$$

and

$$\partial_{\mathcal{F}}(\alpha \otimes_{\Sigma_{n-1}} (\beta \otimes_{\Sigma_2} w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1})$$
$$= j_2\left(\alpha \otimes_{\mathbb{1}\times\Sigma_{n-1}} (\beta(w_1 \otimes w_2) \otimes v_1 \otimes \cdot \ \cdot \otimes v_{n-1})\right),$$

where j_1 reduces the tensor product $\otimes_{\Sigma_2\times\Sigma_{n-1}}$ to $\otimes_{\Sigma_{n+1}}$ and j_2 reduces $\otimes_{\mathbb{1}\times\Sigma_{n-1}}$ to $\otimes_{\Sigma_n}$. Therefore, deleting the subscripts and the morphisms j_1 and j_2 for simplicity, we have

$$\begin{aligned}
&\rho(\partial_C(\alpha \otimes (\beta \otimes w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1})) = \\
&\rho(\partial_{\mathbf{K}}(\alpha \otimes (\beta \otimes w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1}) - \partial_{\mathcal{F}}(\alpha \otimes (\beta \otimes w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1})) \\
&= \mu_V((\alpha \circ_1 \beta) \otimes w_1 \otimes w_2 \otimes v_1 \otimes \cdots \otimes v_{n-1} - \alpha \otimes \beta(w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1}) \\
&= \alpha(\beta(w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1}) - \alpha(\beta(w_1 \otimes w_2) \otimes v_1 \otimes \cdots \otimes v_{n-1}) \\
&= 0.
\end{aligned}$$

The assumption that $\mathcal{P}$ is a Koszul operad implies, by Theorem 3.43, that

$$H^*(Can^{*,m}(V), \partial_{\mathbf{K}}) = 0 \quad \text{for} \quad m \geq 2.$$

A standard double complex argument implies that the only nontrivial cohomology for the total complex comes from the degree 0 term:

$$H^k(Can(V), \partial_C) = \begin{cases} 0 & \text{for } k \leq -1 \text{ and} \\ \uparrow\mathcal{P}(1) \otimes \mathcal{P}'^{\#}(1) \otimes \downarrow V \cong V & \text{for } k = 0. \end{cases}$$

so ρ is a quasi-isomorphism. ∎

THEOREM 3.103. *The $\mathcal{P}$-algebra cohomology of a $\mathcal{P}$-algebra V as defined in Definition 3.99 is the left derived functor of $Der_{\mathcal{P}\text{-}\mathtt{Alg}}(-, V)$*

PROOF. Using the defining properties of a free $\mathcal{P}$-algebra and the cofree nilpotent $\mathcal{P}'$-coalgebra and the definition of the cochain complex for $\mathcal{P}$-algebra cohomology, we have the following sequence of congruences and equalities:

$$\begin{aligned}
Der_{\mathcal{P}\text{-}\mathtt{Alg}}(Can(V), V) &= Der_{\mathcal{P}\text{-}\mathtt{Alg}}(\mathcal{F}_{\mathcal{P}}(\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V)), V) \\
&\cong Hom_{\mathtt{gVec}}(\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V), V) \cong Hom_{\mathtt{gVec}}(\mathcal{F}^c_{\mathcal{P}'}(\downarrow V), \downarrow V) \\
&\cong Coder(\mathcal{F}^c_{\mathcal{P}'}(\downarrow V)) = C_{\mathcal{P}}(V; V).
\end{aligned}$$

We see that indeed $Der_{\mathcal{P}\text{-}\mathtt{Alg}}(Can(V), V) \cong C_{\mathcal{P}}(V; V)$, as graded vector spaces.

We need to compare the differential $\delta_{\mathcal{P}}$ on $C_{\mathcal{P}}(V; V)$ defined in (3.74) with the differential on $Der_{\mathcal{P}\text{-}\mathtt{Alg}}(Can(V), V)$ defined by composition with ∂_C. Let us describe the correspondence between elements of $Der_{\mathcal{P}\text{-}\mathtt{Alg}}(Can(V), V)$ and elements of $C_{\mathcal{P}}(V; V)$ indicated above in more detail.

Each $\alpha \in Hom_{\mathtt{gVec}}(\mathcal{F}^c_{\mathcal{P}'}(\downarrow V), \downarrow V) \cong C_{\mathcal{P}}(V; V)$ determines

$$\uparrow\alpha\downarrow \in Hom_{\mathtt{gVec}}(\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V), V)$$

and a derivation

$$\overline{D}_\alpha \in Der_{\mathcal{P}\text{-}\mathtt{Alg}}(\mathcal{F}_{\mathcal{P}}(\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V)), \mathcal{F}_{\mathcal{P}}(V))$$

given by the ϵ-linear component of the $\mathcal{P}$-algebra morphism $\mathcal{F}_{\mathcal{P}}(\uparrow \pi \downarrow + \epsilon \cdot \uparrow \alpha \downarrow)$, where $\pi : \mathcal{F}^c_{\mathcal{P}'}(\downarrow V) \longrightarrow \downarrow V$ is the projection. Then $D_\alpha := \mu_V \circ \overline{D}_\alpha$ is the derivation in $Der_{\mathcal{P}\text{-}\mathtt{Alg}}(Can(V), V)$ corresponding to α.

Using the fact that $\rho \circ \partial_C = 0$, one verifies immediately that, for each $D \in Der_{\mathcal{P}\text{-Alg}}(Can(V), V)$, $D \circ \partial_C$ satisfies (3.75) (with ρ replacing φ) which characterizes derivations over ρ. Therefore the correspondence $D \mapsto D \circ \partial_C$ is a well-defined differential on $Der_{\mathcal{P}\text{-Alg}}(Can(V), V)$.

LEMMA 3.104. *Given*

$$\alpha \in C^n_{\mathcal{P}}(V; V) \cong Hom_{\mathtt{gVec}}(\mathcal{P}'^{\#}(n+1) \otimes_{\Sigma_{n+1}} (\downarrow V)^{\otimes n+1}, \downarrow V),$$

let D_α be as above. Then $D_\alpha \circ \partial_C$ corresponds, in the above correspondence, to $\delta_{\mathcal{P}}(\alpha)$, that is,

$$D_{\delta_{\mathcal{P}}(\alpha)} = D_\alpha \circ \partial_C.$$

PROOF OF THE LEMMA. The derivation D_α is clearly nonzero only on the components of $\mathcal{F}_{\mathcal{P}}(\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V))$ of the type

$$\mathcal{P}(m) \otimes_{\Sigma_{m-1}} \left((\uparrow \mathcal{P}'^{\#}(n+1) \otimes_{\Sigma_{n+1}} (\downarrow V)^{\otimes n+1}) \otimes V^{\otimes m-1} \right),$$

where we use the isomorphism $\uparrow \mathcal{P}'^{\#}(1) \otimes \downarrow V \cong V$ and once again represent $\otimes_{\Sigma_m}$ as $\otimes_{\Sigma_{m-1}}$ with the distinguished factor on the right of the tensor product moved to the first position On this component, D_α is given by the composition

$$\mathcal{P}(m) \otimes_{\Sigma_{m-1}} \left((\uparrow \mathcal{P}'^{\#}(n+1) \otimes_{\Sigma_{n+1}} (\downarrow V)^{\otimes n+1}) \otimes V^{\otimes m-1} \right)$$
$$\xrightarrow{\mathbb{1} \otimes \uparrow \alpha \downarrow \otimes \mathbb{1}} \mathcal{P}(m) \otimes_{\Sigma_{m-1}} (V \otimes V^{\otimes m-1}) \longrightarrow \mathcal{P}(m) \otimes_{\Sigma_m} V^{\otimes m} \xrightarrow{\mu_V} V,$$

where μ_V is given by the $\mathcal{P}$-algebra structure on V. The derivation $D_\alpha \circ \partial_C$ is determined by its restriction to $\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V)$ and it is easy to see that the only component on which it doesn't vanish is $\uparrow \mathcal{P}'^{\#}(n+2) \otimes_{\Sigma_{n+2}} (\downarrow V)^{\otimes n+2}$. Using the definition in equation (3.76), we write $D_\alpha \circ \partial_C$ as a sum

$$D_\alpha \circ \partial_C = D_\alpha \circ \partial_{\mathbf{K}} + (-1)^{n+1} D_\alpha \circ \partial_{\mathcal{F}}$$

and compute each summand separately. First we have

$$D_\alpha \circ \partial_K : \uparrow \mathcal{P}'^{\#}(n+2) \otimes_{\Sigma_{n+2}} (\downarrow V)^{\otimes n+2} \xrightarrow{Proj(\uparrow^2 \delta_{\mathbf{K}(\mathcal{P})} \downarrow \otimes \mathbb{1})}$$
$$\uparrow^2 \mathcal{P}(2) \otimes \mathrm{sgn}_2 \otimes (\mathcal{P}'^{\#}(n+1) \otimes \mathcal{P}'^{\#}(1)) \otimes_{\Sigma_{n+1}} (\downarrow V)^{\otimes n+2}$$
$$\cong \mathcal{P}(2) \otimes (\uparrow \mathcal{P}'^{\#}(n+1) \otimes \uparrow \mathcal{P}'^{\#}(1)) \otimes_{\Sigma_{n+1}} (\downarrow V)^{\otimes n+2}$$
$$\cong \mathcal{P}(2) \otimes \left(\uparrow \mathcal{P}'^{\#}(n+1) \otimes_{\Sigma_{n+1}} (\downarrow V)^{\otimes n+1}\right) \otimes V$$
$$\xrightarrow{\mathbb{1} \otimes \uparrow \alpha \downarrow \otimes \mathbb{1}} \mathcal{P}(2) \otimes_{\Sigma_2} V^{\otimes 2} \longrightarrow V,$$

where $Proj(-)$ is the projection on the only component of

$$\mathcal{P}(2) \otimes \mathrm{sgn}_2 \otimes \mathcal{P}'^{\#}[2, n+2] \otimes (\downarrow V)^{\otimes n+2}$$

on which D_α doesn't vanish. Comparison with formula (3 72) shows that this term corresponds to $d_{\mathcal{P}} \circ \xi_\alpha$, where $\xi_\alpha \in Coder(\mathcal{F}^c_{\mathcal{P}'}(\downarrow V))$ is the coderivation generated by α.

Next we calculate $D_\alpha \circ \partial_{\mathcal{F}}$ restricted to $\uparrow \mathcal{P}'^{\#}(n+2) \otimes_{\Sigma_{n+2}} (\downarrow V)^{\otimes n+2}$. The morphism $\partial_{\mathcal{F}}$ was defined to be the extension of the map $\uparrow d_{\mathcal{P}} \downarrow$ as a derivation of $\mathcal{F}_{\mathcal{P}}(\uparrow \mathcal{F}^c_{\mathcal{P}'}(\downarrow V))$; therefore, if we restrict to $\uparrow \mathcal{P}'^{\#}(n+2) \otimes_{\Sigma_{n+2}} (\downarrow V)^{\otimes n+2}$, we have the equalities $D_\alpha \circ \partial_{\mathcal{F}} = \uparrow \alpha \circ d_{\mathcal{P}} \downarrow = \uparrow \xi_\alpha \circ d_{\mathcal{P}} \downarrow$, so $D_\alpha \circ \partial_{\mathcal{F}}$ corresponds to $\xi_\alpha \circ d_{\mathcal{P}}$.

We conclude that $D_\alpha \circ \partial_C$ corresponds to $d_{\mathcal{P}} \circ \xi_\alpha + (-1)^{n+1}\xi_\alpha \circ d_{\mathcal{P}} = \delta_{\mathcal{P}}(\alpha)$ which finishes the proof of Lemma 3.104 and also the proof of Theorem 3.103. ∎

REMARK 3.105. We have defined homology of a $\mathcal{P}$-algebra with trivial coefficients and cohomology with coefficients in the $\mathcal{P}$-algebra but neither homology nor cohomology with general coefficients. The definition of cohomology with coefficients in a module over a $\mathcal{P}$-algebra is fairly straightforward (see [**FM97**]) but for homology with coefficients one needs a kind of monoidal structure on the category of modules (homology should be a derived functor of a 'tensor product'). This subtle point is explained and developed in [**Bal98**].

3.9. The pre-Lie structure on $Coder(\mathcal{F}^c_{\mathcal{P}}(X))$

Part of the renewed interest in the theory of operads has come from the deformation theory of associative algebras, in particular, the deformation quantization of Poisson algebras. The (formal) deformation theory of an associative algebra V studies possible extensions of the associative $\mathbf{k}$-algebra structure on V to an associative $\mathbf{k}[[t]]$-algebra structure on $V[[t]]$, where $\mathbf{k}[[t]]$ and $V[[t]]$ are the formal power series in t with coefficients in $\mathbf{k}$ and V respectively. The basic requirement is that there be an isomorphism of $\mathbf{k}$-algebras

$$V[[t]] \otimes_{\mathbf{k}[[t]]} \mathbf{k} \cong V,$$

where the $\mathbf{k}[[t]]$-module structure on $\mathbf{k}$ is given by the augmentation map $\epsilon : \mathbf{k}[[t]] \to \mathbf{k}$. A $\mathbf{k}[[t]]$-algebra structure on $V[[t]]$ is determined by a multiplication $V \otimes_{\mathbf{k}} V \to V[[t]]$, that is, a series of the form

$$a \otimes b \to a *_t b = ab + t\, a *_1 b + t^2\, a *_2 b + \cdots,$$

where ab denotes the original (undeformed) multiplication. If we expand the associativity condition

$$(a *_t b) *_t c = a *_t (b *_t c)$$

in powers of t, we require that the formal power series be equal at each order of t. Modulo t^2 we have

$$(ab)c + t[(a *_1 b)c + (ab) *_1 c] = a(bc) + t[a *_1 (bc) + a(b *_1 c)].$$

This condition on the map $a \otimes b \to a *_1 b$ is precisely the cocycle condition for a Hochschild cochain. If $a \otimes b \to a *_1 b$ satisfies this condition, it is called an *infinitesimal deformation*. To construct the full power series expansion of $a *_t b$, we need to find $*_2, *_3$, etc.

A particularly important application is to the situation known as deformation quantization in which the algebra $V = C^\infty(M)$ is the algebra of smooth functions on a Poisson manifold M. This means $V = C^\infty(M)$ has a Poisson bracket $\{f, g\}$ which is skew, satisfies the Jacobi relation of a Lie algebra and relates to the usual product by $\{f, gh\} = \{f, g\}h + g\{f, h\}$; see Section I.1.17. The bracket $\{f, g\}$ can be chosen as an infinitesimal deformation for the usual commutative product.

M. Gerstenhaber in 1963, [**Ger63**], organized the successive obstructions to the existence of $*_2, *_3$, etc., in terms of the Hochschild cochain complex $CH^*(V, V)$. A fundamental role in this theory is played by the differential graded Lie structure on $CH^*(V, V)$, which was originally discovered by Gerstenhaber in [**Ger63**]. In order to define this dg Lie structure, he first defined the $\circ_i$-operations (part of the prehistory of operads)

$$\circ_i : CH^m(V,V) \otimes CH^n(V,V) \longrightarrow CH^{m+n-1}(V,V).$$

He then defined the $\circ$-product by

$$\alpha \circ \beta = \sum (-1)^{(i-1)|\beta|} \alpha \circ_i \beta,$$

where $|\beta| = n-1$ if $\beta \in CH^n(V,V)$; see also the discussion in Section 2.6. The nonassociativity of the $\circ$-operations is measured by the (graded) *pre-Lie identity*

$$(3.80) \qquad (\alpha \circ \beta) \circ \gamma - \alpha \circ (\beta \circ \gamma) = (-1)^{|\beta||\gamma|} \left((\alpha \circ \gamma) \circ \beta - \alpha \circ (\gamma \circ \beta) \right).$$

Gerstenhaber showed that the pre-Lie identity is a sufficient condition for the commutator

$$[\alpha, \beta] := \alpha \circ \beta - (-1)^{|\alpha||\beta|} \beta \circ \alpha$$

to satisfy the Jacobi identity.

REMARK 3.106. In his study of the cyclic bar complex, Getzler [**Get93**] introduced a generalization of the $\circ$-operations. Voronov and Gerstenhaber [**GV95**] used these operations, which have come to be called *braces*, to prove that the Hochschild cochain complex of an associative algebra has the structure of what they called a homotopy Gerstenhaber algebra.

DEFINITION 3.107. A graded vector space A together with a $\mathbf{k}$-linear product satisfying the pre-Lie identity (3.80) is called a *pre-Lie algebra*.

REMARK 3.108. Chapoton and Livernet [**CL00**] have given an explicit combinatorial description in terms of rooted trees of the operad associated to pre-Lie algebras and have shown that it is a Koszul operad.

Recall that, if V is a $\mathcal{P}^!$-algebra and $X = \downarrow V$, then $Coder(\mathcal{F}^c_{\mathcal{P}}(X))$ is the cochain complex $C_{\mathcal{P}^!}(V,V)$ defining the operadic cohomology of V; see Definition 3.99. In this section we define a pre-Lie product $D\{D'\}$ on $Coder(\mathcal{F}^c_{\mathcal{P}}(X))$ for any operad $\mathcal{P}$. This product which we call the *1-brace* agrees with Gerstenhaber's $\circ$-product on $CH^*(V,V) = C^*_{\mathcal{A}ss}(V,V)$ when $\mathcal{P} = \mathcal{A}ss$. We also will show that the (graded) commutator of 1-braces equals the usual (graded) Lie bracket of coderivations. To simplify the notation, we use the abbreviations

$$\mathcal{F} := \mathcal{F}^c_{\mathcal{P}}(X) \text{ and } \mathcal{F}^q := \mathcal{F}^c_{\mathcal{P}}(X)^q.$$

DEFINITION 3.109. Given coderivations $D_i \in Coder^{p_i,n_i}(\mathcal{F})$, $i = 1,2$, the *1-brace* $D_1\{D_2\} \in Coder^{p_1+p_2,n_1+n_2}(\mathcal{F})$ is the coderivation cogenerated by the linear map $(\pi D_1) \circ D_2 \in Hom_{\mathsf{dgVec}}(\mathcal{F}, X)$, that is,

$$(3.81) \qquad \pi(D_1\{D_2\}) := (\pi D_1) \circ D_2.$$

The fact that $D_1\{D_2\}$ has the asserted bidegree is shown in the course of the proof of Proposition 3.111. First a definition.

DEFINITION 3.110. The *2-brace* of the coderivations D_1, D_2, D_3 is defined as the difference of iterated 1-braces:

$$D_1\{D_2, D_3\} := (D_1\{D_2\})\{D_3\} - D_1\{D_2\{D_3\}\}.$$

PROPOSITION 3.111. *The 1-brace satisfies the pre-Lie identity (3.80), that is, the 2-brace is (graded) symmetric in D_2, D_3,*

$$D_1\{D_2, D_3\} = (-1)^{m_2 m_3} D_1\{D_3, D_2\}, \tag{3.82}$$

where $D_i \in Coder^{p_i, n_i}(\mathcal{F})$, $i = 2, 3$ and $m_i = p_i + n_i$.

PROOF. Representing the value of the cooperad structure map $\overline{\lambda}(r) : \mathcal{F} \to \mathcal{P}^{\#}(r) \otimes_{\Sigma_r} \mathcal{F}^{\otimes r}$ in Sweedler notation

$$\overline{\lambda}(r)(y) = \sum_{(0), \ldots, (r)} \beta_{(0)} \otimes y_{(1)} \otimes \cdots \otimes y_{(r)},$$

for $\beta_{(0)} \in \mathcal{P}^{\#}(r)$ and $y_{(s)} \in \mathcal{F}$, formula (3.68) implies

$$D_i(y) = \sum_{\substack{r, (0), \ldots, (r) \\ 1 \leq s \leq r}} \epsilon_{m_i} \beta_{(0)} \otimes \pi(y_{(1)}) \otimes \cdots \otimes \pi(D_i(y_{(s)})) \otimes \cdots \otimes \pi(y_{(r)}),$$

where ϵ_{m_i} is the Koszul sign factor

$$\epsilon_{m_i} = (-1)^{m_i(|\beta_0| + |y_{(1)}| + \cdots + |y_{(s-1)}|)}, \ i = 1, 2, 3,$$

coming from the transposition of D_i and the factors $\beta_0, y_{(1)}, \ldots, y_{(s-1)}$. Since $D_1 \in Coder^{p_1, n_1}(\mathcal{F})$, $\pi(D_1\{D_2\}(y))$ is given by the sum

$$\begin{aligned}\pi(D_1\{D_2\}(y)) &= (\pi D_1)(D_2(y)) \\ &= \sum_{\substack{(0), \ldots, (n_1+1) \\ 1 \leq s \leq n_1+1}} \epsilon_{m_2} (\pi D_1)\left(\beta_{(0)} \otimes \pi(y_{(1)}) \otimes \cdots \otimes \pi(D_2(y_{(s)})) \otimes \cdots \otimes \pi(y_{(n_1+1)})\right),\end{aligned}$$

and the only nonvanishing terms in this sum come from $y_{(s)} \in \mathcal{F}^{n_2+1}$ and $y_{(t)} \in \mathcal{F}^1$ for $t \neq s$, therefore $y \in \mathcal{F}^{n_1+n_2+1}$.

This shows the additivity of one of the bidegrees in the 1-brace. The additivity of the internal degree is shown similarly. Let us prove (3.82). We have

$$\begin{aligned}&\pi(D_1\{D_2\})\{D_3\}(y) \\ &= \sum_{\substack{(0), \ldots, (n_1+1) \\ 1 \leq s \leq n_1+1}} \epsilon_{m_3} (\pi D_1\{D_2\})(\beta_{(0)} \otimes \pi(y_{(1)}) \otimes \cdots \otimes \pi(D_3(y_{(s)})) \otimes \cdots \otimes \pi(y_{(n_1+n_2+1)}))\end{aligned} \tag{3.83}$$

with $\beta_{(0)} \in \mathcal{P}^{\#}(n_1 + n_2 + 1)$ and $y_{(s)} \in \mathcal{F}^{n_3+1}$. On the other hand

$$\begin{aligned}&\pi(D_1\{D_2\{D_3\}\}(y)) \\ &= \sum_{\substack{(0), \ldots, (n_1+1) \\ 1 \leq s \leq n_1+1}} \epsilon_{m_2+m_3} (\pi D_1)(\beta_{(0)} \otimes \pi(y_{(1)}) \otimes \cdots \otimes \pi(D_2\{D_3\})(y_{(s)}) \otimes \cdots \otimes \pi(y_{(n_1+1)}))\end{aligned} \tag{3.84}$$

with $\beta_{(0)} \in \mathcal{P}^{\#}(n_1 + 1)$ and $y_{(s)} \in \mathcal{F}^{n_2+n_3+1}$.

In order to complete the calculation we have to expand $D_1\{D_2\}$ in (3.83) by applying $\overline{\lambda}(n_1 + n_2 + 1)$ to the tensor product inside the outer parentheses and in (3.84) expand $D_2\{D_3\}(y_{(s)})$ by applying $\overline{\lambda}(n_2 + 1)$ to $y_{(s)}$. The term (3.83) involves the following sequence of coalgebra maps (where we include only the components which do not necessarily vanish when we calculate the composition with

πD_1, πD_2 and πD_3):

$$
\begin{aligned}
(3.85)\quad & \mathcal{P}^{\#}\{n_1+n_2+n_3\}\otimes_{\Sigma_{n_1+n_2+n_3+1}} X^{\otimes n_1+n_2+n_3+1} \\
&\xrightarrow{\overline{\lambda}\{n_1+n_2\}} \mathcal{P}^{\#}\{n_1+n_2\}\otimes_{\Sigma_{n_1+n_2}} \left((\mathcal{P}^{\#}\{n_3\}\otimes_{\Sigma_{n_3+1}} X^{\otimes n_3+1})\otimes X^{\otimes n_1+n_2}\right) \\
&\xrightarrow{\overline{\lambda}\{n_1\}} \mathcal{P}^{\#}\{n_1\}\otimes_{\Sigma_{n_1}}\left((\mathcal{P}^{\#}\{n_2\}\otimes_{\Sigma_{n_2}}(\mathcal{P}^{\#}\{n_3\}\otimes_{\Sigma_{n_3+1}} X^{\otimes n_3+1})\otimes X^{\otimes n_2})\otimes X^{\otimes n_1}\right) \oplus \\
&\qquad \mathcal{P}^{\#}\{n_1\}\otimes_{\Sigma_{n_1-1}}\left((\mathcal{P}^{\#}\{n_2\}\otimes_{\Sigma_{n_2+1}} X^{\otimes n_2+1})\otimes(\mathcal{P}^{\#}\{n_3\}\otimes_{\Sigma_{n_3+1}} X^{\otimes n_3+1})\otimes X^{\otimes n_1-1}\right),
\end{aligned}
$$

where, for a natural number m, here and in (3.86) $\{m\}$ denotes $(m+1)$.

The comultiplication $\overline{\lambda}(n_1+1)$ partitions the tensor product of n_1+n_2+1 factors into n_1+1 subfactors in all possible ways. The only terms which contribute to (3.83) have one subfactor containing n_2+1 of the original factors. The summand in the next to the last row of (3.85) comes from the terms in which the factor of length n_2+1 contains the component $\mathcal{P}^{\#}(n_3+1)\otimes_{\Sigma_{n_3+1}} X^{\otimes n_3+1}$ and the summand in the last row comes from the terms in which it doesn't.

The term (3.84) involves the following sequence of coalgebra maps (where once again we only include the components which do not necessarily vanish when we calculate the composition with πD_1, πD_2 and πD_3:

$$
\begin{aligned}
(3\ 86)\quad & \mathcal{P}^{\#}\{n_1+n_2+n_3\}\otimes_{\Sigma_{n_1+n_2+n_3+1}} X^{\otimes n_1+n_2+n_3+1} \\
&\xrightarrow{\overline{\lambda}\{n_1\}} \mathcal{P}^{\#}\{n_1\}\otimes_{\Sigma_{n_1}}\left((\mathcal{P}^{\#}\{n_2+n_3\}\otimes_{\Sigma_{n_2+n_3+1}} X^{\otimes n_2+n_3+1})\otimes X^{\otimes n_1}\right) \\
&\xrightarrow{1\!\!1\otimes\overline{\lambda}\{n_2\}} \mathcal{P}^{\#}\{n_1\}\otimes_{\Sigma_{n_1}}\left((\mathcal{P}^{\#}\{n_2\}\otimes_{\Sigma_{n_2}}(\mathcal{P}^{\#}\{n_3\}\otimes_{\Sigma_{n_3+1}} X^{\otimes n_3+1})\otimes X^{\otimes n_2})\otimes X^{\otimes n_1}\right).
\end{aligned}
$$

The terms in the next to the last row of (3.85) cancel with the terms in the last row of (3.86) and we are left with

$$
\begin{aligned}
&\pi(((D_1\{D_2\})\{D_3\} - D_1\{D_2\{D_3\}\})(y)) \\
&= \sum_I \epsilon_1(\pi D_1)\left(\beta_{(0)}\otimes\pi(y_{(1)})\otimes\cdots\otimes\pi(D_2(y_{(t)}))\otimes\cdots\otimes\pi(D_3(y_{(s)}))\otimes\cdots\otimes\pi(y_{(n_1+1)})\right) \\
&+\sum_J \epsilon_2(\pi D_1)\left(\beta_{(0)}\otimes\pi(y_{(1)})\otimes\cdots\otimes\pi(D_3(y_{(s)}))\otimes\cdots\otimes\pi(D_2(y_{(t)}))\otimes\cdots\otimes\pi(y_{(n_1+1)})\right),
\end{aligned}
$$

where

$$I = \{(0),\dots,(n_1+1),\ 1\le t<s\le n_1+1\}$$

and

$$J = \{(0),\dots,(n_1+1),\ 1\le s<t\le n_1+1\}.$$

The signs are

$$\epsilon_1 = (-1)^{(|\beta_{(0)}|+|y_{(1)}|+\ \ +|y_{(i-1)}|)m_3+(|\beta_{(0)}|+|y_{(1)}|+\ \ +|y_{(j-1)}|)m_2}$$

and

$$\epsilon_2 = (-1)^{(|\beta_{(0)}|+|y_{(1)}|+\ \cdot+|y_{(i-1)}|)m_3+(|\beta_{(0)}|+|y_{(1)}|+\ \ +|y_{(j-1)}|+m_3)m_2} = (-1)^{m_2m_3}\epsilon_1.$$

The additional factor $(-1)^{m_2m_3}$ in ϵ_2 comes from the fact that, since D_2 is applied after D_3, D_2 is moved past $D_3(y_{(s)})$ which has degree $m_3+|y_{(s)}|$, not $|y_{(s)}|$. Obviously, when we compute $D_1\{D_3,D_2\}$, we will get a sum which has the same form but ϵ_1 and ϵ_2 will be reversed. Therefore, reversing the order of D_2 and D_3 in the 2-brace introduces a factor of $(-1)^{m_2m_3}$ which is what we wanted to prove. ∎

PROPOSITION 3.112. *The graded Lie structure on Coder($\mathcal{F}$) induced by the* 1*-brace of Definition 3.109 agrees with the Lie structure given by the commutator of the* 1*-braces.*

PROOF. If D_1 and D_2 are coderivations, the Lie bracket $[D_1, D_2]$ is given by

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1.$$

Composing with the projection π gives

$$\begin{aligned}\pi([D_1, D_2]) &= \pi(D_1 \circ D_2) - (-1)^{|D_1||D_2|}\pi(D_2 \circ D_1)\\ &= \pi(D_1) \circ D_2 - (-1)^{|D_1||D_2|}\pi(D_2) \circ D_1\\ &= \pi(D_1\{D_2\}) - (-1)^{|D_1||D_2|}\pi(D_2\{D_1\}).\end{aligned}$$

Since a coderivation of $\mathcal{F}$ is determined uniquely by its composition with π, this proves the proposition. ■

3.10. Application: minimal models and homotopy algebras

In this section we study homotopy properties in the category of differential graded operads which are intimately related to the nature of strong homotopy algebras. The approach presented here, pioneered in [**Mar96c**], received recently a new impetus from the work of Kontsevich [**KS00**], Tamarkin and Tsygan [**TT00**], Voronov [**Vor99a**], and others related to the Formality Conjecture (see also Section I.1.19).

We introduce weak equivalence as the relation among differential graded operads generated by maps that induce isomorphism of cohomology – to be compared to weak homotopy equivalence in classical homotopy theory [**Spa66**]. We show that there is a special class of differential graded operads, called minimal operads (Definition 3.117), with the property that they are isomorphic if and only if they are weakly equivalent (Theorem 3.119). We also show that, under some mild assumptions, each differential graded operad has a minimal model (Theorem 3.125), unique up to an isomorphism and functorial 'up to homotopy' (Theorem 3.126). This is the special cofibrant resolution mentioned in Section I.1.18. Finally, we define strong homotopy algebras as algebras over these minimal operads. As an application of homotopy functoriality of the minimal model, we show that minimal models of Hopf operads admit a homotopy Hopf operad structure and interpret this Hopf structure in terms of corresponding algebras.

All algebraic objects in this section are defined over a fixed field $\mathbf{k}$ of characteristic zero. By an operad we mean in this section an operad in the category of differential graded $\mathbf{k}$-vector spaces with $\mathcal{P}(1) = \mathbf{k}$ or, equivalently, we work with pseudo-operads with $\mathcal{P}(1) = 0$. This means in particular that all free operads are generated by Σ-modules with no elements of arity one.

Though in all applications mentioned in this section we use minimal models of operads with trivial differential, that is, operads in `gVec` (but minimal models themselves have very crucially nontrivial differentials even for these operads), we decided to present the theory of minimal models in its full generality for operads in `dgVec`, since restricting to trivial differentials would not simplify the exposition.

The material presented here was inspired by rational homotopy theory. The statements and proofs are analogs of the corresponding classical results of rational homotopy theory as presented, for example, in [**Leh77**] or [**FHT95**]. The situation with operads is, however, in a sense easier than in rational homotopy theory, because here we have one more grading by the 'arity,' preserved by the differential. This may, in some cases, simplify inductive arguments. Most of the material of this section appeared in [**Mar96c**].

DEFINITION 3.113. A *quasi-isomorphism* (abbreviated as *quism* and sometimes also called an *elementary equivalence*) is a morphism $u : \mathcal{S} \to \mathcal{Q}$ of operads that induces an isomorphism of the homology, $H(u) : H(\mathcal{S}) \xrightarrow{\cong} H(\mathcal{Q})$.

A *weak equivalence* is the equivalence relation generated by the elementary equivalences. Weakly equivalent operads are said to have the same *weak homotopy type*.

It is immediate from the definition that operads $\mathcal{S}$ and $\mathcal{Q}$ are weakly equivalent if and only if they are connected by a chain

$$\mathcal{S} \longleftarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longleftarrow \cdots \longrightarrow \mathcal{P}_{s-1} \longleftarrow \mathcal{P}_s \longrightarrow \mathcal{Q} \tag{3.87}$$

of elementary equivalences. The concept of minimal operads introduced later in this section in fact implies that $\mathcal{S}$ and $\mathcal{Q}$ are weakly equivalent if and only if there exist (3.87) with $s = 1$, but we need some auxiliary notions first.

DEFINITION 3.114. Let $\mathcal{P}$ be an operad with $\mathcal{P}(1) = \mathbf{k}$. Define the *decomposables* $D\mathcal{P} = \{D\mathcal{P}(n)\}_{n\geq 1}$ of the operad $\mathcal{P}$ to be the sub-Σ-module of $\mathcal{P}$ consisting of elements

$$\gamma_{\mathcal{P}}(p, p_1, \ldots, p_k),\ p \in \mathcal{P}(k),\ p_i \in \mathcal{P}(n_i),\ 1 \leq i \leq k,$$

such that at least two of $k, n_1, \ldots, n_k$ are greater than or equal to 2.

It is immediate to see that $D\mathcal{P}$ is an ideal in $\mathcal{P}$. A less formal way to define the decomposables, mimicking the definition of decomposables of an augmented algebra, is to recall the ideal $\mathcal{P}^+$ defined by

$$\mathcal{P}^+(n) := \begin{cases} \mathcal{P}(n), & \text{for } n \geq 2, \text{ and} \\ 0, & \text{for } n = 0 \end{cases}$$

and then simply to say that $D\mathcal{P} := (\mathcal{P}^+)^2$. One must however interpret this definition with care, since the γ-operations are not quadratic and $(\mathcal{P}^+)^2$ has, formally speaking, no meaning.

The above analogy can be made even more compelling if we introduce *augmented* operads as operads $\mathcal{P}$ equipped with a homomorphism $\epsilon : \mathcal{P} \to \mathbb{1}$ (the *augmentation*) to the trivial operad $\mathbb{1} = \{\mathbb{1}(n)\}_{n\geq 1}$ defined by

$$\mathbb{1}(n) := \begin{cases} \mathbf{k}, & \text{for } n = 1, \text{ and} \\ 0, & \text{for } n > 1. \end{cases}$$

Operads with $\mathcal{P}(1) = \mathbf{k}$ are clearly augmented, with an augmentation defined in an obvious manner. Decomposables of such an operad are then indeed the 'second power' of the augmentation ideal $\mathcal{P}^+ = Ker(\epsilon)$.

To give an example, recall (Section 1.9) that the free operad $\Gamma(E)$ decomposes as a direct sum indexed by isomorphism classes of trees. The decomposables $D\Gamma(E)$ then correspond to summands over isomorphism classes of trees with at least two (internal) vertices. Let us make another useful definition related to decomposables.

DEFINITION 3.115. Let $\mathcal{P}$ be an operad with $\mathcal{P}(1) = \mathbf{k}$. The *indecomposables* of the operad $\mathcal{P}$ are defined to be the Σ-module $Q\mathcal{P} = \{Q\mathcal{P}(n)\}_{n\geq 1}$ with $Q\mathcal{P}(n) := \mathcal{P}^+(n)/D\mathcal{P}(n)$.

EXAMPLE 3.116. Let us consider the free operad $\Gamma(E)$ on a Σ-module E. Then E is a sub-Σ-module of $\Gamma(E)^+$ and the composition

$$E \hookrightarrow \Gamma(E) \xrightarrow{proj} Q\Gamma(E) \tag{3.88}$$

is an isomorphism. We will use (3.88) to identify $Q\Gamma(E)$ with the Σ-collection of generators E.

Here is the central definition of this section.

DEFINITION 3.117. A *minimal operad* is a differential graded operad $\mathfrak{M}$ of the form $\mathfrak{M} = (\Gamma(M), \partial)$, where $\Gamma(M)$ is the free operad (1.58) on a Σ-module $M = \{M(n)\}_{n\geq 1}$ with $M(1) = 0$, and the differential ∂ is minimal in the sense that the image $\partial(M)$ of M consists of decomposable elements of the free operad $\Gamma(M)$: $\partial(M) \subset D\Gamma(M)$.

EXAMPLE 3.118. Let us consider the dual dg operad $\mathbf{D}(\mathcal{S}) = (\mathbf{D}(\mathcal{S}), \partial_{\mathbf{D}})$ of a differential operad $\mathcal{S} = (\mathcal{S}, \partial_{\mathcal{S}})$ recalled in Definition 3.18. The differential $\partial_{\mathbf{D}}$ consists of two pieces, $\partial_{\mathbf{D}} = \delta + \delta_{\mathcal{S}}$, where the 'internal' part $\delta_{\mathcal{S}}$ is induced by the dual of $\partial_{\mathcal{S}}$.

Recall that $\mathbf{D}(\mathcal{S}) \cong \Gamma(\mathfrak{s}^{-1}{\uparrow}\,\mathcal{S}^{\#})$ and that the image of the differential δ is in the part of $\Gamma(\mathfrak{s}^{-1}{\uparrow}\,\mathcal{S}^{\#})$ corresponding to trees with at least one internal edge and these are decomposable relative to the operad composition in $\Gamma(\mathfrak{s}^{-1}{\uparrow}\,\mathcal{S}^{\#})$. It is also clear that $\delta_{\mathcal{S}}(\mathfrak{s}^{-1}{\uparrow}\,\mathcal{S}^{\#}) \subset \mathfrak{s}^{-1}{\uparrow}\,\mathcal{S}^{\#} \subset \Gamma(\mathfrak{s}^{-1}{\uparrow}\,\mathcal{S}^{\#})$. We conclude that the dual operad $\mathbf{D}(\mathcal{S})$ is minimal if and only if $\partial_{\mathcal{S}} = 0$, i.e. if $\mathcal{S}$ has trivial differential.

The first important statement of this section says:

THEOREM 3.119. *Minimal operads are isomorphic if and only if they are weakly equivalent.*

The theorem will follow from a sequence of statements. Some proofs will contain inductive arguments, for which we need the following notation.

Given a Σ-module $E = \{E(n)\}_{n\geq 1}$, for $k \geq 1$ we denote $E({<}k)$ the Σ-module defined by

$$E({<}k)(n) := \begin{cases} E(n), & \text{for } n < k, \text{ and} \\ 0, & \text{for } n \geq k. \end{cases}$$

Similarly, the Σ-module $E({\leq}k) = \{E({\leq}k)(n)\}_{n\geq 1}$ is defined by

$$E({\leq}k)(n) := \begin{cases} E(n), & \text{for } n \leq k, \text{ and} \\ 0, & \text{for } n > k. \end{cases}$$

Let $\mathfrak{M} = (\Gamma(M), \partial)$ be a minimal operad. Let us denote by

$$\mathfrak{M}^{(n)} := (\Gamma(M({\leq}n)), \partial_n) \tag{3.89}$$

the differential suboperad of $\mathfrak{M}$ generated by $M({\leq}n)$, with the inclusion $i : \mathfrak{M}^{(n)} \hookrightarrow \mathfrak{M}$. We are going to write an exact sequence relating the homologies $H(\mathfrak{M})$ and

$H(\mathfrak{M}^{(n)})$. To this end, observe that the minimality of $\mathfrak{M}$ implies that the differential induced by ∂ on

$$M(n+1) \cong \mathfrak{M}(n+1)/\mathfrak{M}^{(n)}(n+1)$$

is trivial. We thus have the short exact sequence

$$0 \longrightarrow (\mathfrak{M}^{(n)}(n+1), \partial_n) \xrightarrow{\ i\ } (\mathfrak{M}(n+1), \partial) \xrightarrow{j} (M(n+1), \partial = 0) \longrightarrow 0$$

which induces the long exact sequence

$$\begin{aligned} (3.90) \qquad \cdots H(\mathfrak{M}^{(n)})(n+1) &\xrightarrow{H(i)} H(\mathfrak{M})(n+1) \xrightarrow{j} M(n+1) \\ &\xrightarrow{\delta} H(\mathfrak{M}^{(n)})(n+1) \xrightarrow{H(i)} H(\mathfrak{M})(n+1) \cdots \end{aligned}$$

with the connecting morphism δ, for any $n \geq 1$. We are already able to prove the following important proposition.

PROPOSITION 3.120. *A map $\phi : \mathfrak{M} \to \mathfrak{N}$ between minimal operads is an isomorphism if and only if it is a quasi-isomorphism.*

PROOF. An isomorphism is plainly a quasi-isomorphism. Let us prove the opposite implication, that is, that a quasi-isomorphism $\phi : \mathfrak{M} \to \mathfrak{N}$ is an isomorphism. We prove, by induction, that the restriction $\phi_n : \mathfrak{M}^{(n)} \to \mathfrak{N}^{(n)}$, where $\mathfrak{M}^{(n)}$ and $\mathfrak{N}^{(n)}$ are defined as in (3.89), is an isomorphism for each n. This will imply, since $\phi = \varinjlim \phi_n$, that ϕ is an isomorphism, too.

Because $H(\mathfrak{M})(2) \cong M(2)$ and $H(\mathfrak{N})(2) \cong N(2)$, the map $\phi_2 : \mathfrak{M}^{(2)} \to \mathfrak{N}^{(2)}$ is plainly an isomorphism. Suppose we have already proved that ϕ_n is an isomorphism. Let us consider the following commutative diagram:

$$\begin{tikzcd} H(\mathfrak{M}^{(n)})(n+1) \arrow[r] \arrow[d, "H(\phi_n)(n+1)"] & H(\mathfrak{M})(n+1) \arrow[r] \arrow[d, "H(\phi)(n+1)"] & M(n+1) \arrow[r] \arrow[d, "\overline{\phi}(n+1)"] & H(\mathfrak{M}^{(n)})(n+1) \arrow[r] \arrow[d, "H(\phi_n)(n+1)"] & H(\mathfrak{M})(n+1) \arrow[d, "H(\phi)(n+1)"] \\ H(\mathfrak{N}^{(n)})(n+1) \arrow[r] & H(\mathfrak{N})(n+1) \arrow[r] & N(n+1) \arrow[r] & H(\mathfrak{N}^{(n)})(n+1) \arrow[r] & H(\mathfrak{N})(n+1) \end{tikzcd}$$

with the rows the exact sequences (3.90) and $\overline{\phi}$ the induced map of generators. The map $H(\phi)(n+1)$ is an isomorphism because ϕ is a quasi-isomorphism, $H(\phi_n)(n+1)$ is an isomorphism by the induction assumption, thus we conclude from the Five Lemma [**Spa66**, page 185] that $\overline{\phi}(n+1)$ is an isomorphism as well.

We proved that ϕ induces an isomorphism of the generators of $\mathfrak{M}^{(n+1)}$ and $\mathfrak{N}^{(n+1)}$, so ϕ_{n+1} is, by Lemma 3.137 of the Appendix, an isomorphism, too. The induction goes on. ■

We are now in fact very close to a proof of Theorem 3.119 but, in order to put the last stone in place, we need to investigate lifting properties of minimal operads. To this end, we need an appropriate notion of homotopy.

For each operad $\mathcal{Q}$ and for each differential graded commutative $\mathbf{k}$-algebra R, the Σ-module $R \otimes \mathcal{Q} = \{(R\otimes\mathcal{Q})(n)\}_{n\geq 1}$ defined by $(R\otimes\mathcal{Q})(n) := R\otimes_{\mathbf{k}}\mathcal{Q}(n)$, $n \geq 1$, is also an operad, with the composition maps defined by

$$\gamma_{R\otimes\mathcal{Q}}(a\otimes q; a_1\otimes q_1, \ldots, a_l\otimes q_l) := aa_1\cdots a_l \otimes \gamma_{\mathcal{Q}}(q, q_1, \ldots, q_l),$$

for $a, a_1, \ldots, a_l \in R$ and $q, q_1, \ldots, q_l \in \mathcal{Q}$. The procedure we have just described may also be called the *extension of scalars*. We consider, in particular, the differential graded commutative algebra $R := \mathbf{k}[t, \partial t] = \mathbf{k}[t] \otimes_{\mathbf{k}} E(\partial t)$, the tensor product of the polynomial algebra $\mathbf{k}[t]$ on the generator t of degree zero with the exterior algebra $E(\partial t)$ on the generator ∂t of degree -1. There are maps $\rho_i : \mathbf{k}[t, \partial t] \otimes \mathcal{Q} \to \mathcal{Q}$, $i = 0, 1$, defined as follows.

Each $x \in (\mathbf{k}[t, \partial t] \otimes \mathcal{Q})(n)$ is a sum of elements of the form $p(t)q_1 + r(t)\partial t\, q_2$, where $p(t), r(t) \in \mathbf{k}[t]$, $q_1, q_2 \in \mathcal{Q}(n)$. We then define

$$\rho_i(p(t)q_1 + r(t)\partial t\, q_2) := p(i)q_1, \ i = 0, 1$$

(the coefficient $p(i)$ is a scalar, $p(i) \in \mathbf{k}$). It is easy to see that the maps ρ_i are homomorphisms of operads. We define yet another map, $\int : \mathbf{k}[t, \partial t] \otimes \mathcal{Q} \to \mathcal{Q}$, by

$$\int (p(t)q_1 + r(t)\partial t\, q_2) := \left(\int_0^1 r(t)dt \right) q_2.$$

The map $\int$ is a degree $+1$ map of Σ-modules. The following definition is the first step towards a proper notion of homotopy.

DEFINITION 3.121. Two homomorphisms $f_0, f_1 : \mathcal{S} \to \mathcal{Q}$ of differential graded operads are *elementarily homotopic* if there exists a dg homomorphism $F : \mathcal{S} \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ such that $f_i = \rho_i \circ F$, $i = 0, 1$. The dg homomorphism F is called an *elementary homotopy* between f_0 and f_1.

Let us show that elementary homotopy is a symmetric and reflexive relation. Each operadic homomorphism $f : \mathcal{S} \to \mathcal{Q}$ is clearly elementarily homotopic to itself, the elementary homotopy being the map $\tilde{f} : \mathcal{S} \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ given by

$$\mathcal{S}(n) \ni x \longmapsto 1 \otimes f(n)(x) \in \mathbf{k}[t, \partial t] \otimes \mathcal{Q}(n). \tag{3.91}$$

Let $\chi : \mathbf{k}[t, \partial t] \to \mathbf{k}[t, \partial t]$ be an endomorphism given by

$$\chi(t) := (1 - t) \text{ and } \chi(\partial t) := -\partial t.$$

If $F : \mathcal{S} \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ is an elementary homotopy between f_0 and f_1, then $(\chi \otimes 1\!\!1_{\mathcal{Q}})(F)$ is an elementary homotopy between f_1 and f_0, as can be verified easily.

But it is *not* true that elementary homotopy is transitive. In fact, one can prove that it is transitive if the source object is cofibrant in an appropriate sense, but we will not need this result. In general, one must introduce homotopy as the equivalence relation generated by elementary homotopy.

DEFINITION 3.122. Operadic morphisms $f, g : \mathcal{S} \to \mathcal{Q}$ are *homotopic* if there exists a sequence $\{\xi_i : \mathcal{S} \to \mathcal{Q}\}_{i=0}^s$ of operadic morphisms such that $f = \xi_0$, $\xi_s = g$ and ξ_i is elementarily homotopic to ξ_{i+1} for each $0 \leq i \leq s - 1$. The *homotopy* between f and g is the corresponding chain of elementary homotopies.

We claim that, if f_0, f_1 are homotopic in the sense of Definition 3.122, then the chain maps $f_0(n)$ and $f_1(n)$ are homotopic as chain maps, for each $n \geq 1$. It is clearly enough to verify this statement for the case when f_0 and f_1 are elementarily homotopic, with an elementary homotopy F.

Let us define a degree $+1$ map $G : \mathcal{S} \to \mathcal{Q}$ by $G := \int \circ F$. Let us show that $G(n) : \mathcal{S}(n) \to \mathcal{Q}(n)$ is a chain homotopy between $f_0(n)$ and $f_1(n)$.

The nth component $F(n) : \mathcal{S}(n) \longrightarrow \mathbf{k}[t, \partial t] \otimes \mathcal{Q}(n)$ of the homotopy F acts on $x \in \mathcal{S}(n)$ as

$$F(n)(x) = \sum p_i(t) A^i(x) + \sum r_j(t) \partial t \; B^j(x),$$

for some linear Σ_n-equivariant maps $A^i, B^j : \mathcal{S}(n) \longrightarrow \mathcal{Q}(n)$ and polynomials $p_i, q_i \in \mathbf{k}[t]$. It is easy to see that $F(n)$ commutes with the differentials if and only if

$$\begin{aligned} \sum p_i(t)(A^i(\partial_{\mathcal{S}} x) - \partial_{\mathcal{Q}} A^i(x)) &= 0 \text{ and} \\ \sum r_j(t)(B^j(\partial_{\mathcal{S}}(x)) + \partial_{\mathcal{Q}}(B^j(x))) &= \sum \frac{dp_i(t)}{dt} A^i(x). \end{aligned} \tag{3.92}$$

Then $G(n)(x) = \left(\int_0^1 \sum r_j(t) dt\right) B^j(x)$ and

$$\begin{aligned} (\partial_{\mathcal{Q}} G(n) + G(n) \partial_{\mathcal{S}})(x) &= \int_0^1 \sum r_j(t)(B^j(\partial_{\mathcal{S}}(x)) + \partial_{\mathcal{Q}}(B^j(x)))dt \\ &= \int_0^1 \sum \dot{p}_i(t) A^i(x) dt = \sum (p_i(1) - p_i(0)) A^i(x) \text{ (by (3.92))} \\ &= f_1(n)(x) - f_0(n)(x). \end{aligned}$$

In particular, if f_0, f_1 are homotopic in the sense of Definition 3.122, then $H(f_0) = H(f_1)$. The converse is, of course, not true.

Let us formulate the following important lifting property of minimal operads. It suggests that minimal operads are cofibrations of a certain closed monoidal structure on the category of operads. This is indeed true [**Hin97**], but we will not follow that direction in this book.

THEOREM 3.123. *For each quism $\phi : \mathcal{S} \longrightarrow \mathcal{Q}$ and for each morphism $f : \mathfrak{M} \longrightarrow \mathcal{Q}$ of a minimal operad $\mathfrak{M}$ into $\mathcal{Q}$, there exists a map $h : \mathfrak{M} \longrightarrow \mathcal{S}$ such that $\phi \circ h$ and f are homotopic:*

$$\begin{tikzcd} & \mathcal{S} \arrow[d, "\phi"] \\ \mathfrak{M} \arrow[ur, dotted, "h"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\sim"] & \mathcal{Q} \end{tikzcd}$$

where the symbol $\sim$ means that the diagram commutes up to homotopy only. The lift h is unique up to homotopy.

The proof is given in the appendix to this section.

DEFINITION 3.124. Let $\mathcal{S}$ be a differential graded operad. A *minimal model* of $\mathcal{S}$ is a minimal operad $\mathfrak{M}$ together with a quasi-isomorphism $\alpha : \mathfrak{M} \longrightarrow \mathcal{S}$.

The assumption $H(\mathcal{S})(1) = \mathbf{k}$ in the following theorem plays the role of a simple connectivity assumption in rational homotopy theory. As in rational homotopy theory, it can be replaced by a kind of nilpotency, but we will not need this generality in the book.

THEOREM 3.125. *Each differential graded operad $\mathcal{S} = (\mathcal{S}, \partial_{\mathcal{S}})$ such that*

$$H(\mathcal{S})(1) = \mathbf{k}$$

admits a minimal model $\alpha : \mathfrak{M} \longrightarrow \mathcal{S}$.

PROOF. The proof is taken almost literally from [**Mar96c**] where we already translated a proof from [**Leh77**] to operads. We will construct the minimal model $\mathfrak{M} = (\Gamma(M), \partial)$ and the quasi-isomorphism $\alpha : (\Gamma(M), \partial) \to (\mathcal{S}, \partial_{\mathcal{S}})$ inductively.

Let $M(2) := H(\mathcal{S})(2)$ and let $s(2) : M(2) \to Z(\mathcal{S})(2) \subset \mathcal{S}(2)$ be a Σ_2-equivariant section (see the appendix) of the projection $cl : Z(\mathcal{S})(2) \to H(\mathcal{S})(2)$ (here and later, cl will denote the projection of a cycle onto its homology class). Define a differential ∂_2 on $\Gamma(M(2))$ and a map $\alpha_2 : (\Gamma(M(2), \partial_2)) \to (\mathcal{S}, \partial_{\mathcal{S}})$ by

$$\partial_2 := 0 \text{ and } \alpha_2|_{M(2)} := s(2).$$

Let us denote $\mathfrak{M}^{(2)} := (\Gamma(M(2)), \partial_2)$. It is clear that $H(\alpha_2)(n) : H(\mathfrak{M}^{(2)})(n) \to H(\mathcal{S})(n)$ is bijective for $n = 1, 2$.

Suppose we have already constructed a minimal operad

$$\mathfrak{M}^{(n-1)} = (\Gamma(M(<n)), \partial_{n-1})$$

with a morphism $\alpha_{n-1} : \mathfrak{M}^{(n-1)} \to \mathcal{S}$ such that the map

$$H(\alpha_{n-1})(k) : H(\mathfrak{M}^{(n-1)})(k) \to H(\mathcal{S})(k)$$

is an isomorphism for any $k \leq n-1$. We show there is a Σ_n-space $M(n)$, an extension ∂_n of the differential ∂_{n-1} and an extension $\alpha_n : \mathfrak{M}^{(n)} := (\Gamma(M(\leq n)), \partial_n) \to (\mathcal{S}, \partial_{\mathcal{S}})$ of the map α_{n-1} such that

$$H(\alpha_n)(k) : H(\mathfrak{M}^{(n)})(k) \to H(\mathcal{S})(k)$$

is an isomorphism for any $k \leq n$. Let

$$A(n) := H(\mathcal{S})(n)/(Im(H(\alpha_{n-1}))(n)),$$
$$\overline{B}(n) := Ker(H(\alpha_{n-1}))(n) \text{ and } B(n) := \uparrow \overline{B}(n).$$

Let $s(n) : A(n) \to Z(\mathcal{S})(n)$ be an equivariant section of the composition

$$Z(\mathcal{S})(n) \xrightarrow{cl} H(\mathcal{S})(n) \xrightarrow{proj} A(n).$$

Also let $r'(n) : H(\mathfrak{M}^{(n-1)})(n) \to Z(\mathfrak{M}^{(n-1)})(n)$ be an equivariant section of the projection

$$cl : Z(\mathfrak{M}^{(n-1)})(n) \to H(\mathfrak{M}^{(n-1)})(n)$$

and let $r(n) : \overline{B}(n) \to Z(\mathfrak{M}^{(n-1)})(n)$ be the composition

$$B(n) \xrightarrow{\downarrow} \overline{B}(n) \hookrightarrow H(\mathfrak{M}^{(n-1)})(n) \xrightarrow{r'(n)} Z(\mathfrak{M}^{(n-1)}(n)).$$

Since, by definition, $\alpha_{n-1}(r(n)(b))$ is, for $b \in B(n)$, homologous to zero in $H(\mathcal{S})$, there exists a linear Σ_n-equivariant degree zero map $\beta(n) : B(n) \to \mathcal{S}(n)$ such that

$$\alpha_{n-1}(r(n)(b)) = \partial_{\mathcal{S}}(\beta(n)(b)), \ b \in B(n). \qquad (3.93)$$

Let us define $M(n) := A(n) \oplus B(n)$ and extend the differential ∂_{n-1} and the map α_{n-1} by

$$\partial_n|_{A(n)} := 0, \ \partial_n|_{B(n)} := r(n), \ \alpha_n|_{A(n)} := s(n) \text{ and } \alpha_n|_{B(n)} := \beta(n).$$

Then ∂_n is minimal, since

$$Z(\Gamma(M(<n)), \partial_{n-1})(n) \subset \Gamma(M(<n))(n) \subset D\Gamma(M(<n))(n)$$

and since, by definition,

$$Im(r(n)) \subset Z(\Gamma(M(<n)), \partial_{n-1})(n) = Z(\mathfrak{M}^{(n-1)})(n).$$

It is also immediate to see that $\partial_n^2 = 0$. Let us show that $\alpha_n : \mathfrak{M}^{(n)} \to \mathcal{S}$ is a chain map, that is,

$$\alpha_n(\partial_n(a)) = \partial_{\mathcal{S}}(\alpha_n(a)), \text{ for } a \in A(n) \tag{3.94}$$

and

$$\alpha_n(\partial_n(b)) = \partial_{\mathcal{S}}(\alpha_n(b)), \text{ for } b \in B(n). \tag{3.95}$$

Since $\partial_n(a) = 0$, equation (3.94) reduces to $\partial_{\mathcal{S}}(\alpha_n(a)) = \partial_{\mathcal{S}}(s(n)(a)) = 0$ which is true, because $Im(s(n)) \subset Z(\mathcal{S})(n)$. Equation (3.95) is a consequence of

$$\alpha_n(\partial_n(b)) = \alpha_n(r(n)(b)) = \alpha_{n-1}(r(n)(b)) = \partial_{\mathcal{S}}(\beta(n)(b)) = \partial_{\mathcal{S}}(\alpha_n(b))$$

which follows from (3.93).

The map $H(\alpha_n)(n) : H(\mathfrak{M}^{(n)})(n) \to H(\mathcal{S})(n)$ is, by construction, an epimorphism. Let us prove that it is a monomorphism.

Let $z \in Z(\mathfrak{M}^{(n)})(n)$ and write it in the form $z = a + b + \omega$, with $a \in A(n)$, $b \in B(n)$ and $\omega \in D\Gamma(M(<n))$. We show $\partial_n(z) = 0$ implies $b = 0$. By definition $\partial_n(a) = 0$, therefore

$$-\partial_n(b) = -r(n)(b) = \partial_n(\omega).$$

Since ω is decomposable, we know in fact that $\partial_n(\omega) = \partial_{n-1}(\omega)$, therefore $\partial_n(\omega)$ represents a trivial homological class in $H(\mathfrak{M}^{(n-1)})(n)$, thus $b = 0$.

If moreover $[z] \in Ker(H(\alpha_n))(n)$, then $a = 0$, so $z \in Z(\mathfrak{M}^{(n-1)})(n)$ and so, in fact, $[z] \in Ker(H(\alpha_{n-1}))(n)$. But then $[z] \in B(n)$, so $z = \partial_n(\uparrow [z])$ by the construction of ∂_n which implies that z represents a trivial homology class in $H(\mathfrak{M}^{(n)})$.

We verified that $\alpha_n : \mathfrak{M}^{(n)} \to \mathcal{S}$ has the desired properties and the induction may go on, giving rise to the minimal model $\mathfrak{M} := \varinjlim \mathfrak{M}^{(n)}$ and a quism $\alpha := \varinjlim \alpha_n$. ■

The following theorem is an easy corollary to Theorem 3.123.

THEOREM 3.126. *Let $\mathcal{S}$ and $\mathcal{Q}$ be differential graded operads with $\alpha_{\mathcal{S}} : \mathfrak{M}_{\mathcal{S}} \to \mathcal{S}$ and $\alpha_{\mathcal{Q}} : \mathfrak{M}_{\mathcal{Q}} \to \mathcal{Q}$ their minimal models.*

Given a morphism $f : \mathcal{S} \to \mathcal{Q}$, there exists a map $\tilde{f} : \mathfrak{M}_{\mathcal{S}} \to \mathfrak{M}_{\mathcal{Q}}$ such that $f \circ \alpha_{\mathcal{S}}$ is homotopic to $\alpha_{\mathcal{Q}} \circ \tilde{f}$:

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{S}} \arrow[r, "\tilde{f}"] \arrow[d, "\alpha_{\mathcal{S}}"'] \arrow[dr, phantom, "\sim"] & \mathfrak{M}_{\mathcal{Q}} \arrow[d, "\alpha_{\mathcal{Q}}"] \\
\mathcal{S} \arrow[r, "f"] & \mathcal{Q}.
\end{tikzcd}$$

The minimal model is unique up to isomorphism.

PROOF. The first part of the theorem immediately follows from Theorem 3.123. To prove the second part, assume we have two minimal models of the same operad $\mathcal{S}$ and apply the first part of the theorem to construct a lift $\tilde{1\!\!1}_{\mathcal{S}}$ of the identity map $1\!\!1_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$. Since clearly $\tilde{1\!\!1}_{\mathcal{S}}$ is a homology isomorphism, it is, by Proposition 3.120, an isomorphism of the two minimal models. ■

THEOREM 3 127. *Operads $\mathcal{S}$ and $\mathcal{Q}$ have the same weak homotopy type if and only if they have isomorphic minimal models.*

PROOF. It is clear that two operads whose minimal models are isomorphic have the same weak homotopy type.

Let us prove the converse, that is, suppose we have a chain (3.87) of elementary equivalences that connect $\mathcal{P}$ and $\mathcal{Q}$ and prove that their minimal models $\mathfrak{M}_{\mathcal{S}}$ and $\mathfrak{M}_{\mathcal{Q}}$ are isomorphic. To this end, consider the diagram

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{S}} \arrow[d] & \mathfrak{M}_1 \arrow[l] \arrow[r] \arrow[d] \arrow[l, phantom, "\sim", shift right=4] & \mathfrak{M}_2 \arrow[d] \arrow[l, phantom, "\sim", shift right=4] & \cdots \arrow[l] \arrow[r] & \mathfrak{M}_{s-1} \arrow[d] & \mathfrak{M}_s \arrow[l] \arrow[r] \arrow[d] \arrow[l, phantom, "\sim", shift right=4] & \mathfrak{M}_{\mathcal{Q}} \arrow[d] \arrow[l, phantom, "\sim", shift right=4] \\
\mathcal{S} & \mathcal{P}_1 \arrow[l] \arrow[r] & \mathcal{P}_2 & \cdots \arrow[l] \arrow[r] & \mathcal{P}_{s-1} & \mathcal{P}_s \arrow[l] \arrow[r] & \mathcal{Q}
\end{tikzcd}$$

in which the bottom row is the chain (3.87), vertical maps are the minimal models and the top row consists of homotopy lifts of the maps in the bottom row which exist, by Theorem 3.126. Since all maps in the top row are quisms, they are in fact isomorphisms and the theorem is proved. ∎

Theorem 3.119 easily follows from Theorem 3.126 because a minimal operad is its own minimal model. The above results can be summarized by saying that there is exactly one (up to an isomorphism) minimal operad inside each weak homotopy type of operads. Let us introduce our concept of strong homotopy algebras; see also [**Mar00**] for a gentle pedagogical introduction to these objects

DEFINITION 3.128. Let $\mathcal{S}$ be a differential graded operad. A *strongly homotopy $\mathcal{S}$-algebra*, or an *sh $\mathcal{S}$-algebra* for short, is an algebra over the minimal model $\mathfrak{M}_{\mathcal{S}}$ of $\mathcal{S}$.

REMARK 3.129. Observe that we speak about *strong* homotopy algebras when no particular algebraic structure is specified, while we have *strongly* homotopy $\mathcal{S}$-algebras, *strongly* homotopy Lie algebras, etc. This terminology is dictated by historical reasons.

One often denotes strongly homotopy $\mathcal{S}$-algebras as $\mathcal{S}_\infty$-algebras, but there is one important exception, namely the operad B_∞ for 'Baues' algebras, introduced in [**GJ94**], which is not minimal, that is, algebras over this operad are not strong homotopy algebras in the sense of Definition 3.128 (and, moreover, there is no corresponding B). Also the operad $\mathcal{HG}$ discussed in Section I.1.19 is not a strong homotopy algebra in the above sense.

EXAMPLE 3.130. Each graded nondifferential operad $\mathcal{P}$ can be considered as a differential operad with trivial differential. If such an operad is quadratic Koszul in the sense of Definition 3.40 then, by definition, the canonical map $\Theta_{\mathcal{P}} : \mathbf{D}(\mathcal{P}^!) \to \mathcal{P}$ from the dual operad of the quadratic dual $\mathcal{P}^!$ of $\mathcal{P}$ to $\mathcal{P}$ is a quasi-isomorphism. As we observed in Example 3.118, the operad $\mathbf{D}(\mathcal{P}^!)$ is minimal, thus the canonical map $\mathbf{D}(\mathcal{P}^!) \to \mathcal{P}$ is the minimal model of $\mathcal{P}$. By the uniqueness of Theorem 3.126, each minimal model of $\mathcal{P}$ is isomorphic to this one.

REMARK 3.131. Ginzburg and Kapranov defined [**GK94**] strongly homotopy $\mathcal{P}$-algebras for nondifferential quadratic Koszul operads precisely as algebras over $\mathbf{D}(\mathcal{P}^!)$. We see that in this particular case their definition coincides with Definition 3.128. Most classical strong homotopy algebras, such as A_∞-, L_∞-, C_∞- and

G_∞-algebras discussed in Chapter 1, are of this type. In fact, all 'natural' strong homotopy algebras must be of this type, because all 'reasonable' strict algebraic structures are algebras over quadratic Koszul operads. We think that this reflects a deeper gnostic principle that strict algebras over operads which are not Koszul would manifest weird phenomena and thus they would be excluded from the list.

An 'unnatural' example which was not of Ginzburg-Kapranov type was, however, constructed in [**Mar96c**]. In the realm of *colored operads*, 'strong homotopy algebras' typically represent *diagrams* of strong homotopy algebras and their strong homotopy (in an appropriate sense) morphisms and examples which are not of Ginzburg-Kapranov type are abundant; see [**Mar00, Mar01a**].

In the following examples we give axioms of the three classical strong homotopy algebras with all the gory details.

EXAMPLE 3.132. Strong homotopy versions of associative algebras are A_∞-*algebras* (also called *strongly homotopy associative algebras* or $A(\infty)$-*algebras*), introduced by J. Stasheff [**Sta63b**]. An A_∞-algebra $A = (V, d, m_2, m_3, \dots)$ consists of a differential graded vector space $V = (V, d)$ and multilinear operations $m_n : V^{\otimes n} \to V$ of degree $n-2$ that satisfy the following infinite set of axioms:

$$\begin{aligned} 0 &= [m_2, d](a, b),\\ m_2(m_2(a,b)c) - m_2(a, m_2(b,c)) &= [m_3, d](a,b,c),\\ m_3(m_2(a,b), c, d) - m_3(a, m_2(b,c), d) + m_3(a, b, m_2(c,d))&\\ -m_2(m_3(a,b,c), d) - (-1)^{|a|} m_2(a, m_3(b,c,d)) &= [m_4, d](a,b,c,d),\\ &\ \vdots \end{aligned}$$

$$\sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j} (-1)^{\rho} m_i(a_1, \dots, a_s, m_j(a_{s+1}, \dots, a_{s+j}), a_{s+j+1}, \dots, a_n) = [m_n, d](a_1, \dots, a_n),$$

where $[m_n, d]$ denotes the induced differential in the complex $Hom_{\mathbf{k}}(V^{\otimes n}, V)$ of homomorphisms,

$$[m_n, d](a_1, \dots, a_n) := \sum_{1\leq s\leq n} (-1)^{|a_1|+\cdots+|a_{s-1}|} m_n(a_1, \dots, da_s, \dots, a_n) - (-1)^n dm_n(a_1, \dots, a_n),$$

for $a_1, \dots, a_n \in V$. The sign is given by

$$\rho := j + s(j+1) + j(|a_1| + \cdots + |a_{s-1}|).$$

An explicit verification that A_∞-algebras are indeed algebras over the minimal model of the operad $\mathcal{A}ss$ is given in [**Mar96c**, Example 4.8]; see also [**Mar00**, pages 157–159].

EXAMPLE 3.133. Homotopy versions of Lie algebras are L_∞-algebras (sometimes also called *strongly homotopy Lie algebras* or $L(\infty)$-*algebras*). They were introduced and systematically studied in [**LS93**], though they had existed in the literature, in various disguises, even earlier; see also [**LM95**].

Let us introduce some terminology. For graded indeterminates $v_1, \dots, v_n$ and a permutation $\sigma \in \Sigma_n$ define the *Koszul sign* $\epsilon(\sigma) = \epsilon(\sigma; a_1, \dots, a_n) \in \{-1, +1\}$ by

$$a_1 \wedge \cdots \wedge a_n = \epsilon(\sigma; a_1, \dots, a_n) \cdot a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(n)}, \tag{3.96}$$

which has to be satisfied in the free graded commutative algebra $\bigwedge(a_1,\dots,a_n)$. We will also need the *skew-symmetric Koszul sign* defined as

$$\chi(\sigma)=\chi(\sigma;a_1,\dots,a_n):=\operatorname{sgn}(\sigma)\epsilon(\sigma;a_1,\dots,a_n) \tag{3.97}$$

An L_∞-algebra is a differential graded vector space $L=(L,d)$, together with a set $\{l_n\}_{n\geq 2}$ of graded antisymmetric operations $l_n : L^{\otimes n}\to L$ of degree $n-2$. The graded antisymmetry means that, for $\sigma\in\Sigma_n$ and $a_1,\dots,a_n\in L$,

$$l_n(a_1,\dots,a_n)=\chi(\sigma)l_n(a_{\sigma(1)},\cdots,a_{\sigma(n)}).$$

The bilinear product l_2 must be d-invariant and, moreover, the following axiom is satisfied for any $n\geq 2$ and $a_1,\dots,a_n\in L$:

$$\sum_{\substack{i+j=n+1\\ i,j\geq 2}}\sum_{\sigma}\chi(\sigma)(-1)^{i(j-1)}l_j(l_i(a_{\sigma(1)},..,a_{\sigma(i)}),a_{\sigma(i+1)},...,a_{\sigma(n)})$$
$$=(-1)^n[l_n,d](a_1,...,a_n),$$

where the summation is taken over all $(i,n-i)$-unshuffles $\sigma\in\Sigma_n$,

$$\sigma(1)<\cdots<\sigma(i),\ \sigma(i+1)<\cdots<\sigma(n),$$

with $n-1\geq i\geq 1$. We write the first two axioms explicitly (though, to save paper, without signs). The first axiom says that the 'bracket' l_2 satisfies the Jacobi identity up to a homotopy:

$$l_2(l_2(a,b),c)+l_2(l_2(b,c),a)+l_2(l_2(c,a),b)=[l_3,d](a,b,c).$$

The second axiom reads

$$\begin{aligned}l_2(l_3(a,b,c),d)&+l_2(l_3(a,c,d),b)+l_2(l_3(a,b,d),c)+l_2(l_3(b,c,d),a)+l_3(l_2(a,b),c,d)\\&+l_3(l_2(a,c),b,d)+l_3(l_2(a,d),b,c)+l_3(l_2(b,c),a,d)+l_3(l_2(c,d),a,b)\\&+l_3(l_2(b,d),a,c)=[d,l_4](a,b,c,d).\end{aligned}$$

The last equality is related to a Lie-analog L_4 of the associahedron K_4, introduced in [**MS01**] and called the *Lie-hedron.* It is not a polyhedron, but just a graph, and the 10 terms in the left-hand side of the above equation correspond to the vertices (not edges!) of L_4. The Lie-hedron is the Peterson graph in Figure 9 of Section I.1.10.

It can again be proved easily that L_∞-algebras are algebras over the minimal model of the operad $\mathcal{L}ie$; see [**Mar00**, pages 159–160].

EXAMPLE 3.134. Strong homotopy versions of commutative associative algebras are C_∞*-algebras* (also called *commutative* A_∞*-algebras* or *balanced* A_∞*-algebras*), introduced by T. Kadeishvili in [**Kad80**]. Let us recall the following terminology

Let $a_1,\dots,a_n\in V^{\otimes n}$ and $0\leq i\leq n$. The *shuffle product* of $a_1\otimes\cdots\otimes a_i\in V^{\otimes i}$ and $a_{i+1}\otimes\cdots\otimes a_n\in V^{\otimes i-1}$ is defined as

$$Sh(a_1\otimes\cdots\otimes a_i|a_{i+1}\otimes\cdots\otimes a_n):=\sum\epsilon(\sigma)\ a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(n)} \tag{3.98}$$

where the summation is taken over all $(i,n-i)$-shuffles, that is, permutations $\sigma\in\Sigma_n$ with

$$\sigma^{-1}(1)<\cdots<\sigma^{-1}(i),\ \sigma^{-1}(i+1)<\cdots<\sigma^{-1}(n)$$

and $\epsilon(\sigma)$ is the Koszul sign (3.96). There is a skew-symmetric version of the shuffle product defined as

$$hS(a_1 \otimes \cdots \otimes a_i | a_{i+1} \otimes \cdots \otimes a_n) := \sum \chi(\sigma)\, a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} \tag{3.99}$$

where the range of the summation is the same as in (3.98) and $\chi(\sigma)$ is introduced in (3.97).

A C_∞-algebra is then an A_∞-algebra as in Example 3.132 with the additional property that the operations m_k, $k \geq 2$, are trivial on the skew-symmetric shuffle products, that is

$$m_k(\, hS(a_1 \otimes \cdots \otimes a_i | a_{i+1} \otimes \cdots \otimes a_k)) = 0$$

for $1 < i < k$. For instance, the only nontrivial skew-symmetric shuffle product of two elements is $hS(u|v) = u \otimes v - (-1)^{uv} v \otimes u$, therefore

$$m_2(\, hS(u|v)) = m_2(u,v) - (-1)^{uv} m_2(v,u) = 0,$$

which means that m_2 is graded commutative. Similarly, there are two shuffle products of three elements and we see that

$$m_3(\, hS(u|v \otimes w)) = m_3(u,v,w) - (-1)^{uv} m_3(v,u,w) + (-1)^{u(v+w)} m_3(v,w,u) = 0,$$
$$m_3(\, hS(u \otimes v|w)) = m_3(u,v,w) - (-1)^{vw} m_3(u,w,v) + (-1)^{w(u+v)} m_3(w,u,v) = 0.$$

To prove that C_∞-algebras are indeed algebras over the minimal model of the corresponding operad is a bit more difficult than in the previous two cases, since one needs a result of R. Ree [**Ree58**] that relates Lie elements to shuffles in the tensor product.

Let us give an application of the technique developed in this section. Given a differential graded operad $\mathcal{S}$, we introduce the tensor product

$$\mathcal{S} \otimes \mathcal{S} = \{(\mathcal{S} \otimes \mathcal{S})(n)\}_{n \geq 1}$$

to be the Σ-module with $(\mathcal{S} \otimes \mathcal{S})(n) := \mathcal{S}(n) \otimes_{\mathbf{k}} \mathcal{S}(n)$ and the obvious operadic structure. The following definition was taken from [**GJ94**].

DEFINITION 3.135. The operad $\mathcal{P}$ is a *Hopf operad* if there exists an operadic map $\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ (the *diagonal*) which is associative:

$$(\Delta \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \Delta)\Delta : \mathcal{S} \longrightarrow \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S}.$$

Algebras over a Hopf operad form a strict monoidal category, with the monoidal structure $\square$ induced by the diagonal Δ as follows.

Let $A = (A, a_A)$ and $B = (B, a_B)$ be two $\mathcal{S}$-algebras, with structure maps $a_A : \mathcal{S} \to \mathcal{E}nd_A$ and $a_B : \mathcal{S} \to \mathcal{E}nd_B$. Then we define $A \square B$ to be the vector space $A \otimes B$ with the structure map $a_{A \otimes B} : \mathcal{S} \to \mathcal{E}nd_{A \otimes B}$ given as the composition

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S} \otimes \mathcal{S} \xrightarrow{a_A \otimes a_B} \mathcal{E}nd_A \otimes \mathcal{E}nd_B \xrightarrow{j} \mathcal{E}nd_{A \otimes B},$$

where $j : \mathcal{E}nd_A \otimes \mathcal{E}nd_B \to \mathcal{E}nd_{A \otimes B}$ is the homomorphism with components

$$j(n) : (\mathcal{E}nd_A \otimes \mathcal{E}nd_B)(n) \to \mathcal{E}nd_{A \otimes B}(n)$$

defined as the composition

$$\begin{aligned}(\mathcal{E}nd_A \otimes \mathcal{E}nd_B)(n) &= \mathcal{E}nd_A(n) \otimes \mathcal{E}nd_B(n) = Hom(A^{\otimes n}, A) \otimes Hom(B^{\otimes n}, B)\\ &\xrightarrow{\cong} Hom(A^{\otimes n} \otimes B^{\otimes n}, A \otimes B) \xrightarrow{\cong} Hom((A \otimes B)^{\otimes n}, A \otimes B) = \mathcal{E}nd_{A \otimes B}.\end{aligned}$$

As an immediate application of Theorem 3.123 we get the following proposition.

PROPOSITION 3.136. *The minimal model* $\alpha_{\mathcal{S}} : \mathfrak{M} \to \mathcal{S}$ *of a Hopf operad* $\mathcal{S}$ *admits a homotopy Hopf operad structure, that is, a diagonal* $\tilde{\Delta} : \mathfrak{M} \to \mathfrak{M}\otimes\mathfrak{M}$ *such that the maps* $(\tilde{\Delta}\otimes\mathbb{1})\tilde{\Delta}$ *and* $(\mathbb{1}\otimes\tilde{\Delta})\tilde{\Delta}$ *are homotopic as morphisms of operads in the sense of Definition 3.122.*

PROOF. As an immediate consequence of the Künneth theorem, we see that the map $\alpha_{\mathcal{S}}\otimes\alpha_{\mathcal{S}} : \mathfrak{M}\otimes\mathfrak{M} \to \mathcal{S}\otimes\mathcal{S}$ is a quasi-isomorphism. The diagonal $\tilde{\Delta} : \mathfrak{M} \to \mathfrak{M}\otimes\mathfrak{M}$ is then the homotopy lift of $\Delta : \mathcal{S} \to \mathcal{S}\otimes\mathcal{S}$ which exists, by Theorem 3.123.

The homotopy coassociativity of $\tilde{\Delta}$ follows immediately from the uniqueness up to homotopy of the homotopy lift, since both $(\tilde{\Delta}\otimes\mathbb{1})\tilde{\Delta}$ and $(\mathbb{1}\otimes\tilde{\Delta})\tilde{\Delta}$ are lifts of the same map $(\Delta\otimes\mathbb{1})\Delta = (\mathbb{1}\otimes\Delta)\Delta : \mathcal{S} \to \mathcal{S}\otimes\mathcal{S}\otimes\mathcal{S}$. ■

The prominent example of a Hopf operad is the operad $\mathcal{A}ss$ for associative algebras, with the diagonal δ given by

$$\mathcal{A}ss(n) \ni x \longmapsto x\otimes x \in \mathcal{A}ss(n)\otimes\mathcal{A}ss(n).$$

Proposition 3.136 then predicts the existence of a homotopy Hopf structure on the minimal model of $\mathcal{A}ss$, that is, on the operad $\mathcal{A}_\infty$ for A_∞-algebras. An explicit example of such a diagonal was constructed by Saneblidze and Umble in [**SU00**].

3.10.1. Appendix. The following lemma is necessary for the proof of Proposition 3.120.

LEMMA 3.137. *Let* $\phi : \Gamma(M) \to \Gamma(N)$ *be a homomorphism of free operads and let* $\overline{\phi} : M \to N$ *be the induced map of the generators (under identification (3.88)). Suppose that* $M(1) = N(1) = 0$. *Then* ϕ *is an isomorphism of operads if and only if* $\overline{\phi}$ *is an isomorphism of* Σ*-modules.*

PROOF. Let us introduce some notation. Given $n \geq 3$, let

$$\phi_{<n} : \Gamma(M(<n)) \to \Gamma(N(<n))$$

and

$$\phi_n : \Gamma(M(\leq n)) \to \Gamma(N(\leq n))$$

be the restrictions. Since $M(1) = N(1) = 0$, there are canonical decompositions

$$\Gamma(M(\leq n)) \cong M(n) \oplus \Gamma(M(<n)) \tag{3.100}$$

and

$$\Gamma(N(\leq n)) \cong N(n) \oplus \Gamma(N(<n)). \tag{3.101}$$

These decompositions imply that $\phi|_{M(n)}$ is the sum

$$\phi|_{M(n)}(v) = \overline{\phi}(v) + \phi^+(v), \text{ for } v \in M(n), \tag{3.102}$$

where $\overline{\phi}$ is the induced map of generators and $\phi^+(v) \in \Gamma(N(<n))(n)$.

Let us prove first that $\overline{\phi} : M \xrightarrow{\cong} N$ implies that $\phi : \Gamma(M) \to \Gamma(N)$ is also an isomorphism. Since clearly $\phi_2 = \Gamma(\overline{\phi}(2))$, our assumption immediately implies that ϕ_2 is an isomorphism.

Suppose we have already proved that $\phi_{n-1} = \phi_{<n}$ is an isomorphism for some $n \geq 3$ and let us prove ϕ_n is an isomorphism, too. To this end, it is clearly enough to verify that

$$\text{the restriction } \phi|_{M(n)} : M(n) \to \Gamma(N)(n) \text{ is a monomorphism} \tag{3.103}$$

and that

$$N(n) \subset Im(\phi)(n). \tag{3.104}$$

Condition (3.103) immediately follows from (3.102). To prove (3.104), we invoke the induction assumption to see that, for any $v \in M(n)$, there exists some $\omega \in \Gamma(M(<n))$ such that $\phi^+(v) = \phi(\omega)$ Then each $u \in N(n)$ is the image of $v - \omega$ with $v := \overline{\phi}^{-1}(u)$.

Let us prove the opposite implication: ϕ is an isomorphism implies $\overline{\phi}$ is an isomorphism. Assuming that ϕ is an isomorphism and $\overline{\phi}(j)$ is an isomorphism for $j < n$, we will prove that $\overline{\phi}(n)$ is an isomorphism. Using decompositions (3.100) and (3.101) we express

$$\overline{\phi}(n) = proj_{N(n)} \circ \phi|_{M(n)}.$$

As we already know from the first part of the proof, the induction assumption implies that $\phi_{<n} : \Gamma(M(<n)) \to \Gamma(N(<n))$ is an isomorphism. To prove that $\overline{\phi}(n)$ is injective, note that again, for any $v \in M(n)$, there exists some $\omega \in \Gamma(M(<n))$ such that $\phi^+(v) = \phi_{<n}(\omega) = \phi(\omega)$. Assume $v \neq 0$ and $\overline{\phi}(v) = 0$, then $\phi(v - \omega) = 0$ and, since ϕ is an isomorphism, $v = \omega$, contradicting the fact that v is indecomposable. Thus $\overline{\phi}$ is a monomorphism. On the other hand, for any indecomposable $0 \neq u \in N(n)$, there exists some $\nu \in \Gamma(M(\leq n))$ such that $\phi(\nu) = u$. Then

$$\begin{aligned}
\overline{\phi}(proj_{M(n)}(\nu)) &= \overline{\phi}(\nu - proj_{\Gamma(M(<n))}(\nu)) = proj_{N(n)}\phi(\nu - proj_{\Gamma(M(<n))}(\nu)) \\
&= proj_{N(n)}\phi(\nu) - proj_{N(n)}\phi(proj_{\Gamma(M(<n))}(\nu)) \\
&= proj_{N(n)}\phi(\nu)) = u,
\end{aligned}$$

so $\overline{\phi}$ is onto. ■

EQUIVARIANT SECTION. Given two Σ-modules F and G and an epimorphism $\pi : F \to G$ (i.e. a morphism of Σ-modules such that $\pi(n) : F(n) \to G(n)$ is an epimorphism for each $n \geq 1$), there always exists a map of Σ-modules $s : G \to F$ such that $\pi \circ s = 1\!\!1_G$. Indeed, we must show that there exists, for each $n \geq 1$, a Σ_n-equivariant map $s(n) : G(n) \to F(n)$ such that $\pi(n) \circ s(n) = 1\!\!1_{G(n)}$. To construct this map, choose an arbitrary section $\tilde{s}(n) : G(n) \to F(n)$ of $\pi(n)$ and put, for $g \in G(n)$,

$$s(n)(g) := \sum_{\sigma \in \Sigma_n} \frac{1}{n!}[\tilde{s}(n)(g\sigma^{-1})]\sigma.$$

Observe that here the assumption $\mathrm{char}(\mathbf{k}) = 0$ is crucial. We will call such s an *equivariant section* of π.

PROOF OF THEOREM 3.123. We use the fact that minimal operads are limits of special types of inclusions which we now introduce. Suppose we are given a differential graded operad $\mathcal{B} = (\mathcal{B}, \partial_{\mathcal{B}})$, a Σ-module M and a degree -1 map τ :

$M \to \mathcal{B}$ of Σ-modules (a 'generator' for the differential) such that $Im(\tau) \subset Z(\mathcal{B})$. In this situation, we may consider the *free product*

$$\mathcal{B} * \Gamma(M) := \Gamma(\mathcal{B} \oplus M)/\mathcal{J},$$

where the operadic ideal $\mathcal{J}$ is generated by relations given by the composition in $\mathcal{B}$. More precisely, $\mathcal{J}$ is generated by expressions

$$\gamma_\Gamma(b; b_1, \ldots, b_n) = \gamma_\mathcal{B}(b; b_1, \ldots, b_n), \quad \text{for } b, b_1, \ldots, b_n \in \mathcal{B},$$

where γ_Γ denotes the structure operation in $\Gamma(\mathcal{B} \oplus M)$ and $\gamma_\mathcal{B}$ the structure operation in $\mathcal{B}$. Let ∂_τ be the degree -1 derivation of $\Gamma(\mathcal{B} \oplus M)$ defined by $\partial_\tau(b) := \partial_\mathcal{B}(b)$ for $b \in \mathcal{B}$ and $\partial_\tau(v) := \tau(v)$ for $v \in M$. Since $Im(\tau) \subset Z(\mathcal{B})$ by assumption, $\partial_\tau^2 = 0$ and ∂_τ induces a degree -1 differential on $\mathcal{B} * \Gamma(M)$ which we denote again by ∂_τ. Let us finally denote

$$\mathcal{B} *_\tau \Gamma(M) := (\mathcal{B} * \Gamma(M), \partial_\tau).$$

DEFINITION 3.138. A *principal extension* is an inclusion of differential graded operads of the form $\iota : \mathcal{B} \hookrightarrow \mathcal{B} *_\tau \Gamma(M)$.

Observe that $\mathfrak{M}^{(n+1)} = \mathfrak{M}^{(n)} *_\tau \Gamma(M(n+1))$ (the notation of (3.89)) so each minimal operad is a direct limit of principal extensions. Theorem 3.123 will thus follow from the following lemma.

LEMMA 3.139. *For each quasi-isomorphism $\phi : \mathcal{S} \to \mathcal{Q}$, for each principal extension $\iota : \mathcal{B} \hookrightarrow \mathcal{B} *_\tau \Gamma(M)$ and for each couple of maps $f : \mathcal{B} \to \mathcal{S}$ and $g : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{Q}$ such that $\phi \circ f$ is homotopic to $g \circ \iota$, there exists an extension f to $h : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{S}$ such that $\phi \circ h$ is homotopic to g:*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "f"] \arrow[d, "\iota"'] & \mathcal{S} \arrow[d, "\phi"] \\ \mathcal{B} *_\tau \Gamma(M) \arrow[r, "g"'] \arrow[ur, dotted, "h"] \arrow[ur, phantom, "\sim", shift right=3ex] & \mathcal{Q} \end{tikzcd}$$

PROOF. Let us show first that we may in fact assume that the diagram in Lemma 3.139 commutes strictly, that is

$$g \circ \iota = \phi \circ f. \tag{3.105}$$

To be more precise, we show that there exists some $g' : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{Q}$, homotopic to g, for which $g' \circ \iota = \phi \circ f$. This will follow from the following claim.

CLAIM. *Suppose we are given a principal extension $\iota : \mathcal{B} \hookrightarrow \mathcal{B} *_\tau \Gamma(M)$ and maps $\alpha : \mathcal{B} \to \mathcal{Q}$ and $\beta : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{Q}$ such that α is elementarily homotopic to $\beta \circ \iota$. Then there exists a map β', elementarily homotopic to β, for which $\alpha = \beta' \circ \iota$.*

To prove the Claim, let $F : \mathcal{B} \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ be an elementary homotopy between $\beta \circ \iota$ and α, that is, $\rho_0 \circ F = \beta \circ \iota$, $\rho_1 \circ F = \alpha$. We extend F to an elementary homotopy $H : \mathcal{B} *_\tau \Gamma(M) \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ such that $\rho_0 \circ H = \beta$. The map $\beta' := \rho_1 \circ H$ then automatically satisfies $\beta' \circ \iota = \alpha$, because $\beta' \circ \iota = \rho_1 \circ H \circ \iota = \rho_1 \circ F = \alpha$. The situation is depicted in the following diagram:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "\iota"'] & \mathbf{k}[t,\partial t]\otimes\mathcal{Q} \arrow[d, "\rho"] \\ \mathcal{B} *_\tau \Gamma(M) \arrow[r, "\beta"'] \arrow[ur, dotted, "H"] & \mathcal{Q} \end{tikzcd}$$

Let $K := Ker(\rho_0)$. Since ρ_0 is a surjective quism, the Σ-module K is acyclic. Choose an equivariant section $\sigma : Z(K) \to K$ of the epimorphism $\partial : K \to Z(K)$. It is immediate to see, for each $v \in M$, that $F\partial_\tau(v) - 1\otimes(\beta \circ \iota \circ \tau(v))$ is a cycle in K. So we can define the extension H on generators from M by

$$H(1 * v) := 1\otimes(\beta(1 * v)) + \sigma(F\partial_\tau(v) - 1\otimes(\beta \circ \iota \circ \tau(v)))$$

Now $\rho_0 \circ H = \beta$ is verified easily, because the 'correction' $\sigma(F\partial_\tau(v) - 1\otimes(\beta\circ\iota\circ\tau(v)))$ belongs to the kernel K of ρ_0, by the definition of σ. It is equally easy to verify that H commutes with the differentials. The Claim is proved.

Let us turn our attention back to the diagram of Lemma 3.139. The homotopy commutativity of the lower right triangle means that we are given a sequence $\xi_1, \dots, \xi_{s-1}$ of maps such that $g \circ \iota$ is elementarily homotopic to ξ_1, ξ_i is elementarily homotopic to ξ_{i+1} for $1 \leq i < s-1$ and ξ_{s-1} is elementarily homotopic to $\phi \circ f$. Use the Claim (with $\alpha = \xi_1$ and $\beta = g$) to replace g by a homotopic map g_1 such that $g_1 \circ \iota = \xi_1$. Now $g_1 \circ \iota$ is elementarily homotopic to ξ_2. We may use the Claim again to replace g_1 by some g_2, homotopic to g_1, such that $g_2 \circ \iota = \xi_2$. Repeating this process $s-1$ times we end up with a $g' = g_{s-1}$, homotopic to the original g, such that $g' \circ \iota = \phi \circ f$. So we may indeed assume (3.105).

Before we begin to construct the extension h for Lemma 3.139, we need to verify that the image $Im(f \circ \tau)$ consists of boundaries. Indeed, $(f \circ \tau)(v) \in Z(\mathcal{S})$ for $v \in M$ and the equation

$$(\phi \circ f \circ \tau)(v) = (g \circ \iota \circ \tau)(v) = g(\partial_\tau(v)) = \partial g(1 * v)$$

shows that $(\phi \circ f \circ \tau)(v)$ is a boundary. Since ϕ is a quism, $(f \circ \tau)(v)$ must be a boundary as well.

Let γ be an equivariant section of $\partial : \mathcal{S} \to \partial(\mathcal{S})$. As the first approximation to the extension h, consider an extension $\tilde{h} : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{S}$ of f defined by

$$\tilde{h}(1 * v) := (\gamma \circ f \circ \tau)(v) \text{ for } v \in M.$$

It is easy to verify that $\tilde{h}$ commutes with the differentials. Now we must, by adding a correction term from $Z(\mathcal{S})$, modify $\tilde{h}$ to some h such that $\phi \circ h$ and g are homotopic.

Let $\tilde{g} : \mathcal{B} \to \mathbf{k}[t, \partial t]\otimes\mathcal{Q}$ be the trivial homotopy of $g \circ \iota$ to itself (compare (3.91)) defined by

$$\tilde{g}(b) := 1\otimes(g \circ \iota)(b) = 1\otimes(\phi \circ f)(b) \text{ for } b \in \mathcal{B}.$$

Let us extend $\tilde{g}$ naively to a 'homotopy' $\tilde{H} : \mathcal{B} *_\tau \Gamma(M) \to \mathbf{k}[t, \partial t]\otimes\mathcal{Q}$ between g and $\phi \circ \tilde{h}$ by

$$\tilde{H}(1 * v) := (1 - t)g(1 * v) + t(\phi \circ \tilde{h})(1 * v), \; v \in M.$$

Unfortunately, it is not true in general that this naive $\tilde{H}$ commutes with the differentials. Let us denote by δ the deviation from this being true,

$$\delta(v) = \partial\tilde{H}(1 * v) - \tilde{g}(\tau(v)) = \partial t\, (\phi \circ \tilde{h} - g)(1 * v).$$

Now we invoke (3.105) to verify that $((\phi \circ \tilde{h} - g)(1 * v))$ is a cycle in $\mathcal{Q}$:

$$\partial_{\mathcal{Q}}(\phi \circ \tilde{h} - g)(1 * v) = (\phi \circ \tilde{h} - g)(\tau(v)) = (\phi \circ f - g \circ \iota)(\tau(v)) = 0.$$

Since ϕ is a quism, there exist maps $\psi : M \to Z(\mathcal{S})$ and $\eta : M \to \mathcal{Q}$ of Σ-modules such that, for each $v \in M$,

$$(\phi \circ \psi)(v) - (\phi \circ \tilde{h} - g)(1 * v) = (\partial_{\mathcal{Q}} \circ \eta)(v), \ v \in M. \tag{3.106}$$

Let h be the extension of f defined by

$$h(1 * v) := \tilde{h}(1 * v) - \psi(v).$$

Then define the homotopy $H : \mathcal{B} *_\tau \Gamma(M) \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ as the extension of $\tilde{g}$ satisfying

$$H(1 * v) := (1 - t) g(1 * v) + t(\phi \circ \tilde{h})(1 \otimes v) + \partial t\, \eta(v).$$

It follows from (3.106) that the map H commutes with the differentials. It is clear that $\rho_0 \circ H = g$ and $\rho_1 \circ H = \phi \circ h$, therefore $\phi \circ h$ is homotopic to g as claimed.

We will not prove the uniqueness part of Theorem 3.123 here. The proof can be obtained by translating the arguments of [**FHT95**, Theorem 3.7] into the operadic language. ■

CHAPTER 4

Geometry

4.1. Configuration spaces operads and modules

The aim of this section is to provide the reader with a road map of the complicated world of operads and modules over these operads that are constructed from configurations of geometric objects (disks, cubes, points) in manifolds of various types.

LITTLE DISKS OPERADS. We have already seen examples of 'geometric' operads in this book. Let us recall at least the little k-cubes operad $\mathcal{C}_k$ whose detailed description we gave in Definition 2.2. The *little k-disks operad* $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n\geq 1}$ is defined quite analogously, except that we use little disks instead of little cubes. This means that $\mathcal{D}_k(n)$ is, for $n \geq 1$, the space of all maps

$$d : \coprod_{1\leq s\leq n} D^k_s \to D^k$$

from the disjoint union of n numbered standard k-disks $D^k_1, \ldots, D^k_n$ to D^k, where

$$D^k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 \leq 1\},$$

such that d, when restricted to each disk, is a composition of translation and multiplication by a positive real number and the images of the interiors of the disks are disjoint. The symmetric group acts by renumbering the disks. We may interpret $d \in \mathcal{D}_k(n)$ as D^k with n numbered, disjoint circular holes, since all information about d is kept by the shape of its image.

The operadic structure on the Σ-module $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n\geq 1}$ is given exactly as for little cubes in Section 2.2, so we will not give the definition here. We may informally visualize the composition $\gamma_{\mathcal{D}_k}(d; d_1, \ldots, d_n)$ as the gluing of the ith disk d_i into the ith hole of d, for $1 \leq i \leq n$.

Let us introduce a 'framed' version $\mathfrak{f}\mathcal{D}_k = \{\mathfrak{f}\mathcal{D}_k(n)\}_{n\geq 1}$ of the little disk operad $\mathcal{D}_k$. As in the unframed case, elements of $\mathfrak{f}\mathcal{D}_k(n)$ are maps from the disjoint union of n standard disks D^k to D^k, but here we admit *all* conformal linear maps, that is, maps that are compositions of translation, multiplication by a positive real number, and a rotation by an element of the orthogonal group $O(n)$. Now an element $f \in \mathfrak{f}\mathcal{D}_k(n)$ cannot be described just as the disk D^k with n numbered holes, but each hole must also be decorated by a *frame* that encodes the rotation. The operadic composition is defined almost exactly as for the operad $\mathcal{D}_k$. If we visualize the composition as gluing we must, however, rotate each disk before gluing it in. The operad $\mathfrak{f}\mathcal{D}_k$ is called the *framed little k-disks operad*. Both operads $\mathfrak{f}\mathcal{D}_k$ and $\mathcal{D}_k$ are pointed (see Section 2.4), with $\mathfrak{f}\mathcal{D}_k(0) = \mathcal{D}_k(0)$ consisting of the disk D^k with no hole in it.

To understand better the relation between the little disks operad and its framed version, we recall the following definitions introduced in [**Mar99a**].

DEFINITION 4.1. Let G be a topological group. A G-*operad* is an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ such that each $\mathcal{P}(n)$ is a left G-space and the composition map γ satisfies

$$g(\gamma(x; x_1, \ldots, x_n)) = \gamma(gx; x_1, \ldots, x_n), \tag{4.1}$$

for each $x \in \mathcal{P}(n)$, $x_i \in \mathcal{P}(m_i)$, $1 \leq i \leq n$, and $g \in G$.

The framed little k-disks operad $\mathfrak{f}\mathcal{D}_k$ is an example of an $O(k)$-operad, with the orthogonal group acting by rotating the 'big' disk with all its framed holes inside. As we shall see, $\mathfrak{f}\mathcal{D}_k$ is a semidirect product of the orthogonal group $O(k)$ with the little disks operad $\mathcal{D}_k$ in the sense we introduce below. The terminology was suggested by P. Salvatore. These semidirect products were studied by Nathalie Wahl in her PhD thesis [**Wah01**]; see also [**SW01**].

Suppose we have an (ordinary) operad $\mathcal{P}$ such that each $\mathcal{P}(n)$ is a left G-space and such that the composition map satisfies, under the notation of (4.1),

$$g\gamma(x; x_1, \ldots, x_n) = \gamma(gx; gx_1, \ldots, gx_n). \tag{4.2}$$

DEFINITION 4.2. The *semidirect product* $\mathcal{P} \rtimes G = \{(\mathcal{P} \rtimes G)(n)\}_{n\geq 1}$ is the G-operad with $(\mathcal{P} \rtimes G)(n) := \mathcal{P}(n) \times G^{\times n}$, with the diagonal left action of the group G, diagonal right action of the symmetric group Σ_n and the composition map γ_G defined as

$$\begin{aligned}\gamma_G((x, g_1, \ldots, g_n); (x_1, g_1^1, \ldots, g_{m_1}^1), \ldots, (x_n, g_1^n, \ldots, g_{m_n}^n))\\ := (\gamma(x; g_1x_1, \ldots, g_nx_n), g_1g_1^1, \ldots, g_1g_{m_1}^1, \ldots, g_ng_1^n, \ldots, g_ng_{m_n}^n),\end{aligned}$$

for $g_1, \ldots, g_n,\ g_1^1, \ldots, g_{m_1}^1, \ldots, g_1^n, \ldots, g_{m_n}^n \in G$, $x \in \mathcal{P}(n)$, $x_i \in \mathcal{P}(m_i)$, $1 \leq i \leq m$.

It is easy to verify that the semidirect product $\mathcal{P} \rtimes G$ is indeed a G-operad as in Definition 4.1. There is a natural inclusion $\iota_{\mathcal{P}} : \mathcal{P} \hookrightarrow \mathcal{P} \rtimes G$ of operads given by $\iota_{\mathcal{P}}(n)(p) := p \times e^{\times n}$, where e is the unit of G, $p \in \mathcal{P}(n)$ and $n \geq 1$. Observe that the projection $p_{\mathcal{P}} : \mathcal{P} \rtimes G \to \mathcal{P}$ on the first factor is not an operadic morphism.

An example of an operad satisfying (4.2) is the little disk operad $\mathcal{D}_k$ with the $O(k)$-action given by rotating the 'big' disk with all its holes. We leave the proof of the following statement as an easy exercise to the reader.

PROPOSITION 4.3. *There is a natural isomorphism of $O(k)$-operads*

$$\mathcal{D}_k \rtimes O(k) \cong \mathfrak{f}\mathcal{D}_k.$$

Let $\overset{\circ}{D}{}^k$ denote the interior of the standard disk D^k,

$$\overset{\circ}{D}{}^k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 < 1\},$$

and let $\mathsf{Con}(\overset{\circ}{D}{}^k, n)$ be the configuration space of n distinct numbered points in $\overset{\circ}{D}{}^k$, with the obvious right Σ_n-action. P. May proved, in [**May72**, Theorem 4.8], that the nth piece $\mathcal{C}_k(n)$ of the little k-cubes operad has, for each $n \geq 1$, the Σ_n-equivariant homotopy type of $\mathsf{Con}(\overset{\circ}{D}{}^k, n) \cong \mathsf{Con}(\mathbb{R}^k, n)$. Let us prove an analogous statement for the (framed) little k-disks operad.

PROPOSITION 4.4. *For each $n \geq 1$, the space $\mathfrak{f}\mathcal{D}_k(n)$ is Σ_n-equivariantly homotopy equivalent to* $\mathsf{Con}(\overset{\circ}{D}{}^k, n) \times O(k)^{\times n}$.

This homotopy equivalence restricts to an equivariant homotopy equivalence of the Σ_n-subspaces $\mathcal{D}_k(n) \subset \mathfrak{f}\mathcal{D}_k(n)$ and $\mathsf{Con}(\overset{\circ}{D}{}^k, n) = \mathsf{Con}(\overset{\circ}{D}{}^k, n) \times \{1^{\times n}\} \subset \mathsf{Con}(\overset{\circ}{D}{}^k, n) \times O(k)^{\times n}$.

PROOF. For a (framed) little disk $d : D^k \to \mathbb{R}^k$, denote by $c(d) \in D^k$ its *center*, $c(d) := d(0)$. Let $d_* : T_0 D^k \to T_{c(d)} D^k$ be the tangent map at zero and interpret it, using the canonical identifications $T_0 D^k \cong \mathbb{R}^k$ and $T_{c(d)} D^k \cong \mathbb{R}^k$, as a map $d_* : \mathbb{R}^k \to \mathbb{R}^k$. There exists a unique element $ph(d) \in O(k)$ such that, for some fixed scalar $\alpha > 0$,

$$d_*(t) = \alpha \; ph(d)(t), \quad \text{for each } t \in \mathbb{R}^k.$$

We call $ph(d)$ the *phase* of d. Define an equivariant map $a : \mathfrak{f}\mathcal{D}_k(n) \to \mathsf{Con}(\overset{\circ}{D}{}^k, n) \times O(k)^{\times n}$ by

$$a(d_1, \dots, d_n) := (c(d_1), \dots, c(d_n)) \times (ph(d_1), \dots, ph(d_n)),$$

for $(d_1, \dots, d_n) \in \mathfrak{f}\mathcal{D}_k(n)$.

To define a homotopy inverse of a, we need a function $\rho : \mathsf{Con}(\overset{\circ}{D}{}^k, n) \to \mathbb{R}_{>0}$ defined as

$$\rho(c_1, \dots, c_n) := \min \begin{cases} \frac{1}{2}\mathrm{dist}(c_i, c_j), & 1 \leq i \neq j \leq n, \\ 1 - \mathrm{dist}(c_i, 0), & 1 \leq i \leq n, \end{cases}$$

where dist is the standard Euclidean distance in $\mathbb{R}^k$ and $(c_1, \dots, c_n) \in \mathsf{Con}(\overset{\circ}{D}{}^k, n)$. The meaning of the above definition is that $\rho = \rho(c_1, \dots, c_n)$ is the largest number such that the open disks with centers $c_1, \dots, c_n$ and radii ρ are mutually disjoint and all are subsets of D^k.

Next, observe that for each $c \in \overset{\circ}{D}{}^k$ and $g \in O(k)$ there exists exactly one *isometry* $d(c, g) : D^k \to \mathbb{R}^k$ such that $c(d(c, g)) = c$ and $ph(d(c, g)) = g$. The map $b : \mathsf{Con}(\overset{\circ}{D}{}^k, n) \times O(k)^{\times n} \to \mathfrak{f}\mathcal{D}_k(n)$ is defined by the formula

$$b((c_1, \dots, c_n) \times (g_1, \dots, g_n)) := (\rho d(c_1, g_1), \dots, \rho d(c_n, g_n)), \;\; \rho = \rho(c_1, \dots, c_n),$$

for $(c_1, \dots, c_n) \in \mathsf{Con}(\overset{\circ}{D}{}^k, n)$ and $(g_1, \dots, g_n) \in O(k)^{\times n}$. Let us prove that b is a homotopy inverse of a.

It is clear that $a \circ b = \mathbb{1}$, so it remains to prove that $b \circ a$ is homotopic to $\mathbb{1}$. Observe that $(b \circ a)(d_1, \dots, d_n)$ is obtained from $(d_1, \dots, d_n) \in \mathfrak{f}\mathcal{D}_k(n)$ simply by rescaling the disks so that they all have radius $\rho(c(d_1), \dots, c(d_n))$. Thus the most naive interpolation

$$h_t := (1 - t)(a \circ b) + t\mathbb{1}, \;\; 0 \leq t \leq 1, \tag{4.3}$$

is indeed a homotopy between $b \circ a$ and $\mathbb{1}$. It is clear that all constructions above are in fact Σ_n-equivariant, giving rise to an equivariant homotopy equivalence.

The second part of the statement is obtained by restricting to the unframed subspaces. ■

Let us clarify the relation between the little disks and little cubes operads. There is no obvious operadic morphism between them. To compare them, we must introduce an intermediate operad, the suboperad $t\mathcal{C}_k$ of the operad $\mathcal{C}_k$ consisting of 'true' k-cubes, i.e. little k-cubes for which, in the notation of Section 2.2, the difference $y_i - x_i$ is the same for all $1 \leq i \leq n$.

Let $u : \mathcal{D}_k \to t\mathcal{C}_k$ be the map that assigns to each little k-disk the biggest 'true' little k-cube contained in it. Similarly, there is a map $v : t\mathcal{C}_k \to \mathcal{D}_k$ which assigns to each 'true' little k-cube the biggest little k-disk contained in it. Both u and v are maps of operads and it is clear that $v(n)$ is a Σ_n-equivariant homotopy inverse of $u(n)$, for each $n \geq 1$.

The inclusion $\iota : t\mathcal{C}_k \to \mathcal{C}_k$ has a left inverse $j : \mathcal{C}_k \to t\mathcal{C}_k$ which assigns to each little k-cube the biggest 'true' little k-cube contained in it and having the same center. The map j is, however, not a homomorphism of operads, but, as can be easily seen, $j(n)$ is a homotopy inverse of $\iota(n)$, for each $n \geq 1$. The situation is summarized by the following diagram of operad maps:

$$\begin{tikzcd} \mathcal{D}_k \arrow[r, bend left, "u"] & t\mathcal{C}_k \arrow[l, bend left, "v"] \arrow[r, "\iota"] & \mathcal{C}_k \end{tikzcd}$$

As an easy consequence of the above considerations, we get the following proposition.

PROPOSITION 4.5. *The composition $\iota \circ u : \mathcal{D}_k \to \mathcal{C}_k$ induces an isomorphism of differential graded operads $H_*(\mathcal{D}_k)$ and $H_*(\mathcal{C}_k)$.*

We already observed in Section I.1.17 that the homology operad $H_*(\mathcal{C}_2)$ of the little squares operad $\mathcal{C}_2$ describes Gerstenhaber algebras, i.e. algebras with a commutative associative multiplication $\cdot$ and with a degree 1 Lie bracket $[-,-]$ which are related by a distributivity law. By Proposition 4.5, the same is also true for the operad $\mathcal{D} := \mathcal{D}_2$. The homology of the framed little disks operad $\mathfrak{f}\mathcal{D}$ describes another important class of algebras, called Batalin-Vilkovisky algebras.

DEFINITION 4.6. A *Batalin-Vilkovisky algebra* (also called a *BV-algebra*) is a Gerstenhaber algebra $(V, \cdot, [-,-])$ together with a degree 1 linear map $\Delta : V \to V$ such that $\Delta^2 = 0$ and

$$[a,b] = \Delta(ab) - (\Delta(a)b + (-1)^a a\Delta(b)), \tag{4.4}$$

for arbitrary $a, b, c \in V$.

Observe that Δ in the above definition is a differential, but not a derivation with respect to the commutative product, its deviation from being a derivation measured by the Lie bracket as expressed by (4.4). The following characterization of Batalin-Vilkovisky algebras was given in [**Get94a**].

THEOREM 4.7. *Batalin-Vilkovisky algebras are algebras over the operad $\mathcal{BV} := H_*(\mathfrak{f}\mathcal{D})$.*

The space $\mathfrak{f}\mathcal{D}(1)$ has the homotopy type of the circle S^1, so $H_*(\mathfrak{f}\mathcal{D}(1))$ has two generators, one of degree 0 corresponding to the identity $1 \in H_*(\mathfrak{f}\mathcal{D}(1))$ and one of degree $+1$ corresponding to the operation Δ

RIEMANN SPHERES WITH PUNCTURES. All the above examples live in real differential geometry. While there is no obvious complex analog of the unframed little disks operad $\mathcal{D}_k$, the complex analog of the framed little disks operad $f\mathcal{D} := f\mathcal{D}_2$ is the following.

Let Σ be a Riemann sphere, that is, a nonsingular complex projective curve of genus 0. By a *puncture* or a *parametrized hole* we mean a point of Σ together with a holomorphic embedding of the standard disk $U = \{z \in \mathbb{C} \mid |z| \leq 1\}$ to Σ centered at the point. Thus a puncture is a point $p \in \Sigma$ with a holomorphic embedding $u : \tilde{U} \to \Sigma$, where $\tilde{U} \subset \mathbb{C}$ is an open neighborhood of U and $u(0) = p$. We say that two punctures $u_1 : \tilde{U}_1 \to \Sigma$ and $u_2 : \tilde{U}_2 \to \Sigma$ are *disjoint*, if

$$u_1(\overset{\circ}{U}) \cap u_2(\overset{\circ}{U}) = \emptyset,$$

where $\overset{\circ}{U} := \{z \in \mathbb{C} \mid |z| < 1\}$ is the interior of U.

Let $\widehat{\mathcal{M}}_0(n)$ be the moduli space of Riemann spheres Σ with $n+1$ disjoint punctures $u_i : \tilde{U}_i \to \Sigma$, $0 \leq i \leq n$, modulo the action of the complex projective linear group $PGL_2(\mathbb{C})$. The topology of $\widehat{\mathcal{M}}_0(n)$ is a very subtle thing and we are not going to discuss this issue here; see [**Hua97**] for details and compare also Remark 4.10. The constructions below will be made only 'up to topology.'

Renumbering holes defines on each $\widehat{\mathcal{M}}_0(n)$ a natural right Σ_n-action and the Σ-module $\widehat{\mathcal{M}}_0 = \{\widehat{\mathcal{M}}_0(n)\}_{n \geq 1}$ forms an operad under sewing Riemannian spheres at punctures. We already recalled this operad in Section I.1.16.

Let us describe this operadic structure using the $\circ_i$-formalism. Thus, let Σ represent an element $x \in \widehat{\mathcal{M}}_0(m)$ and Δ represent an element $y \in \widehat{\mathcal{M}}_0(n)$. For $1 \leq i \leq m$, let $u_i : \tilde{U}_i \to \Sigma$ be the ith puncture of Σ and let $u_0 : \tilde{U}_0 \to \Delta$ be the 0th puncture of Δ.

There certainly exists some $0 < r < 1$ such that both $\tilde{U}_0$ and $\tilde{U}_i$ contain the disk $U_{1/r} := \{z \in \mathbb{C} \mid |z| < 1/r\}$. Let now $\Sigma_r := \Sigma - u_i(U_r)$ and $\Delta_r := \Delta - u_0(U_r)$. Define finally

$$\Xi := (\Sigma_r \bigsqcup \Delta_r)/\sim,$$

where the relation $\sim$ is given by

$$\Sigma_r \ni u_i(\xi) \sim u_0(1/\xi) \in \Delta_r,$$

for $r < |\xi| < 1/r$. It is immediate to see that Ξ is a well-defined punctured Riemannian sphere, with $n+m-1$ punctures induced in the obvious manner from those of Σ and Δ, and that the class of the punctured surface Ξ in the moduli space $\widehat{\mathcal{M}}_0(m+n-1)$ does not depend on the representatives Σ, Δ and on r. We define $x \circ_i y$ to be the class of Ξ. The symmetric group acts by permuting the labels of little disks.

Before describing the unit $1 \in \widehat{\mathcal{M}}_0(1)$, we introduce the following notation which will be useful in the sequel. Let $\mathbb{CP}^1$ be the complex projective line with homogeneous coordinates $[z, w]$, $z, w \in \mathbb{C}$, [**Wel80**]. Let $0 := [0,1] \in \mathbb{CP}^1$ and $\infty := [1,0] \in \mathbb{CP}^1$. Recall that we have two canonical isomorphisms $p_\infty : \mathbb{CP}^1 - \infty \to \mathbb{C}$ and $p_0 : \mathbb{CP}^1 - 0 \to \mathbb{C}$ given by

$$p_\infty([z,w]) := z/w \text{ and } p_0([z,w]) := w/z.$$

Then $p_\infty^{-1} : \mathbb{C} \to \mathbb{CP}^1$ (respectively $p_0^{-1} : \mathbb{C} \to \mathbb{CP}^1$) defines a puncture at 0 (respectively at ∞). We define $1 \in \widehat{\mathcal{M}}_0(1)$ to be the class of $(\mathbb{CP}^1, p_0^{-1}, p_\infty^{-1})$. Notice that $\widehat{\mathcal{M}}_0$ is in fact a pointed operad (see Section 2.4), with $* \in \widehat{\mathcal{M}}_0(0)$ the class of $(\mathbb{CP}^1, p_0^{-1})$.

PROPOSITION 4.8. *There exists a natural map of topological operads* $j : \mathfrak{f}\mathcal{D} \to \widehat{\mathcal{M}}_0$.

PROOF. Let us describe, following [**KSV95**], the construction of the map j. We start with the observation that elements $f \in \mathfrak{f}\mathcal{D}(n)$ can be interpreted as complex linear maps from the disjoint union of n copies of the standard complex unit disk U to the same U. Given such an element f, the composition $p_\infty^{-1} \circ f$ gives $\mathbb{CP}^1$ with n embedded holomorphic disks numbered by $\{1, \dots, n\}$. In addition, we define the 0th disk to be p_0^{-1}. This gives a point in $\widehat{\mathcal{M}}_0(n)$ and completes the construction of the map $j(n) : \mathfrak{f}\mathcal{D}(n) \to \widehat{\mathcal{M}}_0(n)$. It is clear that the map $j(n)$ is well defined and a moment's reflections shows that it induces an operadic homomorphism. ■

Let $C_*(\widehat{\mathcal{M}}_0)$ be the differential graded operad of singular chains on $\widehat{\mathcal{M}}_0$. Recall (Section I.1.16) that the string background induces an action of the operad $C_*(\widehat{\mathcal{M}}_0)$ on the BRST complex $(\mathcal{H}, Q)$. The following theorem was proved in [**KSV95**].

THEOREM 4.9. *The absolute BRST homology* $H_*(\mathcal{H}, Q)$ *admits a natural structure of a Batalin-Vilkovisky algebra.*

PROOF. The $C_*(\widehat{\mathcal{M}}_0)$-action on the BRST complex $(\mathcal{H}, Q)$ induces a $H_*(\widehat{\mathcal{M}}_0)$-action on its homology $H_*(\mathcal{H}, Q)$. This, composed with the map

$$H_*(j) : H_*(\mathfrak{f}\mathcal{D}) \to H_*(\widehat{\mathcal{M}}_0),$$

where $j : \mathfrak{f}\mathcal{D} \to \widehat{\mathcal{M}}_0$ is the map of Proposition 4.8, gives the advertised structure, because algebras over $H_*(\mathfrak{f}\mathcal{D})$ are, by Theorem 4.7, exactly Batalin-Vilkovisky algebras. ■

REMARK 4.10 It is a commonly accepted fact that the map j of Proposition 4.8 is an equivariant homotopy equivalence. We were, however, not able to find a solid proof in the literature, though discussion with M. Lehm and A. Weber indicated that this might be true if the topology of $\widehat{\mathcal{M}}_0$ is chosen properly.

LITTLE DISKS MODULES. Recall (Definition 3.26) that a topological Σ-module $N = \{N(n)\}_{n\geq 1}$ is a right module over a topological operad $\mathcal{P}$ if we are given operations

$$\nu : N(n) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n) \to N(m_1 + \cdots + m_n),$$

for $n \geq 1$, $m_i \geq 1$, $1 \leq i \leq n$, satisfying appropriate axioms.

In the rest of this section, which has a bit of a speculative character, we show how configuration spaces provide natural examples of modules over operads. Roughly speaking, we show that the Σ-module of framed configuration spaces $\mathfrak{f}\mathsf{Con}(M) = \{\mathfrak{f}\mathsf{Con}(M, n)\}_{n\geq 1}$ (to be introduced in Definition 4.11) has the equivariant homotopy type of a specific right module over the framed little k-disks operad $\mathfrak{f}\mathcal{D}_k$, where k is the dimension of M. For parallelizable M we formulate also an 'unframed' version of the above statement.

As in Proposition 4.4, we are going to compare the configuration space of points in M to a Σ-module of little disks on M, so our first task will be to understand what a little disk in M should be.

The most general choice is that a *little disk* in M is just a smooth map (not even necessarily an embedding) $d : D^k \to M$. As before we denote by $\mathrm{c}(d) := d(0) \in M$ the center of d. We say that two little disks d_1, d_2 are disjoint if $d_1(\overset{\circ}{D}{}^k) \cap d_2(\overset{\circ}{D}{}^k) = \emptyset$. It is clear that there is no way to fix the 'phase' of the disk unless we assume that M is parallelizable, so all disks are 'framed' from the beginning.

Let us denote by $\mathfrak{f}\mathcal{D}M = \{\mathfrak{f}\mathcal{D}M(n)\}_{n\geq 1}$ the Σ-module of numbered little disks in M, with the obvious right action of the symmetric group. The Σ-module $\mathfrak{f}\mathcal{D}M = \{\mathfrak{f}\mathcal{D}M(n)\}_{n\geq 1}$ has a right module structure over the framed little k-disks operad $\mathfrak{f}\mathcal{D}_k$ given as follows.

Let $d : \bigsqcup_{s=1}^{n} D_s^k \to M$ be an element of $\mathfrak{f}\mathcal{D}M(n)$ and $d_s : \bigsqcup_{j=1}^{m_s} D_{j,s}^k \to D^k$ (where D_s^k and $D_{j,s}^k$ are identical copies of the standard k-disk) be elements of $f\mathcal{D}_k(m_s)$, $1 \leq s \leq n$. Then $\nu(d; d_1, \dots, d_n) \in \mathcal{C}_k(m_1 + \cdots + m_n)$ is the map

$$\nu(d; d_1, \dots, d_n) : \bigsqcup_{s=1}^{n} \bigsqcup_{j=1}^{m_s} D_{j,s}^k \longrightarrow M \tag{4.5}$$

given by

$$\nu(d; d_1, \dots, d_n)|_{D_{j,s}^k} := d \circ d_s,\ 1 \leq j \leq m_s,\ 1 \leq s \leq n,$$

where we interpret d_s as a map $d_s : \bigsqcup_{j=1}^{m_s} D_{j,s}^k \to D_s^k$.

This gives a nice example of a right $\mathfrak{f}\mathcal{D}_k$-module, but if we do not put appropriate restrictions on little disks and their configurations, this example is very dull. Indeed, if we allow little disks to be all smooth maps and if we do not require them to be disjoint, then $\mathfrak{f}\mathcal{D}M(n)$ has, for $n \geq 1$, the Σ_n-homotopy type of the cartesian product $M^{\times n}$. To see this, define $a(n) : \mathfrak{f}\mathcal{D}M(n) \to M^{\times n}$ by $a(n)(d_1, \dots, d_n) := (\mathrm{c}(d_1), \dots, \mathrm{c}(d_n))$, for $(d_1, \dots, d_n) \in \mathfrak{f}\mathcal{D}M(n)$. There is also a map $b(n) : M^{\times n} \to \mathfrak{f}\mathcal{D}M(n)$ defined by $b(n)(x_1, \dots, x_n) := (d_{x_1}, \dots, d_{x_n})$ for $(x_1, \dots, x_n) \in M^{\times n}$, where d_{x_i} denotes the constant little disk with $Im(d_i) = \{x_i\}$, $1 \leq i \leq n$. It is immediate to see that $a(n)$ and $b(n)$ are Σ_n-equivariant maps, equivariantly homotopy inverse to each other.

Observe that the Σ-module $\{M^{\times n}\}_{n\geq 1}$ is also a right $\mathfrak{f}\mathcal{D}_k$-module, with the structure given rather trivially by

$$\nu((x_1, \dots, x_n); f_1, \dots, f_n) := (\underbrace{x_1, \dots, x_1}_{m_1 \text{ times}}, \dots, \underbrace{x_n, \dots, x_n}_{m_n \text{ times}}),$$

for $(x_1, \dots, x_n) \in M^{\times n}$ and $f_i \in \mathfrak{f}\mathcal{D}(m_n)$, $1 \leq i \leq n$. The Σ-module map

$$a := \{a(n) : \mathfrak{f}\mathcal{D}M(n) \to M^{\times n}\}_{n\geq 1},$$

commutes with these modular structures.

A better choice that makes $\mathfrak{f}\mathcal{D}M(n)$ less trivial is to assume that the disks are *disjoint*. This means that one must also assume that each little disk is an *embedding* of D^k, otherwise the composition (4.5) need not lead to disjoint disks. Under this choice, we have a map $a(n) : \mathfrak{f}\mathcal{D}M(n) \to \mathsf{Con}(M, n)$ sending each $(d_1, \dots, d_n) \in \mathfrak{f}\mathcal{D}M(n)$ to $(\mathrm{c}(d_1), \dots, \mathrm{c}(d_n)) \in \mathsf{Con}(M, n)$. This $a(n)$ is still far from being a homotopy equivalence, since each little disk could be rotated by an element of $O(k)$ without changing its center, therefore the preimage $a(n)^{-1}(x_1, \dots, x_n)$ of each $(x_1, \dots, x_n) \in M^{\times n}$ contains a highly nontrivial space $O(k)^{\times n}$. This can be improved by considering framed configuration spaces. They will be very useful later in this book.

Let $GL(M)$ denote the principal $GL(k)$-bundle of linear frames on M. We may always assume that M is Riemannian, so $GL(M)$ reduces to orthogonal frames $O(M)$; denote by $\pi_M : O(M) \to M$ the bundle projection. There is a retraction $r_M : GL(M) \to O(M)$ commuting with the bundle projections.

DEFINITION 4.11. The *framed configuration space* $\mathfrak{f}\mathsf{Con}(M,n)$ of n distinct points in M is the space of all $(x_1, \mathfrak{f}_1; \ldots ; x_n, \mathfrak{f}_n)$ where $(x_1, \ldots, x_n) \in \mathsf{Con}(M,n)$ and $\mathfrak{f}_i \in \pi_M^{-1}(x_i)$, $1 \leq i \leq n$.

Observe that the obvious projection $p : \mathfrak{f}\mathsf{Con}(M,n) \to \mathsf{Con}(M,n)$ is a principal $O(k)^{\times n}$-bundle, in fact, $\mathfrak{f}\mathsf{Con}(M,n)$ is the pullback of the diagram:

$$\begin{tikzcd} & O(M)^{\times n} \arrow[d, "\pi_M^{\times n}"] \\ \mathsf{Con}(M,n) \arrow[r, "\text{inclusion}"] & M^{\times n} \end{tikzcd}$$

If the manifold M is parallelizable, then $\mathfrak{f}\mathsf{Con}(M,n)$ is (noncanonically) isomorphic to $\mathsf{Con}(M,n) \times O^{\times n}(k)$. For instance, the space $\mathsf{Con}(\mathring{D}^k, n) \times O(k)^{\times n}$ of Proposition 4.4 is the framed configuration space $\mathfrak{f}\mathsf{Con}(\mathring{D}^k, n)$ in disguise.

Let us introduce the *frame* of a little disk $d : D^k \to M$ as follows. Let $\{e_1, \ldots, e_k\}$ be the canonical basis of the tangent space of $\mathbb{R}^k$ at 0,

$$e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \quad (1 \text{ at the } i\text{th position}),\ 1 \leq i \leq k,$$

and let $\mathfrak{e} := (e_1, \ldots, e_k)$ be the corresponding orthogonal frame at $0 \in \mathbb{R}^k$. Let $d_* : T_0\mathbb{R}^k \to T_{c(d)}M$ be the tangent map to d at $0 \in \mathbb{R}^k$. Then $d_*(\mathfrak{e})$ is an element of the frame bundle $GL(M)$ and we define the frame of d to be

$$fr(d) := r_M(d_*(\mathfrak{e})) \in \pi_M^{-1}(c(d)).$$

We then define a Σ_n-equivariant map $a(n) : \mathfrak{f}\mathcal{D}M(n) \to \mathfrak{f}\mathsf{Con}(M,n)$ by

$$a(n)(d_1, \ldots, d_n) := (c(d_1), fr(d_1); \ldots ; c(d_n), fr(d_n)),$$

for $(d_1, \ldots, d_n) \in \mathfrak{f}\mathcal{D}M(n)$. A candidate for a homotopy inverse of $a(n)$ can be constructed as follows. It is clear that the function $\rho_1 : \mathsf{Con}(M,n) \to \mathbb{R}_{>0}$ defined by

$$\rho_1(x_1, \ldots, x_n) := \min\left\{\tfrac{1}{2}\mathrm{dist}(x_i, x_j),\ 1 \leq i \neq j \leq n\right\},\ (x_1, \ldots, x_n) \in \mathsf{Con}(M,n),$$

where $\mathrm{dist}(-,-)$ is the Riemannian distance in M, is continuous.

Suppose that the manifold M is complete. Then we have, for each point $x \in M$, the exponential map $\exp_x : T_xM \to M$ from the tangent space T_xM at x to M. Let us denote, for $r > 0$, by B_r the open ball in T_xM centered at 0 with radius r. The *injectivity radius* of M at x, denoted $\mathrm{i}(x)$, is the supremum of all $r > 0$ such that $\exp_x$ is an embedding of B_r to M. The following theorem can be found in [**BC64**, page 241].

THEOREM 4.12. *If the manifold M is complete, then the injectivity radius $\mathrm{i} : M \to (0, \infty)$ is a continuous function.*

Thus, for a complete Riemannian manifold M, we have another continuous function $\rho_2 : \mathsf{Con}(M,n) \to \mathbb{R}_{>0}$ defined by

$$\rho_2(x_1,\ldots,x_n) := \min\Big\{ \mathrm{i}(x_i) \mid 1 \le i \le n \Big\}, \quad \text{for } (x_1,\ldots,x_n) \in \mathsf{Con}(M,n).$$

We finally put

$$\rho := \min(\rho_1,\rho_2) : \mathsf{Con}(M,n) \to \mathbb{R}_{>0}.$$

Let $(x_1,\mathfrak{f}_1;\ldots;x_n,\mathfrak{f}_n) \in \mathfrak{f}\mathsf{Con}(M,n)$. Let us introduce a Σ_n-equivariant map $b(n) : \mathfrak{f}\mathsf{Con}(M,n) \to \mathfrak{f}\mathcal{D}M(n)$ by $b(n)(x_1,\mathfrak{f}_1;\ldots;x_n,\mathfrak{f}_n) := (d_1,\ldots,d_n)$, with $d_i : D^k \to M$ defined as follows. There exists, for each $1 \le i \le n$, a unique linear isometric embedding $\xi_i : D^k \hookrightarrow T_{x_i}M$ such that the composition

$$\omega_i : D^k \xrightarrow{\xi_i} T_{x_i}M \xrightarrow{\exp_{x_i}} M$$

has the property that $fr(\omega_i) = \mathfrak{f}_i$. Then

$$d_i(z) := \omega_i(\rho z), \ z \in D^k, \ \rho = \rho(x_1,\ldots,x_n). \tag{4.6}$$

We have constructed equivariant maps

$$a(n) : \mathfrak{f}\mathcal{D}M(n) \to \mathfrak{f}\mathsf{Con}(M,n) \text{ and } b(n) : \mathfrak{f}\mathsf{Con}(M,n) \to \mathfrak{f}\mathcal{D}M(n)$$

clearly satisfying $a(n) \circ b(n) = \mathbb{1}$. Let us see what are our chances to construct a homotopy between $b(n) \circ a(n)$ and $\mathbb{1}$.

To simplify our language, call little disks of the form (4.6) *normal*. By definition, each normal disk d is uniquely determined by its center $\mathrm{c}(d)$, its frame $fr(d)$ and the value of ρ. Each little disk d has its *normalization* $\mathrm{n}(d)$, i.e. the normal disk $\mathrm{n}(d)$ with the same frame and center as d.

The map $b(n) \circ a(n)$ clearly replaces a little disk $d : D^k \to M$ by its normalization, $(b(n) \circ a(n))(d) = \mathrm{n}(d)$. The situation is schematically depicted in Figure 1. Thus a homotopy from $b(n) \circ a(n)$ to $\mathbb{1}$ should interpolate between d and its normalization $\mathrm{n}(d)$, but if the image of d runs off the set $\exp_{c(d)}(B_{\rho_1})$ as it does in Figure 1, it is not clear how to construct such an interpolation.

This can be fixed by introducing, for any $\delta > 0$, the submodule $\mathfrak{f}\mathcal{D}_\delta M = \{\mathfrak{f}\mathcal{D}_\delta M(n)\}_{n\ge1}$ defining $\mathfrak{f}\mathcal{D}_\delta M(n)$ to be the subspace of $\mathfrak{f}\mathcal{D}M(n)$ consisting of all disks in M whose radius (in the obvious sense) is less than δ.

If δ is small enough, the images of little disks d from $\mathfrak{f}\mathcal{D}_\delta M(n)$ are subsets of $\exp_{c(d)}(B_{\rho_1})$, so we may consider them, using the identification via $\exp_{c(d)} : T_{c(d)}M \to M$, as maps to $\mathbb{R}^k \cong T_{c(d)}M$. Now it makes sense to form a linear interpolation

$$d_t := (1-t)\mathrm{n}(d) + td, \ 0 \le t \le 1.$$

There is, however, no reason to expect that the map d_t is injective for all t. This can be achieved by further restricting to those disks in $\mathfrak{f}\mathcal{D}_\delta M(n)$ which are *conformal linear* which means, by definition, that they are compositions

$$D^k \xrightarrow{\xi} T_{c(d)} \xrightarrow{\exp_{c(d)}} M,$$

with some conformal linear ξ. Observe that the classical little k-disks operad $\mathcal{D}_k$ consists of conformal linear disks in the sense of the above definition. We may combine the above observations as:

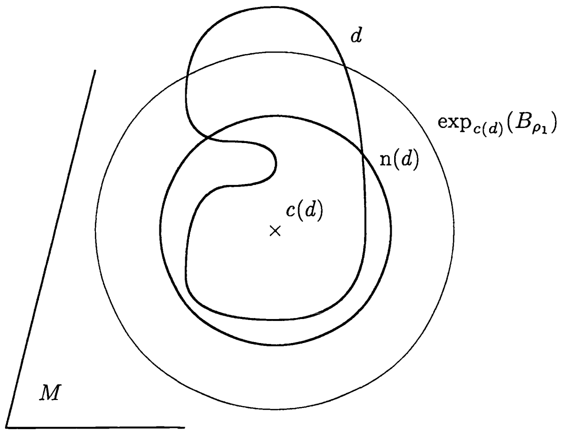


FIGURE 1. Little disk in M and its normalization. The amoeba is the image of d, the small disk is the image of $\mathrm{n}(d)$ and the big disk is the image $\exp_{c(d)}(B_{\rho_1})$. The cross $\times$ indicates the center $c(d)$.

THEOREM 4.13. *Suppose that there is a positive δ such that $\mathrm{i}(x) > \delta$ for each $x \in M$. Then the right Σ_n-space $\mathfrak{f}\mathcal{D}_\delta M(n)$ consisting of conformal linear little disks with radius less than δ is Σ_n-equivariantly homotopy equivalent to the framed configuration space $\mathfrak{f}\mathsf{Con}(M, n)$.*

Since the injectivity radius $\mathrm{i} : M \to \mathbb{R}_{\geq 0}$ is a continuous function (Theorem 4.12), the number δ in Theorem 4.13 always exists if M is compact. A moment's reflection shows, however, that the Σ-module of conformal linear disks $\mathfrak{f}\mathcal{D}_\delta M$ will no longer be closed under the right action (4.5) unless M is flat.

PROPOSITION 4.14. *Suppose that M is a flat Riemannian manifold and $\delta > 0$ as in Theorem 4.13. Then the Σ-module $\mathfrak{f}\mathcal{D}_\delta M$ is a natural right $\mathfrak{f}\mathcal{D}$-module.*

Though the flatness assumption in the previous proposition is very restrictive, there are still interesting examples (such as tori) left. Other, more exciting, geometric examples of right modules will be presented in Section 4.4.

4.2. Deligne-Knudsen-Mumford compactification of moduli spaces

In this section we closely follow the exposition given by V. Ginzburg and M. Kapranov in [**GK94**]. Let $\mathcal{M}_{0,n+1}$ be the moduli space of $(n+1)$-tuples $(x_0, \ldots, x_n)$ of distinct numbered points on the complex projective line $\mathbb{CP}^1$ modulo projective automorphisms, that is, transformations of the form

$$\mathbb{CP}^1 \ni [\xi_1, \xi_2] \mapsto [a\xi_1 + b\xi_2, c\xi_1 + d\xi_2] \in \mathbb{CP}^1,$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

Choose a point $\infty \in \mathbb{CP}^1$ so that $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Since projective automorphisms act transitively and since those automorphisms that fix ∞ are exactly those that restrict to affine automorphisms of $\mathbb{C}$, letting $x_0 = \infty$, one gets an isomorphism of $\mathcal{M}_{0,n+1}$ with the moduli space of n-tuples of distinct numbered points of $\mathbb{C}$ modulo affine automorphisms.

The moduli space $\mathcal{M}_{0,n+1}$ has, for $n \geq 2$, a canonical compactification $\overline{\mathcal{M}}(n) \supset \mathcal{M}_{0,n+1}$ introduced by A. Grothendieck and F.F. Knudsen [**Del72, Knu83**]. The space $\overline{\mathcal{M}}(n)$ is the moduli space of stable $(n+1)$-pointed curves of genus 0:

DEFINITION 4.15. A *stable $(n+1)$-pointed curve of genus* 0 is an object

$$(C; x_0, \dots, x_n),$$

where C is a (possibly reducible) algebraic curve with at most nodal singularities and $x_0, \dots, x_n \in C$ are distinct smooth points such that

(i) each component of C is isomorphic to $\mathbb{CP}^1$,
(ii) the graph of intersections of components of C (i.e. the graph whose vertices correspond to the components of C and edges to the intersection points of the components) is a tree and
(iii) each component of C has at least three special points, where a special point means either one of the x_i, $0 \leq i \leq n$, or a singular point of C (the stability).

REMARK 4.16. It can be easily seen that a stable curve $(C; x_0, \dots, x_n)$ admits no infinitesimal automorphisms that fix marked points $x_0, \dots, x_n$, therefore $(C; x_0, \dots, x_n)$ is 'stable' in the usual sense. Observe also that $\overline{\mathcal{M}}(n) = \emptyset$ for $n \leq 1$ (there are no stable curves with less than three marked points) and that $\overline{\mathcal{M}}(2) =$ the point corresponding to the three-pointed stable curve $(\mathbb{CP}^1; \infty, 1, 0)$

The space $\mathcal{M}_{0,n+1}$ forms an open dense part of $\overline{\mathcal{M}}(n)$ consisting of marked curves $(C; x_0, \dots, x_n)$ such that C is isomorphic to $\mathbb{CP}^1$.

It follows from the results of [**BG92, FM94**] that the space $\overline{\mathcal{M}}(n)$ is a smooth complex projective variety of dimension $n-2$. It has the following elementary construction described, for example, in [**BG92**, pages 64–65]. Let

$$\mathit{Aff} := \{z \mapsto az + b \mid a, b \in \mathbb{C},\ a \neq 0\}$$

be the group of affine transformations of $\mathbb{C}$. The group *Aff* acts diagonally on $\mathbb{C}^n$ preserving the open subset

$$\mathbb{C}^n_* := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } 1 \leq i < j \leq n\}$$

of points with pairwise distinct coordinates. As noted above, one has an isomorphism

$$\mathcal{M}_{0,n+1} \cong \mathbb{C}^n_*/\mathit{Aff}.$$

Let us denote by $\Delta \subset \mathbb{C}^n$ the principal diagonal, that is, the space

$$\Delta := \{(z, \dots, z) \in \mathbb{C}^n \mid z \in \mathbb{C}\}.$$

There is an embedding $\iota_n : \mathcal{M}_{0,n+1} \hookrightarrow \mathbb{CP}^{n-2}$ defined as the composition

$$\mathcal{M}_{0,n+1} \cong \mathbb{C}^n_*/\mathit{Aff} \subset (\mathbb{C}^n - \Delta)/\mathit{Aff} \cong \mathbb{CP}^{n-2},$$

where the isomorphism on the right is induced by

$$\mathbb{C}^n - \Delta \ni (z_1, \dots, z_n) \longmapsto [z_1 - z_n, z_2 - z_n, \dots, z_{n-1} - z_n] \in \mathbb{CP}^{n-2}. \tag{4.7}$$

The coordinate axes in $\mathbb{C}^n$ give, under correspondence (4.7), n distinguished points $p_1, \dots, p_n \in \mathbb{CP}^{n-2}$. Let us blow up all the points p_i, $1 \leq i \leq n$, then

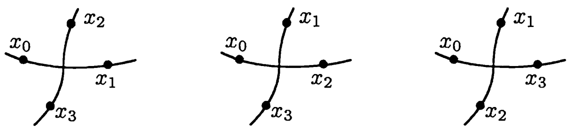


FIGURE 2. Three singular curves compactifying $\mathcal{M}_{0,4}$.

blow up the proper transforms (= closures of the preimages of some open parts (see [**Har77**, page 165])) of the lines $<p_i, p_j>$, $1 \le i < j \le n$, then blow up the proper transforms of the planes $<p_i, p_j, p_k>$, $1 \le i < j < k \le n$, and so on. The resulting space is isomorphic to $\overline{\mathcal{M}}(n)$.

More formally, we define a tower of smooth varieties Y_k, $n \ge k \ge 3$:

$$\mathbb{CP}^{n-2} = Y_n \longleftarrow Y_{n-1} \longleftarrow Y_{n-2} \longleftarrow \cdots \longleftarrow Y_3 = \overline{\mathcal{M}}(n) \tag{4.8}$$

by downward induction on k as follows.

For a subset $I \subset [n] := \{1, \dots, n\}$, let D_I denote the subvariety of $\mathbb{CP}^{n-2}$ corresponding, under (4.7), to the subset

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j \text{ for } i, j \in I\}.$$

Observe that, in this notation, $D_{[n]-\{i\}} = \{p_i\}$, $D_{[n]-\{i,j\}} = \langle p_i, p_j \rangle$, etc. Assume by induction that the tower

$$\mathbb{CP}^{n-2} = Y_n \longleftarrow Y_{n-1} \longleftarrow Y_{n-2} \longleftarrow \cdots \longleftarrow Y_k$$

has already been constructed, for some k, $n \ge k > 3$, and denote by $\pi : Y_k \to \mathbb{CP}^{n-2}$ the composition. Let $\tilde{D}_I$ be the strict transform of D_I in Y_k. The following statement follows from [**BG92**, Lemma 3.2].

LEMMA 4.17. *For each subset $I \subset [n]$ of cardinality $k-1$, $\tilde{D}_I$ is a smooth irreducible subvariety of Y_k. The intersection of any number of subvarieties $\tilde{D}_I$ with $card(I) = k-1$ is either empty or transverse.*

Now, Y_{k-1} is constructed by blowing up all subvarieties $\tilde{D}_I$ with $card(I) = k-1$, in any order. By the transversality property of Lemma 4.17 the result does not depend on the order. This completes the induction step in the inductive construction of (4.8).

EXAMPLE 4.18. For $n = 3$, the image of $\mathcal{M}_{0,4}$ in $\mathbb{CP}^1$ under the inclusion ι_3 is $\mathbb{CP}^1$ with three points $p_1 = \infty$, $p_2 = 0$ and $p_3 = 1$ removed. Tower (4.8) is trivial, so $\overline{\mathcal{M}}(3) = \mathbb{CP}^1$ is obtained from $\mathcal{M}_{0,4}$ by adding three points corresponding to three singular stable 4-curves symbolized in Figure 2.

EXAMPLE 4.19. Let us discuss the case $n = 4$. The image of the embedding $\iota_4 : \mathcal{M}_{0,5} \hookrightarrow \mathbb{CP}^2$ is the complement of the configuration of six lines $\langle p_i, p_j \rangle$, $1 \le i < j \le 4$ as in Figure 3. The line $\langle p_i, p_j \rangle$ corresponds to the degenerate configuration $z_k = z_l$, $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The lines intersect at four triple points p_i, $1 \le i \le 4$, corresponding to the degenerate configurations $z_j = z_k = z_l$, where again $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

The tower $\mathbb{CP}^2 \longleftarrow Y_3 = \overline{\mathcal{M}}(4)$ is given by blowing up the four points p_i, $1 \le i \le 4$. The complement of $\mathcal{M}_{0,5}$ in $\overline{\mathcal{M}}(4)$ consists of strict transforms $\tilde{\ell}_{ij}$ of the lines $\langle p_i, p_j \rangle$, $1 \le i < j \le 4$, and four exceptional lines $\tilde{\ell}_i$, $1 \le i \le 4$, added by

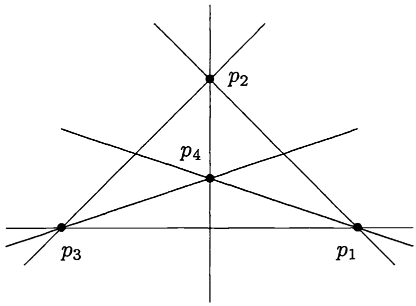
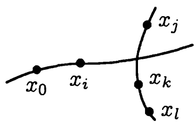


FIGURE 3. A configuration of six lines in $\mathbb{CP}^2$

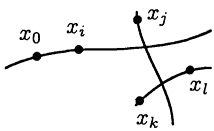
the blow up. In the rest of this example, i, j, k and l will be natural numbers such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Each $\tilde{\ell}_{ij}$ is a smooth curve containing three special points – the point $\tilde{p}_{(ij)}$ (respectively $\tilde{p}_{(ji)}$) at which ℓ_{ij} transversally meets $\tilde{\ell}_i$ (respectively $\tilde{\ell}_j$) and the point $\tilde{q}_{ij} = \tilde{q}_{kl}$ at which $\tilde{\ell}_{ij}$ transversally intersects $\tilde{\ell}_{kl}$. Thus the compactification $\overline{\mathcal{M}}(4)$ adds to $\mathcal{M}_{0,5}$ the following types of singular pointed curves.

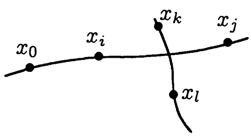
TYPE I. Curves corresponding to points of $\tilde{\ell}_i$ but different from $\tilde{p}_{(ij)}$:



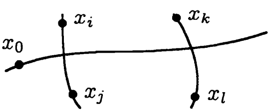
TYPE II. Curves corresponding to $\tilde{p}_{(ij)}$:



TYPE III. Curves corresponding to points of $\tilde{\ell}_{ij}$ but different from $\tilde{p}_{(ij)}$, $\tilde{p}_{(ji)}$ and $\tilde{q}_{ij}$:



TYPE IV. Curves corresponding to $\tilde{q}_{ij} = \tilde{q}_{kl}$:



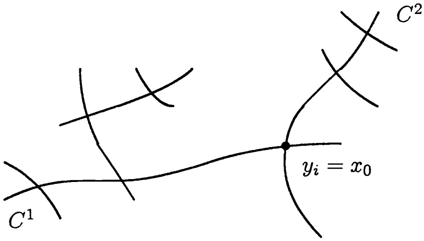


FIGURE 4. The pseudo-operadic composition of $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$.

PSEUDO-OPERAD STRUCTURE. The family of spaces $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$ forms a topological pseudo-operad, with structure operations given as follows. The composition map

$$\circ_i : \overline{\mathcal{M}}(k) \times \overline{\mathcal{M}}(l) \longrightarrow \overline{\mathcal{M}}(k+l-1) \tag{4.9}$$

is, for $k, l \geq 2$, $1 \leq i \leq k$, defined by

$$(C^1; y_0, \ldots, y_k) \times (C^2; x_0, \ldots, x_l) \longmapsto (C; y_0, \ldots, y_{i-1}, x_0, \ldots, x_l, y_{i+1}, \ldots, y_k)$$

where C is the curve obtained from the disjoint union $C^1 \bigsqcup C^2$ by identifying x_0 with y_i, introducing a nodal singularity. The composition is illustrated in Figure 4. The symmetric group acts on $\overline{\mathcal{M}}(n)$ by

$$(C, x_0, x_1, \ldots, x_n) \longmapsto (C, x_0, x_{\sigma(1)}, \ldots, x_{\sigma(n)}),\ \sigma \in \Sigma_n.$$

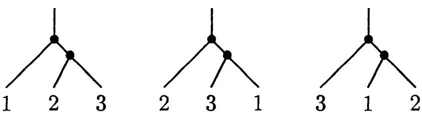
We call $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$ the *configuration pseudo-operad*, since $\overline{\mathcal{M}}(n)$ can be regarded as a (compactified) configuration space of points on $\mathbb{CP}^1$.

One can associate to a point $(C; x_0, \ldots, x_n) \in \overline{\mathcal{M}}(n)$ (an isomorphism class of) a rooted n-tree (see Remark 1.43 for the difference between trees and isomorphism classes of trees)

$$T = \mathit{Tree}(C; x_0, \ldots, x_n) \in \mathit{Tree}(n)$$

whose vertices correspond to irreducible components of C. The vertices corresponding to two components C_1, C_2 are joined by an (internal) edge if and only if $C_1 \cap C_2 \neq \emptyset$. A leaf labelled by i, $1 \leq i \leq n$, is attached to the vertex corresponding to the component containing x_i. The root is attached to the vertex corresponding to the component containing x_0. Property (iii) of stable curves ensures that $\mathit{Tree}(C; x_0, \ldots, x_n)$ is a reduced tree thus, in fact, $\mathit{Tree}(C; x_0, \ldots, x_n) \in \mathit{Rtree}(n)$.

EXAMPLE 4.20. Trees corresponding to singular curves in Figure 2 are



Trees corresponding to singular curves listed in Example 4.19 are given in Figure 5.

For any reduced n-tree T, let $\mathcal{M}(T) \subset \overline{\mathcal{M}}(n)$ be the subset consisting of stable curves $(C; x_0, \ldots, x_n)$ such that $\mathit{Tree}(C; x_0, \ldots, x_n) = T$. In this way we obtain an

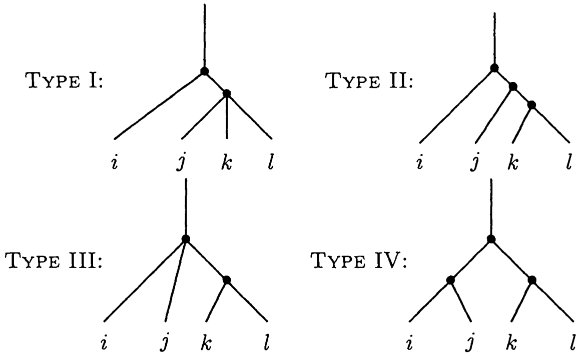


FIGURE 5.

algebraic stratification

$$\overline{\mathcal{M}}(n) = \bigcup_{T \in \mathcal{R}tree(n)} \mathcal{M}(T), \; n \geq 2. \tag{4.10}$$

This stratification has the following properties (see [**BG92**]):

$$\operatorname{codim}(\mathcal{M}(T)) = card(edge(T)),$$

where $edge(T)$ is the set of internal edges of the tree T, and

$$\mathcal{M}(T) \subset \overline{\mathcal{M}(S)} \text{ if and only if } T \geq S.$$

Recall that $T \geq S$ means that S is obtained from T by collapsing one or more internal edges. In the above display, $\overline{\mathcal{M}(S)}$ denotes the closure of $\mathcal{M}(S)$ in $\overline{\mathcal{M}}(n)$.

In particular, 0-dimensional strata are labelled by binary trees. Codimension 1 strata correspond to trees with two vertices. Their closures are precisely the irreducible components of $\overline{\mathcal{M}}(n) - \mathcal{M}_{0,n+1}$, which is a normal crossing divisor In addition, we have the following result.

PROPOSITION 4.21. *For each $n \geq 3$ and $T \in \mathcal{R}tree(n)$, there are canonical direct product decompositions*

$$\mathcal{M}(T) \cong \bigtimes_{v \in Vert(T)} \mathcal{M}_{0,\mathsf{a}(v)+1}$$

and

$$\overline{\mathcal{M}(T)} \cong \bigtimes_{v \in Vert(T)} \mathcal{M}(\mathsf{a}(v)),$$

where $\mathsf{a}(v) = card(In(v))$. In particular, the closure of each stratum is smooth.

PROOF. The first equality means that a pointed curve $(C; x_0, \dots, x_n) \in \mathcal{M}(T)$ is uniquely determined, up to isomorphism, by the projective equivalence classes of the configurations formed on each component by the marked points x_i and the double points which happen to lie on this component. This is obvious.

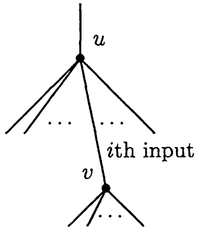


FIGURE 6. The tree $T_i(k,l)$. The vertex u has k inputs; the vertex v has l inputs.

The second equality follows from the first one once we note that

$$\overline{\mathcal{M}(T)} = \bigcup_{S \geq T} \mathcal{M}(S). \qquad \blacksquare$$

For $k, l \geq 2$ and $1 \leq i \leq k$, let $T_i(k,l)$ be the tree in Figure 6.

COROLLARY 4.22. *The structure maps* $\circ_i$ *of the pseudo-operad*

$$\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$$

can be identified with the embedding of the closed stratum

$$\overline{\mathcal{M}}(k) \times \overline{\mathcal{M}}(l) \cong \overline{\mathcal{M}(T_i(k,l))} \hookrightarrow \overline{\mathcal{M}}(k+l-1).$$

4.3. Compactification of configuration spaces of points in $\mathbb{R}^n$

The aim of this section is to construct a compactification of the moduli space $\overset{\circ}{\mathsf{F}}_k(n)$ of configurations of n distinct points in the k-dimensional Euclidean plane $\mathbb{R}^k$ modulo the action of the affine group, described by Getzler and Jones in [**GJ94**] and denoted by $\mathsf{F}_k(n)$. The authors of [**GJ94**] also stated that the Σ-module $\mathsf{F}_k := \{\mathsf{F}_k(n)\}_{n\geq 1}$ has a natural structure of a topological operad. This was a well-known fact for $k = 1$, because, as we will see in Example 4.36, the Σ-module $\mathsf{F}_1 = \{\mathsf{F}_1(n)\}_{n\geq 1}$ is nothing but the symmetrization of the non-Σ operad $\underline{\mathcal{K}} = \{K_n\}_{n\geq 1}$ of the associahedra recalled in Section 1.6. The operad F_2 plays an important rôle in topological closed string field theory. The main results of this section are Theorem 4.35 and Proposition 4.38.

In this and in the following sections we will often work with objects whose nature is similar to that of the free pseudo-operad (Definition 1.77) and whose formal definitions ought to be based on colimits over certain categories of trees. In applications we have in mind it would be, however, difficult to work with these formal definitions, so we prefer to define these objects as direct sums indexed by isomorphism classes of trees, assuming an implicit choice of representatives of these isomorphism classes; see Remark 1.84.

We are going to present a construction that stresses the operadic nature of the compactification. Let V be a k-dimensional vector space. In [**Mar99a**] we observed

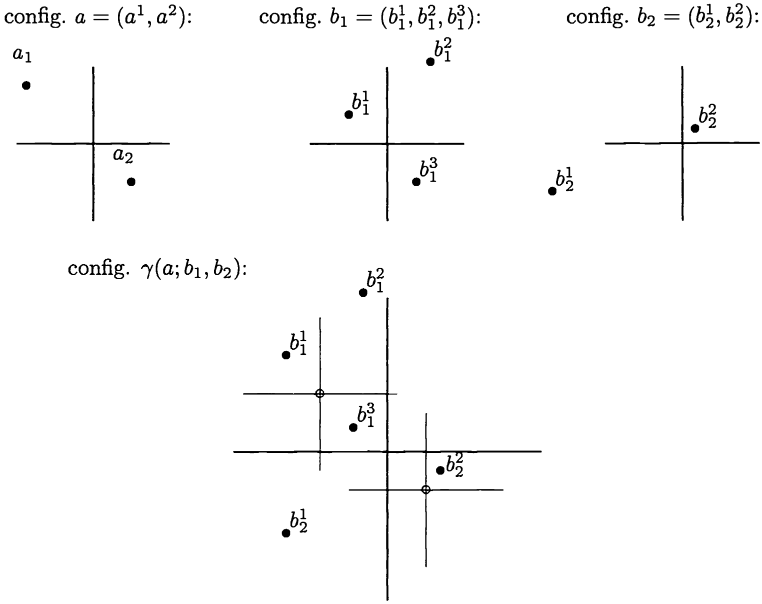


FIGURE 7. The construction of $\gamma(a; b_1, b_2) \in \mathsf{Con}(\mathbb{R}^2, 5)$ from $a \in \mathsf{Con}(\mathbb{R}^2, 2)$, $b_1 \in \mathsf{Con}(\mathbb{R}^2, 3)$ and $b_2 \in \mathsf{Con}(\mathbb{R}^2, 2)$.

that the Σ-module

$$\mathsf{Con}(V) = \{\mathsf{Con}(V, n)\}_{n \geq 1}$$

of configuration spaces possesses a kind of partial operad structure which can be described as follows. We need to specify, for each $a = (a^1, \dots, a^l) \in \mathsf{Con}(V, l)$ and $b_i \in \mathsf{Con}(V, m_i)$, the value of the 'composition map' $\gamma(a; b_1, \dots, b_l) \in \mathsf{Con}(V, m_1 + \cdots + m_l)$. This can be done by putting

$$\gamma(a; b_1, \dots, b_l) := ((\underbrace{a^1, \dots, a^1}_{m_1 \text{ times}}) + b_1, (\underbrace{a^2, \dots, a^2}_{m_2 \text{ times}}) + b_2, \dots, (\underbrace{a^l, \dots, a^l}_{m_l \text{ times}}) + b_l).$$

The configuration $\gamma(a; b_1, \dots, b_l)$ may be viewed as the superposition of the configurations $T_{a_1}(b_1), \dots, T_{a_l}(b_l)$, where $T_a(-)$ means, just here and now, the translation by a vector $a \in V$. This process is visualized, for $V = \mathbb{R}^2$, in Figure 7.

We encourage the reader to verify that all the axioms of an operad are satisfied. The only drawback is that $\gamma(a; b_1, \dots, b_l)$ need not necessarily be an element of the configuration space $\mathsf{Con}(V, m_1 + \cdots + m_l)$, because the components of $\gamma(a; b_1, \dots, b_l)$ need not be different. Thus the structure map is defined only for *some* elements of

$$\mathsf{Con}(V, l) \times \mathsf{Con}(V, m_1) \times \cdots \times \mathsf{Con}(V, m_l).$$

Such an object may be called a *partial operad*; see Section I.1.9.

This partial operad structure motivated us to establish a general theory of partial operads and to show that the compactification is a certain operadic completion of the partial operad of configuration spaces, making it manifestly an operad. Let us point out, however, that our definition of a partial operad indicated above and made precise in [**Mar99a**, page 189] differs from other definitions [**KM95, Hua97**].

We will not repeat the construction of the operadic completion in its full generality here; we just show what it gives for the special case of the partial operad related to the moduli space of configurations $\overset{\circ}{\mathsf{F}}_k(n)$ mentioned above, which we called in [**Mar99a**] the partial operad of *virtual configurations*.

In fact, since we need all our constructions to be 'coordinate-free,' we are going to consider configurations of points in an arbitrary k-dimensional real vector space with a Euclidean metric. Of course, each such V is isomorphic to $\mathbb{R}^k$, therefore all our constructions will also be isomorphic to those made for $\mathbb{R}^k$. The point is that the isomorphism is noncanonical, depending on the identification of V with $\mathbb{R}^k$. In other words, we would like to consider constructions of this section as functors on the category (groupoid) of finite dimensional Euclidean vector spaces and their linear isometries. This will be useful in Section 4.4.

Let us recall some definitions. As before, let $\mathsf{Con}(V, n)$ be the configuration space of n distinct labeled points in a k-dimensional Euclidean vector space V, $n \geq 1$.

DEFINITION 4.23. Let $\mathit{Aff}(V)$ be the affine group of translations and dilatations of V. The group $\mathit{Aff}(V)$ acts in the obvious manner on $\mathsf{Con}(V, n)$. Define $\overset{\circ}{\mathsf{F}}_V(n)$ to be the quotient space

$$\overset{\circ}{\mathsf{F}}_V(n) := \mathsf{Con}(V, n)/\mathit{Aff}(V),\ n \geq 1. \tag{4.11}$$

To keep our notation compatible with the literature [**GJ94**], we denote, for $k \geq 1$, $\overset{\circ}{\mathsf{F}}_k(n) := \overset{\circ}{\mathsf{F}}_{\mathbb{R}^k}(n)$.

The moduli spaces of (4.11) form a Σ-module $\overset{\circ}{\mathsf{F}}_V = \{\overset{\circ}{\mathsf{F}}_V(n)\}_{n\geq 1}$ with the symmetric group permuting the labels of the points. As usual, we can extend $\overset{\circ}{\mathsf{F}}_V$ to a functor on the category of finite sets; for a finite set X, then $\overset{\circ}{\mathsf{F}}_V(X)$ is the moduli space of configurations of distinct points labeled by elements of X. It is immediate to see that each $\eta \in \overset{\circ}{\mathsf{F}}_V(X)$ has a unique representative $\{z^s \in V\}_{s\in X}$ such that

$$\sum_{s\in X} z^s = 0 \text{ and } \sum_{s\in X} |z^s|^2 = 1, \tag{4.12}$$

where $|-|$ is the Euclidean norm of V. Observe that the second equation of (4.12) does not mean that the points $\{z^s\}_{s\in X}$ lie on the unit circle, since $\{z^s\}_{s\in X}$ is a set of points of V, not coordinates of one point.

We call $\{z^s \in V^k\}_{s\in X}$ satisfying (4.12) the *normal representative* of $\eta \in \overset{\circ}{\mathsf{F}}_V(X)$ and we call sequences satisfying (4.12) *normal*.

The above can be rephrased by saying that each sequence $\vec{w} = \{w^s \in V\}_{s\in X}$ has a unique *normalization* $\mathit{nor}(\vec{w})$ which is, by definition, a normal sequence $\vec{z} = \{z^s \in V\}_{s\in X}$ congruent to $\vec{w}$ modulo the action of the affine group. The set of all normal sequences $\vec{z} = \{z^s \in V\}_{s\in X}$ is clearly compact in V^X.

METRIC TREES. In this section, by a *metric tree* we mean (an isomorphism class of) a rooted tree T together with a 'length' function (or a metric) $h : \mathit{edge}(T) \to \mathbb{R}_{\geq 0}$

T:

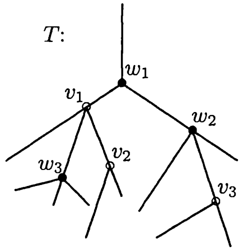


FIGURE 8. Suppose that $\mathfrak{z}(h) = \{w_1, w_2, w_3\}$ (solid vertices). Then T_{w_1} is the subtree with $Vert(T_{w_1}) = \{w_1, v_1, v_2\}$, T_{w_2} is the subtree with $Vert(T_{w_2}) = \{w_2, v_3\}$ and T_{w_3} is the corolla with vertex w_3.

from the set of (internal) edges of T into nonnegative real numbers. See also Remark 1.43 for the relation betweeen trees and isomorphism classes of trees. So metric trees are the same as in Definition 2.19 except that we allow their edges to have an arbitrary nonnegative length. Let us denote by $Met(T)$ the set of all metrics on a given tree T; we are sure that there is no danger of mistaking it with the notation used in Section 2.8. Let us also denote by $\mathcal{R}mtree(n)$ the space of all reduced metric trees in the above sense, that is,

$$\mathcal{R}mtree(n) := \{(T, h) \mid T \in \mathcal{R}tree(n),\ h \in Met(T)\}.$$

So $\mathcal{R}mtree(n)$ has now a slightly different meaning than in Section 2.8.

For any $e \in edge(T)$, let $v_e \in Vert(T)$ be the unique vertex of T such that e is the output of v. Let $rv(T)$ be the vertex adjacent to the root of T. Given $(T, h) \in \mathcal{R}mtree(n)$, define the 'zero set' of h to be the following subset of $Vert(T)$:

$$\mathfrak{z}(h) := \{v_e \in Vert(T) \mid h(e) = 0\} \cup \{rv(T)\}. \tag{4.13}$$

The metric h also determines a decomposition of T into disjoint subtrees $\{T_w\}_{w \in \mathfrak{z}(h)}$ by cutting in two each edge of length zero. More precisely, for each $w \in \mathfrak{z}(h)$, T_w is defined by

$$Vert(T_w) := \{v \in Vert(T) \mid v \geq w \text{ and there is no } u \in \mathfrak{z}(h) \text{ such that } v \geq u > w\},$$

where $<$ is the partial order $<_T$ on $Vert(T)$ introduced in Definition 3.59. The idea of the construction of T_w is illustrated in Figure 8. One may also say that $\{T_w\}_{w \in \mathfrak{z}(h)}$ is the decomposition of T into the biggest subtrees T_w such that the restriction of h to T_w is positive on all (internal) edges of T_w; w is then the root of T_w.

Finally, let $T_h \in \mathcal{R}mtree(n)$ denote the tree obtained from T by shrinking internal edges of each subtree T_w, $w \in \mathfrak{z}(h)$, to a vertex which we denote again by w. The construction of T_h from the metric tree (T, h) of Figure 8 is illustrated in Figure 9.

THE COMPACTIFICATION. The rest of this section is devoted to a construction of the compactification of the Σ-module $\overset{\circ}{\mathsf{F}}_V = \{\overset{\circ}{\mathsf{F}}_V(n)\}_{n \geq 1}$. There is, unfortunately, still a rather lengthy and rough road ahead. For any reduced $T \in \mathcal{R}tree_n$, consider

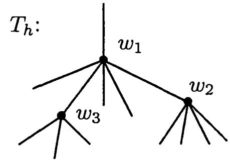


FIGURE 9. Construction of T_h from the metric tree (T, h).

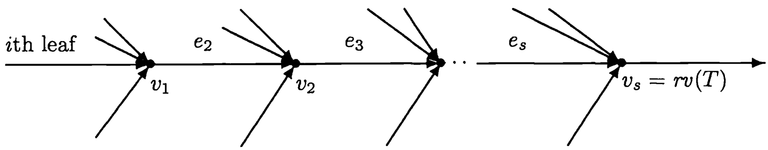


FIGURE 10. A path in the tree T.

the space

$$(4.14)\qquad Y_V[T] := Met(T) \times \overset{\circ}{\mathsf{F}}_V(T).$$

Informally, elements of $Y_V[T]$ are metric trees with vertices colored by the Σ-module $\overset{\circ}{\mathsf{F}}_V$; see Remark 1.71. More formally, elements of $Y_V[T]$ are sequences

$$(4.15)\qquad \xi = \{(t_v, [\vec{z}_v]) \mid v \in Vert(T)\}, \text{ where } (t_v, [\vec{z}_v]) \in \mathbb{R}_{\geq 0} \times \overset{\circ}{\mathsf{F}}_V(In(v)),$$

where $t_v := h(e_v)$, the length of the outgoing edge e_v of $v \in Vert(T)$ if v is not a root, and $t_{rv(T)} := 0$. By $\vec{z}_v$ we denote the normal representative $\vec{z}_v = \{z_v^e \in V\}_{e \in In(v)}$ of an element of $\overset{\circ}{\mathsf{F}}_V(In(v))$.

We define, for $T \in \mathcal{R}tree(n)$, a map $\omega_T : Y_V[T] \to V^n$ as follows. For any $1 \leq i \leq n$ there is in T a unique path from the ith input leaf to the root, as in Figure 10. Using notation (4.15), we put

$$(4.16)\quad \omega_i(\xi) := z_{v_s}^{e_s} + t_{v_{s-1}} \cdot z_{v_{s-1}}^{e_{s-1}} + \cdots + t_{v_2} \cdots t_{v_{s-1}} \cdot z_{v_2}^{e_2} + t_{v_1} \cdots t_{v_{s-1}} \cdot z_{v_1}^{i} \in V^k$$

and

$$(4.17)\qquad \omega_T(\xi) := (\omega_1(\xi), \dots, \omega_n(\xi)) \in V^n.$$

EXAMPLE 4.24. For $T = \mathrm{c}(n)$ (the n-corolla), $n \geq 2$, the map

$$\overset{\circ}{\mathsf{F}}_V(n) \cong Y_V[\mathrm{c}(n)] \xrightarrow{\omega_{\mathrm{c}(n)}} \mathsf{Con}(V, n)$$

maps $\eta \in \overset{\circ}{\mathsf{F}}_V(n)$ to its unique normal representative in $\mathsf{Con}(V, n)$.

The map ω_T in fact reflects a partial operad structure on the configuration space. The 'partiality' means that $\omega_T(\xi)$ is an element of the configuration space $\mathsf{Con}(V, n)$ only for ξ belonging to some subset $U_V[T]$ of $Y_V[T]$,

$$(4.18)\qquad U_V[T] := \{\xi \in Y_V[T] \mid \text{all } \omega_i(\xi) \text{ are distinct, } 1 \leq i \leq n\}.$$

It is thus appropriate to call the subset $U_V[T]$ the set of *composable elements*. We believe that this notation will not be mistaken with the coequalizer in Definition 1.81.

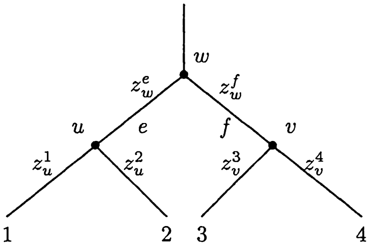


FIGURE 11. An $\overset{\circ}{\mathsf{F}}_V$-colored tree $T \in \mathcal{R}tree(4)$.

Observe that, if $\xi = (h, \zeta) \in U_V[T] \subset Met(T) \times \overset{\circ}{\mathsf{F}}_V(T)$, then $\mathfrak{Z}(h) = \{rv(T)\}$. The opposite implication is not true for other trees than corollae, as illustrated in the following example.

EXAMPLE 4.25. Let $V = \mathbb{R}^1$ and consider the tree T in Figure 11. The root w is colored by $[z_w^e, z_w^f] \in \overset{\circ}{\mathsf{F}}_V(\{e, f\})$ with $z_w^e := -\frac{1}{\sqrt{2}}$, $z_w^f := \frac{1}{\sqrt{2}}$. The colors of the remaining vertices are the same, that is,

$$[z_w^e, z_w^f] = [z_u^1, z_u^2] = [z_v^3, z_v^4].$$

This defines an element $\zeta \in \overset{\circ}{\mathsf{F}}_V(T)$. For two positive parameters $s, t \in \mathbb{R}_{\geq 0}$ define $h_{s,t} \in Met(T)$ by $h_{s,t}(e) := s$, $h_{s,t}(f) := t$ and put $\xi_{s,t} := (h_{s,t}, \zeta) \in Y_V[T]$ with $\zeta \in \overset{\circ}{\mathsf{F}}_V(T)$ as above. An easy calculation shows that

$$\omega_T(\xi_{s,t}) = \frac{1}{\sqrt{2}}(-1 - s, -1 + s, 1 - t, 1 + t) \in \mathbb{R}^4.$$

From this we conclude that $\xi_{s,t} \in U_V[T]$ if and only if

$$s, t \neq 0, \ |s - t| \neq 2 \text{ and } s + t \neq 2. \tag{4.19}$$

Thus, for instance, the element $\xi_{1,1} = (h_{1,1}, \zeta) \in Y_V[T]$ does not belong to $U_V[T]$ though $\mathfrak{Z}(h_{1,1}) = \{rv(T)\}$.

The following lemma, formulated in [**Mar99a**, page 192], will be useful in our proofs.

LEMMA 4.26. *For any $n \geq 2$ and a reduced tree $T \in \mathcal{R}tree(n)$, the composition*

$$\lambda_T : U_V[T] \xrightarrow{\omega_T} \mathsf{Con}(V, n) \xrightarrow{proj} \overset{\circ}{\mathsf{F}}_V(n)$$

is a monomorphism.

We make the set $U_V[T]$ a bit bigger, allowing also some (h, ζ) such that $\mathfrak{Z}(h) \neq \{rv(T)\}$, namely those whose restriction to each subtree T_w, $w \in \mathfrak{Z}(h)$, is composable. This can be formally done as follows.

Any $\xi = (h, \zeta) \in Met(T) \times \overset{\circ}{\mathsf{F}}_V(T) = Y_V[T]$ determines, for each $w \in \mathfrak{Z}(h)$, the restriction

$$r_w(\xi) := (h|_{T_w}, \zeta|_{T_w}) \in Y_V[T_w]. \tag{4.20}$$

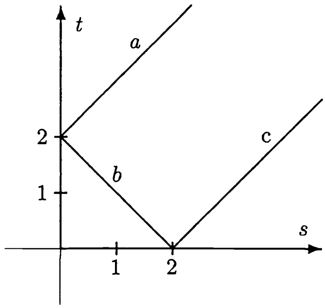


FIGURE 12. The subsets $U_V[T]$ and $\widehat{U}_V[T]$ of the first quadrant $\langle 0,\infty)^{\times 2}$.

Define

$$\widehat{U}_V[T] := \{\xi \in Y_V[T] \mid r_w(\xi) \in U_V[T_w] \text{ for all } w \in \mathfrak{Z}(h)\} \tag{4.21}$$

and call $\widehat{U}_V[T]$ the *extended* set of composables.

EXAMPLE 4.27. In the situation of Example 4.25, $\xi_{s,t} \in \widehat{U}_V[T]$ if and only if

$$|s-t| \neq 2 \text{ and } s+t \neq 2;$$

compare (4.19). It is clear that $Y_V[T]$ is in this case isomorphic to the first quadrant $\langle 0,\infty)^{\times 2}$. Then

$$\begin{aligned} U_V[T] &= (0,\infty)^{\times 2} - (a \cup b \cup c) \quad \text{and} \\ \widehat{U}_V[T] &= \langle 0,\infty)^{\times 2} - (a \cup b \cup c). \end{aligned}$$

See Figure 12.

There are special elements of $Y_V[T]$ of the form $(0,\psi) \in Met(T) \times \overset{\circ}{\mathsf{F}}_V(T)$, where 0 denotes the trivial metric assigning to each edge of T length zero. It is clear that each such element belongs to $\widehat{U}_V[T]$. We call such elements *primitive*. Assigning to each $\psi \in \overset{\circ}{\mathsf{F}}_V(T)$ its related primitive $(0,\psi)$ defines an inclusion

$$\iota_T : \overset{\circ}{\mathsf{F}}_V(T) \hookrightarrow \widehat{U}_V[T]. \tag{4.22}$$

The set $\widehat{U}_V[T]$ was defined in such a way that each $\xi = (h,\zeta) \in \widehat{U}_V[T]$ determines $\gamma(\xi) \in \overset{\circ}{\mathsf{F}}_V(T_h)$ by

$$\gamma(\xi) := \bigtimes_{w \in Vert(T_h) = \mathfrak{Z}(h)} [\omega_{T_w}(r_w(\xi))]. \tag{4.23}$$

Observe that

$$\gamma(0,\psi) = \psi \in \overset{\circ}{\mathsf{F}}_V(T),$$

for each $\psi \in \overset{\circ}{\mathsf{F}}_V(T)$. This can be expressed as saying that 'γ is the identity on primitives.' The following lemma will be used in the proof of Theorem 4.35.

LEMMA 4.28. *Let $\xi_1 = (h_1, \zeta_1)$ and $\xi_2 = (h_2, \zeta_2)$ be elements of $\widehat{U}_V[T]$ such that $\mathfrak{Z}(h_1) = \mathfrak{Z}(h_2)$. If we denote $S := T_{h_1} = T_{h_2}$, then*

$$\gamma(\xi_1) = \gamma(\xi_2) \text{ (equality of elements of } \overset{\circ}{\mathsf{F}}_V(S))$$

implies that $\xi_1 = \xi_2$.

PROOF. By definition (4.23) of γ, it is enough to verify that $[\omega_{T_w}(r_w(\xi_1))] = [\omega_{T_w}(r_w(\xi_2))]$ implies that $r_w(\xi_1) = r_w(\xi_2)$, for any $w \in \mathfrak{Z}(h_1) = \mathfrak{Z}(h_2)$. This immediately follows from Lemma 4.26 applied to the tree T_w. ∎

Define the reduced Σ-module (see Section 1.5 for the terminology) $\widehat{U}_V := \{\widehat{U}_V(n)\}_{n \geq 2}$ by

$$\widehat{U}_V(n) := \bigsqcup_{T \in \mathcal{R}tree(n)} \widehat{U}_V[T] \text{ for } n \geq 2. \tag{4.24}$$

Elements of $\widehat{U}_V(n)$ are called *virtual configurations*.

DEFINITION 4.29. The compactification $\mathsf{F}_V(n)$ is, for $n \geq 2$, defined by

$$\mathsf{F}_V(n) := \widehat{U}_V(n)/\sim \tag{4.25}$$

where the relation $\sim$ is given by

$$\widehat{U}_V(n) \supset \widehat{U}_V[T] \ni \xi \sim (0, \gamma(\xi)) \in \widehat{U}_V(T_h) \subset \widehat{U}_V(n), \tag{4.26}$$

for $T \in \mathcal{R}tree(n)$. We put $\mathsf{F}_V(1) := *$. We also denote $\mathsf{F}_k(n) := \mathsf{F}_{\mathbb{R}^k}(n)$, $k \geq 1$.

We recommend looking at Example 4.32 and Example 4.33 to get an intuitive understanding as to why (4.25) indeed defines a compactification.

LEMMA 4.30. *The projection $\widehat{U}_V(n) \xrightarrow{proj} \mathsf{F}_V(n)$ is, for any $n \geq 1$, an open map.*

PROOF. This is the first of many statements in this section whose 'formal' proof would be, due to the lack of an effective notation, very difficult to give, but whose validity is 'evident.'

By [**Kel57**, Theorem 10, page 97], the projection from $\widehat{U}_V(n)$ to the quotient space $\widehat{U}_V(n)/\sim$ is open if and only if for each open $U \subset \widehat{U}_V(n)$ the set

$$\{\xi \in \widehat{U}_V(n) \mid \xi \sim u \text{ for some } u \in U\}.$$

is also open. The openness of this set follows from the fact that the relation $\sim$ is defined with the help of the maps γ introduced in (4.23) which are continuous (this is clear) and *open*, which can be seen easily. ∎

PROPOSITION 4.31. *The Σ-module $\mathsf{F}_V = \{\mathsf{F}_V(n)\}_{n \geq 1}$ has a natural structure γ_{F} of a topological operad. The natural map*

$$\rho : \Gamma(\overset{\circ}{\mathsf{F}}_V) \to \mathsf{F}_V, \tag{4.27}$$

where $\Gamma(\overset{\circ}{\mathsf{F}}_V)$ is the free operad on the Σ-module $\overset{\circ}{\mathsf{F}}_V$ (Section 1.9), is an isomorphism of Set*-operads.*

PROOF. The operad structure on F_V is given as follows. For $n \geq 2$ define

$$Y_V(n) := \bigsqcup_{T \in \mathcal{R}tree(n)} Y_V[T].$$

The Σ-module $Y_V = \{Y_V(n)\}_{n\geq 2}$ is a topological pseudo-operad, with the composition defined by grafting as in the proof of Proposition 1.78. It is clear that the reduced Σ-module $\widehat{U}_V$ of (4.24) is a sub-pseudo-operad of Y_V. Defining relation (4.26) is obviously compatible with the pseudo-operad structure of $\widehat{U}_V$, thus the reduced Σ-module $\{\mathsf{F}_V(n)\}_{n\geq 2}$ is a pseudo-operad as well. The operad structure on the full Σ-module $\{\mathsf{F}_V(n)\}_{n\geq 1}$ is then given by formally adjoining the identity $1 \in \mathsf{F}_V(1) = *$ to the pseudo-operad $\{\mathsf{F}_V(n)\}_{n\geq 2}$ as in (1.58).

The map (4.22) induces, for the n-corolla $\mathrm{c}(n)$, a Σ_n-equivariant morphism

$$\phi(n) : \overset{\circ}{\mathsf{F}}_V(n) \cong \overset{\circ}{\mathsf{F}}_V(\mathrm{c}(n)) \xrightarrow{\iota_{\mathrm{c}(n)}} \widehat{U}_V[\mathrm{c}(n)] \hookrightarrow \widehat{U}_V(n) \xrightarrow{proj} \mathsf{F}_V(n).$$

By the universal property of the free operad $\Gamma(\overset{\circ}{\mathsf{F}}_V)$, the sequence $\{\phi(n)\}_{n\geq 2}$ gives rise to the desired continuous operadic map $\rho : \Gamma(\overset{\circ}{\mathsf{F}}_V) \to \mathsf{F}_V$.

Let us prove that each $\rho(n)$ is a set isomorphism. This statement is clear for $n = 1$, since $\Gamma(\overset{\circ}{\mathsf{F}}_V)(1) = \mathsf{F}_V(1) = *$, so assume $n \geq 2$. By definition (see (1.52) and (1.58)),

$$\Gamma(\overset{\circ}{\mathsf{F}}_V)(n) = \bigsqcup_{T \in \mathcal{R}tree(n)} \overset{\circ}{\mathsf{F}}_V(T).$$

Further, for $T \in \mathcal{R}tree(n)$, the restriction $\rho|_{\overset{\circ}{\mathsf{F}}_V(T)}$ coincides with the composition

$$\overset{\circ}{\mathsf{F}}_V(T) \xrightarrow{\iota_T} \widehat{U}_V[T] \xrightarrow{proj} \overset{\circ}{\mathsf{F}}_V(n),$$

where ι_T is the map (4.22). In other words, $\rho(n)$ sends an element of $\overset{\circ}{\mathsf{F}}_V(T)$ to the class of the corresponding primitive element. The fact that $\rho(n)$ is a set isomorphism then follows from the following statement whose proof is immediate.

CLAIM. *Suppose A is a set, $P \subset A$ a subset and let $f : A \to P$ be a retraction of sets, i.e. $f|_P = 1\!\!1_P$. Let $X := A/\sim$ with the relation $\sim$ given by*

$$A \ni a \sim f(a) \in P \subset A.$$

Then the composition $\rho : P \hookrightarrow A \xrightarrow{proj} X$ is an isomorphism of sets.

In our concrete situation, we put $A = \widehat{U}_V(n)$, $P \subset \widehat{U}_V(n)$ to be the set of primitive elements and the map f to be induced by γ of (4.23). ■

Following the notation introduced in [**GJ94**], we denote the operad $\mathsf{F}_{\mathbb{R}^k} = \{\mathsf{F}_{\mathbb{R}^k}(n)\}_{n\geq 1}$ by $\mathsf{F}_k = \{\mathsf{F}_k(n)\}_{n\geq 1}$. Proposition 4.31 implies that, forgetting the topology, the compactification F_V is the free operad on the Σ-module $\overset{\circ}{\mathsf{F}}_V$. The role of nonprimitive elements in the quotient (4.25) is to introduce an appropriate topology or, in other words, to glue together the 'strata' $\{\overset{\circ}{\mathsf{F}}_V(T)\}_{T\in\mathcal{R}tree(n)}$ forming the space $\mathsf{F}_V(n)$. This means that the set $\mathsf{F}_V(n)$ can be decomposed into the disjoint

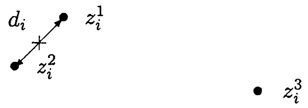


FIGURE 13. Normal configuration (z_i^1, z_i^2, z_i^3) for i very big. The distance $d_i := |z_i^1 - z_i^2|$ is very small compared to $|z_i^1 - z_i^3|$ or $|z_i^2 - z_i^3|$. The $+$ denotes the 'center of gravity' of points z_i^1 and z_i^2.

union of strata,

$$\mathsf{F}_V(n) = \bigsqcup_{T \in \mathcal{R}tree(n)} \mathring{\mathsf{F}}_V(T). \tag{4.28}$$

The moduli space $\mathring{\mathsf{F}}_V(n)$ is a subset of $\mathsf{F}_V(n)$ corresponding, in the above decomposition, to the n-corolla $\mathrm{c}(n)$ (the 'top stratum'). Equivalently, $\mathring{\mathsf{F}}_V(n)$ is the set of equivalence classes of points $\xi = (h, \zeta) \in \hat{U}(T)$ such that

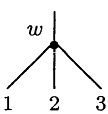
$$\mathfrak{z}(h) = rv(T). \tag{4.29}$$

EXAMPLE 4.32. Let us give some intuition behind formula (4.25) and explain why $\mathsf{F}_V(n)$ indeed compactifies the space $\mathring{\mathsf{F}}_V(n)$. Let us start with the case $\dim(V) = 2$ and $n = 3$. Consider a sequence $\{x_i \in \mathring{\mathsf{F}}_V(3)\}_{i \geq 1}$ and show that it has a cluster point in $\mathsf{F}_2(3)$.

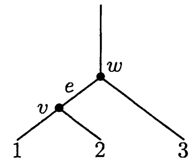
Choose the normal representative $(z_i^1, z_i^2, z_i^3) \in V^3$ for each $x_i \in \mathring{\mathsf{F}}_V(3)$, $i \geq 1$. Since the subspace of all normal triples is compact, the sequence (z_i^1, z_i^2, z_i^3) has a cluster point $(\mathrm{c}^1, \mathrm{c}^2, \mathrm{c}^3)$ in V^3, which is again a normal triple. If all $\mathrm{c}^1, \mathrm{c}^2, \mathrm{c}^3$ are distinct we are done, because then the class $[\mathrm{c}^1, \mathrm{c}^2, \mathrm{c}^3] \in \mathring{\mathsf{F}}_V(3)$ is clearly a cluster point of the sequence $\{x_i\}_{i \geq 1}$.

If the points $\mathrm{c}^1, \mathrm{c}^2, c^3$ are not distinct, then exactly two of them coincide; the case when all three points are the same is excluded by (4.12). Suppose, without the loss of generality, that $\mathrm{c}^1 = \mathrm{c}^2$. This means that, after passing to a subsequence if necessary, the points z_i^1 and z_i^2 are very close for i big enough. The configuration looks as shown in Figure 13.

The point (z_i^1, z_i^2, z_i^3) has several representatives in the 'big' space $\widehat{U}_V(3)$. One such representative is the primitive one $\xi_i^p \in U_V[t(3)]$ attached to the 3-corolla $\mathrm{c}(3)$:



with the vertex w colored by (z_i^1, z_i^2, z_i^3). We will consider another representative $\xi_i^{(\ )} \in U_V[T^{(\ )}]$ attached to the tree $T^{(\ )}$:



with the vertex w colored by the normalization of the configuration

$$\left(\frac{z_i^1 + z_i^2}{2}, z_i^3\right) \in V^3$$

representing a point of $\mathring{\mathsf{F}}_V(In(w)) \cong \mathring{\mathsf{F}}_V(2)$, the vertex v colored by the normalization of the configuration

$$(z_i^1, z_i^2) \in V^3$$

representing a point of $\mathring{\mathsf{F}}_V(In(v)) \cong \mathring{\mathsf{F}}_V(2)$ and the length function $h_i \in Met(T^{(\ )})$ given by

$$h_i(e) := \frac{1}{3}\frac{|z_i^1 - z_i^2|}{|z_i^1 + z_i^2|}. \tag{4.30}$$

It then can be verified directly that

$$\omega_{T^{(\ )}}(\xi_i^{(\ )}) = (z_i^1, z_i^2, z_i^3) = \omega_{c(3)}(\xi_i^p)$$

for each i, therefore, by (4.26), the points $\xi_i^{(\ )}$ and ξ_i^p indeed represent the same element of the space $\mathsf{F}_V(3)$.

Intuitively, we introduced a new vertex v replacing two very close points z_i^1 and z_i^2 by their 'center of gravity' $\frac{1}{2}(z_i^1 + z_i^2)$ and colored this new vertex by the 'microscopic' configuration (z_i^1, z_i^2). The rather complicated form (4.30) of $h_i(e)$ follows from our choice of the normalization. Observe that, for $|z_i^1 - z_i^2|$ very small,

$$h_i(e) \sim \frac{d_i}{6|z_i^1|}, \text{ with } d_i := |z_1^i - z_2^i|. \tag{4.31}$$

The representative $\xi_i^{(\ )}$ belongs to the set

$$U_V[T^{(\ )}] \subset Y_V[T^{(\ )}] \cong \mathring{\mathsf{F}}_V(2) \times \mathring{\mathsf{F}}_V(2) \times \mathbb{R}_{\geq 0} \cong S^1 \times S^1 \times \mathbb{R}_{\geq 0}.$$

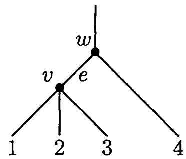
This set is not compact, but since the points z_i^1 and z_i^2 are, for i very big, $i \gg 0$, very close to each other, the third parameter in $\mathbb{R}_{\geq 0}$ representing the length $h_i(e)$ is, by (4.31), very small, so the sequence $\xi_i^{(\ )}$ has a cluster point $\bar{\xi}^{(\ )}$ in $\hat{U}_V[T^{(\ )}]$. The class $\bar{x}$ of $\bar{\xi}^{(\ )}$ in $\mathsf{F}_V(3)$ is then clearly a cluster point of the sequence $\{x_i\}_{i\geq 0}$.

EXAMPLE 4.33. Let us suppose again that $\dim(V) = 2$ and present a similar discussion for the space $\mathring{\mathsf{F}}_V(4)$. So, let $\{x_i\}_{i\geq 1}$ be a sequence of points in $\mathring{\mathsf{F}}_V(4)$ and let $(z_i^1, z_i^2, z_i^3, z_i^4)$ be the normal representative of x_i, $i \geq 1$. Let us show again that the sequence $\{x_i\}_{i\geq 1}$ has a cluster point in $\mathsf{F}_V(4)$. Because the subspace of normal quadruples is compact, we may assume (after passing to a subsequence if necessary) that the sequence $\{(z_i^1, z_i^2, z_i^3, z_i^4)\}_{i\geq 1}$ converges to a point $(c^1, c^2, c^3, c^4) \in V^4$. We distinguish four cases.

CASE 1. All points c^1, c^2, c^3, c^4 are distinct. Then the class $[c^1, c^2, c^3, c^4] \in \mathring{\mathsf{F}}_V(4)$ is a cluster point of the sequence $\{x_i\}_{i\geq 1}$ and we are done.

CASE 2. Exactly two of the points c^1, c^2, c^3, c^4 coincide, say $c^1 = c^2$. We proceed exactly as for $n = 3$, that is, we replace the points z_i^1 and z_i^2 by their 'center of gravity' and introduce a new vertex colored by the normalization of the 'microscopic' configuration (z_i^1, z_i^2).

CASE 3. Exactly three of the points c^1, c^2, c^3, c^4, say $c^1 = c^2 = c^3$, coincide. This is the most difficult case. We represent the point x_i by some $\xi_i^{(\ )} \in U_V[T^{(\ )}]$ assigned to the tree $T^{(\ )}$:



with the vertex w colored by the normalization (p_i^1, p_i^2) of

$$\left(\frac{z_i^1 + z_i^2 + z_i^3}{3}, z^4\right),$$

the vertex v colored by the normalization (q_i^1, q_i^2, q_i^3) of (z_i^1, z_i^2, z_i^3) and the metric $h_i \in Met(T^{(\ )})$ given by

$$h_i(e) := \frac{|2z_i^1 - z_i^2 - z_i^3|^2 + |2z_i^2 - z_i^1 - z_i^3|^2 + |2z_i^3 - z_i^1 - z_i^2|^2}{2|z_i^1 + z_i^2 + z_i^3|^2}.$$

The sequence $\{(p_i^1, p_i^2) \times (q_i^1, q_i^2, q_i^3) \times h_i(e)\}_{i \geq 1}$ of points of $\overset{\circ}{\mathsf{F}}_V(2) \times \overset{\circ}{\mathsf{F}}_V(3) \times \mathbb{R}_{\geq 0}$ has a cluster point

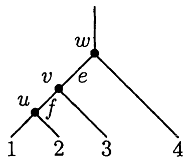
$$(\bar{p}^1, \bar{p}^2) \times (\bar{q}^1, \bar{q}^2, \bar{q}^3) \times 0 \in V^2 \times V^3 \times \mathbb{R}_{\geq 0}.$$

If all $\bar{q}^1, \bar{q}^2, \bar{q}^3$ are distinct, then the point $\bar{\xi}^{(\ )} = (0, \psi^{(\ )}) \in \hat{U}_V[T^{(\ )}]$, with

$$\psi^{(\ )}(w) := [\bar{p}^1, \bar{p}^2] \in \overset{\circ}{\mathsf{F}}_V(In(w)) \quad \text{and} \quad \psi^{(\ )}(v) := [\bar{q}^1, \bar{q}^2, \bar{q}^3] \in \overset{\circ}{\mathsf{F}}_V(In(v))$$

represents a cluster point of the sequence $\{x_i\}_{i \geq 1}$.

Suppose that not all $\bar{q}^1, \bar{q}^2, \bar{q}^3$ are distinct, say $q^1 = q^2$. We introduce yet another new vertex u replacing the tree $T^{(\ )}$ by $T^{((\ ))}$.



and define a representative $\xi^{((\ ))} \in U_V[T^{((\ ))}]$ by coloring the vertex w by the normalization of

$$\left(\frac{z_i^1 + z_i^2 + z_i^3}{3}, z^4\right),$$

the vertex v colored by the normalization of

$$\left(\frac{z_i^1 + z_i^2}{2}, z_i^3\right)$$

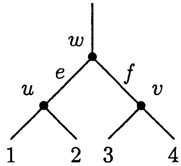
and the vertex u by the normalization of (z_i^1, z_i^2). We leave it as an exercise for the reader to figure out how the length function $h_i \in Met(T^{((\ ))})$ must be defined.

The sequence $\{\xi_i^{((\ ))}\}_{i\geq 1}$ has a cluster point $\bar{\xi}^{((\ ))}$ because

$$\begin{aligned} \xi_i^{((\ ))} \in Y_V[T^{((\ ))}] &\cong \mathring{\mathsf{F}}_V(In(w)) \times \mathring{\mathsf{F}}_V(In(v)) \times \mathring{\mathsf{F}}_V(In(u)) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \\ &\cong (S^1)^3 \times (\mathbb{R}_{\geq 0})^2 \end{aligned}$$

where the last two 'noncompact' parameters, $h_i(e), h_i(f) \in \mathbb{R}_{\geq 0}$, can be assumed very small for $i \gg 0$.

Case 4. The last remaining case is $\mathrm{c}^1 = \mathrm{c}^2 \neq \mathrm{c}^3 = \mathrm{c}^4$ or a situation obtained from this one by a reindexing. We introduce the tree $T^{(\ )(\ )}$:



and color the vertex w by the normalization of

$$\left(\frac{z_i^1 + z_i^2}{2}, \frac{z_i^3 + z_i^4}{2}\right),$$

the vertex u by the normalization of (z_i^1, z_i^2) and the vertex v by the normalization of (z_i^3, z_i^3). We again leave it to the reader to figure out how the lengths $h_i(e)$ and $h_i(f)$ must be defined to get a representative $\xi_i^{(\ )(\ )}$ of x_i, $i \geq 1$, in

$$\begin{aligned} &U_V[T^{(\cdot)(\cdot)}] \\ &\quad\cong \mathring{\mathsf{F}}_V(In(w)) \times \mathring{\mathsf{F}}_V(In(v)) \times \mathring{\mathsf{F}}_V(In(u)) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \cong (S^1)^3 \times (\mathbb{R}_{\geq 0})^2. \end{aligned}$$

The first three relevant factors of the above space are compact, thus the sequence $\{\xi_i^{(\cdot\cdot)(\ )}\}_{i\geq 1}$ has a cluster point $\bar{\xi}^{(\cdot)(\cdot)} \in \hat{U}_V[T^{(\ )(\ )}]$ that represents a cluster point of the sequence $\{x_i\}_{i\geq 1}$.

The existence of a cluster point in $\mathsf{F}_V(n)$ of a sequence $\{x_i\}_{i\geq 1} \in \mathring{\mathsf{F}}_V(n)$ for an arbitrary V and n can be proved using exactly the same arguments as in Example 4.32 and Example 4.33 by introducing a new vertex for each group of two or more points which are, for large i, close to each other and replacing these points by their center of gravity. Intuitively this can be imagined as applying a magnifying glass to 'microscopic' subconfigurations. Observe that we did not need the explicit formula for the metric h_i; we only needed to know that it is small for $i \gg 0$. We may thus formulate:

Lemma 4.34. *Let $k = \dim(V) \geq 1$ and $n \geq 1$. Then each sequence $\{x_i\}_{i\geq 1} \in \mathring{\mathsf{F}}_V(n)$ has a cluster point in $\mathsf{F}_V(n)$.*

The main result of this section reads as follows.

Theorem 4.35. *Let $k = \dim(V), n \geq 1$. Then each $\mathsf{F}_V(n)$ is a compact smooth $(kn - k - 1)$-dimensional manifold-with-corners and the topological operad F_V is in* *fact an operad in the category of manifolds-with-corners. The moduli space* $\overset{\circ}{\mathsf{F}}_V(n)$ *is an open dense subset of* $\mathsf{F}_V(n)$.

PROOF. Let us show that each point $x \in \mathsf{F}_V(n)$ has a neighborhood U_x isomorphic to a neighborhood of a point in the cube I^{kn-k-1}, $I = [0,1]$. Recall that x has a unique primitive representative

$$(0, \psi) \in \widehat{U}_V[S],\ \psi \in \overset{\circ}{\mathsf{F}}_V(S),\ S \in \mathcal{R}tree(n),$$

with

$$\psi = \bigtimes_{w \in Vert(S)} \psi_w, \text{ where } \psi_w \in \overset{\circ}{\mathsf{F}}_V(In(w)).$$

Let us choose, for each $w \in Vert(S)$, neighborhoods U_w of ψ_w in $\overset{\circ}{\mathsf{F}}_V(In(w))$. Choose also a positive number $\epsilon \in \mathbb{R}_{>0}$ and define the 'canonical neighborhood'

$$(4.32)\quad U_x := \left\{ \xi = (h, \zeta) \in Y_V[S] \mid h(e) < \epsilon, e \in edge(S),\ \zeta \in \bigtimes_{w \in Vert(S)} U_w \subset \overset{\circ}{\mathsf{F}}_V(S) \right\}.$$

By a standard continuity argument, the neighborhoods U_w and the number ϵ can be choosen small enough so that U_x is a subset of $\widehat{U}_V[S]$. Define the map $u_x : U_x \to \mathsf{F}_V(n)$ as the composition

$$U_x \hookrightarrow \widehat{U}_V[S] \xrightarrow{proj} \mathsf{F}_V(n).$$

Suppose that $\xi_i = (h_i, \zeta_i) \in U_x$, $i = 1, 2$, are two distinct points. It is clear that if $u_x(\xi_1) = u_x(\xi_2)$, then $\mathfrak{Z}(h_1) = \mathfrak{Z}(h_2)$, so we may apply Lemma 4.28 to conclude that $\xi_1 = \xi_2$. This means that the map u_x is a monomorphism.

The map u_x is clearly continuous. It is also, by Lemma 4.30, an open map. So u_x is an isomorphism of U_x onto its image $u_x(U_x) \subset \mathsf{F}_V(n)$

To simplify the notation, let $N_w := k \cdot \mathsf{a}(w) - k - 1$, for $w \in Vert(S)$. Since $\overset{\circ}{\mathsf{F}}_V(In(w))$ is an N_w-dimensional smooth manifold, we may assume that the subset U_w of $\overset{\circ}{\mathsf{F}}_V(In(w))$ is diffeomorphic to an open subset of $\mathbb{R}^{N_w}$. Then U_x will clearly be diffeomorphic to a neighborhood of the point

$$(0, \dots, 0, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \in I^{kn-k-1},$$

where the number of zeroes is equal to the cardinality of the set $edge(S)$

Choosing the above data for each $x \in \mathsf{F}_V(n)$, we obtain an atlas

$$\mathfrak{U} = \{U_x, u_x\}_{x \in \mathsf{F}_V(n)}.$$

To prove that it defines a structure of a smooth manifold-with-corners, one needs to investigate transition functions and prove that they are smooth. An explicit description of these functions is very clumsy, but the point is that they are combinations of maps ω_T (4.17), projections $proj : \mathsf{Con}(V, m) \to \overset{\circ}{\mathsf{F}}_V(m)$ and normalizations $nor : \overset{\circ}{\mathsf{F}}_V(m) \to \mathsf{Con}(V, m)$, for a choice of trees T and natural numbers m. From this it follows that they are smooth as desired.

It is intuitively clear that each two distinct points x and y of $\mathsf{F}_V(n)$ admit mutually disjoint canonical neighborhoods (4.32) U_x and U_y which proves that the manifold $\mathsf{F}_V(n)$ is Hausdorff. A rigorous proof of this statement would be, however, long and clumsy and we omit it.

Let us prove that $\mathsf{F}_V(n)$ is compact. Since the topological space $\mathsf{F}_V(n)$ obviously has a countable basis, it is enough to prove that each sequence $\{x_i\}_{i\geq 1}$ of points of $\mathsf{F}_V(n)$ has a cluster point. We already know, by Lemma 4.34, that this is true when all $x_i \in \overset{\circ}{\mathsf{F}}_V(n)$. Let us consider the general case.

Let $(0, \psi_i) \in \hat{U}_V[S_i]$, $\psi_i \in \overset{\circ}{\mathsf{F}}_V(S_i)$, be a primitive representative of x_i, $S_i \in \mathcal{R}tree(n)$. Since the set $\mathcal{R}tree(n)$ is finite, there exists $S \in \mathcal{R}tree(n)$ such that $S_i = S$ for infinitely many $i \geq 0$. So we may assume, passing to a subsequence if necessary, that $S_i = S$ for all i. Recall that $\psi_i(w) \in \overset{\circ}{\mathsf{F}}_V(In(w))$ denotes the ψ_i-color of $w \in Vert(S)$.

By Lemma 4.34, the sequence $\{\psi_i(w)\}_{i\geq 1}$ has a cluster point in $\mathsf{F}_V(In(w))$, for each $w \in Vert(S)$. This clearly implies that the sequence $\{\psi_i\}_{i\geq 1}$ has a cluster point

$$\bar{\psi} := \bigtimes_{w \in Vert(S)} \bar{\psi}_w$$

in $\mathsf{F}_V(S) \supset \overset{\circ}{\mathsf{F}}_V(S)$. The operad structure γ_{F} of F_V induces, as usual, the contraction

$$\gamma_S : \mathsf{F}_V(S) \to \mathsf{F}_V(n)$$

along the tree S. Then the point $\bar{x} := \gamma_S(\bar{\psi})$ is a cluster point of the sequence $\{x_i\}_{i\geq 1}$.

To prove that $\overset{\circ}{\mathsf{F}}_V(n)$ is a dense subset of $\mathsf{F}_V(n)$, we observe that each canonical neighborhood U_x of a point $x \in \mathsf{F}_V(n)$ has a nonempty intersection with $\overset{\circ}{\mathsf{F}}_V(n)$. Indeed, U_x clearly contains some $\xi = (h, \zeta)$ such that $h(e) > 0$ for all $e \in edge(S)$. Then $\mathfrak{Z}(h) = \{rv(S)\}$, thus, by (4.29), $[\xi] \in \overset{\circ}{\mathsf{F}}_V(n)$. ■

EXAMPLE 4.36. Let us consider configurations of points on the real line. The space $\mathsf{Con}(\mathbb{R}, n)$ is clearly, for each $n \geq 1$, a free Σ_n-space,

$$\mathsf{Con}(\mathbb{R}, n) \cong \underline{\mathsf{Con}}(\mathbb{R}, n) \times \Sigma_n, \tag{4.33}$$

where

$$\underline{\mathsf{Con}}(\mathbb{R}, n) := \{(z^1, \dots, z^n) \in \mathbb{R}^n \mid z^1 < \cdots < z^n\}.$$

Presentation (4 33) induces the factorization

$$\overset{\circ}{\mathsf{F}}_1(n) \cong \overset{\circ}{\underline{\mathsf{F}}}_1(n) \times \Sigma_n,$$

with

$$\overset{\circ}{\underline{\mathsf{F}}}_1(n) := \underline{\mathsf{Con}}(\mathbb{R}, n)/\mathit{Aff}(\mathbb{R}) \cong \{(z^1, \dots, z^n) \in \mathbb{R}^n \mid 0 = z^1 < \cdots < z^n = 1\},$$

which is the open $(n-2)$-simplex $\overset{\circ}{\Delta}{}^{n-2}$. There is a corresponding decomposition of the compactification:

$$\mathsf{F}_1(n) \cong \underline{\mathsf{F}}_1(n) \times \Sigma_n.$$

The space $\underline{\mathsf{F}}_1(n)$ is, as a manifold-with-corners, isomorphic to the associahedron K_n. In fact, the associahedron can be obtained by blowing up some faces of the

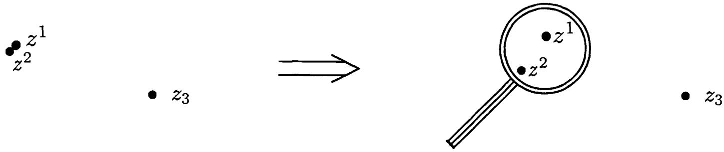


FIGURE 14. Magnifying glass.

simplex Δ^{n-2} or, equivalently, by a truncation of $\overset{\circ}{\Delta}{}^{n-2}$; see [**Sta97**, Appendix B] or Section 1.6.

Recall that each K_n is a cellular chain complex, with cells indexed by reduced planar n-trees. This decomposition corresponds to the stratification (4.28) of the compactification.

The Σ-module $\underline{\mathsf{F}}_1 = \{\underline{\mathsf{F}}_1(n)\}_{n\geq 1}$ is a non-Σ operad, isomorphic to the non-Σ operad of the associahedra $\underline{\mathcal{K}} = \{K_n\}_{n\geq 1}$.

EXAMPLE 4.37. The operad $\mathsf{F}_2 = \{\mathsf{F}_1(n)\}_{n\geq 1}$ was intensively studied as a candidate for a solution of the Deligne conjecture; see also Section I.1.19. Getzler and Jones [**GJ94**] constructed a cell decomposition of $\overset{\circ}{\mathsf{F}}_2$, induced by the Fox-Neuwirth decomposition of the Σ-module of open parts $\overset{\circ}{\mathsf{F}}_2 = \{\overset{\circ}{\mathsf{F}}_2(n)\}_{n\geq 1}$. This cell structure was, unfortunately, discovered not to be compatible with the operad structure, but there still exists, as shown by A. Voronov in [**Vor99a**], a coarser filtration compatible with the operad structure and extending the Fox-Neuwirth decomposition of the open parts.

Since all constructions of this section are clearly functorial, we may formulate the following proposition

PROPOSITION 4.38. *The assignment $V \longmapsto \mathsf{F}_V$ defines a functor from the groupoid of finite dimensional Euclidean spaces and their linear isometries to the category of operads in the symmetric monoidal category of compact smooth manifolds-with-corners.*

REMARK 4.39. The role of trees for the compactification described in this section can also be intuitively explained using the idea of 'macroscopic,' respectively 'microscopic,' configurations. Elements of the moduli space $\overset{\circ}{\mathsf{F}}_V(n)$ are *macroscopic*, by definition. They can be represented by n distinct points $z^1, \dots, z^n$ of V. They belong to the open stratum of the compactification $\mathsf{F}_V(n)$ indexed, in (4.28), by the n-corolla $c(n) \in \mathcal{R}tree(n)$.

Let us illustrate the idea of *microscopic* configurations on elements of the stratum $\overset{\circ}{\mathsf{F}}_V(T^{(\ )\cdot})$, where $T^{(\ )} \in \mathcal{R}tree(3)$ is as in Example 4.32. From the macroscopic point of view, the first two points z^1 and z^2 are so close that they almost coincide. The root vertex of $T^{(\ )}$ is then colored by this 'almost macroscopic' configuration. To distinguish between z^1 and z^2, one must apply a 'magnifying glass' as indicated in Figure 14. The vertex v of $T^{(\ )}$ (the unique nonroot vertex (see Example 4.32 for the notation)) is then colored by this microscopic configuration, which can be made 'macroscopic' by applying our magnifying glass and it is thus in fact an element of (in this case) $\overset{\circ}{\mathsf{F}}_V(2)$

More complex strata are indexed by more complex trees such as $T^{((\ ))}$ from Example 4.33 One must then consider microscopic configurations of points that themselves are decorated by (higher-order) microscopic configurations, etc.

4.4. Compactification of configurations of points in a manifold

The aim of this section is to modify the approach of Section 4.3 and review a construction of a compactification $\overline{\mathsf{Con}}(M,n)$ of the configuration space $\mathsf{Con}(M,n)$ of n distinct labeled points in a complete k-dimensional Riemannian manifold M, originally due to S. Axelrod and I.M. Singer [**AS94**]. As explained by Proposition 4.44 and Remark 4.45, the Σ-module $\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ has, for a general M, no 'reasonable' algebraic structure, because there is no canonical identification of the tangent space T_xM at a point $x \in M$ with the 'model space' $\mathbb{R}^k$. Such an identification exists if M is parallelizable; the Σ-module $\overline{\mathsf{Con}}(M)$ is then a right module over the operad $\mathsf{F}_k = \{\mathsf{F}_k(n)\}_{n\geq 1}$ constructed in Section 4.3 (see also Theorem 4.46).

More suitable for a nonparallelizable manifold M is the framed configuration space $\mathfrak{f}\mathsf{Con}(M,n)$ introduced in Definition 4.11. We construct a compactification $\mathfrak{f}\overline{\mathsf{Con}}(M,n)$ of this space and show that the Σ-module $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ forms a right module over a framed version $\mathfrak{f}\mathsf{F}_k$ of the operad F_k (Theorem 4.49).

This section is very technical and consists mostly of constructions of the above mentioned objects though it also contains some material of independent interest, e.g. the cyclohedra (Example 4.47).

COMPACTIFICATION OF $\mathsf{Con}(M,n)$. Let us describe a compactification of the (unframed) configuration space $\mathsf{Con}(M,n)$ of n distinct labeled points in a complete Riemannian manifold M. The steps of our construction will be analogous to the steps of the construction of the compactification $\mathsf{F}_V(n)$ in Section 4.3.

Instead of the set $\mathcal{R}tree(n)$ of (isomorphism classes of) reduced rooted n-trees, we use here the set $\mathcal{E}tree(n)$ of all (isomorphism classes of) n-trees such that all vertices except possibly the vertex adjacent to the root are at least binary (the root vertex may have arity one). In other words, $\mathcal{E}tree(n)$ denotes the set of trees with a *special vertex* – the root – such that all vertices which are not special are of arity ≥ 2.

The reason why we need these extended trees and not $\mathcal{R}tree$ as in Section 4.3 is the following. Strata of the compactification $\mathsf{F}_V(n)$ described in the previous section were indexed by trees T whose root vertices were colored by macroscopic configurations, while the remaining vertices were colored by microscopic configurations. Both macroscopic and microscopic configurations were elements of the moduli space $\overset{\circ}{\mathsf{F}}_V(n)$ for some n, as explained in Remark 4.39. Since the affine group $Aff(V)$ acts transitively, $\overset{\circ}{\mathsf{F}}_V(1) =$ the point, therefore vertices of arity one play no role and we may in fact assume that T is reduced, $T \in \mathcal{R}tree$.

There is a similar decomposition (4.44) into strata also for the compactification $\overline{\mathsf{Con}}(M,n)$ described in this section. These strata are again indexed by trees T with the root vertices colored by macroscopic configurations and remaining vertices colored by microscopic configurations. While microscopic configurations are, as before, elements of the moduli spaces $\overset{\circ}{\mathsf{F}}_V(n)$, macroscopic configurations are elements of the configuration spaces $\mathsf{Con}(M,n)$ which are nontrivial even for $n=1$, $\mathsf{Con}(M,1)\cong M$. Therefore we must consider also trees with root vertices of arity one (or higher, of course), that is, $T \in \mathcal{E}tree$.

Observe that the grafting defines a natural right $\mathcal{R}tree$-module structure on $\mathcal{E}tree$, which provides a 'philosophical' explanation of the module structures considered in this section.

Observe that while $\mathcal{R}tree(1)=\emptyset$, $\mathcal{E}tree(1)=\{\phi\}$, the 1-corolla with one vertex of arity one. It will be useful to think of a tree $T \in \mathcal{E}tree(n)$ as obtained by grafting subtrees $T_i \in \mathcal{R}tree(n_i)$ to the s-corolla, for $s \leq n$ and some (not necessarily all) $1 \leq i \leq s$, as indicated in Figure 15.

Let $Met(T)$ denote, as in Section 4.3, the space of all metrics $h \cdot edge(T) \to \mathbb{R}_{\geq 0}$ on the set of (internal) edges of a tree T. We will also need the following notation.

NOTATION 4.40. For a tree $T \in \mathcal{E}tree(n)$, let $ir\,(T)$ (inputs of root) denote the set $In(rv(T))$ of input edges of the root vertex $rv(T)$ of T. We also denote by $iVert(T)$ the set

$$iVert(T) := Vert(T) - \{rv(T)\}$$

(the set of internal vertices).

For any vertex $v \in iVert(T)$, there exists a unique path in T connecting v and $rv(T)$; this path contains exactly one element of $ir\,(T)$ which we denote by $pa(v)$ (path of v). Similarly, for each $1 \leq i \leq n$ there exists a unique edge $pa(i) \in ir\,(T)$ (path of i) such that $pa(i)$ lies on the path from the ith input leaf of T to the root. Sometimes we will write more explicitly $pa_T(v)$ and $pa_T(i)$ instead of $pa(v)$ and $pa(i)$.

Let us come back to the manifold M. Since we assume M to be Riemannian, the tangent space $T_x(M)$ at each point $x \in M$ has an $\mathbb{R}$-linear positive definite metric, so we may consider the moduli spaces $\overset{\circ}{\mathsf{F}}_{T_x(M)}(n)$ and their compactifications $\mathsf{F}_{T_x(M)}(n)$, $n \geq 1$; see (4.11) and (4.25) for the definitions. We simplify the clumsy notation $\mathsf{F}_{T_x(M)}(n)$ by introducing

$$\mathsf{F}_x(n) := \mathsf{F}_{T_x(M)}(n), \text{ for } x \in M \text{ and } n \geq 1, \tag{4.34}$$

and similarly, for the open parts, $\overset{\circ}{\mathsf{F}}_x(n) := \overset{\circ}{\mathsf{F}}_{T_x(M)}(n)$. The system $\{\overset{\circ}{\mathsf{F}}_x(n)\}_{x\in M}$ can clearly be pasted together to form a smooth fibration

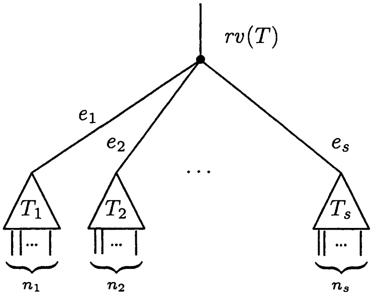
$$\mathsf{f}_n : \overset{\circ}{\mathsf{F}}(n) \to M$$

with fiber $\mathsf{f}_n^{-1}(x) = \overset{\circ}{\mathsf{F}}_x(n)$, $x \in M$. The standard extension trick enables one to consider similar fibrations $\mathsf{f}_S : \overset{\circ}{\mathsf{F}}(S) \to M$ with an arbitrary nonempty finite set S.

For each $T \in \mathcal{E}tree(n)$, there is a map $a_T : \mathsf{Con}(M, ir\,(T)) \to M^{iVert(T)}$ given by

$$a_T(\vec{x})(v) := x^{pa(v)}$$

for $v \in iVert(T)$ and $\vec{x} = \{x^e\}_{e \in ir\,(T)} \in \mathsf{Con}(M, ir\,(T))$. Recall that $pa(v) \in ir\,(T)$ was introduced in Notation 4.40.



A tree $T \in \mathcal{E}tree(n)$ decomposed to the s-corolla and trees $T_i \in \mathcal{E}tree(n_i)$, $1 \leq i \leq s$. The set $ir\,(T)$ equals $\{e_1, \ldots, e_s\}$. Some of the trees T_i may be absent, as is T_2 in the following tree with $s = 3$:

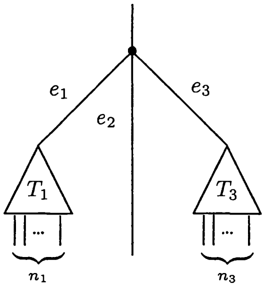


FIGURE 15. Examples of trees from $\mathcal{E}tree(n)$.

Using the schematic picture Figure 15, $a_T(\vec{x})$ can be seen as the function whose value at $v \in Vert(T_i)$ is x^{e_i}, $1 \leq i \leq s$. Let $\mathsf{Con}(M)(T)$ denote the pullback of the diagram

$$\begin{array}{ccc} & & \displaystyle\bigtimes_{v \in iVert(T)} \mathring{\mathsf{F}}(In(v)) \\ & & \Big\downarrow \displaystyle\bigtimes_{v \in iVert(T)} \mathsf{f}_{In(v)} \\ \mathsf{Con}(M, ir\,(T)) & \xrightarrow{\;\;a_T\;\;} & M^{iVert(T)}. \end{array}$$

The set $\mathsf{Con}(M)(T)$ is the set of all colorings of the tree T such that the root is colored by an element $\vec{x} = \{x^e\}_{e \in ir\,(T)}$ of the configuration space $\mathsf{Con}(M, ir\,(T))$ while the remaining vertices $v \in iVert(T)$ (if any) are colored by elements of the space $\mathring{\mathsf{F}}_{x^{pa(v)}}(In(v))$. Let us define, for $T \in \mathcal{E}tree(n)$,

$$Z[T] := Met(T) \times \mathsf{Con}(M)(T);$$

compare (4.14). Elements of $Z[T]$ are sets

$$\eta = \{\lambda_v \mid v \in Vert(T)\} \tag{4.35}$$

with

$$\lambda_{rv(T)} = \vec{x} = \{x^e\}_{e \in ir\,(T)} \in \mathsf{Con}(M, ir\,(T))$$

and, for $v \in iVert(T)$,

$$\lambda_v = (t_v, [\vec{z}_v]) \in \mathbb{R}_{\geq 0} \times \overset{\circ}{\mathsf{F}}_{x^{pa(v)}}(In(v));$$

compare (4.15). Observe that $\lambda_{rv(T)}$ has no '$Met(T)$'-coordinate, since the metric is defined on internal edges of T only.

For each $1 \leq i \leq n$, there is in the tree T a unique path from the ith input leaf to the root. Using the notation introduced in Figure 10 of Section 4.3, we define a map $\varphi_i : Z[T] \to M$ by the formula

$$\varphi_i(\eta) := \exp_{x^{e_s}}(t_{v_{s-1}} z_{v_{s-1}}^{e_{s-1}} + t_{v_{s-2}} t_{v_{s-1}} z_{v_{s-2}}^{e_{s-2}} + \cdots + t_{v_1} \cdots t_{v_{s-1}} z_{v_1}^{i}), \tag{4.36}$$

where $\exp_{x^{e_s}} : T_{x^{e_s}}(M) \to M$ is the exponential map at x^{e_s} whose existence is guaranteed by the completeness of M.

The idea behind this formula is similar to that of (4.16), that is, $\varphi_i(\eta)$ is a combination of $x^{e_s} \in M$ with the image under the exponential map of $z_{v_{s-1}}^{e_{s-1}}$ scaled by $t_{v_{s-1}}$, with $z_{v_{s-2}}^{e_{s-2}}$ scaled by $t_{v_{s-2}} t_{v_{s-1}}$, ..., with $z_{v_{s-1}}^{i}$ scaled by $t_{v_1} \cdots t_{v_{s-1}}$. Finally, let $\varphi_T : Z[T] \to M^n$ be given as

$$\varphi_T(\eta) := (\varphi_1(\eta), \dots, \varphi_n(\eta)) \in M^n,$$

to be compared with the definition (4.17) of the map $\omega_T : Y_V[T] \to V^n$ The subset $W[T] \subset Z[T]$ of *composable elements* (an analog of the set $U_V[T]$ of 4.18) is defined by

$$W[T] := \{\eta \in Z[T] \mid \text{ all } \varphi_i(\eta) \in M \text{ are distinct, } 1 \leq i \leq n\}. \tag{4.37}$$

Each $\eta = (h, \phi) \in Z[T]$ with a metric $h \in Met(T)$ determines the 'zero set' $\mathfrak{Z}(h) \subset Vert(T)$ as in (4.13). Recall that the tree T then decomposes into disjoint subtrees $\{T_w\}_{w \in \mathfrak{Z}(h)}$, see also Figure 8. For $w \in \mathfrak{Z}(h)$ and η as above we have, as in (4.20), obvious restrictions

$$r_w(\eta) \in \begin{cases} Z[T_w], & \text{for } w = rv(T), \text{ and} \\ Y_{x^{pa_T(w)}}[T_w], & \text{for } w \in \mathfrak{Z}(h),\ w \neq rv(T) \end{cases} \tag{4.38}$$

where

$$Y_{x^{pa_T(w)}}[T_w] := Y_{T_{x^{pa_T(w)}}(M)}[T_w]$$

with $Y_V[T]$, for a vector space V, introduced in (4.14). The meaning of (4.38) is the following. The decomposition $\{T_w\}_{w \in \mathfrak{Z}(h)}$ is the decomposition of T into the biggest subtrees such that the metric h is positive on all (internal) edges of T_w. As in Section 4.3 this means that $\{T_w\}_{w \in \mathfrak{Z}(h)}$ is obtained by cutting in two each edge of length zero.

The tree T_w with $w = rv(T)$ contains the special vertex, $T_w \in \mathcal{E}tree$, so the restriction of the 'coloring' η is an element of $Z[T_w]$. The remaining trees T_w with $w \neq rv(T)$ have no special vertices, so all their vertices v are 'colored' by $\overset{\circ}{\mathsf{F}}_{x^{pa(v)}}(In(v))$ and the restriction of η is an element of $Y_V[T]$ with $V = T_{x^{pa(w)}}(M)$.

The *extended set* $\widehat{W}[T]$ (an analog of the extended set $\widehat{U}_V[T]$ of (4.21)) is the set of all $\eta \in Z[T]$ such that all restrictions $r_w(\eta)$ are composable, that is

$$r_w(\eta) \in \begin{cases} W[T_w], & \text{for } w = rv(T), \text{ and} \\ U_{x^{pa_T(w)}}[T_w], & \text{for } w \in \mathfrak{Z}(h),\ w \neq rv(T), \end{cases} \tag{4.39}$$

where, in accord with our conventions, we denoted

$$U_{x^{pa_T(w)}}[T_w] := U_{T_{x^{pa_T(w)}}(M)}[T_w]$$

with the set $U_V[T_w]$ defined, for a vector space V, in (4.18).

As in Section 4.3, there are special *primitive* elements of the space $Z[T] = Met(T) \times \mathsf{Con}(M)(T)$ of the form $(0, \phi)$, where 0 denotes the trivial metric and $\phi \in \mathsf{Con}(M)(T)$. It is clear that each such element belongs to $\widehat{W}[T]$.

Let T_h denote, as in Section 4.3, the tree obtained from T by shrinking each subtree T_w, $w \in \mathfrak{Z}(T)$, to a vertex which we denote again by w. We would like, as in (4.23), to assign to each $\eta \in \widehat{W}[T]$ some $\delta(\eta) \in \mathsf{Con}(M)(T_h)$. The formula

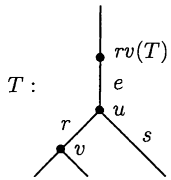
$$\delta(\eta) := \vec{y} \times \bigtimes_{w \in iVert(T_h)} [\omega_{T_w}(r_w(\eta))] \tag{4.40}$$

with

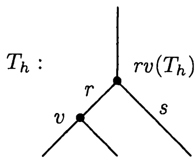
$$\vec{y} := \varphi_{T_{rv(T)}}(r_{rv(T)}(\eta)) \in \mathsf{Con}(M, ir\,(T_h)) \tag{4.41}$$

analogous to (4.23) unfortunately does not define an element of $\mathsf{Con}(M)(T)$. The point is that $[\omega_{T_w}(r_w(\eta))] \in \mathring{\mathsf{F}}_{x^{pa_T(w)}}(In_{T_h}(w))$ while it must be an element of $\mathring{\mathsf{F}}_{y^{pa_{T_h}(w)}}(In_{T_h}(w))$. The following example shows that the points $x^{pa_T(w)}$ and $y^{pa_{T_h}(w)}$ are, in general, different.

EXAMPLE 4.41. Let us consider the following tree $T \in \mathcal{E}tree(3)$:



with $ir\,(T) = \{e\}$, $In(u) = \{r, s\}$ and $pa_T(v) = e$. Define a metric $h \in Met(T)$ by $h(e) := t > 0$ and $h(r) = 0$. Then $\mathfrak{Z}(h) = \{rv(T), v\}$, $T_{rv(T)}$ is the subtree with vertices $rv(T)$ and u and T_v is the subtree (2-corolla) with vertex v. The corresponding tree $T_h \in \mathcal{E}tree(3)$ is obtained by collapsing $T_{rv(T)}$ and T_v to corollae,



where we used the letters s and r to denote the edges corresponding to the edges of T with the same name. Clearly $ir\,(T_h) = \{r, s\}$ and $pa_{T_h}(v) = r$.

Let us consider $\eta \in Z[T]$ (notation of (4.35)) with $\lambda_u = (t, [z^r, z^s])$, where (z^r, z^s) is a normal pair of points of $T_x M$, $x := x^e$. For this η we clearly have $x^{pa_T(v)} = x$. On the other hand, the element $\vec{y} = (y^r, y^s)$ of (4.41) is given by

$$y^r = \exp_x(tz^r) \text{ and } y^s = \exp_x(tz^s),$$

as follows from (4.36). We conclude that, at least for some if not for all $t > 0$,

$$x^{pa_T(v)} = x \neq \exp_x(tz^r) = y^{pa_{T_h}(v)}. \tag{4.42}$$

So we need to move the factor $[\omega_{T_w}(r_w(\eta))]$ in (4.40) from the point $x^{pa_T(w)}$ to $y^{pa_{T_h}(w)}$. We use the 'parallel transport' $\Phi_{x,y} : T_x(M) \to T_y(M)$ defined, for $x \in M$, $a \in T_x(M)$ and $y := \exp_x(a)$, by

$$\Phi_{x,y}(\xi) := \frac{d}{ds}\bigg|_{s=0} \exp_x(a + s\xi), \ \xi \in T_x(M);$$

compare [**AS94**, (5.80)]. Strictly speaking, the notation $\Phi_{x,y}$ is not correct because there might be several a's with $y := \exp_x(a)$, but in our applications the a will always be as explicit as in (4.42) where $a = tz^s$.

Parallel transport can clearly be used to move all objects functorially depending on the tangent spaces of M. In particular, parallel transport induces the map

$$\Phi_{x,y} : \mathring{\mathsf{F}}_x(S) \to \mathring{\mathsf{F}}_y(S),$$

for an arbitrary finite set S. The correct formula for $\delta : \widehat{W}[T] \to \mathsf{Con}(M)(T_h)$ is

$$\delta(\eta) := \vec{y} \times \bigtimes_{w \in iVert(T_h)} \Phi_{x^{pa_T(w)}, y^{pa_{T_h}(w)}}[\omega_{T_w}(r_w(\eta))]. \tag{4.43}$$

We then proceed exactly as in Section 4.3, that is, we define

$$\overline{\mathsf{Con}}(M, n) := \widehat{W}(n)/\sim,$$

where

$$\widehat{W}(n) := \bigsqcup_{T \in \mathcal{E}tree(n)} \widehat{W}[T]$$

and the relation $\sim$ given by

$$\widehat{W}(n) \supset \widehat{W}[T] \ni \eta \sim (0, \delta(\eta)) \in \widehat{W}[T_h] \subset \widehat{W}(n),$$

for $T \in \mathcal{E}tree(n)$ and $n \geq 1$. The following theorem can be proved exactly as was Theorem 4.35 of Section 4.3.

THEOREM 4.42. *The space* $\overline{\mathsf{Con}}(M, n)$ *is a smooth manifold-with-corners of dimension kn containing the configuration space* $\mathsf{Con}(M, n)$ *as an open dense subset. It is compact if and only if M is.*

As proved in [**Mar99a**], the space $\overline{\mathsf{Con}}(M, n)$ coincides with the Axelrod-Singer compactification of the configuration space [**AS94**]. As in (4.28), the set $\overline{\mathsf{Con}}(M, n)$ is the disjoint union of strata,

$$\overline{\mathsf{Con}}(M, n) = \bigsqcup_{T \in \mathcal{E}tree(n)} \mathsf{Con}(M)(T), \ n \geq 1. \tag{4.44}$$

EXAMPLE 4.43. It is immediate to see that $\overline{\mathsf{Con}}(M, 1) \cong M$. In fact, the decomposition (4.44) consist, for $n = 1$, of a single strata corresponding to the 1-corolla ϕ,

$$\mathsf{Con}(M)(\phi) \cong M.$$

So even $\overline{\mathsf{Con}}(M,1)$ is not compact if M is not. Loosely speaking, the 'compactification' $\overline{\mathsf{Con}}(M,n)$ takes care of points that come close together, not of those that 'diverge to infinity' in M. Therefore a more appropriate name for $\overline{\mathsf{Con}}(M,n)$ would be a 'resolution of diagonals' (following [**Gin95**]) or a 'partial compactification.'

Let us try to understand what kind of algebraic structure the Σ-module

$$\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$$

may possibly have. Observe first that there is a *'blowing down' map* (terminology borrowed from algebraic geometry)

$$bld : \overline{\mathsf{Con}}(M,n) \to M^{\times n},\ bld(u) = (bld_1(u), \dots, bld_n(u)),\ n \geq 1. \tag{4.45}$$

The most efficient way to define this map is to say that it is an extension of the inclusion $\mathsf{Con}(M,n) \hookrightarrow M^{\times n}$ to the compactification $\overline{\mathsf{Con}}(M,n)$. This extension is necessarily unique, but its existence must be proved.

In terms of primitive representatives, the blowing down map is given as follows. First, define $bld_T : \mathsf{Con}(M)(T) \to M^n$ by

$$bld_T\left(\{x^e\}_{e\in ir(T)} \times \bigtimes_{v\in iVert(T)} [\vec{z}_v]\right) := (x^{pa(1)}, \dots, x^{pa(n)}), \tag{4.46}$$

where $pa(i)$, $1 \leq i \leq n$, was introduced in Notation 4.40. Then we put $bld(0,\phi) := bld_T(\phi)$.

For each $x \in M$, we have the operad $\mathsf{F}_x := \mathsf{F}_{T_x(M)}$. This collection in fact forms a 'fibration'

$$\mathsf{f} : \mathsf{F} \to M,\ \mathsf{f} = \{\mathsf{f}_n : \mathsf{F}(n) \to M\}_{n\geq 1}$$

whose fiber over $x \in M$ is the operad F_x. For each $n \geq 1$ and $k_1, \dots, k_n \geq 1$, let $X(n; k_1, \dots, k_n)$ be the pullback of the diagram

$$\begin{array}{ccc} & & \bigtimes_{1\leq i\leq n} \mathsf{F}(k_i) \\ & & \Big\downarrow {\scriptstyle \bigtimes_{1\leq i\leq n} \mathsf{f}_{k_i}} \\ \overline{\mathsf{Con}}(M,n) & \xrightarrow{bld} & M^{\times n} \end{array}$$

Elements of $X(n; k_1, \dots, k_n)$ are arrays $(u; v_1, \dots, v_n)$, where $u \in \overline{\mathsf{Con}}(M,n)$ and $v_i \in \mathsf{F}_{bld_i(u)}(k_i)$, for $1 \leq i \leq n$. The algebraic structure of the Σ-module $\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ is described in the following proposition, which can be proved by arguments similar to those of Proposition 4.31.

PROPOSITION 4.44. *For each $n \geq 1$ and $k_1, \dots, k_n \geq 1$, there exists a natural 'composition'*

$$\begin{array}{rcl} \nu : X(n; k_1, \dots, k_n) & \longrightarrow & \overline{\mathsf{Con}}(M, k_1 + \cdots + k_n), \\ X(n; k_1, \dots, k_n) \ni (u; v_1, \dots, v_n) & \longmapsto & \nu(u; v_1, \dots, v_n) \in \overline{\mathsf{Con}}(M, k_1 + \cdots + k_n) \end{array}$$

satisfying obvious associativity, equivariance and unit axioms analogous to that of a right module over an operad.

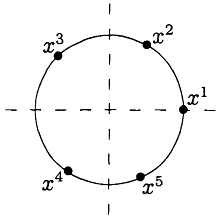


FIGURE 16. A configuration from $\underline{\mathsf{Con}}(S^1, 5)$.

REMARK 4.45 A proper name for the structure on $\overline{\mathsf{Con}}(M)$ described in Proposition 4.44 would probably be a *colored* right module over the colored operad F, with a smooth set of colors M. While in an ordinary module N over an ordinary operad $\mathcal{P}$, each 'input' of $u \in N$ is acted on by elements of the same operad $\mathcal{P}$, in this case the 'ith input' of $\overline{\mathsf{Con}}(M,n)$ is acted on by elements of the operad $\mathsf{F}_{bld_i(u)}$ whose 'color' $bld_i(u) \in M$ depends on both i and (smoothly) also on $u \in \overline{\mathsf{Con}}(M,n)$. Of course, all operads $\mathsf{F}_{bld_i(u)}$ are isomorphic to F_k but, in general, noncanonically. One way to obtain a canonical identification is to assume that the manifold M is parallelizable, as in the following theorem.

THEOREM 4.46. *Let M be a complete parallelizable Riemannian manifold. The Σ-module $\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ is then a right F_k-module. The module structure is functorial up to a choice of the trivialization of the tangent bundle of M.*

PROOF. The theorem will follow from a more general Theorem 4.49. ■

EXAMPLE 4.47. Consider the configuration space of points on the unit circle S^1. Clearly

$$\mathsf{Con}(S^1, n) \cong \underline{\mathsf{Con}}(S^1, n) \times S^1 \times \Sigma_n/\mathbb{Z}_n, \tag{4.47}$$

where $\underline{\mathsf{Con}}(S^1, n)$ is the space of all n-tuples $(x^1, \dots, x^n)$ of points of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ such that $x^1 = 1$ and

$$0 < \arg(x^2) < \arg(x^3) < \cdots < \arg(x^n) < 2\pi;$$

see Figure 16. In (4.47), the group $\mathbb{Z}_n$ is embedded in Σ_n as the subgroup of cyclic permutations. Factorization (4.47) induces the decomposition of the compactification

$$\overline{\mathsf{Con}}(S^1, n) = W_n \times S^1 \times \Sigma_n/\mathbb{Z}_n,$$

where W_n, the compactification of $\underline{\mathsf{Con}}(S^1, n)$, is the compact $(n-1)$-dimensional polyhedron introduced by R. Bott and C. Taubes [**BT94**, page 5249] in connection with the study of nonperturbative link invariants (but denoted there by W_{n-1}), and dubbed, in [**Sta97**], the *cyclohedron*. The polyhedron W_1 is just the point, W_2 is the interval, W_3 is the hexagon portrayed in Figure 17 and the three-dimensional polyhedron W_4 is portrayed in Figure 18.

Since S^1 is a compact parallelizable manifold of dimension 1, the Σ-module $\overline{\mathsf{Con}}(S^1) := \{\overline{\mathsf{Con}}(S^1, n)\}_{n\geq 1}$ is, by Theorem 4.46, a right module over the operad F_1 which was discussed at great length in Example 4.36. This induces an action of the nonsymmetric operad $\underline{\mathcal{K}} = \{K_n\}_{n\geq 1}$ of the associahedra on $W = \{W_n\}_{n\geq 1}$.

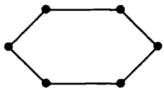


FIGURE 17. The cyclohedron W_3

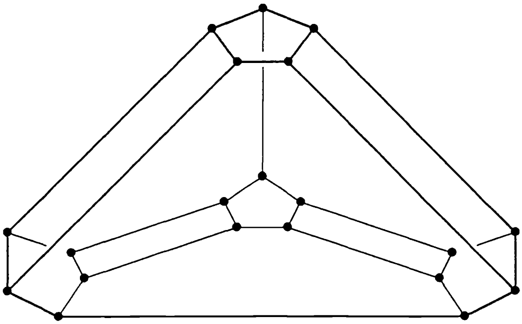


FIGURE 18. The cyclohedron W_4.

This action was studied and very explicitly described in [**Mar99d**]. It was also shown in the same paper that the contractibility of the cyclohedra reflects a certain property of Koszulness for modules over operads.

For a general, nonparallelizable manifold, one needs extra data that would identify the tangent space $T_x(M)$ with $\mathbb{R}^k$, for any point $x \in M$. These data are provided by a *framing*, that is, one must consider the framed configuration space $\mathsf{fCon}(M,n)$ which we introduced in Definition 4.11. The compactification $\overline{\mathsf{fCon}}(M) = \{\overline{\mathsf{fCon}}(M,n)\}_{n\geq 1}$ is then indeed a right F_k-module. It is, in fact, a module over a bigger operad fF_k which we introduce now.

FRAMED OPERAD fF_k. The natural action of the orthogonal group $O(k)$ on $\mathbb{R}^k$ induces a natural left action on the configuration space $\mathsf{Con}(\mathbb{R}^k,n)$ which in turn induces a left action on the quotient $\mathring{\mathsf{F}}_k(n) = \mathsf{Con}(\mathbb{R}^k,n)/\mathit{Aff}$. In terms of normal representatives introduced in Section 4.3, this action is described by

$$(z^1,\ldots,z^n) \longmapsto (gz^1,\ldots,gz^n),$$

where $(z^1,\ldots,z^n)$ is a normal representative of a point of $\mathring{\mathsf{F}}_k(n)$ and $g \in O(n)$. Observe that the configuration $(gz^1,\ldots,gz^n)$ is normal, too.

The above action induces a left $O(k)$-action on the compactification $\mathsf{F}_k(n)$ satisfying (4.2) and we define

$$\mathsf{fF}_k := \mathsf{F}_k \rtimes O(k),$$

the semidirect product introduced in Definition 4.2.

We will need for the proof of Theorem 4.49 a more detailed description obtained by decorating the steps of the construction of $\mathsf{F}_k(n)$ by orthogonal frames. We use the notation introduced in Section 4.3, but we drop the subscript V since V will always be the space $\mathbb{R}^k$ here. Namely, for $n \geq 2$ and $T \in \mathcal{R}tree(n)$ we introduce

$$\mathfrak{f}Y[T] := Y[T] \times O(k)^{\times n}, \tag{4.48}$$

with $Y[T]$ defined in (4.14) as $Y[T] = Met(T) \times \overset{\circ}{\mathsf{F}}(T)$. The elements of (4.48) will be written as

$$\xi \times (g_1, \ldots, g_n),\ \xi \in Y[T],\ (g_1, \ldots, g_n) \in O(k)^{\times n}. \tag{4.49}$$

Observe that the Σ-module $\mathfrak{f}Y = \{\mathfrak{f}Y(n)\}_{n \geq 1}$ with

$$\mathfrak{f}Y(n) := \bigsqcup_{T \in \mathcal{R}tree(n)} \mathfrak{f}Y[T]$$

is an operad. Indeed, we already know from the proof of Proposition 4.31 that the Σ-module $Y = \{Y(n)\}_{n \geq 1}$ with

$$Y(n) := \bigsqcup_{T \in \mathcal{R}tree(n)} Y[T]$$

is an operad. Each $Y(n)$ clearly admits a natural left action of the orthogonal group $O(k)$ which satisfies (4.2). The operad structure of $\mathfrak{f}Y$ is then given by the identification

$$\mathfrak{f}Y \cong Y \rtimes O(k).$$

Let us introduce the *extended set* of composable elements by

$$\mathfrak{f}\widehat{U}[T] := \widehat{U}[T] \times O(k)^{\times n},$$

where the set $U[T] = U_V[T]$ was defined in (4.21). Then the framed compactification $\mathfrak{f}\mathsf{F}_k(n)$ can be defined also as

$$\mathfrak{f}\mathsf{F}_k(n) := \mathfrak{f}\widehat{U}(n)/\sim,$$

where

$$\mathfrak{f}\widehat{U}(n) := \bigsqcup_{T \in \mathcal{R}tree(n)} \mathfrak{f}\widehat{U}[T],$$

and the equivalence $\sim$ is given by

$$\mathfrak{f}\widehat{U}(n) \supset \mathfrak{f}\widehat{U}[T] \ni \xi \times \vec{g} \sim (0, \gamma(\xi)) \times \vec{g} \in \mathfrak{f}\widehat{U}[T_h] \subset \mathfrak{f}\widehat{U}(n), \tag{4.50}$$

for $T \in \mathcal{R}tree(n)$, $\xi \in \widehat{U}[T]$ and $\vec{g} \in O(k)^{\times n}$, with $\gamma(\xi)$ defined in (4.23).

COMPACTIFICATION OF $\mathfrak{f}\mathsf{Con}(M, n)$. The shortest way to introduce the compactification $\mathfrak{f}\overline{\mathsf{Con}}(M, n)$ of the framed configuration space $\mathfrak{f}\mathsf{Con}(M, n)$ is to say that it is the pullback of the diagram

$$\begin{tikzcd} & O(M)^{\times n} \arrow[d, "\pi_M^{\times n}"] \\ \overline{\mathsf{Con}}(M,n) \arrow[r, "bld"] & M^{\times n} \end{tikzcd} \tag{4.51}$$

where $bld : \overline{\mathsf{Con}}(M,n) \to M^{\times n}$ is the blowing down map (4.45) and $\pi_M : O(M) \to M$ is the orthogonal frame bundle of M. The following proposition follows immediately from Theorem 4.42, since $\mathfrak{f}\overline{\mathsf{Con}}(M,n)$ is, as the pullback of (4.51), 'locally' the product $\overline{\mathsf{Con}}(M,n) \times O(k)^{\times n}$.

PROPOSITION 4.48. *The space $\mathfrak{f}\overline{\mathsf{Con}}(M,n)$ is a smooth manifold-with-corners of dimension*

$$\dim(\overline{\mathsf{Con}}(M,n)) + n\dim(O(k)) = kn + \frac{kn(k-1)}{2} = \frac{kn(k+1)}{2}$$

containing the framed configuration space $\mathfrak{f}\mathsf{Con}(M,n)$ as an open dense subset. It is compact if and only if the manifold M is.

In order to prove that the Σ-module $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ is a right $\mathfrak{f}\mathsf{F}_k$-module, one must understand better the structure of these objects. Let $\mathfrak{f}\mathsf{Con}(M)(T)$ be the pullback of the diagram

$$\begin{tikzcd} & O(M)^{\times n} \arrow[d, "\pi_M^{\times n}"] \\ \mathsf{Con}(M)(T) \arrow[r, "bld_T"] & M^n \end{tikzcd}$$

where the blowing down map bld_T was defined in (4.46). For $T \in \mathcal{E}tree(n)$ let

$$\mathfrak{f}Z[T] := Met(T) \times \mathfrak{f}\mathsf{Con}(M)(T); \tag{4.52}$$

compare (4.14). An element of $\mathfrak{f}\eta \in \mathfrak{f}Z[T]$ is of the form

$$\mathfrak{f}\eta = \eta \times \vec{\mathfrak{f}} \quad \text{with} \quad \vec{\mathfrak{f}} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_n), \tag{4.53}$$

where η is as in (4.35) and $\mathfrak{f}_i$ is a frame at $x^{pa(i)}$, $1 \leq i \leq n$. Define the composable elements and the extended set by

$$\begin{aligned} \mathfrak{f}W[T] &:= \{\eta \times \vec{\mathfrak{f}} \in \mathfrak{f}W[T] \mid \eta \in W[T]\} \text{ and} \\ \mathfrak{f}\widehat{W}[T] &:= \{\eta \times \vec{\mathfrak{f}} \in \mathfrak{f}W[T] \mid \eta \in \widehat{W}[T]\} \end{aligned}$$

with $W[T]$ (respectively $\widehat{W}[T]$) as in (4.37) (respectively in (4.39)). Finally, each $\mathfrak{f}\eta \in \mathfrak{f}\widehat{W}[T]$ determines $\delta_{\mathfrak{f}}(\mathfrak{f}\eta) \in \mathfrak{f}\mathsf{Con}(M)(T)$ by the formula

$$\delta_{\mathfrak{f}}(\mathfrak{f}\eta) := \delta(\eta) \times \Phi(\vec{\mathfrak{f}}),$$

where $\delta(\eta) \in \mathsf{Con}(M,T)$ was defined in (4.43) and

$$\Phi(\vec{\mathfrak{f}}) := (\Phi_{x^{pa_T(1)}, y^{pa_{T_h}(1)}}(\mathfrak{f}_1), \ldots, \Phi_{x^{pa_T(n)}, y^{pa_{T_h}(n)}}(\mathfrak{f}_n)),$$

where $\vec{y}$ has the same meaning as in (4.41). The space

$$\mathfrak{f}\overline{\mathsf{Con}}(M,n) := \mathfrak{f}\widetilde{W}(n)/\sim$$

with

$$\mathfrak{f}\widetilde{W}(n) := \bigsqcup_{T \in \mathcal{E}tree(n)} \mathfrak{f}\widetilde{W}[T]$$

and the relation $\sim$ given by

$$\mathfrak{f}\widehat{W}(n) \supset \mathfrak{f}\widehat{W}[T] \ni \mathfrak{f}\eta \sim (0, \delta_{\mathfrak{f}}(\mathfrak{f}\eta)) \in \mathfrak{f}\widehat{W}[T_h] \subset \mathfrak{f}\widehat{W}(n), \tag{4.54}$$

for $T \in \mathcal{E}tree(n)$ and $n \geq 1$, then coincides with the pull back of the diagram (4.51).

THEOREM 4.49. *Let M be a complete Riemannian manifold. The Σ-module $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ is then a natural right $\mathfrak{f}\mathsf{F}_k$-module.*

PROOF. The scheme of the proof is similar to that of Proposition 4.31. For $m, n \geq 1$ define

$$\mathfrak{f}Z(n) := \coprod_{T\in\mathcal{E}tree(n)} \mathfrak{f}Z[T]$$

with $\mathfrak{f}Z[T]$ as in (4.52) and

$$\mathfrak{f}Y(m) := \coprod_{S\in\mathcal{R}tree(m)} \mathfrak{f}Y[S]$$

with $\mathfrak{f}Y[S]$ as in (4.48).

The most important step of the proof is to observe that the Σ-module $\mathfrak{f}Z = \{\mathfrak{f}Z(n)\}_{n\geq 1}$ is a right module over the operad $\mathfrak{f}Y = \{\mathfrak{f}Y(n)\}_{n\geq 1}$. This module structure is defined by grafting underlying trees, using the frames to identify tangent spaces at various points. It is described, in terms of $\circ_i$-operations, as follows.

Let $T \in \mathcal{E}tree(n)$, $S \in \mathcal{R}tree(m)$ and $1 \leq i \leq n$. Let $\eta \times (\mathfrak{f}_1, \dots, \mathfrak{f}_n) \in \mathfrak{f}Z[T] \subset \mathfrak{f}Z(n)$ be as in (4.53) and $\xi \times (g_1, \dots, g_m) \in \mathfrak{f}Y[S] \subset \mathfrak{f}Y(m)$ as in (4.49). The ith frame $\mathfrak{f}_i$ defines an identification of $T_{x^{pa(i)}}M$ and $\mathbb{R}^k$. So it also identifies $\xi \in Y[S]$ with a point $\mathfrak{f}_i(\xi) \in Y_{x^{pa(i)}}[S]$ and $(g_1, \dots, g_m) \in O(k)^{\times m}$ with a vector $(\mathfrak{f}_i(g_1), \dots, \mathfrak{f}_i(g_m))$ of frames at $x^{pa(i)} \in M$. This vector can of course be described, using the natural left action of the orthogonal group on frames, as

$$(\mathfrak{f}_i(g_1), \dots, \mathfrak{f}_i(g_m)) = (g_1^{-1}(\mathfrak{f}_i), \dots, g_m^{-1}(\mathfrak{f}_i)).$$

We then define

$$\begin{aligned}(4.55)\quad &[\eta \times (\mathfrak{f}_1, \dots, \mathfrak{f}_n)] \circ_i [\xi \times (g_1, \dots, g_m)] \\ &:= \mu \times (\mathfrak{f}_1, \dots, \mathfrak{f}_{i-1}, \mathfrak{f}_i(g_1), \dots, \mathfrak{f}_i(g_m), \mathfrak{f}_{i+1}, \dots, \mathfrak{f}_m) \in \mathfrak{f}Z(m+n-1),\end{aligned}$$

where μ is the coloring of $T \circ_i S$ given by η on T and ξ on S.

It is evident that the above action restricts to an action of the extended Σ-module $\mathfrak{f}\widehat{U} = \{\mathfrak{f}\widehat{U}(n)\}_{n\geq 1}$ on the extended Σ-module $\mathfrak{f}\widehat{W} = \{\mathfrak{f}\widehat{W}(n)\}_{n\geq 1}$ and that it commutes with the defining relations (4.54) and (4.50). Therefore it induces the requisite action of the quotient spaces. This finishes the proof of the theorem. ■

REMARK 4.50. The unframed operad F_k is a suboperad of $\mathfrak{f}\mathsf{F}_k$; therefore the Σ-module $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ is also a right F_k-module, with the action given by the obvious modification of (4.55):

$$\begin{aligned}(4.56)\quad &[\eta \times (\mathfrak{f}_1, \dots, \mathfrak{f}_n)] \circ_i \xi \\ &:= \mu \times (\mathfrak{f}_1, \dots, \mathfrak{f}_{i-1}, \underbrace{\mathfrak{f}_i, \dots, \mathfrak{f}_i}_{i \text{ times}}, \mathfrak{f}_{i+1}, \dots, \mathfrak{f}_m) \in \mathfrak{f}Z(m+n-1).\end{aligned}$$

PROOF OF THEOREM 4.46. If the manifold M is parallelizable, then clearly

$$\mathsf{f}\overline{\mathsf{Con}}(M)(n) \cong \overline{\mathsf{Con}}(M)(n) \times O(k)^{\times n}, \tag{4.57}$$

with the isomorphism depending on the choice of the trivialization of the tangent bundle of M. The action (4.56) clearly does not affect the second factor of (4.57), therefore it induces an action on the compactification of the unframed configuration space. ∎

CHAPTER 5

Generalization of Operads

5.1. Cyclic operads

In Section I.1.16 we recalled the operad $\widehat{\mathcal{M}}_0 = \{\widehat{\mathcal{M}}_0(n)\}_{n\geq 1}$ of Riemann spheres with parametrized labeled holes. Each $\widehat{\mathcal{M}}_0(n)$ was a right Σ_n-space, with the operadic right Σ_n-action permuting labels $\{1,\dots,n\}$ of holes. But each $\widehat{\mathcal{M}}_0(n)$ obviously admits a higher type of symmetry which interchanges labels $\{0,\dots,n\}$ of *all* holes, including the label of the 'output.' A similar example admitting a higher symmetry is the configuration pseudo-operad $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$ of the Deligne-Knudsen-Mumford compactification of configuration spaces of points on $\mathbb{CP}^1$; see Section 4.2.

These examples indicate that, for some (pseudo-)operads, there is no clear distinction between 'inputs' and the 'output,' so instead of the '$\circ_i$-formalism' of Section 1.3 one has to consider operations ${}_i\circ_j$ as in Remark 5.10. Cyclic operads, introduced by E. Getzler and M.M. Kapranov, formalize this phenomenon. They are, roughly speaking, operads with an extra symmetry that interchanges the output with one of the inputs. It is helpful to interpret the output of such an operad as being indexed by 0, as indicated in Figure 1.

The material of this section and of Section 5.2 is taken mostly from [**GK95**]. Recall that the symmetric group Σ_n is the group of automorphisms of the set $\{1,\dots,n\}$. As observed above, operations of arity n in a cyclic operad admit an action of the group of automorphisms of $\{0,\dots,n\}$. Let us give a name to it.

DEFINITION 5.1. Let Σ_n^+ be the group of automorphisms of the set $\{0,\dots,n\}$.

The group Σ_n^+ is, of course, isomorphic to the symmetric group Σ_{n+1}, but the isomorphism is canonical only up to an identification $\{0,\dots,n\} \cong \{1,\dots,n+1\}$. We interpret Σ_n as the subgroup of Σ_n^+ consisting of permutations $\sigma \in \Sigma_n^+$ with $\sigma(0) = 0$. If $\tau_n \in \Sigma_n^+$ denotes the cycle $(0,\dots,n)$, that is, the permutation with $\tau_n(0) = 1$, $\tau_n(1) = 2$, $\dots, \tau_n(n) = 0$, then τ_n and Σ_n generate Σ_n^+.

Let $\mathcal{C} = (\mathcal{C}, \odot, s, \mathbf{1})$ be a strict symmetric monoidal category as in Section 1.1.

DEFINITION 5.2. A *cyclic operad* in $\mathcal{C}$ is an ordinary operad $\mathcal{P}$ as in Definition 1.4 such that the right Σ_n-action on $\mathcal{P}(n)$ extends, for $n \geq 1$, to an action of Σ_n^+ fulfilling the following three axioms.

(i) If $\eta : \mathbf{1} \to \mathcal{P}(1)$ is the unit of $\mathcal{P}$, then the following diagram commutes.

$$\begin{tikzcd} \mathcal{P}(1) \arrow[rr, "\tau_1"] & & \mathcal{P}(1) \\ & \mathbf{1} \arrow[ul, "\eta"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

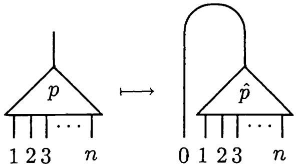


FIGURE 1. An n-ary operation p of a cyclic operad. The output of p is interpreted as the input of $\hat{p}$ labeled 0.

(ii) For each $m, n \geq 1$, $\circ_1 = \gamma_{\mathcal{P}}(\mathbb{1}; \mathbb{1} \odot \eta^{\odot(m-1)})$ and $\circ_n = \gamma_{\mathcal{P}}(\mathbb{1}; \eta^{\odot(n-1)} \odot \mathbb{1})$ the diagram

$$\begin{tikzcd}
\mathcal{P}(m) \odot \mathcal{P}(n) \arrow[r, "\circ_1"] \arrow[d, "\tau_m \odot \tau_n"] & \mathcal{P}(m+n-1) \arrow[dd, "\tau_{m+n-1}"] \\
\mathcal{P}(m) \odot \mathcal{P}(n) \arrow[d, "s_{\mathcal{P}(m),\mathcal{P}(n)}"] & \\
\mathcal{P}(n) \odot \mathcal{P}(m) \arrow[r, "\circ_n"] & \mathcal{P}(m+n-1)
\end{tikzcd}$$

commutes.

(iii) For each $m, n \geq 1$, $2 \leq i \leq m$, $\circ_i = \gamma_{\mathcal{P}}(\mathbb{1}; \eta^{\odot(i-1)} \odot \mathbb{1} \odot \eta^{\odot(n-i)})$ and $\circ_{i-1} = \gamma_{\mathcal{P}}(\mathbb{1}; \eta^{\odot(i-2)} \odot \mathbb{1} \odot \eta^{\odot(n-i-1)})$ the diagram

$$\begin{tikzcd}
\mathcal{P}(m) \odot \mathcal{P}(n) \arrow[r, "\circ_i"] \arrow[d, "\tau_m \odot \mathbb{1}"] & \mathcal{P}(m+n-1) \arrow[d, "\tau_{m+n-1}"] \\
\mathcal{P}(m) \odot \mathcal{P}(n) \arrow[r, "\circ_{i-1}"] & \mathcal{P}(m+n-1)
\end{tikzcd}$$

commutes.

Observe that (iii) is not demanded in the original paper [**GK95**], though, as observed by P. van der Laan, it should follow from the definition of cyclic operads in terms of the triple of unrooted trees (Theorem 5.8). We were not able to verify that (iii) follows from the other axioms of a cyclic operad.

REMARK 5.3. One may rewrite, for some concrete categories, axioms (i), (ii) and (iii) of Definition 5.2 in terms of elements. When $\mathcal{C} = \mathtt{dgVec}$, the axioms read

(i) $\tau_1(1) = 1$, where $1 \in \mathcal{P}(1)$ is the unit and

(ii$\frac{1}{2}$) for $p \in \mathcal{P}(m)$ and $q \in \mathcal{P}(n)$, the composition maps satisfy

$$\gamma(p; 1, \ldots, 1, \overset{i}{q}, 1, \ldots, 1) \cdot \tau_{m+n-1} = \begin{cases} (-1)^{|p|\,|q|} \gamma(q \cdot \tau_n; 1, \ldots, 1, p \cdot \tau_m), & \text{for } i = 1, \text{ and} \\ \gamma(p \cdot \tau_m; 1, \ldots, 1, \overset{i-1}{q}, 1, \ldots, 1), & \text{for } 2 \leq i \leq m, \end{cases}$$

where $\overset{i}{q}$ (respectively $\overset{i-1}{q}$) indicates that q is at the ith (respectively $(i-1)$th) position.

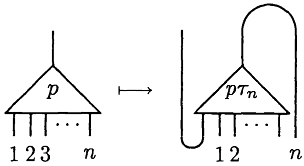


FIGURE 2. A 'visualization' of the action of τ_n. The element τ_n redraws $p \in \mathcal{P}(n)$, represented as an 'operation' with n inputs and one output, so that the first input becomes the output and the output becomes the last input of $p \cdot \tau_n$.

For $\mathcal{C}$ the category of topological spaces, the second axiom of course does not contain the sign factor.

Because our conventions are based on the *right* action of the symmetric group, our form of axioms of cyclic operads slightly differs from [**GK95**] which uses the left action.

An intuitive feeling for the action of τ_n is suggested by Figure 2. In terms of Figure 1, the action of τ_n is (up to sign) just the cyclic permutation of the labels on the inputs of $\hat{p}$ from $(0, 1, 2, \ldots, n)$ to $(n, 0, 1, \ldots, n-1)$. We denote by $\mathtt{Op}^+_{\mathcal{C}}$ or simply by $\mathtt{Op}^+$ when $\mathcal{C}$ is understood the category of cyclic operads in $\mathcal{C}$.

As operads were Σ-modules with an additional structure, cyclic operads are *cyclic* Σ-modules with an extra structure.

DEFINITION 5.4. A *cyclic Σ-module* or a *Σ^+-module* in a category $\mathcal{C}$ is a sequence

$$W = \{W(n)\}_{n \geq 1}$$

of objects of $\mathcal{C}$ such that each $W(n)$ is a (right) Σ^+_n-module. Let $\Sigma^+\text{-}\mathtt{Mod}_{\mathcal{C}}$ or simply $\Sigma^+\text{-}\mathtt{Mod}$ when $\mathcal{C}$ is understood denote the category of cyclic Σ-modules.

REMARK 5.5. There is an obvious forgetful functor $U^- : \Sigma^+\text{-}\mathtt{Mod} \to \Sigma\text{-}\mathtt{Mod}$ from the category of cyclic Σ-modules to the category of Σ-modules. A cyclic operad is then a cyclic Σ-module $\mathcal{P}$ such that the Σ-module $U^-(\mathcal{P})$ is an ordinary operad with an endomorphism τ_n acting on each $U^-(\mathcal{P})(n)$, $n \geq 1$, whose structure maps satisfy (i), (ii) and (iii) of Definition 5.2.

We saw in Theorem 1.105 that an (ordinary) operad is an algebra over the triple $\Gamma : \Sigma\text{-}\mathtt{Mod} \to \Sigma\text{-}\mathtt{Mod}$ of rooted trees. We show next that cyclic operads are algebras over a similar triple based on *unrooted* trees, the 'free cyclic operad' functor $\Gamma_+ : \Sigma^+\text{-}\mathtt{Mod} \to \Sigma^+\text{-}\mathtt{Mod}$.

Let us assume, as in Section 1.7, that $\mathcal{C}$ has all small limits and colimits and that for any object A the endofunctor $A \odot - : \mathcal{C} \to \mathcal{C}$ preserves colimits. Recall (1.24) that each Σ-module E determines the functor $E \longmapsto E(X)$ from the category of finite sets and their bijections to $\mathcal{C}$. Because $\Sigma^+_n \cong \Sigma_{n+1}$ for each n, this functor can be used to define as similar functor for *cyclic* Σ-modules as follows.

Each cyclic Σ-module $W = \{W(n)\}_{n \geq 1}$ induces a natural Σ-module $E_W = \{E_W(n)\}_{n \geq 1}$ with

$$E_W(n) := \begin{cases} W(n-1), & \text{for } n \geq 2 \text{ and} \\ \mathbf{0} \text{ (the initial object of } \mathcal{C}), & \text{for } n = 1. \end{cases}$$

The action of Σ_n on $E_W(n)$ is given by the identification $\Sigma_{n-1}^+ \cong \Sigma_n$ induced by the isomorphism

$$\{0, \dots, n-1\} \cong \{1, \dots, n\},\ i \mapsto i+1 \text{ for } 0 \le i \le n-1.$$

We can thus define, for any cyclic Σ-module W, a functor $W \longmapsto W(\!(X)\!)$ on the category of nonempty finite sets and their bijections by

$$W(\!(X)\!) := E_W(X). \tag{5.1}$$

For instance, if $\mathcal{C}$ is the category of graded vector spaces and X a set with $n+1$ elements, then

$$W(\!(X)\!) = \left(\bigoplus_{g\ \{1,\dots,n+1\}\to X} E_W(n+1) \right)_{\Sigma_{n+1}} \cong \left(\bigoplus_{f\ \{0,\dots,n\}\to X} W(n) \right)_{\Sigma_n^+};$$

compare (1.25).

REMARK 5.6. Double brackets in $W(\!(X)\!)$ remind us that the nth piece of the cyclic Σ-module $W = \{W(n)\}_{n\ge 1}$ is applied on a set with $n+1$ elements, using the extended Σ_n^+-symmetry. We thus have, for example, noncanonical isomorphisms

$$W(\!(\{0, \dots, n\})\!) \cong W(n)$$

of Σ_n^+-spaces and

$$W(\{0, \dots, n\}) \cong W(n+1)$$

of Σ_{n+1}-spaces.

Recall (Section 1.5) that an (unrooted) tree is a finite contractible graph. We denoted by $Vert(T)$ the set of vertices of a tree T and, for a vertex $v \in Vert(T)$, by $edge(v)$ the set of edges incident with v. Let $\mathtt{Tree}_n^+$ denote the category of unrooted trees with legs (it makes no sense to speak about leaves of an unrooted tree) labeled by $0, \dots, n$. That is, elements of $\mathtt{Tree}_n^+$ are unrooted trees T with legs labeled by $\ell : Leg(T) \to \{0, \dots, n\}$. Of course, each tree $T \in \mathtt{Tree}_n^+$ can also be considered as a rooted tree, with root the leg labeled 0.

For a cyclic Σ-module W and a labeled unrooted tree (T, ℓ) we have the following cyclic version of the unordered product of Definition 1.70:

$$W(\!(T, \ell)\!) := \bigodot_{Vert(T)} W(\!(edge(v))\!).$$

The main difference between the formula in Definition 1.70 and the above formula is that instead of the set $In(v)$ of incoming edges of a vertex v of a rooted tree, here we use the set $edge(v)$ of all edges incident with v (it makes no sense to speak about incoming edges of a vertex of an unrooted tree). As before, we will usually omit ℓ and write simply $W(\!(T)\!)$ instead of $W(\!(T, \ell)\!)$.

If $*_n$ is the $(n+1)$-*star*, i.e. the unrooted tree with one vertex and $n+1$ legs, then clearly

$$W(\!(*_n)\!) \cong W(n), \text{ for all } n \ge 1. \tag{5.2}$$

Finally (cf. Definition 1.77), let

$$\Psi_+(W)(n) := \operatorname*{colim}_{T \in Iso(\mathtt{Tree}_n^+)} W(\!(T)\!). \tag{5.3}$$

The grafting of trees defines a natural map $\mu_{\Psi_+} : \Psi_+\Psi_+W \longrightarrow \Psi_+W$. We also have an embedding of cyclic Σ-modules $v_{\Psi_+} : W \longrightarrow \Psi_+W$ given by the canonical map of $W(n) \cong W((*_n))$ to the colimit (5.3). For $n \geq 1$, the Σ_n^+-action on $\Psi_+(W)(n)$ is given by renumbering legs of trees of the indexing category $\mathtt{Tree}_n^+$ in (5.3). Thus Ψ_+ is an endofunctor on the category of cyclic Σ-modules.

In fact, Ψ_+ is the 'free cyclic pseudo-operad' on W, but we are not going to study such objects here (see, however, Remark 5.10). As in (1.58), the free cyclic *operad* is obtained by adjoining the unit,

$$\Gamma_+(W) := \mathbf{1} \sqcup \Psi_+(W),$$

that is, taking the coproduct with the cyclic Σ-module $\mathbf{1}$ defined by

$$\mathbf{1}(n) := \begin{cases} \mathbf{1}, & \text{for } n = 1, \text{ and} \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where $\mathbf{1}$ is the unit and $\mathbf{0}$ the initial objects of the category $\mathcal{C}$. The Σ^+-action on $\mathbf{1}$ is trivial.

To show that $\Gamma_+(W)$ is a cyclic operad we need, by Remark 5.5, to check that $U^-\Gamma_+(W)$ is an ordinary operad and that the Σ^+-action is compatible with composition operations. The operad structure on $U^-\Gamma_+(W)$ is induced by the natural equivalence

$$U^-(\Gamma_+(W)) \cong \Gamma(W).$$

It is easy to verify that it commutes with the Σ^+-action constructed above.

PROPOSITION 5.7. *The functor $\Gamma_+ : \Sigma^+\text{-}\mathtt{Mod} \longrightarrow \mathtt{Op}^+$, $W \longmapsto \Gamma_+(W)$, from the category of cyclic Σ-modules to the category of cyclic operads defines the* free cyclic operad *on the cyclic Σ-module W.*

This proposition can be proved exactly as was Proposition 1.92 and we omit the proof. The following theorem [**GK95**] is a 'cyclic version' of Theorem 1.105.

THEOREM 5.8. *The natural transformations $\mu : \Gamma_+\Gamma_+ \longrightarrow \Gamma_+$ and $\eta : \mathbb{1} \longrightarrow \Gamma_+$ make the functor $\Gamma_+ : \Sigma^+\text{-}\mathtt{Mod} \longrightarrow \Sigma^+\text{-}\mathtt{Mod}$ into a triple. An algebra over the triple Γ_+ is the same as a cyclic operad.*

The proof is analogous to the proof of Theorem 1.105 and we also omit it. We saw in Section 1.9 that, for an ordinary operad $\mathcal{P}$, the space $\mathcal{P}(n)$ is identified with $\mathcal{P}(\mathrm{c}(n))$, where $\mathrm{c}(n)$ is the 'n corolla,' i.e. the rooted tree with one vertex and input legs labeled by $1, \ldots, n$. This is expressed by the idea of viewing elements of $\mathcal{P}(n)$ as operations with n inputs and one output and the structure map as the composition of these operations, as indicated in Figure 1 of Section I.1.3.

Similarly, for a cyclic operad $\mathcal{P}$, the identification $\mathcal{P}(n) \cong \mathcal{P}((*_n))$ of (5.2) suggests imagining elements of $\mathcal{P}(n)$ as 'spiders' with $n+1$ legs and the structure map as joining legs of these spiders; see Figure 3.

By drawing up-rooted trees in the plane, we specify what is the output and what are the inputs of a vertex v and our preferred orientation (say from left to right) gives an order on the inputs $In(v)$; see also Appendix 1.9.1. For unrooted trees, there is no distinction between inputs and the output, but our preferred orientation of the plane (anticlockwise in our case) still defines a cyclic orientation of the set $edge(e)$.

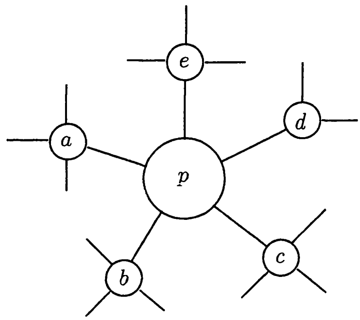
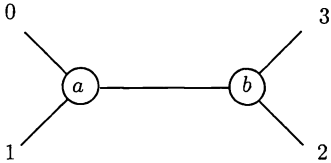
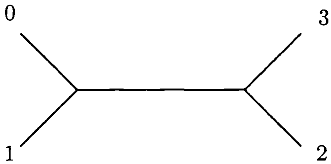


FIGURE 3. The composition of $p \in \mathcal{P}(4)$ with $a, b, c, e \in \mathcal{P}(3)$ and $d \in \mathcal{P}(2)$.

EXAMPLE 5.9. Consider the following picture:

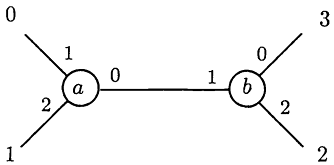


where $a, b \in \mathcal{P}(2)$. This picture should represent an element of $\mathcal{P}((T)) \in \mathcal{C}$, where $T \in \mathtt{Tree}_3^+$ is the tree

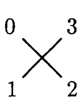


This is, however, not exactly so, unless we choose labels for the edges of *edge*(v) for both vertices of T, for example

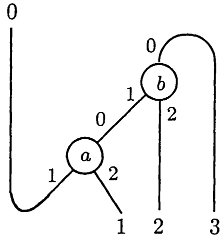
(5.4)



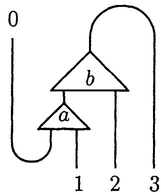
Denote the corresponding element $x(a,b) \in \mathcal{P}((T))$. The composition $\mu_{\mathcal{P}}(x(a,b))$ is an element of $\mathcal{P}((S)) \cong \mathcal{P}(3)$, where S is the 3-star $*_3$:



In order to describe the element of $\mathcal{P}(3)$ represented by this picture, redraw (5.4) as:



Now replace the 'spiders' by 'operations' (since a cyclic operad is, after all, an operad) and invoke the standard convention that the inputs are numbered from the left to the right:



The 'visualized' action of Figure 2 identifies $\mu_{\mathcal{P}}(x(a,b))$ with $[b(a,1)]\cdot\tau_3$. The last expression uses only the structure data of Definition 5.2.

REMARK 5.10. There is another axiomatization of cyclic operads, closer to the 'spider' intuition. Namely, we may say that a *cyclic pseudo-operad* is a cyclic Σ-module $\mathcal{P}=\{\mathcal{P}(n)\}_{n\geq 1}$ together with operations

$$_i\circ_j : \mathcal{P}(m)\odot\mathcal{P}(n)\longrightarrow\mathcal{P}(m+n-1),\ 0\leq i\leq m,\ 0\leq j\leq n,$$

which formalize the 'joining of spider's legs.' We use the convention with $a\,{}_i\circ_j\, b$ denoting the result of joining the ith leg of a with the jth leg of b, so the structure operations $\circ_i$ of an ordinary operad correspond to ${}_i\circ_0$. The operations ${}_i\circ_j$ have to satisfy certain obvious axioms whose written form is very complicated due to the linear structure of human language. A cyclic operad is then a cyclic pseudo-operad with a *unit*, that is, with a morphism $\eta:\mathbf{1}\to\mathcal{P}(1)$ for which the diagrams

$$\begin{tikzcd} \mathcal{P}(1)\odot\mathcal{P}(n) \arrow[rr, "{}_\epsilon\circ_j"] & & \mathcal{P}(n) \\ & \mathbf{1}\odot\mathcal{P}(n)\cong\mathcal{P}(n) \arrow[ul, "\eta\odot 1\!\!1"] \arrow[ur, "\eta"'] & \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathcal{P}(n)\odot\mathcal{P}(1) \arrow[rr, "{}_j\circ_\epsilon"] & & \mathcal{P}(n) \\ & \mathcal{P}(n)\cong\mathcal{P}(n)\odot\mathbf{1} \arrow[ul, "1\!\!1\odot\eta"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

where $\epsilon\in\{0,1\}$ and $0\leq j\leq n$, commute. In terms of 'elements,' this means that there exists an element $1\in\mathcal{P}(1)$ satisfying

$$1_\epsilon\circ_j\, p = p_j\circ_\epsilon\, 1 = p,$$

for all $\epsilon\in\{0,1\}$, $p\in\mathcal{P}(n)$ and $0\leq j\leq n$.

REMARK 5.11. We may mimic constructions of Section 1.7 and express the axioms of cyclic pseudo-operads in terms of arbitrary finite sets. For each pair of finite sets X and Y and elements $x\in X$ and $y\in Y$, let

$$X_x\sqcup_y Y := X\sqcup Y-\{x,y\}.$$

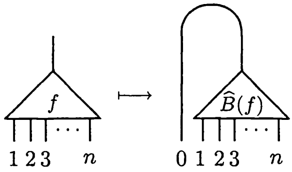


FIGURE 4. A visualization of the correspondence $f \mapsto \widehat{B}(f)$.

Then a cyclic pseudo-operad structure on a cyclic Σ-module $\mathcal{P}$ is determined by operations

$$_x\circ_y : \mathcal{P}((X)) \odot \mathcal{P}((Y)) \to \mathcal{P}((X\,{}_x\sqcup_y Y))$$

satisfying appropriate axioms.

EXAMPLE 5.12. Let V be a finite dimensional (graded) vector space and $B : V \otimes V \to \mathbf{k}$ a nondegenerate symmetric bilinear form. The form B induces the identification

$$Hom(V^{\otimes n}, V) \ni f \longmapsto \widehat{B}(f) := B(-, f(-)) \in Hom(V^{\otimes(n+1)}, \mathbf{k}) \tag{5.5}$$

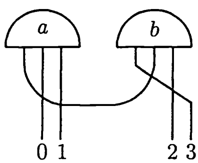
of $Hom(V^{\otimes n}, V)$ and $Hom(V^{\otimes(n+1)}, \mathbf{k})$. This correspondence can be visualized as shown in Figure 4. The standard right Σ_n^+-action

$$\widehat{B}(f)\sigma(v_0, \dots, v_n) = \epsilon(\sigma) \cdot \widehat{B}(f)(v_{\sigma^{-1}(0)}, \dots, v_{\sigma^{-1}(n)}),\ \sigma \in \Sigma_n^+,\ v_0, \dots, v_n \in V,$$

where $\epsilon(\sigma)$ is the Koszul sign (3.96), defines, via this identification, a right Σ_n^+-action on $Hom(V^{\otimes n}, V)$, that is, on the nth piece of the endomorphism operad $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 1}$. In terms of Figure 4 representing $\widehat{B}$, the action of τ_n is just the cyclic permutation of the inputs, changing the labels under the right-hand object from the sequence $(0, 1, 2, \dots, n)$ to $(n, 0, 1, 2, \dots, n-1)$.

It is easy to show that, with the above action, $\mathcal{E}nd_V$ is a cyclic operad in the monoidal category of graded vector spaces, called the *cyclic endomorphism operad* on the pair $V = (V, B)$. We will call such pairs *bilinear spaces*.

Cyclic endomorphism operads of Example 5.12 are archetypes of cyclic operads. Formula (5.5) suggests another way of viewing elements of the nth piece $\mathcal{P}(n)$ of a cyclic operad, as 'operations' with $n+1$ inputs and no output. The following picture shows how to represent the element $\mu_{\mathcal{P}}(x(a, b))$ of Example 5.9 in this way, which is very close to the intuition of Figure 1:



Consider now a more general situation of a bilinear form $B : A \otimes A \to U$ with values in a (graded) vector space U. Suppose we are given a cyclic operad $\mathcal{P}$ and a $\mathcal{P}$-algebra structure $\alpha : \mathcal{P} \to \mathcal{E}nd_A$ on A (Definition 1.20). Define, for each $n \geq 0$, a map $\widetilde{B}(\alpha) = \widetilde{B}_n(\alpha) : \mathcal{P}(n) \otimes A^{\otimes(n+1)} \to U$ by

$$\widetilde{B}(\alpha)(p \otimes a_0 \otimes \cdots \otimes a_n) := \widehat{B}(\alpha(p))(a_0, \dots, a_n), \tag{5.6}$$

for $a_0, ..., a_n \in A$, $p \in \mathcal{P}(n)$, where

$$\widehat{B}(\alpha(p))(a_0, \dots, a_n) = (-1)^{|a_0||p|} B(a_0, \alpha(p)(a_1, \dots, a_n))$$

was introduced in (5.5).

DEFINITION 5.13. A bilinear form $B : A \otimes A \to U$ is *invariant* (on the $\mathcal{P}$-algebra A) if and only if the maps $\widetilde{B}_n(\alpha)$ of (5.6) are, for each $n \geq 1$, invariant under the diagonal action of Σ_n^+ on $\mathcal{P}(n) \otimes A^{\otimes(n+1)}$. Equivalently, $\widetilde{B}_n(\alpha)$ factors through the projection $\mathcal{P}(n) \otimes A^{\otimes(n+1)} \to \mathcal{P}(n) \otimes_{\Sigma_n^+} A^{\otimes(n+1)}$:

$$\begin{tikzcd}
\mathcal{P}(n) \otimes A^{\otimes(n+1)} \arrow[rr, "\widetilde{B}_n(\alpha)"] \arrow[d] & & U \\
\mathcal{P}(n) \otimes_{\Sigma_n^+} A^{\otimes(n+1)} \arrow[urr, dotted] & &
\end{tikzcd}$$

Since the map $\widetilde{B}_n(\alpha)$ is always Σ_n-invariant, it is enough to check whether it commutes with τ_n. Taking, in (5.6), $n = 1$ and $p = 1 \in \mathcal{P}(1)$ we learn that an invariant form is necessary symmetric, $B(a, b) = (-1)^{|a||b|} B(b, a)$, for all $a, b \in A$. We have the following easy proposition.

PROPOSITION 5.14. *Let $\mathcal{P}$ be a cyclic operad, $\alpha : \mathcal{P} \to \mathcal{E}nd_A$ a $\mathcal{P}$-algebra structure on a vector space A and $B : A \otimes A \to \mathbf{k}$ a nondegenerate symmetric bilinear form. Let us consider $\mathcal{E}nd_A$ as a cyclic operad, with the cyclic structure given by B as in Example 5.12. Then $\alpha : \mathcal{P} \to \mathcal{E}nd_A$ is a map of cyclic operads if and only if B is invariant in the sense of Definition 5.13.*

PROOF. Since α is always a map of ordinary operads, it is enough to check the Σ^+-equivariance. Let us analyze, for $n \geq 1$, when the map $\alpha = \alpha(n) : \mathcal{P}(n) \to \mathcal{E}nd_A(n)$ is Σ_n^+-equivariant. Under the identifications of Example 5.12, this means that $\widehat{B}(\alpha(p)) \cdot \sigma = \widehat{B}(\alpha(p \cdot \sigma))$, for $p \in \mathcal{P}(n)$ and $\sigma \in \Sigma_n^+$ or, in 'coordinates,'

$$\widehat{B}(\alpha(p))(\sigma(a_0, \dots, a_n)) = \widehat{B}(\alpha(p \cdot \sigma))(a_0, \dots, a_n), \ n \geq 1.$$

But this clearly means that B is invariant in the sense of Definition 5.13. ∎

In the situation of Proposition 5.14, we say that A is a *cyclic $\mathcal{P}$-algebra*. An example of this structure can be found in Example 5.86.

Recall (Definition 3.31) that an (ordinary) operad $\mathcal{P}$ is quadratic if it has a presentation $\mathcal{P} = \langle E; R \rangle = \Gamma(E)/(R)$, where $E = \mathcal{P}(2)$ and $R \subset \Gamma(E)(3)$. The action of Σ_2 on E extends to an action of Σ_2^+, via the sign representation sgn : $\Sigma_2^+ \to \{\pm 1\} = \Sigma_2$. It can be easily verified that this action induces a cyclic operad structure on the free operad $\Gamma(E)$. In particular, $\Gamma(E)(3)$ is a right Σ_3^+-module.

DEFINITION 5.15. We say that an operad $\mathcal{P}$ is a *cyclic quadratic operad* if, in the above presentation, R is a Σ_3^+-invariant subspace.

If the condition of the above definition is satisfied, $\mathcal{P}$ has an induced cyclic operad structure. The proof of the following lemma is easy and we leave it as an exercise for the reader.

LEMMA 5.16. *The quadratic dual $\mathcal{P}^!$ of a cyclic quadratic operad $\mathcal{P}$ is again a cyclic quadratic operad.*

The following theorem was proved in [**GK95**].

THEOREM 5.17. *Each quadratic operad $\mathcal{P} = \langle E; R\rangle$ with $\dim(E) = 1$ is cyclic quadratic.*

PROOF. The representation of Σ_2 on E may be either the trivial representation $\mathbb{1}$ or the signum representation sgn. This means that algebras over the operad $\mathcal{P}$ have either a commutative or an anticommutative product.

Commutative case. A direct calculation shows that

$$\Gamma(E)(3) \cong \ydiagram{4} \oplus \ydiagram{2,2}$$

as a right Σ_3^+-module. On restriction to the subgroup $\Sigma_3 \subset \Sigma_3^+$ it becomes

$$\Gamma(E)(3) \cong \ydiagram{3} \oplus \ydiagram{2,1}$$

We see that the decomposition of $\Gamma(E)(3)$ into Σ_3^+-invariant irreducible subspaces is the same as the decomposition into Σ_3-invariant irreducible subspaces. Therefore each Σ_3-invariant subspace R of $\Gamma(E)(3)$ is also Σ_3^+-invariant and thus it defines a cyclic quadratic operad.

Anticommutative case. The arguments are similar. We have

$$\Gamma(E)(3) \cong \ydiagram{1,1,1,1} \oplus \ydiagram{2,2}$$

over Σ_3^+ and

$$\Gamma(E)(3) \cong \ydiagram{1,1,1} \oplus \ydiagram{2,1}$$

as Σ_3-modules. This finishes the proof. ■

An immediate consequence of Theorem 5.17 is that the operads $\mathcal{C}om$ (Example 3.33) and $\mathcal{L}ie$ (Example 3.34) are cyclic quadratic. A symmetric bilinear form $B : A \otimes A \to U$ on a commutative associative algebra $A = (A, \cdot)$ is invariant if

$$B(a, b \cdot c) = B(a \cdot b, c), \quad \text{for all } a, b, c \in A, \tag{5.7}$$

which is the Σ_2^+-invariance of (5.6) for $n = 2$. This follows from an obvious fact that, for quadratic operads, the Σ_n^+-invariance of (5.6) for $n = 1$ (which is the symmetry of B) and $n = 2$ implies the Σ_n^+-invariance of (5.6) for an arbitrary n.

The same arguments imply that a symmetric bilinear form $B : L \otimes L \to U$ on a Lie algebra $L = (L, [-,-])$ is invariant if and only if

$$B([u, v], w) = B(u, [v, w]), \quad \text{for } u, v, w \in U.$$

Also the operad $\mathcal{A}ss$ (Example 3.35) for associative algebras and the operad $\mathcal{P}oiss$ for Poisson algebras are cyclic quadratic. Observe that this fact does not follow from Theorem 5.17, since now $\dim(E) = 2$,

$$E = \text{the regular representation of } \Sigma_2 = \mathbb{1} \oplus \text{sgn}, \tag{5.8}$$

but it can be verified directly, by a similar calculation as in the proof of Theorem 5.17. Equation (5.8) expresses the elementary but surprising fact that, in characteristic zero, one bilinear operation is the same as one symmetric and one antisymmetric operation.

A symmetric bilinear form $B : A \otimes A \to U$ on an associative algebra $A = (A, \cdot)$ is invariant if $B(a \cdot b, c) = B(a, b \cdot c)$ (formally the same as (5.7)). If A has a unit 1, there is a one-to-one correspondence between such invariant bilinear forms and *traces* on A, that is, linear maps $T : A \to U$ such that $T(a \cdot b) = T(b \cdot a)$, for $a, b \in A$ The correspondence is given by $T(a) = B(1, a)$, respectively $B(a, b) = T(a \cdot b)$.

In [**GK95**], a complete classification of cyclic quadratic operads with $\mathcal{P}(2) = \mathbb{1} \oplus \mathrm{sgn}$ is given – there are 80 of those. An example of an operad which is quadratic but not cyclic quadratic is the operad $\mathcal{L}eib$ for Leibniz algebras; see [**GK95**] for the details.

EXAMPLE 5.18. The topological operad $\mathcal{K} = \{\mathcal{K}(n)\}_{n \geq 1}$ for A_∞-spaces is the symmetrization of the non-Σ topological operad $\underline{\mathcal{K}} = \{K_n\}_{n \geq 1}$, where K_n is the nth Stasheff associahedron (see Section 1.6). The vertices of K_n correspond to complete parenthesizations of the string $1 \ldots n$. Such parenthesizations are in bijective correspondence with triangulations of an $(n+1)$-gon P_{n+1} into triangles whose vertices are among vertices of P_{n+1}, while faces of K_n correspond to decompositions of P_{n+1} into polygons; see, for example, [**Kap93**]. Taking P_{n+1} to be a regular polygon, we obtain an action of $\mathbb{Z}_{n+1}$ on P_{n+1} by rotation through multiples of the angle $2\pi/(n+1)$, inducing an action on K_n. Now $\mathcal{K}(n)$ is the induced Σ_n^+-space $\mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_n^+} K_n$. This interpretation shows that each $\mathcal{K}(n)$ admits a natural right Σ_n^+-action. This action defines a cyclic operad structure on $\mathcal{K}$.

As indicated above, the polyhedron K_n has a very rich combinatorial structure. Its faces can be described either in terms of parenthesizations of n nonassociative symbols, in terms of decompositions of P_{n+1} into polygons or in terms of n-trees. All this shows that the combinatorics of the faces of K_n is similar to that of Catalan numbers

$$ {}_p a_k := \frac{1}{k} \binom{pk}{k-1}, \ p \geq 1, \ k \geq 2, $$

defined as the number of p-ary rooted trees with k vertices; see [**HP91**], [**Arn83**, Supplement]. For example, the number of vertices of K_n equals

$$ {}_2 a_{n-1} = \frac{1}{n-1} \binom{2(n-1)}{n-2}. $$

For recent results in this direction see also [**DR00**].

EXAMPLE 5.19. We already indicated at the beginning of this section that the operad $\widehat{\mathcal{M}}_0$ of genus zero Riemann surfaces with parametrized punctures is a topological cyclic operad. This immediately implies that the operad $\mathcal{BV} = H_*(\widehat{\mathcal{M}}_0; \mathbf{k})$ for Batalin-Vilkovisky algebras is a cyclic operad in the category of graded vector spaces; see Definition 4.6, Theorem 4.7 and Remark 4.10.

VARIANTS. Like ordinary operads, cyclic operads also have a non-Σ variant. We say that a non-Σ operad $\mathcal{P}$ (Definition 1.14) is a *non-Σ cyclic operad*, if each $\mathcal{P}(n)$ has a right $\mathbb{Z}_{n+1}$-action such that axioms (i), (ii) and (iii) of Definition 5.2, with τ_n interpreted as an element of $\mathbb{Z}_{n+1}$, are satisfied. The prominent examples of nonsymmetric cyclic operads are the non-Σ operad $\underline{\mathcal{A}ss}$ for associative algebras (Remark 1.15), the non-Σ topological operad $\underline{\mathcal{K}}$ for A_∞-spaces (Section 1.6) and the non-Σ operad $\underline{\mathcal{A}ss}_\infty$ for A_∞-algebras (Section I.1.17). Non-Σ cyclic operads, as simpler versions of cyclic operads, were studied in [**Mar99d**].

Recall (Definition 3.15) that the operadic suspension $\mathfrak{s}\mathcal{P}$ of a dg operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ was defined by

$$(\mathfrak{s}\mathcal{P})(n) := \uparrow^{n-1} \mathrm{sgn}_n \otimes \mathcal{P}(n),\ n \geq 1. \tag{5.9}$$

If the operad $\mathcal{P}$ is cyclic, then the suspension $\mathfrak{s}\mathcal{P}$ has a natural Σ^+-module structure defined by (5.9) with sgn_n replaced by sgn_{n+1}. It turns out that the resulting object is not a cyclic operad, but rather a modified version introduced by Getzler and Kapranov:

DEFINITION 5.20. An *anticyclic operad* in $\mathcal{C}$ is an ordinary operad $\mathcal{P}$ as in Definition 1.4 such that the right Σ_n-action on $\mathcal{P}(n)$ extends, for $n \geq 1$, to an action of Σ_n^+ such that diagram (iii) of Definition 5.2 commutes and diagrams (i) and (ii) commute up to -1.

REMARK 5.21. We will need to consider, in Section 5.3, slightly more general cyclic operads $\mathcal{P}$ with nontrivial components $\mathcal{P}(0)$ and $\mathcal{P}(-1)$. Elements of $\mathcal{P}(0)$ represent operations with one output and no inputs, with $\circ_i : \mathcal{P}(m) \odot \mathcal{P}(0) \to \mathcal{P}(m-1)$ for $1 \leq i \leq n$.

Elements of $\mathcal{P}(-1)$ have no inputs and no outputs. Although there are no $\circ_i$-operations involving them, these elements still can be in the image of modular contractions ξ_{ij} of Definition 5.35. We leave as an exercise to write the axioms for these 'extended' cyclic operads.

An example is provided by the operad $\widehat{\mathcal{M}}_0$ considered at the beginning of this section. For this operad, there are naturally defined spaces $\widehat{\mathcal{M}}_0(0)$ (moduli space of Riemann spheres with one parametrized hole) and $\widehat{\mathcal{M}}_0(-1)$ (one-point space of Riemann sphere with no hole).

5.2. Application: cyclic (co)homology

The cyclic cohomology of an associative unital algebra was introduced in 1983 by A. Connes [**Con83**] as one of the basic tools of his noncommutative geometry. In the same year, B.L. Tsygan [**Tsy83**] used cyclic homology for the computation of homology groups of some Lie algebras; this approach was further developed by J.-L. Loday and D. Quillen in [**LQ84**]. See also [**Cun00, AK00**] for a recent account.

Let us briefly recall the definitions. For a (graded) unital associative algebra A over $\mathbf{k}$ (which can be now any commutative ring), let $C_n(A) := A^{\otimes(n+1)}$ with the standard grading

$$\deg(a_0 \otimes \cdots \otimes a_n) = \sum_{0\leq i\leq n} \deg(a_i) + n.$$

Observe that we use the convention with $C_n(A)$ consisting of products of $(n+1)$ elements of A. Let us define degree -1 linear operators $b, b' : C_n(A) \to C_{n-1}(A)$

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & {} \arrow[l, "1-t"']
\end{tikzcd}$$

FIGURE 5. The bicomplex defining the cyclic homology.

by the formulas

$$\begin{aligned}
b'(a_0,\dots,a_n) &:= \sum_{i=0}^{n-1}(-1)^i(a_0,\dots,a_ia_{i+1},\dots,a_n) \text{ and}\\
b(a_0,\dots,a_n) &:= b'(a_0,\dots,a_n)+(-1)^n(a_n,a_0,\dots,a_{n-1}),
\end{aligned}$$

for $a_0,\dots,a_n \in A$. The complex $(C_*(A),b')$ is the standard unnormalized bar resolution of the algebra A as a left A-module [**Mac63a**, X(2.6)], hence it is acyclic. The complex $CH_*(A;A) := (C_*(A),b)$ is the complex calculating the *Hochschild homology* $HH_*(A;A)$ of A with coefficients in itself [**Mac63a**, X(4.2)],

$$H_*(C_*(A),b) := Tor_*^{A\otimes A^{\mathrm{op}}}(A,A).$$

The cyclic group $\mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$ acts on $C_n(A)$ by

$$t(a_0,\dots,a_n) := (-1)^n(a_n,a_0,\dots,a_{n-1}),$$

where t is the generator. Let $N := 1+t+\cdots+t^n \in \mathbf{k}[\mathbb{Z}_{n+1}]$ be the norm operator in the group algebra. A crucial fact is that

$$b(1-t) = (1-t)b' \text{ and } b'N = Nb. \tag{5.10}$$

Thus it makes sense to define the *cyclic homology* $H_*^\lambda(A)$ of A as the homology of the total complex of the bicomplex in Figure 5.

Let us assume from now on that the characteristic of $\mathbf{k}$ is zero. Under this assumption, the acyclicity of rows implies that the cyclic homology can be calculated as the homology of the complex

$$C_*^\lambda(A) := C_*(A)/(1-t)$$

with the differential induced by b (and denoted by the same symbol), $H_*^\lambda(A) = H_*(C_*^\lambda(A),b)$.

The cyclic homology has many miraculous properties. For example, let $\mathfrak{gl}_r(A)$ be the Lie algebra of $r\times r$ matrices with coefficients in A, with the commutator bracket. Denote by

$$\mathfrak{gl}(A) := \varinjlim \mathfrak{gl}_r(A)$$

the direct limit of the inclusions $\mathfrak{gl}_r(A) \hookrightarrow \mathfrak{gl}_{r+1}(A)$ given, for $r \geq 1$, by

$$\mathfrak{gl}_r(A) \ni M \longmapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{r+1}(A).$$

Then the space of primitive elements of the *Chevalley-Eilenberg homology*

$$H_*^{\mathrm{CE}}(\mathfrak{gl}(A)) := Tor^{\mathcal{U}(\mathfrak{gl}(A))}(\mathbf{k}, \mathbf{k})$$

of $\mathfrak{gl}(A)$ is isomorphic to the shifted cyclic homology of A,

$$Prim\ H_*(\mathfrak{gl}(A)) \cong H_{*-1}^{\lambda}(A);$$

see [**LQ84**, Theorem 6.2]. This should be compared with the isomorphism of the primitives of the Hochschild homology of the infinite matrix general linear group $\mathrm{GL}(A)$ of A and the rationalized algebraic K-theory of A [**Qui70**],

$$Prim\ H_*(\mathrm{GL}(A)) \cong K_*(A) \otimes \mathbb{Q}.$$

This explains why the cyclic cohomology is sometimes also called the *additive algebraic K-theory*.

Another interesting property is the relation with the *de Rham cohomology* of a smooth algebra. Recall that, for a commutative unital $\mathbf{k}$-algebra A, the *module of differentials* Ω_A^1 is the left A-module defined as the quotient of $A \otimes A$ by the left A-submodule generated by $\{1 \otimes xy - x \otimes y - y \otimes x \mid x, y \in A\}$,

$$\Omega_A^1 := (A \otimes A)/\langle 1 \otimes xy - x \otimes y - y \otimes x \rangle. \tag{5.11}$$

It is characterized by the property that, for any left A-module M,

$$Der_{\mathbf{k}}(A, M) \cong Hom_A(\Omega_A^1, M),$$

where $Der_{\mathbf{k}}(A, M)$ is the linear space of derivations of the algebra A with values in M, and $Hom_A(\Omega_A^1, M)$ is the space of A-linear maps from Ω_A^1 to M. A direct verification shows that in fact Ω_A^1 coincides with the first Hochschild homology group of the algebra A, $\Omega_A^1 \cong HH_1(A; A)$.

The map $d : A \to \Omega_A^1$ given, in the description of (5.11), by $d(a) := 1 \otimes a$, is a derivation that corresponds to the identity map in $Hom_A(\Omega_A^1, \Omega_A^1)$. It is called the *universal differential* and extends to a degree 1 differential (denoted again by d) on the exterior A-linear algebra $\Omega_A^* := \wedge_A(\Omega_A^1)$ on Ω_A^1. The cohomology $H_{\mathrm{DR}}^*(A) := H^*(\Omega_A^*, d)$ is called the de Rham cohomology of A; see [**Bou**, 2.10].

The de Rham cohomology is related to the cyclic homology of a smooth algebra A in the sense of Grothendieck (a prominent example of this type of algebra is the coordinate ring of a smooth variety) by the formula [**LQ84**, Theorem 2.9]

$$H_n^{\lambda}(A) \cong \Omega_A^n / d\Omega_A^{n-1} \oplus H_{\mathrm{DR}}^{n-2}(A) \oplus H_{\mathrm{DR}}^{n-4}(A) \oplus \cdots$$

which holds for any $n \geq 0$. An interested reader may find all this and much more in the overview [**Car**].

Technically, the existence of the cyclic homology of an associative algebra is a consequence of the stunning fact that the action of the cyclic group on $C_*(A)$ behaves well with respect to the differentials, as expressed by (5.10). E. Getzler and M.M. Kapranov realized that this follows from the cyclicity of the operad $\mathcal{A}ss$ which governs associative algebras. They then defined the cyclic homology for an arbitrary category of algebras over a cyclic operad. Their definition is based on the notion of a *universal bilinear invariant form*.

Recall (Definition 5.13) that, given a $\mathcal{P}$-algebra A, $\alpha : \mathcal{P} \to \mathcal{E}nd_A$, over a cyclic operad $\mathcal{P}$, a bilinear form $B : A \otimes A \to U$ is invariant if and only if, for any $n \geq 1$,

$\sigma \in \Sigma_{n+1}$, $a_0, \ldots, a_1 \in A$ and $p \in \mathcal{P}(n)$,

$$(5.12) \qquad \begin{aligned}&(-1)^{|a_0||p|} B(a_0, \alpha(p)(a_1, \ldots, a_n)) \\ &= (-1)^{|a_{\sigma(0)}||p|} \epsilon(\sigma) B(a_{\sigma(0)}, \alpha(p)(a_{\sigma(1)}, \ldots, a_{\sigma(n)})),\end{aligned}$$

where $\epsilon(\sigma) = \epsilon(\sigma; a_0, \ldots, a_n)$ is the Koszul sign (3.96) of the permutation σ.

DEFINITION 5.22. Let A be an algebra over a cyclic operad $\mathcal{P}$, $\alpha : \mathcal{P} \to \mathcal{E}nd_A$. We say that an invariant bilinear form $T : A \otimes A \to \Lambda$ with values in a vector space Λ is *universal* if, for any invariant bilinear form $B : A \otimes A \to U$, there exist a unique linear map $\omega : \Lambda \to U$ such that $B = \omega \circ T$.

The universal bilinear invariant form for $\mathcal{P}$ can be constructed as follows. Take $\Lambda = \Lambda(\mathcal{P}, A)$ to be the coequalizer of the composite maps

$$\mathcal{P}(n) \otimes A^{\otimes(n+1)} \xrightarrow{\sigma} \mathcal{P}(n) \otimes A^{\otimes(n+1)} \xrightarrow{R_n} A \otimes A, \ n \geq 1, \ \sigma \in \Sigma_{n+1},$$

where

$$R_n(p \otimes a_0 \otimes \cdots \otimes a_n) := (-1)^{|a_0||p|} a_0 \otimes \alpha(p)(a_1, \ldots, a_n)$$

and Σ_{n+1} acts diagonally. In other words, $\Lambda(\mathcal{P}, A)$ is the quotient of the product $A \otimes A$ modulo the identifications

$$(-1)^{|a_0||p|} a_0 \otimes \alpha(p)(a_1, \ldots, a_n) \sim (-1)^{|a_{\sigma(0)}||p|} \epsilon(\sigma) a_{\sigma(0)} \otimes \alpha(p)(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

required by (5.12). Then we define $T : A \otimes A \to \Lambda$ to be the canonical projection. The universal property of the map $T : A \otimes A \to \Lambda(\mathcal{P}, A)$ is immediately obvious. For $\mathcal{P} = \mathcal{A}ss$, $\mathcal{C}om$ and $\mathcal{L}ie$, the space $\Lambda(\mathcal{P}, A)$ was introduced by M. Kontsevich [**Kon93**] who denoted it $\Omega^0(A)$.

EXAMPLE 5.23. If $\mathcal{P} = \mathcal{A}ss$ and $A = (A, \cdot)$ is a unital associative algebra, then $\Lambda = A/[A, A]$ is the abelianization A_{ab} of A and $T(a \otimes b) \in \Lambda$ is the equivalence class of $a \cdot b$ in A_{ab}.

If $\mathcal{P} = \mathcal{C}om$ and $C = (C, \cdot)$ is a commutative unital algebra, then $\Lambda = C$ and $T : C \otimes C \to C$ is the multiplication.

If $\mathcal{P} = \mathcal{P}oiss$ and $P = (P, \{-,-\}, \cdot)$ is a unital Poisson algebra (this means that there exists a unit $1 \in P$ for the multiplication such that $\{1, P\} = 0$), then $\Lambda(\mathcal{P}oiss, P) = P/\{P, P\}$ is the quotient of P by the span of all Poisson brackets.

For a given cyclic operad $\mathcal{P}$, the correspondence $A \longmapsto \Lambda(P, A)$ defines a functor from the category $\mathcal{P}$-$\mathtt{dgAlg}$ of differential graded $\mathcal{P}$-algebras to the category of differential graded vector spaces. The category $\mathcal{P}$-$\mathtt{dgAlg}$ is a Quillen model category [**Qui67**] and the functor $\Lambda(\mathcal{P}, -)$ preserves weak equivalences between cofibrations - a quite nontrivial fact proved in [**GK95**, Theorem 5.3]. Thus the following definition, due to E. Getzler and M.M. Kapranov, makes sense.

DEFINITION 5.24. Let A be an algebra over a cyclic operad $\mathcal{P}$. The *ith cyclic homology group $HA_i(A)$ of A* is the ith left nonabelian derived functor of the functor $\Lambda(\mathcal{P}, -)$, $HA_i(A) := H_i(\Lambda(\mathcal{P}, F_*))$, where F_* is a cofibrant model of the algebra A.

Since the theory of nonabelian derived functors would go far beyond the scope of the book, we will not discuss the cyclic homology in the sense of this general definition, but only for algebras over quadratic Koszul operads (Section 3.3), in which case we define the cyclic cohomology as the cohomology of a very explicit complex. From now on we assume that $\mathbf{k}$ is a field of characteristic zero.

So, let $\mathcal{P} = \langle E; R\rangle$ be a cyclic quadratic operad in the sense of Definition 5.15. Then also, by Lemma 5.16, the quadratic dual $\mathcal{P}^!$ of $\mathcal{P}$ is a cyclic quadratic operad, thus $\mathcal{P}^\perp := \mathfrak{s}(\mathcal{P}^!)^\#$, the operadic suspension (Definition 3.15) of the linear dual $(\mathcal{P}^!)^\#$ of the operad $\mathcal{P}^!$, is an anticyclic cooperad – an obvious object dual to an anticyclic operad recalled in Definition 5.20; compare also Remark 3.4. As a cyclic Σ-module,

$$\mathcal{P}^\perp(n) \cong \mathrm{sgn}_{n+1} \otimes \uparrow^{n-1} (\mathcal{P}^!(n))^\#,\ n \geq 1. \tag{5.13}$$

Let $\mathbf{k}^{n+1}$ denote the standard permutation representation of Σ_n^+ (see Definition 5.1 for the notation), i.e. the vector space of $(n+1)$-tuples $(k_0, \ldots, k_n)$ of elements of the ground field $\mathbf{k}$, with the left action of Σ_n^+ given by

$$\sigma \cdot (k_0, \ldots, k_n) = (k_{\sigma^{-1}(0)}, \ldots, k_{\sigma^{-1}(n)}).$$

Then the n-dimensional representation $V_{n,1}$ of Σ_n^+ can be described as

$$V_{n,1} = \{(k_0, \ldots, k_n) \in \mathbf{k}^{n+1} \mid k_0 + \cdots + k_n = 0\},$$

so there is an exact sequence

$$0 \longrightarrow V_{n,1} \longrightarrow \mathbf{k}^{n+1} \longrightarrow \mathbb{1} \longrightarrow 0 \tag{5.14}$$

of Σ_n^+-modules. Combining these data, for any $\mathcal{P}$-algebra A, there is a natural short exact sequence of chain complexes

$$0 \longrightarrow CA_*(A) \longrightarrow CB_*(A) \longrightarrow CC_*(A) \longrightarrow 0. \tag{5.15}$$

The underlying graded vector spaces of these complexes defined as follows:

$$\begin{aligned} CA_*(A) &:= \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_n^+} (V_{n,1} \otimes A^{\otimes(n+1)}), \\ CB_*(A) &:= \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_n^+} (\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)}) \text{ and} \\ CC_*(A) &:= \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_n^+} (\mathbb{1} \otimes A^{\otimes(n+1)}) \cong \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_n^+} A^{\otimes(n+1)}, \end{aligned}$$

with Σ_n^+ acting diagonally on the factors in parentheses on the right. The maps in (5.15) are induced by (5.14). We are going to define a differential δ on $CB_*(A)$ which preserves $CA_*(A)$ and projects to a differential on $CC_*(A)$.

For any quadratic operad $\mathcal{P} = \langle E; R\rangle$, there is a cyclic quadratic *twisting cochain* (an operadic analog of twisting cochains for associative algebras introduced in [**GK95**, Example 5.7]), i.e. the degree -1 map $\Psi : \mathcal{P}^\perp \to \mathcal{P}$ of Σ-modules given by

$$\Psi(2) : \mathcal{P}^\perp(2) = \uparrow E \xrightarrow{\downarrow} E = \mathcal{P}(2),$$

and $\Psi(n) = 0$, for $n \neq 2$.

By dualizing the construction in Remark 5.11 we see that structure maps of an (anti)cyclic cooperad $\mathcal{Q}$ written in terms of finite sets give, for each finite set K, a map

$$\mathcal{Q}((K)) \to \bigoplus_{K = L \sqcup M} \mathcal{Q}((L^+)) \otimes \mathcal{Q}((M^+)),$$

where $L^+ := L \sqcup \{M\}$ and $M^+ := M \sqcup \{L\}$. In particular, the cocomposition of the anticyclic cooperad $\mathcal{P}^\perp$ gives, for any decomposition $L \sqcup M = \{0, \dots, n\}$, a map

$$\mathcal{P}^\perp(n) \longrightarrow \mathcal{P}^\perp((L^+)) \otimes \mathcal{P}^\perp((M^+))$$

which induces a map

$$(5.16) \quad \mathcal{P}^\perp(n) \otimes \mathbf{k}^{n+1} \otimes A^{\otimes(n+1)} \longrightarrow (\mathcal{P}^\perp((L^+)) \otimes A^{\otimes L}) \otimes (\mathcal{P}^\perp((M^+)) \otimes A^{\otimes M}) \otimes \mathbf{k}^{n+1}.$$

In the above formula, we also moved the factor $\mathbf{k}^{n+1}$ to the right. This will formally simplify definitions of the maps below. Let us remark that $A^{\otimes L}$ and $A^{\otimes M}$ are unordered products introduced in (1.31). On the other hand, the twisting cochain Ψ induces a map

$$\Psi : \mathcal{P}^\perp((L^+)) \to \mathcal{P}((L^+))$$

(which, by definition, is nonzero only for $card(L) = 2$). This map, when composed with the structure map of the $\mathcal{P}$-algebra A,

$$\mathcal{P}((L^+)) \otimes A^{\otimes L} \cong \mathcal{P}(L) \otimes A^{\otimes L} \longrightarrow A,$$

gives a map

$$\mathcal{P}^\perp((L^+)) \otimes A^{\otimes L} \longrightarrow A$$

whose composition with (5.16) gives

$$\mathcal{P}^\perp(n) \otimes \mathbf{k}^{n+1} \otimes A^{\otimes(n+1)} \longrightarrow A \otimes \mathcal{P}^\perp((M^+)) \otimes A^{\otimes M} \otimes \mathbf{k}^{n+1}.$$

Moreover,

$$A \otimes \mathcal{P}^\perp((M^+)) \otimes A^{\otimes M} \otimes \mathbf{k}^{n+1} \cong \mathcal{P}^\perp(M) \otimes A^{\otimes M^+} \otimes \mathbf{k}^{n+1}.$$

The 'summation' $\Sigma : \mathbf{k}^L \to \mathbf{k}$ defined by $\Sigma(\oplus_{l \in L} k_l) := \sum_{l \in L} k_l$ induces the map

$$\mathbf{k}^{n+1} \cong \mathbf{k}^L \oplus \mathbf{k}^M \xrightarrow{\Sigma \oplus \mathbb{1}} \mathbf{k} \oplus \mathbf{k}^M \cong \mathbf{k}^{M^+}.$$

Composing all these maps, we obtain a map

$$\delta_M : \mathcal{P}^\perp(n) \otimes \mathbf{k}^{n+1} \otimes A^{\otimes(n+1)} \to \mathcal{P}^\perp(M) \otimes A^{\otimes M^+} \otimes \mathbf{k}^{M^+},$$

which is in fact nonzero only for $card(M) = n - 1$. Let us denote by $\tilde{\delta}_I$ the sum

$$(5.17) \qquad \mathcal{P}^\perp(n) \otimes \mathbf{k}^{n+1} \otimes A^{\otimes(n+1)} \xrightarrow{\sum_M \delta_M} \bigoplus_{card(M) = n-1} \mathcal{P}^\perp(M) \otimes A^{\otimes M^+} \otimes \mathbf{k}^{M^+}.$$

The group Σ_n^+ acts on both sides of (5.17) and $\tilde{\delta}_I$ is Σ_n^+-equivariant. Thus, it descends to a map (denoted again by $\tilde{\delta}_I$)

$$(5.18) \qquad \tilde{\delta}_I : \mathcal{P}^\perp(n) \otimes_{\Sigma_n^+} (\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)}) \longrightarrow \Big(\bigoplus_{card(M) = n-1} \mathcal{P}^\perp(M) \otimes A^{\otimes M^+} \otimes \mathbf{k}^{M^+} \Big)_{\Sigma_n^+}.$$

On the other hand, every subset M of $\{0, \dots, n\}$ is naturally ordered, thus there is a specified order $M \cong \{1, \dots, n\}$ for any M in the summation above, which means that there are identifications

$$\mathcal{P}^\perp(M) \otimes A^{\otimes M^+} \otimes \mathbf{k}^{M^+} \cong \mathcal{P}^\perp(n-1) \otimes (\mathbf{k}^n \otimes A^{\otimes n}).$$

These identifications induce the projection

$$\pi : \Big(\bigoplus_{card(M)=n-1} \mathcal{P}^{\perp}(M) \otimes A^{\otimes M^+} \otimes \mathbf{k}^{M^+} \Big)_{\Sigma_n^+} \longrightarrow \mathcal{P}^{\perp}(n-1) \otimes_{\Sigma_n} (\mathbf{k}^n \otimes A^{\otimes n}).$$

Composing π with the map $\tilde{\delta}_I$ of (5.18), we finally obtain a map δ_I (I from 'internal'),

$$\begin{aligned} \delta_I : CB_n(A) = \mathcal{P}^{\perp}(n) \otimes_{\Sigma_n^+} (\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)}) \\ \longrightarrow \mathcal{P}^{\perp}(n-1) \otimes_{\Sigma_n} (\mathbf{k}^n \otimes A^{\otimes n}) = CB_{n-1}(A). \end{aligned}$$

If $A = (A, d)$ is a *differential* graded algebra, then the differential d of A induces another differential δ_A on the complex $CB_*(A)$, by the standard formula

$$\begin{aligned} &\delta_A(p \otimes_{\Sigma_n^+} ((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n))) \\ &:= \sum_{0 \leq i \leq n} \eta\, p \otimes_{\Sigma_n^+} ((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n)), \end{aligned}$$

where $p \otimes_{\Sigma_n^+} ((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n)) \in CB_n(A)$ and

$$\eta := (-1)^{(|a_0| + \quad + |a_{i-1}| + |p|)}.$$

The following lemma, which is a special case of [**GK95**, Lemma 5.11] for quadratic twisting cochains, can be verified directly.

LEMMA 5.25. *The endomorphism $\delta := \delta_I + \delta_A$ is a differential on $CB_*(A)$. It preserves $CA_*(A)$ and projects to a differential (denoted again by δ) on $CC_*(A)$.*

We can now define the cyclic homology $HA_*(A)$ of A as the homology of the complex (CA_*, δ). The following theorem, which was proved in [**GK95**], shows that this definition is consistent with Definition 5.24.

PROPOSITION 5.26. *Let A be an algebra over a cyclic operad $\mathcal{P}$. Suppose $\mathcal{P}$ is cyclic quadratic and Koszul. Then the homology of the complex $(CA_*(-), \delta)$ calculates the left derived functor of $\Lambda(\mathcal{P}, -)$.*

If we denote the homology of the remaining two complexes in (5.15) by $HB_*(A)$ and $HC_*(A)$, respectively, then we have the long exact sequence

$$(5.19) \quad \cdots \to HC_{n+1}(A) \to HA_n(A) \to HB_n(A) \to HC_n(A) \to HA_{n-1}(A) \to \cdots$$

EXAMPLE 5.27. Let us identify the complexes of (5.15) for the associative algebra case, $\mathcal{P} = \mathcal{A}ss$. Recall that the operad $\mathcal{A}ss$ is self-dual, $\mathcal{A}ss^{!} = \mathcal{A}ss$, and observe that, as a Σ_n^+-module, $\mathcal{A}ss(n) = \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_n^+}(\mathbb{1})$, where $\mathbb{1}$ is the one-dimensional trivial representation of $\mathbb{Z}_{n+1}$ and $\mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_n^+}(\mathbb{1}) := \mathbb{1} \otimes_{\mathbb{Z}_{n+1}} \mathbf{k}[\Sigma_n^+]$ is the representation induced along $\mathbb{Z}_{n+1} \hookrightarrow \Sigma_n^+$. By (5.13) this means that

$$\mathcal{A}ss^{\perp}(n) = \mathrm{sgn}_{n+1} \otimes \uparrow^{n-1} \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_n^+}(\mathbb{1}) \cong \uparrow^{n-1} \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_n^+}(\mathbb{1}).$$

It follows from the definition of the induced representation that for any left Σ_n^+-module M

$$\mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_n^+}(\mathbb{1}) \otimes_{\Sigma_n^+} M \cong \mathbb{1} \otimes_{\mathbb{Z}_{n+1}} M \cong M_{\mathbb{Z}_{n+1}} \cong M/(1-t),$$

where $t \in \mathbb{Z}_{n+1}$ is the generator. This means that sequence (5.15) becomes, for $\mathcal{P} = \mathcal{A}ss$, the sequence

$$0 \longrightarrow \uparrow^{n-1} (V_{n,1} \otimes A^{\otimes(n+1)})/(1-t)$$
$$\longrightarrow \uparrow^{n-1} (\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)})/(1-t) \longrightarrow \uparrow^{n-1} A^{\otimes(n+1)}/(1-t) \longrightarrow 0.$$

The map $\alpha : \mathbf{k}^{n+1} \otimes A^{\otimes(n+1)} \longrightarrow A^{\otimes(n+1)}$ given by

$$\alpha((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n)) := (k_0 + k_1 t^{-1} + \cdots + k_n t^{-n})(a_0 \otimes \cdots \otimes a_n)$$

is invariant under the diagonal action of t:

$$\begin{aligned} \alpha\,(t((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n))) \\ &= ((k_0 + k_1 t^{-1} + \cdots + k_n t^{-n})t^{-1})(t(a_0 \otimes \cdots \otimes a_n)) \\ &= (k_0 + k_1 t^{-1} + \cdots + k_n t^{-n})(a_0 \otimes \cdots \otimes a_n) \\ &= \alpha((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n)), \end{aligned}$$

thus it descends to an isomorphism $(\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)})/(1-t) \cong A^{\otimes n+1}$ and converts the above sequence to

$$(5.20) \quad 0 \to \uparrow^{n-1} (1-t) A^{\otimes(n+1)} \xrightarrow{\iota} \uparrow^{n-1} A^{\otimes(n+1)} \xrightarrow{\pi} \uparrow^{n-1} A^{\otimes(n+1)}/(1-t) \to 0,$$

where ι is the inclusion and π the projection. If one goes through the definition of δ on the middle term of (5.20), one finds that it may be identified with the differential b of the shifted Hochschild complex $CH_*(A, A)$, thus

$$(CB_*(A), \delta) = (\downarrow C_*(A), b).$$

It is also immediately clear from this that the rightmost term of (5.20) is isomorphic to the shifted complex $C^\lambda_*(A)$ calculating the cyclic homology of A, $CC_*(A) = C^\lambda_{*+1}(A)$, therefore $HC_n(A) \cong H^\lambda_{n+1}(A)$. To identify the leftmost term of (5.20), consider the bicomplex in Figure 5 as an exact sequence of chain complexes

$$(5.21) \quad (C_*(A), b) \xleftarrow{(1-t)} (C_*(A), -b') \xleftarrow{N} (C_*(A), b) \xleftarrow{(1-t)} (C_*(A), -b') \xleftarrow{N} \cdots .$$

Consider also the short exact sequence

$$(5.22) \qquad 0 \longrightarrow \mathrm{Ker}(1-t) \hookrightarrow (C_*(A), -b') \xrightarrow{(1-t)} \mathrm{Im}(1-\mathrm{t}) \longrightarrow 0.$$

The exactness of (5.21) gives the isomorphisms

$$\mathrm{Ker}(1-t) \cong \mathrm{Im}(N) \cong \mathrm{Coker}(1-t) = C_*(A)/(1-t)$$

which enables us to identify the sequence (5.22) with

$$(5.23) \qquad 0 \longrightarrow (C^\lambda_*(A), b) \xrightarrow{N} (C_*(A), -b') \longrightarrow \uparrow CA_*(A) \longrightarrow 0.$$

PROPOSITION 5.28. *If the algebra A is unital, then the complex $CA_*(A)$ is weakly equivalent to $C^\lambda_*(A)$.*

PROOF. The proposition follows from the short exact sequence (5.23) and the contractibility of the bar complex $(C_*(A), b')$. ∎

To summarize, for a unital associative algebra A, both complexes $CA_*(A)$ and $CC_*(A)$ calculate the cyclic homology, while $CB(A)_*$ calculates the Hochschild homology. The long exact sequence (5.19) can then be identified with the *Connes-Tsygan periodicity exact sequence*

$$\cdots \longrightarrow H^{\lambda}_{n+2}(A) \xrightarrow{S} H^{\lambda}_{n}(A) \xrightarrow{B} H_{n+1}(A,A) \xrightarrow{I} H^{\lambda}_{n+1}(A) \longrightarrow \cdots$$

where the boundary map S is an additive analog of the Bott periodicity map. This sequence can also be interpreted as a consequence of the periodicity of the bicomplex in Figure 5. More precisely, observe that if we delete the first two columns of this bicomplex, we obtain an isomorphic complex, while the cohomology of the quotient is the Hochschild cohomology. These remarks also imply the sequence above.

EXAMPLE 5.29. Consider the Lie algebra case. Since, for any $n \geq 1$, $\mathcal{L}ie^{!}(n) = \mathcal{C}om(n) = 1\!\!1$, the one-dimensional trivial representation of Σ^{+}_{n}, we obtain $\mathcal{L}ie^{\perp}(n) \cong \uparrow^{n-1} \mathrm{sgn}_{n+1}$, the shifted signum representation of Σ^{+}_{n}. By similar methods to those in Example 5.27 we infer that, for any Lie algebra $\mathfrak{g}$,

$$CB_n(\mathfrak{g}) = \mathfrak{g}\otimes\uparrow^{n-1}\wedge^n\mathfrak{g}.$$

Thus $CB_*(\mathfrak{g})$ is identified with the shifted Chevalley-Eilenberg chain complex of the Lie algebra $\mathfrak{g}$ with coefficients in itself [**HS71**, Chapter VII], $CB_*(\mathfrak{g}) \cong\downarrow C_n(\mathfrak{g},\mathfrak{g})$. Similarly,

$$CC_*(\mathfrak{g}) \cong \uparrow^{n-1}\wedge^{n+1}\mathfrak{g} = \downarrow^2 C_*(\mathfrak{g},\mathbf{k}),$$

the shifted Chevalley-Eilenberg chain complex of $\mathfrak{g}$ with trivial coefficients. As a consequence, $CA_*(\mathfrak{g})$ is the kernel of the map $m : C_{*+1}(\mathfrak{g},\mathfrak{g}) \to C_{*+2}(\mathfrak{g},\mathbf{k})$ given by the formula

$$m(g_0\otimes(g_1 \wedge \cdots \wedge g_n)) := g_0 \wedge g_1 \wedge \cdots \wedge g_n.$$

The map m was studied by T. Pirashvili in [**Pir94**]. He denoted

$$CR_n(\mathfrak{g}) := \mathrm{Ker}(m : C_{n+1}(\mathfrak{g},\mathfrak{g}) \to C_{n+2}(\mathfrak{g},\mathbf{k}))$$

and $HR_*(\mathfrak{g})$ the corresponding homology. The dimension shift is chosen because the map m is an isomorphism in degree 0, thus, with this degree convention, $CR_0(\mathfrak{g})$ is the first nontrivial piece of the complex. We may thus identify $CA_*(\mathfrak{g}) = CR_*(\mathfrak{g})$. The associated long exact sequence

$$\cdots \longrightarrow HR_{n-1}(\mathfrak{g}) \longrightarrow H^{\mathrm{CE}}_{n}(\mathfrak{g},\mathfrak{g}) \longrightarrow H^{\mathrm{CE}}_{n+1}(\mathfrak{g},\mathbf{k}) \longrightarrow HR_n(\mathfrak{g}) \longrightarrow \cdots$$

is that of [**Pir94**, Proposition 1.2].

EXAMPLE 5.30. Let us briefly mention also the case of commutative associative algebras. If C is such an algebra, then it can be shown that the complex $(CA_*(C),\delta)$ calculates the shifted *Harrison homology* of C with trivial coefficients,

$$HA_*(C) \cong Harr_{*+1}(C,\mathbf{k})$$

and that

$$HC_*(C) := Harr_{*+1}(C,C),$$

the Harrison cohomology of C with coefficients in itself. The exact sequence (5.19) now takes the form

$$\cdots \longrightarrow Harr_{n+2}(C,C) \longrightarrow Harr_{n+1}(C,\mathbf{k}) \longrightarrow HB_n(C) \longrightarrow Harr_{n+1}(C,C) \longrightarrow \cdots$$

We refer to the original paper [**GK95**, Example 6.5] for the details of this example.

REMARK 5.31. We defined the cyclic homology of A as the homology of the complex $CA_*(A)$. It is not difficult to see that the complex $CB_*(A)$ calculates the operadic homology [**Bal98**] of the algebra A with coefficients in A.

The interpretation of $CC_*(A)$ is more subtle. For a $\mathcal{P}$-algebra A, the bar construction $C^*_{\mathcal{P}}(A) = (C^*_{\mathcal{P}}(A), \partial_{\mathcal{P}})$ is a dg $\mathcal{P}^!$-coalgebra (see Definition 3.91 and recall that we assume $\mathcal{P}$ to be Koszul). Our definitions can easily be dualized to introduce the cyclic homology for (differential) *co*algebras. Then $HC_*(A)$ is the cyclic cohomology of the differential $\mathcal{P}^!$-coalgebra $C^*_{\mathcal{P}}(A)$. The details can be found in [**GK95**, 6.12].

REMARK 5.32. We may consider a unit of an algebra either as an extra structure or build it into the governing operad. Thus we have, for example, the operad $\mathcal{U}\mathcal{A}ss$ for associative algebras with unit, which coincides with $\mathcal{A}ss$ with the exception of $\mathcal{U}\mathcal{A}ss(0)$ which is $\mathbf{k}$ and, similarly, the operads $\mathcal{U}\mathcal{C}om$ and $\mathcal{U}\mathcal{P}oiss$. There is a parallel cyclic homology theory for these operads. This theory agrees with the theory presented here except for the lowest term, thus we will not treat it separately here. For more details, see [**GK95**].

5.3. Modular operads

Let us consider again the Σ^+-module $\widehat{\mathcal{M}}_0 = \{\widehat{\mathcal{M}}_0(n)\}_{n\geq 1}$ of Riemann spheres with holes. Recall (Section I.1.16) that $\widehat{\mathcal{M}}_0(n)$ denotes the moduli space of objects $(C; f_0, \ldots, f_n)$, where C is a complex manifold biholomorphic to $\mathbb{CP}^1$ and f_i are biholomorphic maps from the unit disk D^2 into C with disjoint images. We saw in Section 5.1 that the operation $M, N \mapsto M_i\circ_j N$ of sewing the ith hole of the surface M to the jth hole of the surface N, defined on $\widehat{\mathcal{M}}_0$ a cyclic operad structure.

In the same manner, we may consider a single surface $M \in \widehat{\mathcal{M}}_0(n)$, choose labels i, j, $0 \leq i \neq j \leq n$, and sew the ith hole of M along the jth hole of the *same* surface. The result is a new surface $\xi_{ij}(M)$, with $n-2$ holes and genus 1.

This leads us to consider the system $\widehat{\mathcal{M}} = \{\widehat{\mathcal{M}}\}_{g\geq 0, n\geq -1}$, where $\widehat{\mathcal{M}}(g,n)$ denotes now the moduli space of genus g Riemann surfaces with $n+1$ holes. The above two families of operations act on $\widehat{\mathcal{M}}$. Clearly, for $M \in \widehat{\mathcal{M}}(g,m)$ and $N \in \widehat{\mathcal{M}}(h,n)$,

$$M_i\circ_j N \in \widehat{\mathcal{M}}(g+h, m+n-1)$$

and

$$\xi_{ij}(M) \in \widehat{\mathcal{M}}(g+1, m-2).$$

Observe that it makes sense to consider $\widehat{\mathcal{M}}(g,n)$ also for $n = 0$ and $n = -1$; see Remark 5.21.

Modular operads are abstractions of the above structure. Our exposition follows [**GK98**]. The following definitions are made for the category of graded vector spaces, but they can be generalized to an arbitrary monoidal category with finite limits and colimits.

DEFINITION 5.33. A *modular Σ-module* is a sequence $\mathcal{E} = \{\mathcal{E}(g,n)\}_{g\geq 0, n\geq -1}$ of graded vector spaces such that each $\mathcal{E}(g,n)$ has, for $n \geq 0$, a right Σ_n^+-action. We say that the modular Σ-module $\mathcal{E}$ is *stable* if

$$\mathcal{E}(g,n) = 0 \text{ for } 2g + n - 1 \leq 0. \tag{5.24}$$

Let MMod denote the category of stable modular Σ-modules. For $e \in \mathcal{E}(g,n)$, we call the number n the *arity* and the number g the *genus* of the element e. We also sometimes call $n+1$ the *valence* of $e \in \mathcal{E}(g,n)$.

Stability (5.24) says that $\mathcal{E}(g,n)$ is trivial for $(g,n) = (0,-1)$, $(1,-1)$, $(0,0)$ and $(0,1)$. We will sometimes stress the stability of $\mathcal{E}$ by writing $\mathcal{E} = \{\mathcal{E}(g,n)\}_{(g,n)\in\mathfrak{S}}$, where

$$\mathfrak{S} := \{(g,n) \mid g \geq 0,\ n \geq -1 \text{ and } 2g+n-1 > 0\}.$$

Recall that a genus g Riemann surface with k punctures is called stable if $2(g-1)+k > 0$. Thus the stability property of modular Σ-modules is analogous to the stability of Riemann surfaces, that is, excluded is the torus with no marked points and the sphere with less than three marked points.

REMARK 5.34. Our notation differs from that of **[GK98]** where our $\mathcal{E}(g,n)$ would be denoted $\mathcal{E}((g,n+1))$. Therefore also the form of our stability condition (5.24) formally differs from the one in **[GK98]**.

Thus a modular Σ-module is a cyclic Σ-module (Definition 5.4) with an extra grading given by the genus. More precisely, each modular Σ-module $\mathcal{E}$ determines a cyclic Σ-module $\mathcal{E}^\flat = \{\mathcal{E}^\flat(n)\}_{n\geq -1}$ by

$$\mathcal{E}^\flat(n) := \bigoplus_{g\geq 0} \mathcal{E}(g,n).$$

As before, the extension functor (5.1) enables us to work with $\mathcal{E}((g,S))$ for an abitrary finite set S; see also Remark 5.6 where the meaning of double brackets $((-))$ is explained. By a cyclic (pseudo-)operad we mean in this section an 'extended' cyclic (pseudo-)operad in the sense of Remark 5.21.

DEFINITION 5.35. A *modular operad* is a stable modular Σ-module

$$\mathcal{A} = \{\mathcal{A}(g,n)\}_{(g,n)\in\mathfrak{S}}$$

together with a cyclic pseudo-operad structure on the cyclic Σ-module

$$\mathcal{A}^\flat = \{\mathcal{A}^\flat(n)\}_{n\geq -1}, \tag{5.25}$$

homogeneous with respect to the grading given by the genus, and a family of 'contractions'

$$\xi_{ij} : \mathcal{A}((g,S)) \to \mathcal{A}((g+1, S - \{i,j\})),$$

defined for each finite set S and distinct $i,j \in S$. The contractions are equivariant, meaning that for any $a \in \mathcal{A}((g,S))$ and any automorphism σ of S such that $\sigma(\{i,j\}) = \{i,j\}$,

$$\xi_{\sigma(i)\sigma(j)}(a) \cdot \sigma_{ij} = \xi_{ij}(a \cdot \sigma),$$

where σ_{ij} is the restriction $\sigma|_{S-\{i,j\}}$. We also require that, for any distinct labels $i,j,k,l \in S$, the contractions ξ_{ij} and ξ_{kl} mutually commute,

$$\xi_{ij} \circ \xi_{kl} = \xi_{kl} \circ \xi_{ij}. \tag{5.26}$$

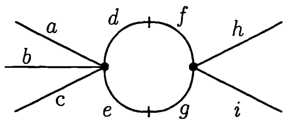


FIGURE 6. The sputnik.

The contractions must be compatible with the cyclic operad structure $\{_x\circ_y\}$ on $\mathcal{A}^\flat$ in the sense that, for any two finite sets S, T, for any $x \in S$, $y \in T$, for any distinct $i, j \in S \sqcup T - \{x, y\}$ and any $a \in \mathcal{A}((g, S))$, $b \in \mathcal{A}((h, T))$,

$$\xi_{ij}(a_x\circ_y b) = \begin{cases} (\xi_{ij}(a))_x\circ_y b, & \text{for } i, j \in S, \\ a_x\circ_y(\xi_{ij}(b)), & \text{for } i, j \in T, \text{ and} \\ \xi_{xy}(a_i\circ_j b), & \text{for } i \in S,\ j \in T. \end{cases} \tag{5.27}$$

REMARK 5.36. Observe that the stability condition is build into the definition of modular operads. Very crucially, modular operads do not have a unit, because such a unit ought to be an element of the space $\mathcal{A}(0, 1)$ which is empty, by (5.24). So a better name would probably be modular *pseudo-operads*, but we will respect the terminology introduced in [**GK98**].

As we will see below, the stability condition is necessary for having control over the size of the free modular operad functor.

We saw in Theorem 1.105 that ordinary operads were algebras over the triple of rooted trees while cyclic operads were algebras over the triple of unrooted trees (Theorem 5.8). We now show that also modular operads are algebras over a certain triple of graphs. The naive notion of a graph as we have used it up to this point is not subtle enough; we need to replace it by a more sophisticated notion.

DEFINITION 5.37. A *graph* Γ is a finite set $Flag(\Gamma)$ (whose elements are called flags or half-edges) together with an involution σ and a partition λ.

The *vertices* $Vert(\Gamma)$ of a graph Γ are the blocks of the partition λ. The *edges* $edge(\Gamma)$ are pairs of flags forming a two-cycle of σ relative to the decomposition of a permutation into disjoint cycles. The *legs* $Leg(\Gamma)$ are the fixed points of σ.

We also denote by $Leg(v)$ the flags belonging to the block v or, in common speech, half-edges adjacent to the vertex v. We say that two flags $x, y \in Flag(\Gamma)$ *meet* if they belong to the same block of the partition λ. In plain language, this means that they share a common vertex.

We may associate to a graph Γ a finite one-dimensional cell complex $|\Gamma|$, obtained by taking one copy of $[0, \frac{1}{2}]$ for each flag and imposing the following equivalence relation: The points $0 \in [0, \frac{1}{2}]$ are identified for all flags in a block of the partition λ and the points $\frac{1}{2} \in [0, \frac{1}{2}]$ are identified for pairs of flags exchanged by the involution σ. We call $|\Gamma|$ the *geometric realization* of the graph. We will sometimes make no distinction between the graph in the sense of Definition 5.37 and its geometric realization.

As an example (taken from [**GK98**]), consider the graph with $\{a, b, \ldots, i\}$ as the set of flags, the involution $\sigma = (df)(eg)$ and the partition $\{a, b, c, d, e\} \cup \{f, g, h, i\}$. The geometric realization is then the 'sputnik' in Figure 6.

Intuitively, a morphism $f : \Gamma_0 \to \Gamma_1$ of graphs is given by a permutation of vertices, followed by a contraction of some edges of the graph Γ_0, leaving the legs untouched. Translated into the language of Definition 5.37, this means an injection $f^* : Flag(\Gamma_1) \to Flag(\Gamma_0)$ that commutes with the involutions. Moreover, the involution σ_0 of Γ_0 must act freely on the complement of the image of f^* in $Flag(\Gamma_0)$ (i.e. the legs of the graphs are preserved by the map f) and two flags a and b in Γ_1 meet either if $f^*(a)$ and $f^*(b)$ meet in Γ_0 or there is a chain of flags $(x_1, \dots, x_k)$ in the complement $Flag(\Gamma_0) - f^*(Flag(\Gamma_1))$ of f^* in $Flag(\Gamma_0)$ such that $f^*(a)$ meets x_1, $\sigma_0 x_i$ meets x_{i+1}, for $1 \le i \le k-1$, and $\sigma_0 x_k$ meets $f^*(b)$ (i.e. there exists a chain of edges in Γ_0 from a to b).

A morphism $f : \Gamma_0 \to \Gamma_1$ clearly defines a surjective cellular map $|f| : |\Gamma_0| \to |\Gamma_1|$ of geometric realizations, which is bijective on the legs.

There is a special class of morphisms which are given by contracting a subset of edges, without permuting the vertices. First of all, for any subset I of $edge(\Gamma)$, there is a unique graph Γ/I such that $Flag(\Gamma/I)$ is obtained from $Flag(\Gamma)$ by deleting the flags constituting the edges in I and combining blocks of the partition that contain flags connected by a chain in I. Then the inclusion $Flag(\Gamma/I) \hookrightarrow Flag(\Gamma)$ is a morphism of graphs, which we denote by $\pi_I : \Gamma \to \Gamma/I$. An important special case is when I consists of a single edge e. We then simplify our notation by writing Γ/e instead of $\Gamma/\{e\}$ and π_e instead of $\pi_{\{e\}}$.

The graph Γ/I is called the *contraction of* Γ along the set of edges I. Any morphism $f : \Gamma \to \Gamma'$ of graphs is isomorphic to a morphism of this form. This means that there exists a subset $I \subset edge(\Gamma)$ and an isomorphism $\phi : \Gamma/I \to \Gamma'$ such that the following diagram of graph maps commutes:

$$\begin{tikzcd} & \Gamma/I \arrow[dr, "\phi"] & \\ \Gamma \arrow[ur, "\pi_I"] \arrow[rr, "f"'] & & \Gamma' \end{tikzcd}$$

Let us introduce labeled versions of the above objects. A *labeled graph* is a connected graph Γ together with a map g from $Vert(\Gamma)$ to the set $\{0, 1, 2, \dots\}$. In other places in the book we consider also graphs with labeled edges, like the metrized graphs of Section 5.5, but we believe that our terminology will cause no confusion.

The *genus* $g(\Gamma)$ of a labeled graph Γ is defined by the formula

$$g(\Gamma) := b_1(\Gamma) + \sum_{v \in Vert(\Gamma)} g(v), \tag{5.28}$$

where $b_1(\Gamma) := \dim H_1(|\Gamma|)$ is the first Betti number of the graph $|\Gamma|$, i.e. the number of independent circuits of Γ. The classical Euler formula says that $|Vert(\Gamma)| - |edge(\Gamma)| = 1 - b_1(\Gamma)$, therefore

$$b_1(\Gamma) = 1 - |Vert(\Gamma)| + |edge(\Gamma)|.$$

This easily implies the following expression for $g(\Gamma)$:

$$g(\Gamma) = \sum_{v \in Vert(\Gamma)} (g(v) - 1) + |edge(\Gamma)| + 1. \tag{5.29}$$

Let us derive another useful formula involving the genus. Since a flag forms either a leg or half an edge,

$$\sum_{v\in \mathit{Vert}(\Gamma)} |\mathit{Leg}(v)| = 2|\mathit{edge}(\Gamma)| + |\mathit{Leg}(\Gamma)|. \tag{5.30}$$

Combining this with (5.29) we derive that

$$2(g(\Gamma)-1) + |\mathit{Leg}(\Gamma)| = \sum_{v\in \mathit{Vert}(\Gamma)} \Big(2(g(v)-1) + |\mathit{Leg}(v)|\Big). \tag{5.31}$$

Similarly, adding three times equation (5.29) to equation (5.30) gives the equation

$$3(g(\Gamma)-1) + |\mathit{Leg}(\Gamma)| = |\mathit{edge}(\Gamma)| + \sum_{v\in \mathit{Vert}(\Gamma)} (3(g(v)-1) + |\mathit{Leg}(v)|). \tag{5.32}$$

To modify the notion of a morphism to the category of labeled graphs, we need to observe that the preimage $f^{-1}(v)$ of a vertex $v \in \mathit{Vert}(\Gamma_1)$ under a morphism $f : \Gamma_0 \to \Gamma_1$ is the graph consisting of those flags in Γ_0 that are connected to a flag in $\mathit{Leg}(v)$ by a chain of edges in Γ_0 contracted by the morphism.

A *morphism* $f : \Gamma_0 \to \Gamma_1$ *of labeled graphs* is a morphism of the underlying graphs such that the genus $g(v)$ of a vertex v of Γ_1 is equal to the genus of the graph $f^{-1}(v)$, for each $v \in \mathit{Vert}(\Gamma_1)$.

DEFINITION 5.38. A graph Γ is called *stable* if

$$2(g(v)-1) + |\mathit{Leg}(v)| > 0,$$

at each vertex $v \in \mathit{Vert}(\Gamma)$.

For a finite set S, let $\mathbf{\Gamma}((g,S))$ be the category whose objects are pairs (Γ,ρ) consisting of a stable labeled graph Γ of genus g and an isomorphism $\rho : \mathit{Leg}(\Gamma) \to S$ labeling the legs of Γ by elements of S. If $S = \{0,\dots,n\}$, then we will write simply $\mathbf{\Gamma}(g,n)$ instead of $\mathbf{\Gamma}((g,S))$. Morphisms of $\mathbf{\Gamma}((g,S))$ are morphisms of labeled graphs preserving the labelling of the legs. The category $\mathbf{\Gamma}((g,S))$ has a terminal object $*_{g,S}$, the 'modular corolla' with no edges, one vertex v of genus g and legs labeled by S. The category of stable graphs $\mathbf{\Gamma}((g,S))$ has the property that, for each fixed $g \geq 0$ and a finite set S, there is only a finite number of isomorphism classes of stable graphs $\Gamma \in \mathbf{\Gamma}((g,S))$ [**GK98**, Lemma 2.16].

For a modular Σ-module $\mathcal{E} = \{\mathcal{E}(g,n)\}_{g\geq 0, n\geq -1}$ and a labeled graph Γ, let $\mathcal{E}((\Gamma))$ be the unordered tensor product

$$\mathcal{E}((\Gamma)) := \bigotimes_{v\in \mathit{Vert}(\Gamma)} \mathcal{E}((g(v), \mathit{Leg}(v))). \tag{5.33}$$

For a category $\mathcal{D}$, let $\mathit{Iso}(\mathcal{D})$ denote the subcategory all of whose morphisms are isomorphisms. Evidently, the correspondence $\Gamma \mapsto \mathcal{E}((\Gamma))$ defines a functor from the category $\mathit{Iso}(\mathbf{\Gamma}((g,S)))$ to the category of vector spaces and their isomorphisms. We may thus define an endofunctor $\mathbb{M}$ on the category $\mathtt{MMod}$ of modular Σ-modules by the formula

$$\mathbb{M}\mathcal{E}((g,S)) = \operatorname*{colim}_{\Gamma\, \in\, \mathit{Iso}\,\mathbf{\Gamma}((g,S))} \mathcal{E}((\Gamma)).$$

This, by definition, means that the space $\mathbb{M}\mathcal{E}((g,S))$ is the quotient of the space $\bigoplus_{\Gamma\in\mathbf{\Gamma}((g,S))} \mathcal{E}((\Gamma))$ modulo the identifications $\mathcal{E}((\Gamma_0)) \ni x \sim \mathcal{E}((f))(x) \in \mathcal{E}((\Gamma_1))$, for any isomorphism $f : \Gamma_0 \to \Gamma_1$; see Example 1.76. Choosing a representative Γ_γ for any isomorphism class γ in $\mathbf{\Gamma}((g,S))$, we obtain a noncanonical identification

$$\mathbb{M}\mathcal{E}((g,S)) \cong \bigoplus_{\gamma\in\{\mathbf{\Gamma}((g,S))\}} \mathcal{E}((\Gamma_\gamma))_{\mathrm{Aut}(\Gamma_\gamma)}, \tag{5.34}$$

where $\{\mathbf{\Gamma}((g,S))\}$ denotes the set of isomorphism classes of objects of the category $\mathbf{\Gamma}((g,S))$ and the subscript $\mathrm{Aut}(\Gamma_\gamma)$ denotes the space of coinvariants. Stability (5.24) implies that the summation in the right-hand side of (5.34) is, for each S and g, finite. Formula (5.34) generalizes (1.52) which does not contain coinvariants because there are no nontrivial automorphisms of labeled trees.

We will show that the functor $\mathbb{M}$ is a triple in the sense of Definition 1.102. Our arguments here, as everywhere in this section, closely follow [**GK98**].

Recall that the *nerve* of a category $\mathcal{C}$ is a simplicial category $\mathtt{Nerve}_\bullet(\mathcal{C})$ whose category of k-simplices, $\mathtt{Nerve}_k(\mathcal{C})$, is the category of diagrams

$$(f_1,\ldots,f_k) = [\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} \Gamma_{k-1} \xrightarrow{f_k} \Gamma_k]$$

in $\mathcal{C}$, while the morphisms are morphisms of such diagrams. The face functors

$$\partial_i : \mathtt{Nerve}_k(\mathcal{C}) \to \mathtt{Nerve}_{k-1}(\mathcal{C})$$

are given, for $0 \le i \le k$, by the usual formulas

$$\partial_i(f_1,\ldots,f_k) =: \begin{cases} (f_2,\ldots,f_k), & i=0,\\ (f_1,\ldots,f_{i+1}f_i,\ldots,f_k), & 1\le i\le k-1,\\ (f_1,\ldots,f_{k-1}), & i=k. \end{cases} \tag{5.35}$$

For $k=1$ we interpret the above formulas as $\partial_0(\Gamma_0 \xrightarrow{f} \Gamma_1) := \Gamma_1$ and $\partial_1(\Gamma_0 \xrightarrow{f} \Gamma_1) := \Gamma_0$. Similarly, the degeneracy functors $\sigma_j : \mathtt{Nerve}_k(\mathcal{C}) \to \mathtt{Nerve}_{k+1}(\mathcal{C})$ are, for $0 \le j \le k$, given by

$$\sigma_j(f_1,\ldots,f_k) := (f_1,\ldots,f_j,\mathbb{1}_{\Gamma_j},f_{j+1},\ldots,f_k).$$

Finally, let $\mathit{Iso}_*(\mathcal{C})$ denote the simplicial category $\mathit{Iso}(\mathtt{Nerve}_\bullet(\mathcal{C}))$. The following proposition is due to Getzler and Kapranov [**GK98**].

PROPOSITION 5.39. *For each $k \ge 0$, the $(k+1)$th iterate of the functor $\mathbb{M}$ is given by*

$$(\mathbb{M}^{k+1}\mathcal{E})((g,S)) = \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \cdot \quad \xrightarrow{f_k}\Gamma_k] \,\in\, \mathit{Iso}_k\Gamma((g,S))} \mathcal{E}((\Gamma_0)). \tag{5.36}$$

PROOF. We prove the proposition by induction. For $k=0$, it is the definition. Fix an $m \ge 1$ and suppose we have proved (5.36) for $k < m$. Let us prove it for $k=m$.

For $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \Gamma_{m-1} \xrightarrow{f_m} \Gamma_m] \in \mathit{Iso}_m(\mathbf{\Gamma}((g,S)))$ and $u \in \mathit{Vert}(\Gamma_m)$, let

$$[\Gamma^u_0 \xrightarrow{f^u_1} \Gamma^u_1 \xrightarrow{f^u_2} \cdots \xrightarrow{f^u_{m-1}} \Gamma^u_{m-1}] \in \mathit{Iso}_{m-1}(\mathbf{\Gamma}((g(u),\mathit{Leg}(u)))) \tag{5.37}$$

be defined by $\Gamma^u_i =: (f_m \cdots f_{i+1})^{-1}(u)$, while the maps $f^u_i : \Gamma^u_{i-1} \to \Gamma^u_i$ are the restrictions, $0 \le i \le m-1$. Then we have, for the colimit over $[\Gamma_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} \Gamma_m] \in$

$Iso_m\mathbf{\Gamma}((g,S))$,

$$\begin{aligned}\mathcal{E}((\Gamma_0)) &= \operatorname{colim}\left\{\bigotimes_{v\in Vert(\Gamma_0)} \mathcal{E}((g(v), Leg(v)))\right\}\\ &= \operatorname{colim}\left\{\bigotimes_{u\in Vert(\Gamma_m)}\ \bigotimes_{v\in Vert(\Gamma_0^u)} \mathcal{E}((g(v), Leg(v)))\right\} \qquad (5.38)\\ &= \operatorname{colim}\left\{\bigotimes_{u\in Vert(\Gamma_m)} \mathcal{E}((\Gamma_0^u))\right\}.\end{aligned}$$

Observe now that the correspondence

$$[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \Gamma_{m-1} \xrightarrow{f_m} \Gamma_m] \longmapsto \Gamma_m \times \mathop{\times}_{u\in Vert(\Gamma_m)} [\Gamma_0^u \xrightarrow{f_1^u} \Gamma_1^u \xrightarrow{f_2^u} \cdots \xrightarrow{f_{m-1}^u} \Gamma_{m-1}^u]$$

defines an isomorphism of categories,

$$Iso_m\mathbf{\Gamma}((g,S)) \cong \coprod_{\Gamma\,\in\, Iso\,\mathbf{\Gamma}((g,S))}\ \mathop{\times}_{u\in Vert(\Gamma_n)} Iso_{m-1}\mathbf{\Gamma}((g(u), Leg(u))).$$

So, we can rewrite the last expression of (5.38) as

$$\operatorname*{colim}_{\Gamma_m\,\in\, Iso\,(\mathbf{\Gamma}((g,S)))}\left\{\bigotimes_{u\in Vert(\Gamma_m)}\left[\operatorname*{colim}_{[\Gamma_0^u \xrightarrow{f_1^u} \quad \xrightarrow{f_{m-1}^u} \Gamma_{m-1}^u]\,\in\, Iso_{k-1}\Gamma((g(u),Leg(u)))} \mathcal{E}((\Gamma_0^u))\right]\right\}$$

$$= \operatorname*{colim}_{\Gamma_m\,\in\, Iso\,(\mathbf{\Gamma}((g,S)))}\left\{\bigotimes_{u\in Vert(\Gamma_m)} \mathbb{M}^m\mathcal{E}((g(v), Leg(u)))\right\} = \mathbb{M}^{m+1}\mathcal{E}((g,S)). \qquad \blacksquare$$

It might be helpful to compare the construction of Proposition 5.39 with the iteration of the tensor algebra triple for associative **k**-algebras, where there is a 'nesting' of tensor algebras: tensor algebra of tensor algebra of tensor algebra ...

An m-simplex $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \Gamma_{m-1} \xrightarrow{f_m} \Gamma_m]$ defines a similar nesting of tensor products given by the nested chains of subsets of the vertices of Γ_0:

$$(f_m \cdots f_1)^{-1}(u_m) \supset (f_{m-1}\cdots f_1)^{-1}(u_{m-1}) \supset \cdots f_1^{-1}(u_1) \supset \{u_0\},$$

where $u_m \in Vert(\Gamma_m)$, $u_{m-1} \in f_m^{-1}(u_m)$, ..., $u_1 \in f_2^{-1}(u_2)$ and $u_0 \in f_1^{-1}(u_1)$.

Let us recall (Definition 1.102) that a triple in a category $\mathcal{C}$ is an endofunctor T on $\mathcal{C}$ together with natural transformations $\mu : T^2 \to T$ and $v : \mathbb{1}_{\mathcal{C}} \to T$ such that (T,μ,v) is an associative monoid in the category of endofunctors of $\mathcal{C}$.

Define $\mu : \mathbb{M}^2\mathcal{E} \to \mathbb{M}\mathcal{E}$ to be the transformation induced by $\partial_1 : Iso_1\mathbf{\Gamma}((g,S)) \to Iso_0\mathbf{\Gamma}((g,S))$, which sends $[\Gamma_0 \xrightarrow{f_1} \Gamma_1]$ to Γ_0. The unit $v : \mathcal{E} \to \mathbb{M}\mathcal{E}$ is defined as the inclusion of the summand $\mathcal{E}((*_{g,S})) \cong \mathcal{E}((g,S))$ of $\mathbb{M}\mathcal{E}((g,S))$ associated to the corolla $*_{g,S}$.

THEOREM 5.40. *The transformations* $\mu : \mathbb{M}^2 \to \mathbb{M}$ *and* $v : \mathbb{1}_{\mathsf{MMod}} \to \mathbb{M}$ *defined above form a triple* $(\mathbb{M},\mu,v)$ *in the category of stable modular* Σ*-modules.*

PROOF. The natural transformations

$$\mathbb{M}\mu \text{ and } \mu\mathbb{M} : (\mathbb{M}^3\mathcal{E})((g,S)) \to (\mathbb{M}^2\mathcal{E})((g,S))$$

are clearly induced by the functors $\partial_1, \partial_2 : \mathit{Iso}_2\mathbf{\Gamma}((g,S)) \to \mathit{Iso}_1\mathbf{\Gamma}((g,S))$, which send the sequence $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$ respectively to $\Gamma_0 \xrightarrow{f_2 f_1} \Gamma_2$ and $\Gamma_0 \xrightarrow{f_1} \Gamma_1$. In the same manner, the natural transformations $\mu(\mu\mathbb{M})$ and $\mu(\mathbb{M}\mu)$ are induced respectively by $\partial_1\partial_2$ and $\partial_1\partial_1$. Since $\partial_1\partial_2 = \partial_1\partial_1$ as must be in any simplicial category (in fact, both compositions send $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$ to Γ_0), $\mu(\mu\mathbb{M}) = \mu(\mathbb{M}\mu)$ which proves the associativity of μ. To see that v is a unit, i.e. that $\mu(v\mathbb{M}) = \mu(\mathbb{M}v) = \mathbb{M}$, is immediate. ∎

An algebra over the triple $(\mathbb{M}, \mu, v)$ is, by Definition 1.103, a stable modular Σ-module $\mathcal{A} = \{\mathcal{A}(g,n)\}_{(g,n)\in\mathfrak{S}}$ together with a morphism of modular Σ-modules

$$\alpha : \mathbb{M}(\mathcal{A}) \to \mathcal{A}$$

satisfying

$$\alpha \circ \mathbb{M}(\alpha) = \alpha \circ \mu_{\mathcal{A}} \text{ and} \tag{5.39}$$

$$\alpha v_{\mathcal{A}} = 1\!\!1_{\mathcal{A}}. \tag{5.40}$$

The following theorem is a modular version of Theorem 1.105 or Theorem 5.8.

THEOREM 5.41. *Modular operads are algebras over the triple* $(\mathbb{M}, \mu, v)$ *in the category of stable modular* Σ*-modules.*

Before proving the theorem, we formulate a technical proposition. Suppose we have a map $\alpha : \mathbb{M}\mathcal{A} \to \mathcal{A}$ of modular Σ-modules satisfying (5.40), but not necessarily (5.39). Each such map defines, for a graph $\Gamma \in \mathbf{\Gamma}((g,S))$, a map $\alpha_\Gamma : \mathcal{A}((\Gamma)) \to \mathcal{A}((g,S))$, as the composition of the universal map $\mathcal{A}((\Gamma)) \to \mathbb{M}\mathcal{A}((g,S))$ with α. We call this map the *composition along the graph* Γ. Observe that (5.40) requires that the composition along the corolla be given by the identification $\mathcal{A}((*_{g,S})) \cong \mathcal{A}((g,S))$.

We may define, for each morphism $f : \Gamma_0 \to \Gamma_1$, a morphism $\alpha((f)) : \mathcal{A}((\Gamma_0)) \to \mathcal{A}((\Gamma_1))$ as the composition

$$\begin{aligned} \mathcal{A}((\Gamma_0)) &= \bigotimes_{u\in \mathit{Vert}(\Gamma_0)} \mathcal{A}((g(u), \mathit{Leg}(u))) \cong \bigotimes_{v\in \mathit{Vert}(\Gamma_1)} \mathcal{A}((f^{-1}(v))) \\ &\xrightarrow{\bigotimes_v \alpha_{f^{-1}(v)}} \bigotimes_{v\in \mathit{Vert}(\Gamma_1)} \mathcal{A}((g(v), \mathit{Leg}(v))) = \mathcal{A}((\Gamma_1)). \end{aligned} \tag{5.41}$$

The following statement is a 'modular' version of Theorem 1.73.

THEOREM 5.42. *Let* $\alpha : \mathbb{M}\mathcal{A} \to \mathcal{A}$ *be a map of modular* Σ*-modules as above. Such a map satisfies (5.39), i.e. it is an algebra over the triple* $(\mathbb{M}, \mu, v)$, *if and only if the correspondence* $\Gamma \mapsto \mathcal{A}((\Gamma))$, $f \mapsto \alpha(f)$ *defines, for each* $g \geq 0$ *and a finite set* S, *a functor from the category* $\mathbf{\Gamma}((g,S))$ *to the category of graded vector spaces, that is, if*

$$\alpha((f_2 f_1)) = \alpha((f_2))\alpha((f_1)) \tag{5.42}$$

for any diagram $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$ *in* $\mathbf{\Gamma}((g,S))$.

PROOF. Consider a diagram $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$. By definition, $\alpha((f_2))\alpha((f_1))$ is the composition

$$(5.43)\quad \begin{aligned} \mathcal{A}((\Gamma_0)) &= \bigotimes_{w\in Vert(\Gamma_0)} \mathcal{A}((g(w), Leg(w))) \cong \bigotimes_{v\in Vert(\Gamma_1)} \mathcal{A}((f_1^{-1}(v))) \\ &\xrightarrow{\otimes_v \alpha_{f_1^{-1}(v)}} \bigotimes_{v\in Vert(\Gamma_1)} \mathcal{A}((g(v), Leg(v))) \cong \bigotimes_{u\in Vert(\Gamma_2)} \mathcal{A}((f_2^{-1}(u))) \\ &\xrightarrow{\otimes_u \alpha_{f_2^{-1}(u)}} \bigotimes_{u\in Vert(\Gamma_2)} \mathcal{A}((g(u), Leg(u))) = \mathcal{A}((\Gamma_2)). \end{aligned}$$

Our next task is to interpret $\alpha((f_2))\alpha((f_1))$ in terms of $\alpha\circ\mathbb{M}$. For $u \in Vert(\Gamma_2)$, let $[\Gamma_0^u \xrightarrow{f_1^u} \Gamma_1^u] \in Iso_1\mathbf{\Gamma}((g(u), Leg(u)))$ be as in (5.37), i.e. $\Gamma_1^u := f_2^{-1}(u)$, $\Gamma_0^u := (f_2f_1)^{-1}(u)$ and $f_1^u =$ the restriction of f_1 to Γ_0^u. Recall from Proposition 5.39 that, for each $u \in Vert(\Gamma_2)$,

$$(\mathbb{M}^2\mathcal{A})((g(u), Leg(u))) = \operatorname*{colim}_{[\Upsilon_0 \xrightarrow{g} \Upsilon_1] \in Iso_1\mathbf{\Gamma}((g(u), Leg(u)))} \mathcal{A}((\Upsilon_0)).$$

This means that there is a map $\rho_u : \mathcal{A}((\Gamma_0^u)) \to (\mathbb{M}^2\mathcal{A})((g(u), Leg(u)))$ given by mapping $\mathcal{A}((\Gamma_0^u))$ to the component indexed by $[\Gamma_0^u \xrightarrow{f_1^u} \Gamma_1^u] \in Iso_1\mathbf{\Gamma}((g(u), Leg(u)))$. Let us define $[\alpha \circ \mathbb{M}(\alpha)]_{f_1^u} : \mathcal{A}((\Gamma_0^u)) \to \mathcal{A}((g(u), Leg(u)))$ as the composition

$$\mathcal{A}((\Gamma_0^u)) \xrightarrow{\rho_u} (\mathbb{M}^2\mathcal{A})((g(u), Leg(u))) \xrightarrow{\alpha\circ\mathbb{M}(\alpha)} \mathcal{A}((g(u), Leg(u)))$$

A moment's reflection on (5.43) persuades one that

$$(5.44)\qquad \alpha((f_2))\alpha((f_1)) = \bigotimes_{u\in Vert(\Gamma_0)} [\alpha \circ \mathbb{M}(\alpha)]_{f_1^u}.$$

Similarly, $\alpha((f_2f_1))$ is, by definition, given by

$$\begin{aligned} \mathcal{A}((\Gamma_0)) &= \bigotimes_{w\in Vert(\Gamma_0)} \mathcal{A}((g(w), Leg(w))) \cong \bigotimes_{u\in Vert(\Gamma_2)} \mathcal{A}(((f_2f_1)^{-1}(u))) \\ &\xrightarrow{\otimes_u \alpha_{(f_2f_1)^{-1}(u)}} \bigotimes_{u\in Vert(\Gamma_2)} \mathcal{A}((g(u), Leg(u))) = \mathcal{A}((\Gamma_2)). \end{aligned}$$

As above, we see that if we denote by $[\alpha \circ \mu_{\mathcal{A}}]_{f_1^u}$ the composition

$$\mathcal{A}((\Gamma_0^u)) \xrightarrow{\rho_u} (\mathbb{M}^2\mathcal{A})((g(u), Leg(u))) \xrightarrow{\alpha\circ\mu_{\mathcal{A}}} \mathcal{A}((g(u), Leg(u))),$$

then

$$(5.45)\qquad \alpha((f_2f_1)) = \bigotimes_{u\in Vert(\Gamma_2)} [\alpha \circ \mu_{\mathcal{A}}]_{f_1^u}.$$

This together with (5.44) implies that if $\alpha \circ \mathbb{M}(\alpha) = \alpha \circ \mu_{\mathcal{A}}$, then $\alpha((f_2f_1)) = \alpha((f_2))\alpha((f_1))$, that is $\alpha((-))$ is functorial.

Suppose, on the other hand, that $\alpha((-))$ is functorial. Let

$$[\Gamma_0 \xrightarrow{f} \Gamma_1] \in Iso_1\mathbf{\Gamma}((g, S))$$

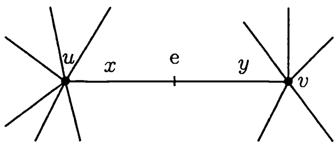
$$\begin{tikzcd}
(\mathbb{M}^2\mathcal{A})((g,S)) \arrow[d, "\alpha\circ\mathbb{M}(\alpha)"'] & \mathcal{A}((\Gamma_0)) \arrow[l] \arrow[r] \arrow[d, "\alpha((f_2 f_1))=\alpha((f_2))\alpha((f_1))"] & (\mathbb{M}^2\mathcal{A})((g,S)) \arrow[d, "\alpha\circ\mu_{\mathcal{A}}"] \\
\mathcal{A}((g,S)) \arrow[r, "\cong"'] & \mathcal{A}((*_{g,S})) & \mathcal{A}((g,S)) \arrow[l, "\cong"]
\end{tikzcd}$$

FIGURE 7.

and apply the above considerations to $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]$ with $f_1 := f$, $\Gamma_2 := *_{g,S}$ and $f_2 : \Gamma_1 \to \Gamma_2$, the contraction. For the unique vertex $u \in \mathit{Vert}(*_{g,S})$, we of course have $[\Gamma_0^u \xrightarrow{f_1^u} \Gamma_1^u] = [\Gamma_0 \xrightarrow{f} \Gamma_1]$. Equations (5.44) and (5.45) imply that the diagram in Figure 7 is commutative. To finish the proof, observe that $(\mathbb{M}^2\mathcal{A})((g,S))$ is the colimit of $\mathcal{A}((\Gamma_0))$ over all such $[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathit{Iso}_1\mathbf{\Gamma}((g,S))$. ∎

PROOF OF THEOREM 5.41. Suppose that $\alpha : \mathbb{M}\mathcal{A} \to \mathcal{A}$ is an algebra over the triple $(\mathbb{M}, \mu, v)$. By Theorem 5.42, α then defines a functor from the category $\mathbf{\Gamma}((g,S))$ to $\mathtt{MMod}$. Let us explain how α induces the data of Definition 5.35 by evaluating α on $\mathcal{A}((\Upsilon))$ for graphs Υ with one edge e. The composition ${}_x\circ_y$ is defined using Υ of the form:

(5.46)

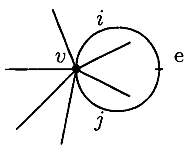


with two distinct vertices u and v and flags $x \in \mathit{Leg}(u)$ and $y \in \mathit{Leg}(v)$ forming the edge e, as the composition along Υ:

$$\begin{aligned}
{}_x\circ_y := \mathcal{A}((g_1, \mathit{Vert}(u)))\otimes\mathcal{A}((g_2, \mathit{Vert}(v))) &\cong \mathcal{A}((\Upsilon)) \\
&\xrightarrow{\alpha_\Upsilon} \mathcal{A}((g_1+g_2, \mathit{Leg}(u)\sqcup \mathit{Leg}(v) - \{x,y\})),
\end{aligned}$$

where $g_1 := g(u)$ and $g_2 := (v)$. Similarly, the contractions ξ_{ij} are defined as the composition along the graph Θ:

(5.47)



with one vertex v of genus g and flags $i, j \in \mathit{Leg}(v)$ forming the edge e, by

$$\xi_{ij} := \mathcal{A}((g, \mathit{Vert}(v))) \cong \mathcal{A}((\Theta)) \xrightarrow{\alpha_\Theta} \mathcal{A}(g+1, \mathit{Vert}(v) - \{i,j\}).$$

We must prove that the operations defined above satisfy the axioms of Definition 5.35, i.e. that the compositions $\{{}_x\circ_y\}$ form an operad structure graded by the genus and that the contractions $\{\xi_{ij}\}$ satisfy (5.26) and (5.27).

Let $\Gamma \in \mathbf{\Gamma}((g, S))$ be a graph with two distinct edges $\mathrm{e}, \mathrm{e}' \in edge(\Gamma)$. The projections π_{e} and $\pi_{\mathrm{e}'}$ obviously commute:

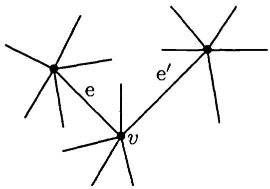
$$\begin{tikzcd} \Gamma \arrow[r, "\pi_{e'}"] \arrow[d, "\pi_e"'] & \Gamma/e' \arrow[d, "\pi_e"] \\ \Gamma/\mathrm{e} \arrow[r, "\pi_{e'}"] & \Gamma/\{\mathrm{e}, \mathrm{e}'\} \end{tikzcd}$$

where we of course identify $(\Gamma/\mathrm{e})/\mathrm{e}' \cong (\Gamma/\mathrm{e}')/\mathrm{e} \cong \Gamma/\{\mathrm{e}, \mathrm{e}'\}$. Since α is a functor, the commutativity of the above diagram implies that

$$\alpha((\pi_e))\alpha((\pi_{e'})) = \alpha((\pi_{e'}))\alpha((\pi_e)). \tag{5.48}$$

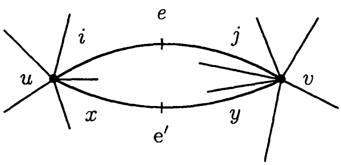
Evaluating (5.48) on graphs such as

(5.49)



where both e and e′ have two distinct vertices and they meet in one vertex v, we prove that $\mathcal{A}^\flat$ with operations $\{{}_x\circ_y\}$ form a cyclic operad graded by the genus. The argument is exactly the same as in the proof of [**GK95**, Theorem 2.2]; the only difference is that we must also check that the operations ${}_x\circ_y$ are homogeneous with respect to the genus, but this is obvious. Consider a graph Γ as in the following picture:

(5.50)

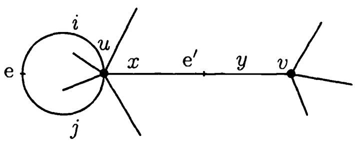


The evaluation of the left-hand side of equation (5.48) on

$$a \otimes b \in \mathcal{A}((g, Leg(u))) \otimes \mathcal{A}((h, Leg(v))) \cong \mathcal{A}((\Gamma))$$

for Γ gives $\xi_{ij}(a_x\circ_y b)$, while the evaluation of the right-hand side gives $\xi_{xy}(a_i\circ_j b)$, and we obtain the third case of (5.27). The evaluation of the left-hand side of (5.48) on the graph

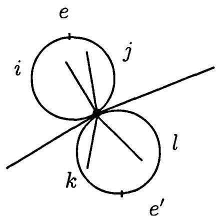
(5.51)



gives $\xi_{ij}(a_x\circ_y b)$, while the right-hand side gives $(\xi_{ij}(a))_x\circ_y b$, which is the first case of (5.27). The evaluation of (5.48) on the same graph but with the indices u and

v, x and y interchanged gives the second case of (5.27). Finally, the evaluation of (5.48) on

(5.52)



gives (5.26).

On the other hand, suppose that $\mathcal{A}^\flat$ is a graded (with respect to the genus) cyclic operad equipped with contraction maps $\{\xi_{ij}\}$ that satisfy (5.26) and (5.27). We will show how, for each $g \geq 0$ and each finite set S, the modular operad structure defines a functor from the category $\mathbf{\Gamma}((g,S))$ to MMod. On objects, we define the functor to be $\mathcal{A}((\Gamma))$ as in (5.33). This definition is clearly functorial on isomorphisms $\phi : \Gamma_0 \to \Gamma_1$.

As the next step we define $\mathcal{A}((f))$ for f a contraction along an edge e of a graph Γ, $f = \pi_e : \Gamma \to \Gamma/e$. Let Υ be the minimal subgraph of Γ containing e and all half-edges (flags) meeting e.

If the edge e has two distinct ends u and v, then Υ looks like the graph in (5.46). We define $\alpha((f))$ to be the operadic composition ${}_x\circ_y$ on Υ and the identity outside Υ. Formally this means that we realize that

$$\begin{aligned} \mathcal{A}((\Gamma)) &\cong \mathcal{A}((g_1,u))\otimes\mathcal{A}((g_2,v))\otimes \bigotimes_{w\neq u,v} \mathcal{A}((g(w),Leg(w))) \text{ and} \\ \mathcal{A}((\Gamma/e)) &\cong \mathcal{A}((g_1+g_2,Leg(\overline{v}))) \otimes \bigotimes_{w\neq u,v} \mathcal{A}((g(w),Leg(w))), \end{aligned}$$

where $\overline{v}$ is the vertex of Γ/e created by the contraction of e. Then we define

$$\mathcal{A}((\pi_e)) := {}_x\circ_y \otimes \bigotimes_{w\neq u,v} \mathbb{1}_{\mathcal{A}((g(w),Leg(w)))}. \tag{5.53}$$

Another possibility is that the edge e forms a loop with one vertex v. The graph Υ then looks like the graph in (5.47). Exactly as in (5.53), we define $\mathcal{A}((f))$ to be the contraction ξ_{ij} on Υ and the identity outside Υ.

A general morphism $f : \Gamma_0 \to \Gamma_1$ of graphs decomposes into a composition

$$\Gamma_0 \xrightarrow{\pi_I} \Gamma_0/I \xrightarrow{\phi} \Gamma_1,$$

where I is the set of edges contracted by f and $\phi : \Gamma_0/I \to \Gamma_1$ is an isomorphism. Choosing an ordering $\{e_1,\ldots,e_k\}$ of the edges in I, we obtain a factorization

$$\Gamma_0 \xrightarrow{\pi_{e_1}} \Gamma_0/e_1 \xrightarrow{\pi_{e_2}} \Gamma_0/\{e_1,e_2\} \xrightarrow{\pi_{e_3}} \cdots \xrightarrow{\pi_{e_k}} \Gamma_0/I \xrightarrow{\phi} \Gamma_1, \tag{5.54}$$

where each morphism is a contraction along one edge except the last, which is an isomorphism. We define

$$\alpha((f)) := \mathcal{A}((\phi))\mathcal{A}((\pi_{e_k}))\cdots\mathcal{A}((\pi_{e_1}))$$

We must prove that this definition is independent of the ordering of the elements of I. Of course, it suffices to prove the composition does not change if we interchange two consecutive edges e_i and e_{i+1}. This is evident if e_i and e_{i+1} have no common vertex. If they have, then they form a subgraph of Γ with two edges, which must be one of the graphs listed in (5.50)–(5.52). But then the maps $\mathcal{A}((\pi_{e_i}))$ and $\mathcal{A}((\pi_{e_{i+1}}))$ commute, because the axioms of modular operads were derived exactly to assure the commutativity over these graphs.

Thus, we have defined a functor $\Gamma \mapsto \mathcal{A}((\Gamma))$, $f \mapsto \alpha((f))$ on the category $\mathbf{\Gamma}((g,S))$, using the operad structure of $\mathcal{A}^\flat$ and the contractions ξ_{ij}. By Theorem 5.42, this functor is an algebra over the triple $(\mathbb{M},\mu,v)$. ∎

EXAMPLE 5.43. Let $V = (V,B)$ be a differential graded vector space with a graded symmetric inner product $B : V \otimes V \to \mathbf{k}$. Recall that graded symmetry means that, for any homogeneous $u,v \in V$,

$$B(u,v) = (-1)^{|u||v|}B(v,u).$$

Let us define, for each genus $g \geq 0$ and a finite set S,

$$\mathcal{E}nd_V((g,S)) := V^{\otimes S}.$$

It follows from definition that, for any graph $\Gamma \in \mathbf{\Gamma}((g,S))$, $\mathcal{E}nd_V((\Gamma)) = V^{\otimes Flag(\Gamma)}$.

Let $B^{\otimes edge(\Gamma)} : V^{\otimes Flag(\Gamma)} \to V^{\otimes Leg(\Gamma)}$ be the multilinear form which contracts the factors of $V^{\otimes Flag(\Gamma)}$ corresponding to the flags which are paired up as edges of Γ. Then we define $\alpha_\Gamma : \mathcal{E}nd_V((\Gamma)) \to \mathcal{E}nd_V((g,S))$ to be the map

$$\alpha_\Gamma : \mathcal{E}nd_V((\Gamma)) \cong V^{\otimes Flag(\Gamma)} \xrightarrow{B^{\otimes edge(\Gamma)}} V^{\otimes Leg(\Gamma)} \cong V^{\otimes S} = \mathcal{E}nd_V((g,S)).$$

It is easy to show that the compositions $\{\alpha_\Gamma |\ \Gamma \in \mathbf{\Gamma}((g,S))\}$ define on $\mathcal{E}nd_V$ the structure of a modular operad. The underlying cyclic operad (5.25) coincides with the cyclic endomorphism operad introduced in Example 5.12.

Strictly speaking, $\mathcal{E}nd_V$ is not a modular operad in the sense of Definition 5.35, because the modular Σ-module $\mathcal{E}nd_V$ is not stable, but it will not really matter, because the only use for these endomorphism operads we have is to define algebras. If the reader feels this trick is not appropriate, he might define $\mathcal{E}nd_V((g,S)) = 0$ for $2g + \#S - 2 \leq 0$.

DEFINITION 5.44. A modular algebra over a modular operad $\mathcal{A}$ is given by a graded vector space with a symmetric inner product, $V = (V,B)$, and a map $a : \mathcal{A} \to \mathcal{E}nd_V$ of modular operads.

5.4. The Feynman transform

In this section we prepare the background machinery and introduce the Feynman transform of a modular operad. It is an analog of the cobar complex of an ordinary operad (Definition 3.9). The Feynman transform was introduced by E. Getzler and M.M. Kapranov in **[GK98]** and called so because it is given as a sum over graphs, as is Feynman's expansion for amplitudes in quantum field theory. This section contains a lot of technical material, so we provide the reader with a 'road map' first.

The conceptual difference between the operadic cobar complex $\mathbf{C}(\mathcal{P})$ of an ordinary operad and the Feynman transform $\mathsf{F}(\mathcal{A})$ of a modular operad is that while $\mathbf{C}(\mathcal{P})$ is again an ordinary operad, $\mathsf{F}(\mathcal{A})$ is not a modular operad, but a certain type of a 'twisted' modular operad. To give a precise meaning to this, we need to introduce 'coefficient systems' (Definition 5.45). For such a coefficient system $\mathfrak{D}$

we construct a $\mathfrak{D}$-twisted version $\mathbb{M}_{\mathfrak{D}}$ of the functor $\mathbb{M}$ and prove that it is in fact a triple (Theorem 5.47). Modular $\mathfrak{D}$-operads (or $\mathfrak{D}$-twisted modular operads) are then algebras for this triple (Definition 5.55). We will also study formal properties of coefficient systems and give a couple of examples, the most important being the dualizing cocycle $\mathfrak{K}$ of Example 5.52 necessary for the Feynman transform which is eventually defined in Definition 5.58. We close this section with a proposition stating that the Feynman transform is a homotopy functor (Proposition 5.60).

Suppose we are given, for each $g \geq 0$, $n \geq 1$, a functor

$$\mathfrak{D} : \mathit{Iso}\,\mathbf{\Gamma}(g,n) \to \mathtt{gVec}$$

from $\mathit{Iso}\,\mathbf{\Gamma}(g,n)$ to the category $\mathtt{gVec}$ of graded vector spaces. For each $k \geq 0$, there is a natural extension of $\mathfrak{D}$ to a functor $\mathfrak{D}_k : \mathit{Iso}_k\mathbf{\Gamma}(g,n) \to \mathtt{gVec}$ given by the formula

$$\mathfrak{D}_k([\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} \Gamma_k]) \\ := \mathfrak{D}(\Gamma_k) \otimes \bigotimes_{v_k \in \mathit{Vert}(\Gamma_k)} \mathfrak{D}(f_k^{-1}(v_k)) \otimes \cdots \otimes \bigotimes_{v_1 \in \mathit{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)).$$

By our standard extension trick we may in fact assume that $\mathfrak{D}_k$ is defined on $\mathit{Iso}_k\mathbf{\Gamma}((g,S))$ for an arbitrary finite set S. We will also assume that

$$\mathfrak{D}(*_{g,S}) = \mathbf{k}, \text{ for any } g, S. \tag{5.55}$$

Under this assumption, $\mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) = \mathfrak{D}(\Gamma_1) \otimes \bigotimes_{v \in \mathit{Vert}(\Gamma_1)} \mathfrak{D}(f^{-1}(v))$ is naturally isomorphic to $\mathfrak{D}(\Gamma_1)$,

$$\mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \cong \mathfrak{D}(\Gamma_1), \tag{5.56}$$

whenever f is an isomorphism.

Let $\partial_1 : \mathit{Iso}_1\mathbf{\Gamma}((g,S)) \to \mathit{Iso}_0\mathbf{\Gamma}((g,S))$ be as in (5.35) and suppose we are given a transformation $\nu : \mathfrak{D}_1 \to \mathfrak{D} \circ \partial_1$. This means that to each $[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathit{Iso}_0\mathbf{\Gamma}((g,S))$ is assigned a natural morphism

$$\nu_f : \mathfrak{D}(\Gamma_1) \otimes \bigotimes_{v \in \mathit{Vert}(\Gamma_1)} \mathfrak{D}(f^{-1}(v)) \longrightarrow \mathfrak{D}(\Gamma_0).$$

Such ν induces, for $1 \leq i \leq k$, transformations $\nu_{ki} : \mathfrak{D}_k \to \mathfrak{D}_{k-1} \circ \partial_i$ as follows. For $i = k$ and $\mathbf{\Gamma} := [\Gamma_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} \Gamma_k] \in \mathit{Iso}_k\mathbf{\Gamma}((g,S))$, $(\nu_{kk})_{\mathbf{\Gamma}} : \mathfrak{D}_k(\mathbf{\Gamma}) \to (\mathfrak{D}_{k-1} \circ \partial_k)(\mathbf{\Gamma})$ is the composition

$$\begin{aligned} \mathfrak{D}_k(\mathbf{\Gamma}) &= \mathfrak{D}(\Gamma_k) \otimes \bigotimes_{v_k \in \mathit{Vert}(\Gamma_k)} \mathfrak{D}(f_k^{-1}(v_k)) \otimes \bigotimes_{v_{k-1} \in \mathit{Vert}(\Gamma_{k-1})} \mathfrak{D}(f_{k-1}^{-1}(v_{k-1})) \otimes \cdots \otimes \bigotimes_{v_1 \in \mathit{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\ &\xrightarrow{\nu_{f_k} \otimes \mathbb{1}} \mathfrak{D}(\Gamma_{k-1}) \otimes \bigotimes_{v_{k-1} \in \mathit{Vert}(\Gamma_{k-1})} \mathfrak{D}(f_{k-1}^{-1}(v_{k-1})) \otimes \cdots \otimes \bigotimes_{v_1 \in \mathit{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\ &= (\mathfrak{D}_{k-1} \circ \partial_k)(\mathbf{\Gamma}). \end{aligned}$$

For $i < k$, $(\nu_{ki})_{\mathbf{\Gamma}} : \mathfrak{D}_k(\mathbf{\Gamma}) \to (\mathfrak{D}_{k-1} \circ \partial_i)(\mathbf{\Gamma})$ is given by

$$\begin{aligned}
\mathfrak{D}_k(\mathbf{\Gamma}) &= \mathfrak{D}(\Gamma_k) \otimes \cdots \otimes \bigotimes_{v_{i+1} \in Vert(\Gamma_{i+1})} \mathfrak{D}(f_{i+1}^{-1}(v_{i+1})) \otimes \bigotimes_{v_i \in Vert(\Gamma_i)} \mathfrak{D}(f_i^{-1}(v_i)) \otimes \cdots \otimes \bigotimes_{v_1 \in Vert(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\
&\cong \mathfrak{D}(\Gamma_k) \otimes \cdots \otimes \bigotimes_{v_{i+1} \in Vert(\Gamma_{i+1})} [\mathfrak{D}(f_{i+1}^{-1}(v_{i+1})) \otimes \bigotimes_{v_i \in f_i^{-1}(v_{i+1})} \mathfrak{D}(f_i^{-1}(v_i))] \otimes \cdots \otimes \bigotimes_{v_1 \in Vert(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\
&\xrightarrow{\mathbb{1} \otimes \bigotimes_{v_{i+1}} \nu_{f_i | f_{i+1}^{-1}(v_{i+1})} \otimes \mathbb{1}} \mathfrak{D}(\Gamma_k) \otimes \cdots \otimes \bigotimes_{v_{i+1} \in Vert(\Gamma_{i+1})} \mathfrak{D}((f_{i+1} f_i)^{-1}(v_{i+1})) \otimes \cdots \otimes \bigotimes_{v_1 \in Vert(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\
&= (\mathfrak{D}_k \circ \partial_i)(\mathbf{\Gamma}).
\end{aligned}$$

DEFINITION 5.45. Let $\mathfrak{D} : Iso\,\mathbf{\Gamma}((g,n)) \to \mathsf{gVec}$ and $\nu : \mathfrak{D}_1 \to \mathfrak{D} \circ \partial_1$ be as above. We call $(\mathfrak{D}, \nu)$ a *coefficient system*, (or a *hyperoperad* in the original terminology of [**GK98**]), if

(i) the diagram

$$\begin{tikzcd}
\mathfrak{D}_2 \arrow[rr, "\nu_{22}"] \arrow[d, "\nu_{21}"'] & & \mathfrak{D}_1 \circ \partial_2 \arrow[d, "\nu \circ \partial_2"] \\
\mathfrak{D}_1 \circ \partial_1 \arrow[r, "\nu \circ \partial_1"] & \mathfrak{D}_1 \circ (\partial_1 \partial_1) = \mathfrak{D}_1 \circ (\partial_1 \partial_2) &
\end{tikzcd}$$

commutes, and

(ii) for each isomorphism $f : \Gamma_1 \to \Gamma_0$, the diagram

$$\begin{tikzcd}
\mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \arrow[r, "\nu_f"] \arrow[dr, "\cong"'] & \mathfrak{D}(\Gamma_0) \arrow[d, "\mathfrak{D}(f)"] \\
& \mathfrak{D}(\Gamma_1)
\end{tikzcd}$$

where $\cong$ is the identification of (5.56), commutes.

For a coefficient system $\mathfrak{D}$, define the twisted version $\mathbb{M}_{\mathfrak{D}}$ of the endofunctor $\mathbb{M}$ by

(5.57) $$(\mathbb{M}_{\mathfrak{D}} \mathcal{E})((g,n)) := \operatorname*{colim}_{\Gamma \in Iso\,\mathbf{\Gamma}((g,n))} \mathfrak{D}(\Gamma) \otimes \mathcal{E}((\Gamma)), \text{ for } \mathcal{E} \in \mathsf{MMod}.$$

The following proposition is an innocuous twisted version of Proposition 5.39 and we leave the proof to the reader.

PROPOSITION 5.46. *For each $k \geq 0$, the $(k+1)$th iterate of the functor $\mathbb{M}_{\mathfrak{D}}$ is given by*

(5.58) $$(\mathbb{M}_{\mathfrak{D}})^{k+1} \mathcal{E}((g,S)) = \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} \Gamma_k] \in Iso_k \Gamma((g,S))} \mathcal{E}((\Gamma_0)) \otimes \mathfrak{D}_k([\Gamma_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} \Gamma_k]).$$

Using the description of (5.58), we may define a natural transformation μ · $\mathbb{M}_{\mathfrak{D}}^2 \to \mathbb{M}_{\mathfrak{D}}$ by setting $\mu_{\mathcal{E}} : (\mathbb{M}_{\mathfrak{D}})^2 \mathcal{E} \to \mathbb{M}_{\mathfrak{D}} \mathcal{E}$ to be the colimit of the system of maps

$$\mathcal{E}((\Gamma_0)) \otimes \mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \xrightarrow{\mathbb{1} \otimes \nu_f} \mathcal{E}((\Gamma_0)) \otimes \mathfrak{D}(\Gamma_0), \; [\Gamma_0 \xrightarrow{f} \Gamma_1] \in Iso_1 \mathbf{\Gamma}((g,S)).$$

Let also $v : \mathbb{1}_{\mathsf{MMod}} \to \mathbb{M}_{\mathfrak{D}}$ be the transformation given by the inclusion of $\mathcal{E}((g,S))$ into the component $\mathcal{E}((*_{g,S}))\otimes\mathfrak{D}(*_{g,S}) \cong \mathcal{E}((g,S))$ of $(\mathbb{M}_{\mathfrak{D}}\mathcal{E})((g,S))$.

THEOREM 5.47. *The transformations* $\mu : \mathbb{M}_{\mathfrak{D}}^2 \to \mathbb{M}_{\mathfrak{D}}$ *and* $v : \mathbb{1}_{\mathsf{MMod}} \to \mathbb{M}_{\mathfrak{D}}$ *define on the functor* $\mathbb{M}_{\mathfrak{D}}$ *a structure of a triple.*

PROOF. As in the proof of Theorem 5.40 we see that $\mu(\mu\mathbb{M}_{\mathfrak{D}})_{\mathcal{E}} : \mathbb{M}_{\mathfrak{D}}^3\mathcal{E} \to \mathbb{M}_{\mathfrak{D}}\mathcal{E}$ is the colimit, over $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]$, of the system of maps

$$\mathcal{E}((\Gamma_0))\otimes\mathfrak{D}([\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]) \xrightarrow{\mathbb{1}\otimes(\nu\circ\partial_2)\nu_{22}} \mathcal{E}((\Gamma_0))\otimes\mathfrak{D}(\Gamma_0),$$

while $\mu(\mathbb{M}_{\mathfrak{D}}\mu)_{\mathcal{E}} : \mathbb{M}_{\mathfrak{D}}^3\mathcal{E} \to \mathbb{M}_{\mathfrak{D}}\mathcal{E}$ is the colimit of

$$\mathcal{E}((\Gamma_0))\otimes\mathfrak{D}([\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]) \xrightarrow{\mathbb{1}\otimes(\nu\circ\partial_1)\nu_{21}} \mathcal{E}((\Gamma_0))\otimes\mathfrak{D}(\Gamma_0).$$

Thus the first diagram of Definition 5.45 assures that $\mu(\mu\mathbb{M}_{\mathfrak{D}}) = \mu(\mathbb{M}_{\mathfrak{D}}\mu)$. The unitarity $\mu(v\mathbb{M}_{\mathfrak{D}}) = \mu(\mathbb{M}_{\mathfrak{D}}v) = \mathbb{M}_{\mathfrak{D}}$ is an easy consequence of the commutativity of the second diagram of Definition 5.45. ∎

For two coefficient systems $\mathfrak{D} = (\mathfrak{D},\nu)$ and $\mathfrak{C} = (\mathfrak{C},\omega)$, define a new coefficient system $\mathfrak{D}\otimes\mathfrak{C} = (\mathfrak{D}\otimes\mathfrak{C},\nu\otimes\omega)$ by $(\mathfrak{D}\otimes\mathfrak{C})(\Gamma) := \mathfrak{D}(\Gamma)\otimes\mathfrak{C}(\Gamma)$, with $(\nu\otimes\omega)_f : (\mathfrak{D}\otimes\mathfrak{C})_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \to (\mathfrak{D}\otimes\mathfrak{C})(\Gamma_0)$ given as the composition

$$\begin{aligned}(\mathfrak{D}\otimes\mathfrak{C})_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) &= \mathfrak{D}(\Gamma_1)\otimes\mathfrak{C}(\Gamma_1)\otimes \bigotimes_{v\in Vert(\Gamma_1)} \mathfrak{D}(f^{-1}(v))\otimes \bigotimes_{v\in Vert(\Gamma_1)} \mathfrak{C}(f^{-1}(v))\\ &\cong \Big\{\mathfrak{D}(\Gamma_1)\otimes \bigotimes_{v\in Vert(\Gamma_1)} \mathfrak{D}(f^{-1}(v))\Big\}\otimes\Big\{\mathfrak{C}(\Gamma_1)\otimes \bigotimes_{v\in Vert(\Gamma_1)} \mathfrak{C}(f^{-1}(v))\Big\}\\ &\xrightarrow{\nu_f\otimes\omega_f} \mathfrak{D}(\Gamma_0)\otimes\mathfrak{C}(\Gamma_0) = (\mathfrak{D}\otimes\mathfrak{C})(\Gamma_0).\end{aligned}$$

Finally, let $\mathfrak{J} = (\mathfrak{J},\iota)$ be the coefficient system with $\mathfrak{J}(\Gamma) = \mathbf{k}$ for each $\Gamma \in \mathbf{\Gamma}((g,S))$ and ι given in the evident way.

LEMMA 5.48. *The tensor product $\otimes$ introduced above defines a monoidal structure on the set of coefficient systems. The system $\mathfrak{J}$ is the unit for $\otimes$.*

The proof of the lemma is a direct verification and we leave it to the reader. Let us say that a graded vector space V is *invertible*, if there exists another graded vector space V^{-1} such that $V\otimes V^{-1} \cong \mathbf{k}$. Clearly, V is invertible if and only if it is of the form $\uparrow^n\mathbf{k}$ for some $n \in \mathbb{Z}$; then $V^{-1} = \downarrow^n\mathbf{k}$. According to **[GK98]** we call a coefficient system $\mathfrak{D} = (\mathfrak{D},\nu)$ a *cocycle* if $\mathfrak{D}(\Gamma)$ is invertible for each $\Gamma \in Iso\,\mathbf{\Gamma}((g,S))$ and the maps ν_f are isomorphisms for each $f : \Gamma_0 \to \Gamma_1$. The inverse of a cocycle $\mathfrak{D}$ is again a cocycle which we denote $\mathfrak{D}^{-1}$.

A special class of cocycles can be constructed as follows. Let $\mathfrak{l}$ be a modular Σ-module such that each graded vector space $\mathfrak{l}(g,n)$ is invertible. Define, for $\Gamma \in Iso\,\mathbf{\Gamma}((g,S))$,

$$\mathfrak{D}_{\mathfrak{l}}(\Gamma) := \mathfrak{l}((g,S))\otimes \bigotimes_{v\in Vert(\Gamma)} \mathfrak{l}((g(v),Leg(v)))^{-1}. \tag{5.59}$$

For $f : \Gamma_0 \to \Gamma_1$, let $\nu_{\mathfrak{l}} : \mathfrak{D}_{\mathfrak{l}}(\Gamma_1)\otimes\bigotimes_{v\in Vert(\Gamma_1)}\mathfrak{D}_{\mathfrak{l}}(f^{-1}(v)) \to \mathfrak{D}_{\mathfrak{l}}(\Gamma_0)$ be given by the identification

$$
\begin{aligned}
&\mathfrak{D}_{\mathfrak{l}}(\Gamma_1)\otimes \bigotimes_{v\in Vert(\Gamma_1)} \mathfrak{D}_{\mathfrak{l}}(f^{-1}(v)) \\
&= \mathfrak{l}((g,S))\otimes\bigotimes_{v\in Vert(\Gamma_1)}\mathfrak{l}((g(v),Leg(v)))^{-1}\otimes\Big\{\bigotimes_{v\in Vert(\Gamma_1)}\mathfrak{l}((g(v),Leg(v)))\otimes\bigotimes_{w\in f^{-1}(v)}\mathfrak{l}((g(w),Leg(w)))^{-1}\Big\} \\
&\cong \mathfrak{l}((g,S))\otimes\bigotimes_{v\in Vert(\Gamma_1)}\Big\{\mathfrak{l}((g(v),Leg(v)))^{-1}\otimes\mathfrak{l}((g(v),Leg(v)))\Big\}\otimes\bigotimes_{w\in Vert(\Gamma_0)}\mathfrak{l}((g(w),Leg(w)))^{-1} \\
&\cong \mathfrak{l}((g,S))\otimes\bigotimes_{w\in Vert(\Gamma_0)}\mathfrak{l}((g(w),Leg(w)))^{-1} = \mathfrak{D}_{\mathfrak{l}}(\Gamma_0).
\end{aligned}
$$

Each modular Σ-module $\mathfrak{l} \in \mathtt{MMod}$ defines an endofunctor

$$\mathfrak{l}(-) : \mathtt{MMod} \to \mathtt{MMod} \tag{5.60}$$

by $\mathfrak{l}(\mathcal{E})((g,S)) := \mathfrak{l}((g,S))\otimes\mathcal{E}((g,S))$.

LEMMA 5.49. *The object $\mathfrak{D}_{\mathfrak{l}} = (\mathfrak{D}_{\mathfrak{l}},\nu_{\mathfrak{l}})$ constructed above is a cocycle, called the* coboundary *of* $\mathfrak{l}$. *There is a natural isomorphism of triples*

$$\mathbb{M}_{\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}} \cong \mathfrak{l}\circ\mathbb{M}_{\mathfrak{D}}\circ\mathfrak{l}^{-1}. \tag{5.61}$$

PROOF. The verification that $\mathfrak{D}_{\mathfrak{l}}$ is a cocycle is easy and we leave it to the reader. The isomorphism (5.61) is given as the colimit, over $\Gamma \in \mathbf{\Gamma}((g,S))$, of the identifications

$$
\begin{aligned}
(\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}})(\Gamma)\otimes\mathcal{E}((\Gamma)) &= \mathfrak{D}(\Gamma)\otimes\mathfrak{D}_{\mathfrak{l}}(\Gamma)\otimes\mathcal{E}((\Gamma)) \\
&= \mathfrak{D}(\Gamma)\otimes\mathfrak{l}((g,S))\otimes\bigotimes_{v\in Vert(\Gamma)}\mathfrak{l}((g(v),Leg(v)))^{-1}\otimes\mathcal{E}((\Gamma)) \\
&\cong \mathfrak{l}((g,S))\otimes\Big\{\mathfrak{D}(\Gamma)\otimes(\mathfrak{l}^{-1}\mathcal{E})((\Gamma))\Big\}.
\end{aligned}
$$

■

EXAMPLE 5.50. Let $\mathfrak{s}(g,n) :=\uparrow^{2(g-1)+n+1}\mathrm{sgn}_n^+$, where sgn_n^+ is the signum representation of Σ_n^+. For $\mathcal{E} \in \mathtt{MMod}$ we call $\mathfrak{s}\mathcal{E} = \mathfrak{s}\otimes\mathcal{E}$ the *modular suspension* of the modular Σ-module $\mathcal{E}$. The associated cocycle $\mathfrak{D}_{\mathfrak{s}}$ is concentrated in degree zero. Indeed, we obtain from (5.59) that, for $\Gamma \in \mathbf{\Gamma}((g,S))$,

$$\deg(\mathfrak{D}_{\mathfrak{s}}(\Gamma)) = 2(g-1) + |S| - \sum_{v\in Vert(\Gamma)}\Big(2(g(v)-1) + |Leg(v)|\Big),$$

which is zero, by (5.31). Let us remark that the original definition of $\mathfrak{s}$ in [**GK98**] uses desuspension instead of suspension, $\mathfrak{s}(g,n) :=\downarrow^{2(g-1)+n+1}\mathrm{sgn}_n^+$.

For a finite dimensional vector space V of dimension n, let

$$Det(V) := \wedge^n(\downarrow V),$$

the one-dimensional top component of the n-fold exterior power of V concentrated in degree $-n$. If S is a finite set, define $Det(S) := Det(\mathbf{k}^S) = \wedge^{|S|}(\downarrow\mathbf{k}^S)$. Explicitly this means that, if $\mathrm{card}(S) = k$,

$$Det(S) = \left(\bigoplus_{f\ \{1,\ \ ,k\}\cong S} Span(\downarrow s_{f(1)}\wedge\cdots\wedge\downarrow s_{f(k)})\right)_{\Sigma_k},$$

with Σ_k acting by $\sigma(\downarrow s_{f(1)} \wedge \cdots \wedge \downarrow s_{f(k)}) := \mathrm{sgn}(\sigma)(\downarrow s_{f(\sigma(1))} \wedge \cdots \wedge \downarrow s_{f(\sigma(k))})$. The following lemma is an easy consequence of definitions.

LEMMA 5.51. *For any finite set S, there is a natural isomorphism*

$$Det(S)^2 \cong \downarrow^{2|S|} \mathbf{k}. \tag{5.62}$$

Given a decomposition $S = \bigsqcup_{i \in I} S_i$, of a finite set S into a disjoint union, then there is a natural identification

$$Det(S) \cong \bigotimes_{i \in I} Det(S_i). \tag{5.63}$$

EXAMPLE 5.52. Let us introduce the cocycle $\mathfrak{K} = (\mathfrak{K}, \mathfrak{k})$ by

$$\mathfrak{K}(\Gamma) := Det(edge(\Gamma)).$$

The structure morphism $\mathfrak{k}_f : \mathfrak{K}(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \mathfrak{K}(f^{-1}(v)) \to \mathfrak{K}(\Gamma_0)$ is, for $f : \Gamma_0 \to \Gamma_1$, given by the natural isomorphism

$$\begin{aligned}\mathfrak{K}(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \mathfrak{K}(f^{-1}(v)) &= Det(edge(\Gamma_1)) \otimes \bigotimes_{v \in Vert(\Gamma_1)} Det(edge(f^{-1}(v)))\\ &\cong Det\Big(edge(\Gamma_1) \sqcup \bigsqcup_{v \in Vert(\Gamma_1)} edge(f^{-1}(v))\Big) \cong Det(edge(\Gamma_0)) = \mathfrak{K}(edge(\Gamma_0))\end{aligned}$$

based on (5.63). We call $\mathfrak{K}$ the *dualizing cocycle*. Given a cocycle $\mathfrak{D}$, we denote

$$\check{\mathfrak{D}} := \mathfrak{K} \otimes \mathfrak{D}^{-1} \tag{5.64}$$

and call $\check{\mathfrak{D}}$ the *dual of* $\mathfrak{D}$. This duality will be crucial in the definition of the Feynman transform for modular operads.

Let $Or(\mathrm{e})$ be the orientation line of an edge e in Γ, that is, the determinant $\uparrow^2 Det(\{s, t\})$, where s and t are the pair of flags making up the edge e. The *orientation cocycle* $\mathfrak{T}$ of a graph Γ is given by

$$\mathfrak{T}(\Gamma) := Det\Big(\bigoplus_{e \in edge(\Gamma)} Or(\mathrm{e})\Big)$$

with the structure morphism $\mathfrak{t}_f : \mathfrak{T}(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \mathfrak{T}(f^{-1}(v)) \to \mathfrak{T}(\Gamma_0)$ given, for $f : \Gamma_0 \to \Gamma_1$, by the identification

$$\begin{aligned}\mathfrak{T}(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \mathfrak{T}(f^{-1}(v)) &= Det\Big(\bigoplus_{e \in edge(\Gamma_1)} Or(\mathrm{e})\Big) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \Big(\bigoplus_{e \in edge(f^{-1}(v))} Or(\mathrm{e})\Big)\\ &\cong Det\Big(\bigoplus \Big\{ Or(\mathrm{e}) \mid \mathrm{e} \in edge(\Gamma_1) \sqcup \bigsqcup_{v \in Vert(\Gamma_1)} edge(f^{-1}(v)) \Big\}\Big)\\ &\cong Det\Big(\bigoplus_{e \in edge(\Gamma_0)} Or(e)\Big) = \mathfrak{T}(\Gamma_0)\end{aligned}$$

based on an obvious vector-space variant of (5.63) saying that, for a decomposition $V = \bigoplus_{i \in I} V_i$ of a finite dimensional vector space,

$$Det(V) \cong \bigotimes_{i \in I} Det(V_i).$$

PROPOSITION 5.53. *There is a natural isomorphism*

$$\mathfrak{D}_{\mathfrak{s}} \otimes \mathfrak{K} \cong \mathfrak{T},$$

where $\mathfrak{D}_{\mathfrak{s}}$ is the coboundary of the suspension $\mathfrak{s}$ introduced in Example 5.50.

PROOF. For symbols x and y, let $\downarrow x \wedge \downarrow y$ be the generator of $Det(Span\{x,y\})$. If s and t are two half-edges forming an edge $e \in edge(\Gamma)$, then the orientation line $Or(e)$ is, by definition, spanned by $\uparrow^2 (\downarrow s \wedge \downarrow t)$, therefore $Det(Or(e))$ is spanned by $\uparrow (\downarrow s \wedge \downarrow t)$, because the determinant of a one-dimensional space is just its desuspension. Let us identify $\uparrow (\downarrow s \wedge \downarrow t)$ with $\uparrow e \otimes (\downarrow s \wedge \downarrow t) \in (Det\{e\})^{-1} \otimes Det(\{s,t\})$. This means that for each $e = \{s,t\} \in edge(\Gamma)$, we defined a canonical identification

$$Det(Or(e)) \cong Det(\{e\})^{-1} \otimes Det(\{s,t\}), \tag{5.65}$$

compatible with the action of the automorphism group $\mathbb{Z}_2$ which interchanges s and t.

Let $IFlag(\Gamma)$ denote the set of internal flags of the graph Γ, that is, the set of flags that are not the legs of Γ. Then, multiplying (5.65) over $IFlag(\Gamma)$ and using (5.63), we derive that

$$\mathfrak{T}(\Gamma) \cong Det(edge(\Gamma))^{-1} \otimes Det(IFlag(\Gamma)). \tag{5.66}$$

On the other hand, $Flag(\Gamma) = IFlag(\Gamma) \sqcup Leg(\Gamma)$, thus (5.66) gives

$$\mathfrak{T}(\Gamma) \cong Det(edge(\Gamma))^{-1} \otimes Det(Flag(\Gamma)) \otimes Det(Leg(\Gamma))^{-1}.$$

It remains to show that

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) \cong Det(edge(\Gamma))^{-2} \otimes Det(Flag(\Gamma)) \otimes Det(Leg(\Gamma))^{-1} \tag{5.67}$$

which, because $\mathfrak{K}(\Gamma) = Det(edge(\Gamma))$, implies the proposition. To prove (5.67), observe that, for $v \in Vert(\Gamma)$,

$$\mathfrak{s}((g(v), Leg(v))) \cong \uparrow^{2(g(v)-1)} Det^{-1}(Leg(v)).$$

Thus

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) \cong \uparrow^{2(g-1)} Det^{-1}(Leg(\Gamma)) \otimes \bigotimes_{v \in Vert(\Gamma)} \downarrow^{2(g(v)-1)} Det(Leg(v)).$$

Because $\downarrow^{2(g(v)-1)}$ shifts the degree by an even number, we may move it to the left and write

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) = \uparrow^{2[g-1-\sum_{v \in Vert(\Gamma)}(g(v)-1)]} \Big(Det^{-1}(Leg(\Gamma)) \otimes \bigotimes_{v \in Vert(\Gamma)} Det(Leg(v)) \Big).$$

By (5.29), $g - 1 - \sum_{v \in Vert(\Gamma)} (g(v) - 1) = |edge(\Gamma)|$, thus

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) = \uparrow^{2|edge(\Gamma)|} \Big(Det^{-1}(Leg(\Gamma)) \otimes \bigotimes_{v \in Vert(\Gamma)} Det(Leg(v)) \Big).$$

This gives (5.67), because, by (5.62), $Det(edge(\Gamma))^{-2} \cong \uparrow^{2|edge(\Gamma)|} \mathbf{k}$. ■

In Section 5.6 we use the coboundary $\mathfrak{D}_{\mathfrak{p}}$ associated to the stable modular Σ-module $\mathfrak{p}$ with $\mathfrak{p}(g,n) = \downarrow^{6(g-1)-2n} \mathbf{k}$. The following proposition, taken from [**GK98**], relates $\mathfrak{D}_{\mathfrak{p}}$ to the square of the dualizing cocycle introduced in Example 5.52.

PROPOSITION 5.54. *There is a natural isomorphism of cocycles $\mathfrak{K}^2 \cong \mathfrak{D}_{\mathfrak{p}}$.*

PROOF. It follows from (5.59) that, for $\Gamma \in \mathbf{\Gamma}((g,n))$, $\mathfrak{D}_{\mathfrak{p}}(\Gamma) \cong \downarrow^{2l}\mathbf{k}$, where

$$l = 3(g(\Gamma)-1)+n-\sum_{v\in Vert(\Gamma)}(3(g(v)-1)+|Leg(v)|).$$

Equation (5.32) implies that $l = |edge(\Gamma)|$ and the result follows from the definition of $\mathfrak{K}$. ∎

Let us recall that, by Theorem 5.41, modular operads are algebras over the triple $\mathbb{M} = (\mathbb{M}, \mu, v)$. 'Twisted' modular operads are defined in a similar way.

DEFINITION 5.55. Let $\mathfrak{D}$ be a coefficient system. A *modular $\mathfrak{D}$-operad* (also called a *$\mathfrak{D}$-twisted modular operad*) is an algebra over the triple $\mathbb{M}_{\mathfrak{D}} = (\mathbb{M}_{\mathfrak{D}}, \mu, v)$.

Lemma 5.49 implies that, for any modular Σ-module $\mathfrak{l}$, the functor $\mathfrak{l} : \mathtt{MMod} \to \mathtt{MMod}$ induces an equivalence between the category of modular $\mathfrak{D}$-operads and the category of modular $\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}$-operads as follows. By definition, a $\mathfrak{D}$-modular operad structure on $\mathcal{A}$ is given by a map $\alpha : \mathbb{M}_{\mathfrak{D}}\mathcal{A} \to \mathcal{A}$, which in turn induces the map

$$\mathfrak{l}\alpha : (\mathfrak{l}\circ\mathbb{M}_{\mathfrak{D}})\mathcal{A} \to \mathfrak{l}\mathcal{A}.$$

But $(\mathfrak{l}\circ\mathbb{M}_{\mathfrak{D}})\mathcal{A} = (\mathfrak{l}\circ\mathbb{M}_{\mathfrak{D}}\circ\mathfrak{l}^{-1})\mathfrak{l}\mathcal{A}$ is, by (5.61), naturally isomorphic to $\mathbb{M}_{\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}}$, thus $\mathfrak{l}\alpha$ may be interpreted as a map

$$\mathfrak{l}\alpha : \mathbb{M}_{\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}}\mathfrak{l}\mathcal{A} \to \mathfrak{l}\mathcal{A}$$

that defines a $\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}$-modular operad structure on $\mathfrak{l}\mathcal{A}$.

The above observation can be reformulated intuitively as saying that changing $\mathfrak{D}$ by a coboundary does not change the category of $\mathfrak{D}$-modular operads.

Modular $\mathfrak{J}$-operads are just ordinary modular operads, with the corresponding notion of modular algebras as representations in the modular endomorphisms operad; see Definition 5.44. Modular algebras exist also for $\mathfrak{T}$-modular operads, since there is the '$\mathfrak{T}$-endomorphism' operad, as shown in the following example.

EXAMPLE 5.56. Let $V = (V, B)$ be a chain complex with a graded antisymmetric nondegenerate bilinear form $B : V\otimes V \to V$ of degree -1. Define the modular Σ-module $\mathcal{E}nd_V^{\mathfrak{T}}$ by

$$\mathcal{E}nd_V^{\mathfrak{T}}((g,S)) := V^{\otimes S},$$

for $g \geq 0$ and a finite set S. To define, for $\Gamma \in \mathbf{\Gamma}((g,S))$, the 'composition map'

$$\alpha_\Gamma^{\mathfrak{T}} : \mathcal{E}nd_V^{\mathfrak{T}}((\Gamma))\otimes\mathfrak{T}(\Gamma) \to \mathcal{E}nd_V^{\mathfrak{T}}((g,S))$$

choose labels s_e, t_e such that $e = \{s_e, t_e\}$ for each edge $e \in edge(\Gamma)$. Then $\alpha_\Gamma^{\mathfrak{T}}$ is the following composition:

$$\begin{aligned}
&\mathcal{E}nd_V^{\mathfrak{T}}((\Gamma))\otimes\mathfrak{T}(\Gamma)\\
&\quad\cong V^{\otimes Flag(\Gamma)}\otimes Det\Big(\bigoplus_{e\in edge(\Gamma)} Or(e)\Big) \cong V^{\otimes Leg(\Gamma)}\otimes\bigotimes_{e\in edge(\Gamma)}\Big(V^{\otimes\{s_e,t_e\}}\otimes Det(Or(e))\Big)\\
&\quad\cong V^{\otimes Leg(\Gamma)}\otimes\bigotimes_{e\in edge(\Gamma)}\Big(V_{s_e}\otimes V_{t_e}\otimes Span(\uparrow(\downarrow s_e\wedge\downarrow t_e))\Big)\\
&\quad\xrightarrow{\mathbb{1}\otimes\bigotimes_e B_e} V^{\otimes Leg(\Gamma)}\otimes\mathbf{k}^{\otimes edge(\Gamma)} \cong \mathcal{E}nd_V^{\mathfrak{T}}((g,S)),
\end{aligned}$$

where B_e is the map that sends $u \otimes v \otimes \uparrow(\downarrow s_e \wedge \downarrow t_e) \in V_{s_e} \otimes V_{t_e} \otimes Span(\uparrow(\downarrow s_e \wedge \downarrow t_e))$ to $B(u,v) \in \mathbf{k}$.

We must show that the definition of $\alpha_\Gamma^{\mathfrak{T}}$ does not depend on the choice of labels. If we interchange labels s_e, t_e of an edge e, we obtain

$$(-1)^{|u|\,|v|+1}(v \otimes u \otimes \uparrow(\downarrow t_e \wedge \downarrow s_e)) \in V_{t_e} \otimes V_{s_e} \otimes Span(\uparrow(\downarrow t_e \wedge \downarrow s_e))$$

instead of

$$u \otimes v \otimes \uparrow(\downarrow s_e \wedge \downarrow t_e) \in V_{s_e} \otimes V_{t_e} \otimes Span(\uparrow(\downarrow s_e \wedge \downarrow t_e)).$$

The form B_e maps $(-1)^{|u|\,|v|+1}(v \otimes u \otimes \uparrow(\downarrow t_e \wedge \downarrow s_e))$ to $(-1)^{|u|\,|v|+1}B(v,u)$, which is the same as $B(u,v)$, because B is assumed to be graded antisymmetric. Thus the value of $\alpha_\Gamma^{\mathfrak{T}}$ does not depend on the choice of labels forming up the edge e. It is also clear that the degree of B_e is zero, thus also $\deg(\alpha_\Gamma^{\mathfrak{T}}) = 0$.

It is easy to verify that $\{\alpha_\Gamma^{\mathfrak{T}} \mid \Gamma \in \mathbf{\Gamma}((g,S))\}$ induces on $\mathcal{E}nd_V^{\mathfrak{T}}$ the structure of a $\mathfrak{T}$-modular operad. See also the similar Example 5.87.

EXAMPLE 5.57. Let $\mathcal{Q}$ be a cyclic pseudo-operad. Define a stable modular Σ-module (denoted again $\mathcal{Q}$) by

$$\mathcal{Q}(g,n) = \begin{cases} \mathcal{Q}(n), & \text{for } g = 0,\ n \geq 2, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathfrak{Det}$ be the coefficient system defined by $\mathfrak{Det}(\Gamma) := Det(H_1(|\Gamma|;\mathbf{k}))$. It is easily seen that it is in fact a cocycle; see [**GK98**, 4.13]. Since $Det(H_1(|\Gamma|;\mathbf{k}))$ is trivial if Γ is a tree, each cyclic operad can be considered either as an ordinary modular operad or as an $\mathfrak{D}$-modular operad with $\mathfrak{D} = \mathfrak{Det}$.

We are ready to define, for each cocycle $\mathfrak{D}$ and $\mathfrak{D}$-modular operad $\mathcal{A}$, the Feynman transform $\mathsf{F}_{\mathfrak{D}}\mathcal{A} = (\mathsf{F}_{\mathfrak{D}}\mathcal{A}, \partial_{\mathsf{F}_{\mathfrak{D}}})$. Ignoring the differential, $\mathsf{F}_{\mathfrak{D}}\mathcal{A}$ equals $\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}^{\#}$, the free $\check{\mathfrak{D}}$-modular operad on the linear dual $\mathcal{A}^{\#}$ of $\mathcal{A}$, where $\check{\mathfrak{D}}$ is the cocycle introduced in (5.64). The differential $\partial = \partial_{\mathsf{F}_{\mathfrak{D}}}$ is the sum $\partial_I + \partial_E$, where ∂_I (the internal part) is induced in the standard way from the differential of $\mathcal{A}$ and ∂_E is defined as follows.

Since $\mathcal{A}$ is, by assumption, a $\mathfrak{D}$-modular operad, we are given, for each graph $\Gamma \in \mathbf{\Gamma}((g,S))$, the structure map $\alpha_\Gamma : \mathfrak{D}(\Gamma) \otimes \mathcal{A}((\Gamma)) \to \mathcal{A}((g(\Gamma), Leg(\Gamma)))$. This structure map induces, for each $f : \Gamma_0 \to \Gamma_1$, a morphism (a 'twisted' analog of the morphism of (5.41))

$$\alpha((f)) : \mathfrak{D}(\Gamma_0) \otimes \mathcal{A}((\Gamma_0)) \to \mathfrak{D}(\Gamma_1) \otimes \mathcal{A}((\Gamma_1))$$

given as the composition

$$\begin{aligned}
\mathfrak{D}(\Gamma_0) \otimes \mathcal{A}((\Gamma_0)) &\xrightarrow{\nu_f^{-1} \otimes \mathbb{1}} \mathfrak{D}(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \mathfrak{D}(f^{-1}(v)) \otimes \mathcal{A}((\Gamma_0)) \\
&\cong \mathfrak{D}(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \Big(\mathfrak{D}(f^{-1}(v)) \otimes \mathcal{A}((f^{-1}(v))) \Big) \\
&\xrightarrow{\mathbb{1} \otimes \bigotimes_v \nu_{f|f^{-1}(v)}} \mathfrak{D}(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} \mathcal{A}((g(v), Leg(v))) \cong \mathfrak{D}(\Gamma_1) \otimes \mathcal{A}((\Gamma_1)).
\end{aligned}$$

The above definition makes sense, since $\mathfrak{D}$ is, by assumption, a cocycle, thus the inverse of ν_f exists. The functoriality of the map $\alpha((-))$ is obvious. Let us denote

$$\alpha_{\Gamma,e} := \alpha((\pi_e)) : \mathfrak{D}(\Gamma)\otimes\mathcal{A}((\Gamma))\longrightarrow\mathfrak{D}(\Gamma/e)\otimes\mathcal{A}((\Gamma/e)).$$

Using the identification $\mathfrak{D}(\Gamma)^{\#} \cong \mathfrak{D}^{-1}(\Gamma)$, we may write the dual of $\alpha_{\Gamma,e}$ as

$$(\alpha_{\Gamma,e})^{\#} : \mathfrak{D}(\Gamma/e)^{-1}\otimes\mathcal{A}((\Gamma/e))^{\#}\longrightarrow\mathfrak{D}(\Gamma)^{-1}\otimes\mathcal{A}((\Gamma))^{\#}.$$

There is another natural degree -1 map $\varepsilon_{\Gamma,e} : \mathfrak{K}(\Gamma/e) \to \mathfrak{K}(\Gamma)$ given by tensoring with the basic element $\downarrow$e of $\mathfrak{K}(\{e\}) = Det(\{e\})$ as

$$\begin{aligned}\varepsilon_{\Gamma,e} : \mathfrak{K}(\Gamma/e) &\xrightarrow{\downarrow e\otimes 1\!\!1} Det(\{\mathrm{e}\})\otimes\mathfrak{K}(\Gamma/\mathrm{e}) \cong Det(\{e\})\otimes Det(edge(\Gamma/e)) \\ &\cong Det\Big(\{\mathrm{e}\} \sqcup edge(\Gamma/\mathrm{e})\Big) \cong Det(edge(\Gamma)) = \mathfrak{K}(\Gamma).\end{aligned}$$

Let us denote the tensor product of $\varepsilon_{\Gamma,e}$ with the dual of $\alpha_{\Gamma,e}$ by $\partial_{\Gamma,e}$:

$$\begin{aligned}\partial_{\Gamma,e} : \check{\mathfrak{D}}(\Gamma/e)\otimes\mathcal{A}((\Gamma/\mathrm{e}))^{\#} &= \mathfrak{K}(\Gamma/\mathrm{e})\otimes\mathfrak{D}^{-1}(\Gamma/\mathrm{e})\otimes\mathcal{A}((\Gamma/\mathrm{e}))^{\#} \\ &\xrightarrow{\varepsilon_{\Gamma,e}\otimes(\alpha_{\Gamma,e})^{\#}} \mathfrak{K}(\Gamma)\otimes\mathfrak{D}^{-1}(\Gamma)\otimes\mathcal{A}((\Gamma))^{\#} = \check{\mathfrak{D}}(\Gamma)\otimes\mathcal{A}((\Gamma))^{\#}.\end{aligned}$$

The vector space $\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}((g,S))$ is the sum

$$\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}((g,S)) \cong \bigoplus_{\gamma\in\{\mathbf{\Gamma}((g,S))\}} \Big(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^{\#}\Big)_{Aut(\Gamma_\gamma)} \tag{5.68}$$

over representatives of the isomorphism classes of elements of $\mathbf{\Gamma}((g,S))$ (compare formula (5.34) for $\mathbb{M}\mathcal{A}$). For two classes $\gamma,\delta \in \{\mathbf{\Gamma}((g,S))\}$, the 'matrix element'

$$\partial_{\gamma,\delta} : \mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^{\#} \to \mathfrak{K}(\Gamma_\delta)\otimes\mathfrak{D}^{-1}(\Gamma_\delta)\otimes\mathcal{A}((\Gamma_\delta))^{\#}$$

is defined as the summation of all $\partial_{\Gamma_\delta,e}$ over all edges e of Γ_δ such that $\Gamma_\gamma \cong \Gamma_\delta/\mathrm{e}$. If no such an edge exists, we put $\partial_{\gamma,\delta} := 0$. Of course, $\partial_{\gamma,\delta}$ vanishes if $|edge(\Gamma_\gamma)| \neq |edge(\Gamma_\delta)| - 1$. The map $\partial_{\gamma,\delta}$ descends to the map (denoted by the same symbol)

$$\begin{aligned}\partial_{\gamma,\delta}: &\Big(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^{\#}\Big)_{Aut(\Gamma_\gamma)} \\ &\longrightarrow \Big(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^{\#}\Big)_{Aut(\Gamma_\delta)}\end{aligned} \tag{5.69}$$

because any automorphism ϕ of $\Gamma_\gamma \cong \Gamma_\delta/\mathrm{e}$ clearly lifts to some automorphism $\tilde{\phi} \in Aut(\Gamma_\delta)$. The map in (5.69) also obviously does not depend on the choice of an isomorphism $\Gamma_\gamma \cong \Gamma_\delta/\mathrm{e}$, because any two such isomorphisms differ by an automorphism of Γ_γ.

Finally, define ∂_E on the component

$$\Big(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^{\#}\Big)_{Aut(\Gamma_\gamma)}$$

of (5.68) as the sum

$$\partial_E := \sum_{\delta\in\{\mathbf{\Gamma}((g,S))\}} \partial_{\gamma,\delta}.$$

This sum is finite, since, for each Γ_γ, there clearly exist only finitely many couples (δ,e), $\delta \in \{\mathbf{\Gamma}((g,S))\}$, $\mathrm{e} \in edge(\Gamma_\delta)$, such that $\Gamma_\gamma \cong \Gamma_\delta/\mathrm{e}$. Here is the main definition of this section.

DEFINITION 5.58. The object $\mathsf{F}_{\mathfrak{D}}(\mathcal{A}) = (\mathbb{M}_{\check{\mathfrak{D}}}(\mathcal{A}^{\#}), \partial)$ with $\partial := \partial_E + \partial_I$ is called the *Feynman transform* of the $\mathfrak{D}$-modular operad $\mathcal{A}$.

THEOREM 5.59. *The Feynman transform* $\mathsf{F}_{\mathfrak{D}}$ *is a modular* $\check{\mathfrak{D}}$*-operad in the category of chain complexes.*

PROOF. It is clear that the 'matrix element'

$$(\partial^2)_{\gamma,\omega} : \Big(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^{\#}\Big)_{Aut(\Gamma_\gamma)} \longrightarrow \Big(\mathfrak{K}(\Gamma_\omega)\otimes\mathfrak{D}^{-1}(\Gamma_\omega)\otimes\mathcal{A}((\Gamma_\omega))^{\#}\Big)_{Aut(\Gamma_\omega)}$$

can be written as

$$(\partial^2)_{\gamma,\omega} = \sum_{e_1,e_2} (\partial_{\Gamma_\omega,e_1} \circ \partial_{\Gamma_\omega/e_1,e_2} + \partial_{\Gamma_\omega,e_2} \circ \partial_{\Gamma_\omega/e_2,e_1}),$$

where the summation is taken over pairs $\{e_1, e_2\}$ of distinct edges of Γ_ω such that $\Gamma_\gamma \cong \Gamma_\omega/\{e_1, e_2\}$. Thus, in order to prove that $\partial_E^2 = 0$, it is enough to show that

$$\partial_{\Gamma_\omega,e_1} \circ \partial_{\Gamma_\omega/e_1,e_2} = -\partial_{\Gamma_\omega,e_2} \circ \partial_{\Gamma_\omega/e_2,e_1} \tag{5.70}$$

which is, by definition, the same as to show that

$$\Big(\varepsilon_{\Gamma_\omega,e_1}\otimes(\alpha_{\Gamma_\omega,e_1})^{\#}\Big) \circ \Big(\varepsilon_{\Gamma_\omega/e_1,e_2}\otimes(\alpha_{\Gamma_\omega/e_1,e_2})^{\#}\Big) = -\Big(\varepsilon_{\Gamma_\omega,e_2}\otimes(\alpha_{\Gamma_\omega,e_2})^{\#}\Big) \circ \Big(\varepsilon_{\Gamma_\omega/e_2,e_1}\otimes(\alpha_{\Gamma_\omega/e_2,e_1})^{\#}\Big). \tag{5.71}$$

The functoriality of $\alpha((-))$ implies that

$$\alpha_{\Gamma_\omega/e_1,e_2} \circ \alpha_{\Gamma_\omega,e_1} = \alpha_{\Gamma_\omega/e_2,e_1} \circ \alpha_{\Gamma_\omega,e_2}$$

thus (5.71) will follow from

$$\varepsilon_{\Gamma_\omega,e_1} \circ \varepsilon_{\Gamma_\omega/e_1,e_2} = -\varepsilon_{\Gamma_\omega,e_2} \circ \varepsilon_{\Gamma_\omega/e_2,e_1}. \tag{5.72}$$

By definition, $\varepsilon_{\Gamma_\omega,e_1} \circ \varepsilon_{\Gamma_\omega/e_1,e_2}$ maps $x \in \mathfrak{K}(\Gamma_\omega/\{e_1, e_2\})$ to $\downarrow e_1\otimes \downarrow e_2\otimes x$, which is an element of

$$Det(\{e_1\})\otimes Det(\{e_2\})\otimes Det(edge(\Gamma_\omega/\{e_1, e_2\})) \cong Det(edge(\Gamma_\omega)) = \mathfrak{K}(\Gamma_\omega).$$

Similarly, $\varepsilon_{\Gamma_\omega,e_2} \circ \varepsilon_{\Gamma_\omega/e_2,e_1}$ sends $x \in \mathfrak{K}(\Gamma_\omega/\{e_1, e_2\})$ to $\downarrow e_2\otimes \downarrow e_1\otimes x$, which is an element of

$$Det(\{e_2\})\otimes Det(\{e_1\})\otimes Det(edge(\Gamma_\omega/\{e_1, e_2\})) \cong Det(edge(\Gamma_\omega)) = \mathfrak{K}(\Gamma_\omega)$$

Equation (5.72) now follows from $\downarrow e_1\otimes \downarrow e_2 = - \downarrow e_2\otimes \downarrow e_1$. This also explains why we had to introduce the coefficient system $\mathfrak{K}$ into the definition of the Feynman transform.

Since all constructions involved in the definition of ∂_E were made in the category of graded differential vector spaces, ∂_E and ∂_I must commute as degree -1 objects, i.e. $\partial_E\partial_I + \partial_I\partial_E = 0$. Thus

$$\partial^2 = (\partial_E + \partial_I)^2 = \partial_E^2 + (\partial_E\partial_I + \partial_I\partial_E) + \partial_I^2 = 0.$$

The last thing that remains to be proved is that the differential ∂ is compatible with the $\check{\mathfrak{D}}$-operad structure on $\mathbb{M}_{\check{\mathfrak{D}}}$, that is, the triple multiplication $\mu_{\mathcal{A}^{\#}} : (\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}^{\#} \to \mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}^{\#}$ commutes with the differentials. The proof is quite straightforward.

By Proposition 5.46, $(\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}^{\#}((g,S))$ is the colimit

$$\operatorname*{colim}_{[\Gamma_0 \xrightarrow{f} \Gamma_1] \in Iso_1\mathbf{\Gamma}((g,S))} \Big(\check{\mathfrak{D}}(\Gamma_1)\otimes \bigotimes_{v\in Vert(\Gamma_1)} \check{\mathfrak{D}}(f^{-1}(v))\otimes\mathcal{A}((\Gamma_0))^{\#}\Big).$$

Because $\check{\mathfrak{D}}$ is a cocycle, for each $[\Gamma_0 \xrightarrow{f} \Gamma_1] \in Iso_1\mathbf{\Gamma}((g,S))$ is the product

$$\check{\mathfrak{D}}(\Gamma_1)\otimes \bigotimes_{v\in Vert(\Gamma_1)} \check{\mathfrak{D}}(f^{-1}(v))$$

canonically and functorially isomorphic to $\check{\mathfrak{D}}(\Gamma_0)$, thus the above colimit can be written as

$$(\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}((g,S))^{\#} = \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f} \Gamma_1] \in Iso_1\mathbf{\Gamma}((g,S))} \Big(\check{\mathfrak{D}}(\Gamma_0)\otimes\mathcal{A}((\Gamma_0))^{\#}\Big).$$

The triple product to

$$\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}((g,S))^{\#} = \operatorname*{colim}_{[\Gamma_0] \in Iso\,\mathbf{\Gamma}((g,S))} \Big(\check{\mathfrak{D}}(\Gamma_0)\otimes\mathcal{A}((\Gamma_0))^{\#}\Big)$$

is then given by 'forgetting' the graph Γ_1 in the colimit. The statement is now almost tautological, because the definition of ∂ on $(\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}^{\#}$ does not involve any data related to the graph Γ_1. ■

Let us state without proof, which can be found in [**GK98**], an important homotopy property of the Feynman transform. We call a morphism $f : \mathcal{A} \to \mathcal{B}$ of differential $\mathfrak{D}$-modular operads a *weak homotopy equivalence* if

$$f(g,n) : \mathcal{A}(g,n) \to \mathcal{B}(g,n)$$

is a homology isomorphism, for each g and n.

PROPOSITION 5.60. *The Feynman transform is a homotopy functor. This means that, given a weak homotopy equivalence $f : \mathcal{A} \to \mathcal{B}$, the induced map $\mathsf{F}_{\mathfrak{D}}(f) : \mathsf{F}_{\mathfrak{D}}(\mathcal{A}) \to \mathsf{F}_{\mathfrak{D}}(\mathcal{B})$ is also a weak homotopy equivalence. Moreover, there is a canonical weak homotopy equivalence $\tau : \mathsf{F}_{\check{\mathfrak{D}}}\mathsf{F}_{\mathfrak{D}}(\mathcal{A}) \to \mathcal{A}$.*

5.5. Application: graph complexes

The aim of this section is to study moduli spaces of graphs and objects they represent. We then recall the graph complexes of M. Kontsevich which calculate rational homology of these spaces and show that these complexes are in fact particular cases of the Feynman transform of Section 5.4. We also explain how the graph complexes are related to Chevalley-Eilenberg cohomology of certain infinite dimensional Lie algebras. The exposition is based on the original papers of M. Kontsevich [**Kon94, Kon93**] as well as on [**GK98**] and [**Mar99b**].

RIBBON GRAPHS AND RIEMANN SURFACES. We first show that ribbon graphs describe certain moduli spaces of decorated Riemann surfaces. We then introduce a chain complex calculating rational cohomology of these spaces.

Let us fix a smooth oriented genus g surface $F_{g,s}$ with s distinct numbered points $p_1,\dots,p_s$. Let us assume the usual stability condition

$$2g-2+s>0$$

which excludes the cases $g = 0$ and $s \leq 2$ and $(g,s) = (1,0)$. Let us denote $F_{g,s}^- := F_{g,s} - \{p_1, \dots, p_s\}$. Recall that the *Teichmüller space* $\mathcal{T}_{g,s}$ is the space of all pairs $(X, [f])$, where $[f]$ is the homotopy class of a homeomorphism $f : X \longrightarrow F_{g,s}^-$ and X is a finite-area complete hyperbolic surface. The homotopy class $[f]$ is called the *marking* of the surface X. Two elements $(X, [f])$ and $(Y, [g])$ of $\mathcal{T}_{g,s}$ are taken to be equivalent if there exists an isometry $h : X \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & F_{g,s}^- & \end{tikzcd} \tag{5.73}$$

is homotopy commutative. The topology of $\mathcal{T}_{g,s}$ is induced by Fenchel-Nielson coordinates which identify $\mathcal{T}_{g,s}$ with the space $(\mathbb{R} \times \mathbb{R}_{>0})^{3g-3+s}$; see [**Har88**, Theorem 1.1].

A *decoration* of a point $(X, [f])$ of $\mathcal{T}_{g,s}$ is a choice of a horocycle around each puncture of X. A *horocycle* about a puncture is, by definition, a simple closed geodesic that divides X into two components in such a way that one of these components is a disc minus the puncture. Let us denote by $\mathcal{T}_{g,s}^{\mathrm{dec}}$ the space of equivalence classes of these decorated surfaces and call it the *decorated Teichmüller space*.

There is an obvious fibration $\varphi : \mathcal{T}_{g,s}^{\mathrm{dec}} \longrightarrow \mathcal{T}_{g,s}$ given by forgetting the decorations. Since a horocycle around a puncture is determined by its length, a decoration is in fact given by a choice of positive real numbers $(a_1, \dots, a_s)$. This means that the fibration φ is trivial with fiber $\mathbb{R}_{>0}^{\times s}$.

The dimension of $\mathcal{T}_{g,s}^{\mathrm{dec}}$ is easy to determine – there are $6g - 6 + 2s$ Fenchel-Nielson coordinates for $\mathcal{T}_{g,s}$ plus s coordinates for the decorations, so the dimension must be $6g - 6 + 3s$. We see that in fact (see also [**Pen87**, Theorem 3.1]):

THEOREM 5.61. *The decorated Teichmüller space $\mathcal{T}_{g,s}^{\mathrm{dec}}$ is homeomorphic to the space $\mathbb{R}_{>0}^{\times 6g-6+3s}$.*

There is an equivalent alternative description of Teichmüller space $\mathcal{T}_{g,s}$ in terms of complex structures. Elements are again pairs $(X, [f])$, but X is now a Riemann surface with distinct ordered points $(q_1, \dots, q_s)$ and $[f]$ is a homotopy class of a homeomorphism $f : X \longrightarrow F_{g,s}$ (observe that we now use $F_{g,s}$ and not $F_{g,s}^-$ as in the hyperbolic case) such that $f(q_i) = p_i$ for each $1 \leq i \leq s$. The equivalence is defined as in (5.73) except that h is now a conformal homeomorphism preserving marked points.

A decoration of a pair $(X, [f])$ is given, as before, by a choice of positive real numbers $(a_1, \dots, a_s)$. Such a decoration is now equivalent to a choice of a *Strebel differential* on X whose definition we briefly recall. A *quadratic differential* is a meromorphic section of the second symmetric power $K_R^{\otimes 2}$ of the canonical bundle K_R, that is, it locally behaves as $\zeta(z)(dz)^2$, with a meromorphic function ζ.

As proved by K. Strebel in [**Str84**], for each Riemann surface X with distinct numbered points $q_1, \dots, q_s$ and for each choice of positive real numbers $a_1, \dots, a_s$ there exists a unique quadratic differential ω on R which is holomorphic outside

$\{q_1, \ldots, q_s\}$ and which has a quadratic pole at each distinguished point q_i. Moreover for each sufficiently small simple loop circling around the pole q_i one assumes that

$$a_j = \oint_\alpha \sqrt{\omega};$$

see [**MP98**, Theorem 4.2] for details. Such a quadratic differential is called a Strebel differential.

Let $MC_{g,s}$ be the mapping class group, that is, the group of isotopy classes of orientation-preserving homeomorphisms of $F_{g,s}$ (which may permute the punctures). The mapping class group acts on $\mathcal{T}_{g,s}$ by

$$[g](X, [f]) := (X, [gf]), \quad \text{for } [g] \in MC_{g,s} \text{ and } (X, [f]) \in \mathcal{T}_{g,s}.$$

This action clearly extends over $\mathcal{T}^{\mathrm{dec}}_{g,s}$ and the fibration φ is $MC_{g,s}$-equivariant. The moduli space $\mathfrak{M}^{\mathrm{dec}}_{g,s} := \mathcal{T}^{\mathrm{dec}}_{g,s}/MC_{g,s}$ is the moduli space of genus g Riemann surfaces with s distinct <u>unlabeled</u> punctures decorated by positive real numbers.

There is of course also the standard nondecorated version of the above moduli space, $\mathfrak{M}_{g,s} := \mathcal{T}_{g,s}/MC_{g,s}$, the base of the obvious fibration $\pi : \mathcal{M}_{g,s} \to \mathfrak{M}_{g,s}$, where $\mathcal{M}_{g,s}$ is the moduli space of genus g Riemann surfaces with s distinct <u>labeled</u> punctures which we will discuss in more detail in Section 5.6. Both fibrations mentioned above:

$$\begin{tikzcd} \mathbb{R}^{\times s}_{>0} \arrow[r, hook] & \mathfrak{M}^{\mathrm{dec}}_{g,s} \arrow[dr] & & \mathcal{M}_{g,s} \arrow[dl] & \Sigma_s \arrow[l, hook'] \\ & & \mathfrak{M}_{g,s} & & \end{tikzcd}$$

induce isomorphisms of rational cohomology

$$H^*(\mathfrak{M}^{\mathrm{dec}}_{g,s}) \cong H^*(\mathfrak{M}_{g,s}; \mathbb{Q}) \cong H^*(\mathcal{M}_{g,s}; \mathbb{Q}). \tag{5.74}$$

We are going to describe a neat combinatorial model for $\mathfrak{M}^{\mathrm{dec}}_{g,s}$ whose existence follows from the results of [**Har88**] and [**Pen87**] and which is defined in terms of ribbon graphs. Let us say first what a ribbon graph is.

By a *ribbon graph* (also called a *fat graph*) we mean a connected graph Γ with fixed cyclic orders on the set $edge(v)$ of half-edges attached to each vertex $v \in Vert(\Gamma)$. We assume that all vertices are at least ternary, that is, $|edge(v)| \geq 3$ for each $v \in Vert(\Gamma)$. We also assume that graphs considered in this section have no legs (i.e. all edges are internal), though this assumption can be relaxed at the cost of minor technical complications.

One can associate an oriented surface with boundary, $Surf(\Gamma)$, to each ribbon graph Γ by replacing edges by thin oriented rectangles (ribbons) and gluing them together at all vertices according to the chosen cyclic order. Let $RGr_{g,s}$ denote the set of all isomorphism classes of ribbon graphs Γ such that the surface $Surf(\Gamma)$ has s holes and genus g. See Figure 8 for examples of ribbon graphs. Each graph $\Gamma \in RGr_{g,s}$ clearly satisfies

$$|Vert(\Gamma)| - |edge(\Gamma)| = 2 - 2g - s.$$

A *metric* on a ribbon graph Γ is a map l from the set of edges of Γ to the set of positive real numbers $\mathbb{R}_{>0}$. We denote by $RGr^{\mathrm{met}}_{g,s}$ the set of ribbon graphs $\Gamma \in RGr_{g,s}$ with a metric. There is an obvious fibration of $RGr^{\mathrm{met}}_{g,s}$ over $RGr_{g,s}$;

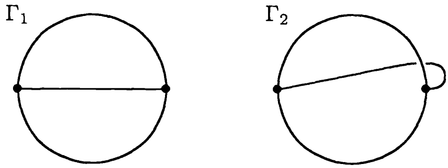


FIGURE 8. Two examples of ribbon graphs. The cyclic order of edges at vertices is induced by the anticlockwise orientation of the plane. The left graph Γ_1 represents the sphere minus three distinct points, $\Gamma_1 \in RGr_{0,3}$, the right graph Γ_2 the torus minus a point, $\Gamma_2 \in RGr_{1,1}$.

the fiber σ_Γ over a point $\Gamma \in RGr_{g,s}$ is called in [**MP98**] a *rational cell*,

$$\sigma_\Gamma \cong \frac{\mathbb{R}_{>0}^{|edge(\Gamma)|}}{Aut(\Gamma)}.$$

We will also need the cover $\tilde{\sigma}_\Gamma$ of σ_Γ,

$$\tilde{\sigma}_\Gamma := \mathbb{R}_{>0}^{|edge(\Gamma)|} \longrightarrow \sigma_\Gamma. \tag{5.75}$$

Intuitively, when a length of an edge of Γ tends to zero, the graph degenerates to a new graph Γ', which determines a cell $\sigma_{\Gamma'}$ on the boundary of σ_Γ together with the gluing map. This gives $RGr_{g,s}^{\mathrm{met}}$ a structure of an *orbifold* (also called a *V-manifold*), that is, a topological space locally modeled by a Euclidean space (in this case by $\mathbb{R}^{\times 6g-6+3s}$) modulo a finite group action; see [**MP98, Sat56**] for details.

The following theorem can be proved by either using methods of hyperbolic geometry as in [**Pen87**, Theorem 5.5] or using Strebel differentials as in [**MP98**].

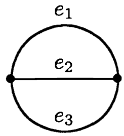
THEOREM 5.62. *The space $RGr_{g,s}^{\mathrm{met}}$ is, as an orbifold, canonically isomorphic to $\mathfrak{M}_{g,s}^{\mathrm{dec}}$.*

EXAMPLE 5.63. Let us discuss, following [**MP98**], Teichmüller space $\mathcal{T}_{0,3}^{\mathrm{dec}}$ and the related decorated moduli space $\mathfrak{M}_{0,3}^{\mathrm{dec}}$ Since there is only one complex structure on the sphere S^2 and the group of holomorphic automorphisms acts simply transitively on triples of distinct points of S^2, the decorated Teichmüller space $\mathcal{T}_{0,3}^{\mathrm{dec}}$ is just $\mathbb{R}_{>0}^{\times 3}$ with coordinates (a_0, a_1, a_∞) and an obvious action of the symmetric group Σ_3. The moduli space $\mathfrak{M}_{0,3}^{\mathrm{dec}}$ is then $\mathbb{R}_{>0}^{\times 3}/\Sigma_3$. Types of the corresponding metric ribbon graphs are classified by the *discriminant*

$$D := (a_0 + a_1 - a_\infty)(a_1 + a_\infty - a_0)(a_\infty + a_0 - a_1) \in \mathbb{R}.$$

We distinguish three cases, two regular and one singular.

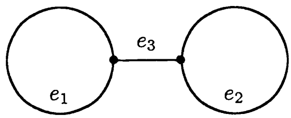
CASE 1. $D > 0$. Clearly this happens if and only if all three factors of D are positive. The corresponding metric ribbon graph graph is



$l(e_1) = a_0 + a_1 - a_\infty$,
$l(e_2) = a_1 + a_\infty - a_0$ and
$l(e_3) = a_\infty + a_0 - a_1$,

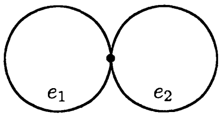
where $l(-)$ denotes the length of the corresponding edge.

CASE 2. $D < 0$. This happens if precisely one factor of D is negative and two factors are positive. Suppose for instance that $a_0 + a_1 - a_\infty < 0$. Then the corresponding metric ribbon graph is



$l(e_1) = a_0$,
$l(e_2) = a_1$ and
$l(e_3) = \frac{1}{2}(a_\infty - a_0 - a_1)$.

THE DEGENERATE CASE $D = 0$. This is the case when precisely one of the factors of D, say $a_0 + a_1 - a_\infty$, is zero. The corresponding graph is



$l(e_1) = a_0$ and
$l(e_2) = a_1$.

We conclude that $RGr_{0,3}^{\mathrm{met}} \cong \mathbb{R}_{>0}^{\times 3}/\Sigma_3$ has four open rational three-dimensional cells, one corresponding to Case 1 and three corresponding to Case 2. They intersect at three two-dimensional rational cells corresponding to the degenerate case. See again [**MP98**] for more details.

We are going to describe an algebraic chain complex that calculates the rational cohomology of $RGr_{g,s}^{\mathrm{met}}$. Our exposition follows [**Kon94, Kon93**] where the details can be found.

The space $RGr_{g,s}^{\mathrm{met}}$ is a noncompact but smooth orbifold of virtual dimension $6g-6+2s$. It is not oriented, but its orientation sheaf ϵ is easy to describe – its fiber over the covering cell $\tilde{\sigma}_\Gamma$ of (5.75) is $\wedge^s(H_1(|\Gamma|;\mathbb{Q}))$. Poincaré duality for orbifold (co)homology with closed supports ([**Bre67**, V.9.2], [**Sat56**]) gives the isomorphism

$$H^{\mathrm{closed}}_{6g-6+3s-*}(RGr_{g,s}^{\mathrm{met}};\epsilon) \cong H^*(RGr_{g,s}^{\mathrm{met}};\mathbb{Q}). \tag{5.76}$$

The homology $H_*^{\mathrm{closed}}(RGr_{g,s}^{\mathrm{met}};\epsilon)$ can be computed using a chain complex $RGC_*^{g,s} = (RGC^{g,s}, \partial)$ arising from the spectral sequence of the stratification given by rational cells. This chain complex is generated by ribbon graphs $\Gamma \in RGr_{g,s}$ endowed with an orientation that reflects the orientation induced by the orientation sheaf ϵ of the corresponding covering cell $\tilde{\sigma}_\Gamma$. To describe this orientation in terms of the graph Γ itself, observe that, by definition, the ϵ-orientation is given by an orientation of $H_1(|\Gamma|;\mathbb{Q})$, while the cell $\tilde{\sigma}_\Gamma$ has coordinates 'indexed' by edges of the graph Γ, that is, its orientation is given by an orientation of the vector space $\mathbb{Q}^{edge(\Gamma)}$. Thus an orientation of Γ must be given by the superposition of these two data. We formulate

DEFINITION 5.64. An *orientation of a graph* Γ is an orientation of the vector space

$$\mathbb{Q}^{edge(\Gamma)} \oplus H_1(|\Gamma|;\mathbb{Q}).$$

The orientation described in Definition 5.64 of course differs from what is usually meant by an orientation of a graph. While it is easy to say what an orientation of $\mathbb{Q}^{edge(\Gamma)}$ is – it might be given for example by an order of edges of Γ – it is more difficult to work with $H_1(|\Gamma|;\mathbb{Q})$. Therefore the following proposition which in fact compares orientations used by Kontsevich [**Kon94, Kon93**] to those used by Penkava [**Pen96**] will be useful.

PROPOSITION 5.65. *The orientation in Definition 5.64 is given by ordering the vertices of* Γ *and assigning an orientation to each edge of* Γ. *Two orientations are, of course, assumed to be the same if they differ by an even number of changes.*

PROOF. Consider the complex of oriented cells of the cellular complex $|\Gamma|$, $\mathbb{Q}^{Vert(\Gamma)} \longleftarrow \mathbb{Q}^{\mathrm{or}(\Gamma)}$, where $\mathrm{or}(\Gamma)$ is the set of oriented edges of Γ. It is a part of the long exact sequence

$$0 \longleftarrow \mathbb{Q} \longleftarrow \mathbb{Q}^{Vert(\Gamma)} \longleftarrow \mathbb{Q}^{\mathrm{or}(\Gamma)} \longleftarrow H_1(|\Gamma|;\mathbb{Q}) \longleftarrow 0. \tag{5.77}$$

A moment's reflection convinces us that an orientation of the vector space $\mathbb{Q}^{\mathrm{or}(\Gamma)}$ is given by an orientation of the edges of Γ and by an orientation of $\mathbb{Q}^{edge(\Gamma)}$. On the other hand, the exact sequence (5.77) implies that an orientation of $\mathbb{Q}^{\mathrm{or}(\Gamma)}$ is given by an orientation of $H_1(|\Gamma|;\mathbb{Q})$ and an orientation of $\mathbb{Q}^{Vert(\Gamma)}$. The proposition is then a combination of these two pieces of information. ∎

Let us finally introduce the *ribbon graph complex* $RGC^{g,s}_* = (RGC^{g,s}_*, \partial)$ so that the degree k piece $RGC^{g,s}_k$ is the $\mathbb{Q}$-vector space generated by isomorphism classes of oriented ribbon graphs $\Gamma \in RGr_{g,s}$ with k edges, normalized subject to the relation

$$\Gamma^- = -\Gamma, \tag{5.78}$$

where Γ^- is the graph Γ taken with the opposite orientation. The differential $\partial : RGC^{g,s}_{*+1} \to RGC^{g,s}_*$ is defined as follows.

For each generator $\Gamma \in RGr_{g,s}$ and an edge e joining two distinct vertices of Γ, let Γ/e be the graph obtained by collapsing e to a point. Observe that this new graph is again an element of $RGr_{g,s}$, because contracting e does not change the topology of $Surf(\Gamma)$. Suppose that Γ is oriented in the above sense, with vertices numbered by $1,\dots,m$, that is, $Vert(\Gamma) = \{v_1,\dots,v_m\}$.

We define the orientation of Γ/e as follows. We may assume, applying a number of changes if necessary, that the edge e points from v_{m-1} towards v_m. The set $Vert(\Gamma/e)$ is isomorphic to $\{v_1,\dots,v_{m-2},v_e\}$, where v_e is the vertex created by collapsing e. See Figure 9.

We assign to v_e label $m-1$ (thus the vertex with label m disappears). Since $edge(\Gamma/e) \cong edge(\Gamma) - \{e\}$, the orientation of edges of Γ induces an orientation of edges of Γ/e in an obvious manner. Now ∂ is given by

$$\partial(\Gamma) := \sum_e \Gamma/e, \tag{5.79}$$

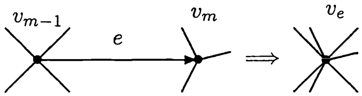


FIGURE 9. Contracting an edge e of an oriented ribbon graph Γ.

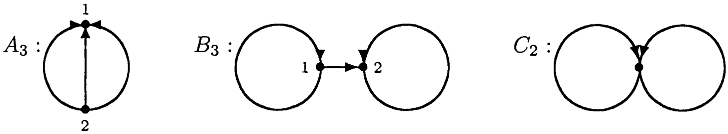


FIGURE 10. Generators of $RGC_*^{0,3}$.

the summation of oriented graphs Γ/e over all edges joining two distinct vertices of Γ. The loops of the graph Γ do not contribute to the sum. To prove that $\partial^2 = 0$ is an easy exercise and we can leave it to the reader. The following proposition follows from the above remarks.

PROPOSITION 5.66. *The complex* $RGC_*^{g,s} = (RGC_*^{g,s}, \partial)$ *calculates the rational cohomology of* $RGr_{g,s}^{\mathrm{met}}$,

$$H^*(RGr_{g,s}^{\mathrm{met}}; \mathbb{Q}) \cong H_{6g-6+3s-*}(RGC_*^{g,s}, \partial),\ 2g - 2 + s > 0.$$

Proposition 5.66, equation (5.74) and the fact that $\mathcal{M}_{g,s}$ is 'rationally' the classifying space $BMC_{g,s}$ of the mapping class group [**Har88**, p. 143] imply the following theorem which summarizes the above calculations.

THEOREM 5.67. *There are the following isomorphisms of rational cohomology:*

$$\begin{aligned} H_{6g-6+3s-*}(RGC_*^{g,s}, \partial) &\cong H^*(\mathfrak{M}_{g,s}^{\mathrm{dec}}; \mathbb{Q}) \\ &\cong H^*(\mathfrak{M}_{g,s}; \mathbb{Q}) \cong H^*(\mathcal{M}_{g,s}; \mathbb{Q}) \cong H^*(BMC_{g,s}, \mathbb{Q}). \end{aligned}$$

Maxim Kontsevich proved [**Kon94, Kon93**] that each strongly homotopy associative algebra with a scalar product (a cyclic A_∞-algebra) determines a class in the homology of $RGC_*^{g,s}$; see also [**Pen96**]. By Theorem 5.67, his construction gives cohomology classes in the rational cohomology of the moduli space $\mathcal{M}_{g,s}$ itself.

EXAMPLE 5.68. Let us calculate the homology of the complex $RGC_*^{0,3}$. As a vector space, $RGC_*^{0,3} \cong Span_{\mathbb{Q}}(A_3, B_3, C_2)$, with generators given by graphs in Figure 10, $\deg(A_3) = \deg(B_3) = 3$ and $\deg(C_2) = 2$.

Clearly $\partial(A_3) = \partial(B_3) = C_2$ and $\partial(C_2) = 0$, thus $H_*(RGC_*^{0,3}; \partial)$ is one-dimensional, concentrated in degree 3. Theorem 5.67 gives $H^*(\mathcal{M}_{0,3}; \mathbb{Q}) \cong \mathbb{Q}$ concentrated in degree zero which agrees with the fact that $\mathcal{M}_{0,3} =$ the point.

The graph complex $RGC_*^{g,s}$ also calculates the Chevalley-Eilenberg homology of a certain infinite dimensional Lie algebra a_∞ defined as follows. Let a_n be the Lie algebra of derivations θ of the free associative algebra without unit on $p_1, \ldots, p_n, q_1, \ldots, q_n$ that satisfy the condition

$$\theta\left(\sum_{i=1}^{n} p_i q_i - q_i p_i\right) = 0, \tag{5.80}$$

and let $a_\infty := \varinjlim a_n$ be the direct limit. M. Kontsevich proved [**Kon93**, Theorem 1.1]:

THEOREM 5.69. *The Chevalley-Eilenberg homology of a_∞ is a primitively generated Hopf-algebra structure whose primitives Prim $H_*(a_\infty)$ are*

$$\text{(5.81)} \qquad \mathit{Prim}\ H_k(a_\infty) = \mathit{Prim}\ H_k(sp(\infty)) \oplus \bigoplus_{\substack{s>0 \\ 2g-2+s>0}} H_{2g+s-1+k}(RGC_*^{g,s};\mathbb{Q}),$$

$k \geq 1$, where the primitive homology of the infinite symplectic group $sp(\infty)$ is well known:

$$\text{(5.82)} \qquad \mathit{Prim}\ H_k(sp(\infty)) = \begin{cases} \mathbb{Q}, & \text{for } k \equiv 3 \bmod 4, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

We are going to sketch the proof. To this end we need, in the first place, an alternative description of the algebra a_n. According to M. Kontsevich's interpretation, a_n is an algebra of Hamiltonian vector fields on a flat symplectic manifold in noncommutative geometry. He derived from this observation the following proposition, which we prove directly.

PROPOSITION 5.70. *Let $V = V_n := \mathit{Span}_{\mathbb{Q}}(p_1,\dots,p_n,q_1,\dots,q_n)$. Then, as an $sp(2n)$-module*

$$a_n \cong \bigoplus_{k\geq 2} (V^{\otimes k})^{\mathbb{Z}_k},$$

where the cyclic group $\mathbb{Z}_k$ acts on $V^{\otimes k}$ by permuting the factors and $(-)^{\mathbb{Z}_k}$ denotes, as usual, the space of invariants.

PROOF. Derivations of the free nonunital associative algebra

$$F_{\mathrm{nu}}(V) \cong \bigoplus_{m\geq 1} V^{\otimes m}$$

on V are uniquely determined by their restriction on the space of generators V. This clearly implies that the following formula defines a one-to-one correspondence between derivations $\theta \in \mathit{Der}(F_{\mathrm{nu}}(V))$ and elements ρ of the space $\bigoplus_{k\geq 2} V^{\otimes k}$:

$$\mathit{Der}(F_{\mathrm{nu}}(V)) \ni \theta \longleftrightarrow \rho := \sum_{1\leq i\leq n} p_i\theta(q_i) - q_i\theta(p_i) \in \bigoplus_{k\geq 2} V^{\otimes k}.$$

Let us show that the above correspondence maps a_n to the invariant subspace $\bigoplus_{k\geq 2}(V^{\otimes k})^{\mathbb{Z}_k}$, i.e. that $\theta \in a_n$ if and only if the corresponding ρ is invariant. By definition, such a θ belongs to a_n if and only if

$$\sum_{1\leq i\leq n} \theta(p_i)q_i + p_i\theta(q_i) - \theta(q_i)p_i - q_i\theta(p_i) = 0$$

which is the same as

$$\text{(5.83)} \qquad \sum_{1\leq i\leq n} p_i\theta(q_i) - q_i\theta(p_i) = \sum_{1\leq i\leq n} \theta(q_i)p_i - \theta(p_i)q_i.$$

We may safely assume that θ is homogeneous, $\theta(V) \subset V^{\otimes m}$. Let τ be the generator of $\mathbb{Z}_k$, $k = m+1$, then clearly

$$\tau\rho = \sum_{1\le i\le n} \theta(q_i)p_i - \theta(p_i)q_i,$$

therefore (5.83) says that $\tau\rho = \rho$, which means that ρ is $\mathbb{Z}_k$-invariant. ■

Let V be as in Proposition 5.70 and N a natural number. We will also need a description of $sp(2n)$-invariant elements on $V^{\otimes N}$. Let $\omega := \sum_{1\le i\le n} p_i \otimes q_i - q_i \otimes p_i \in V^{\otimes 2}$ be the symplectic form on the dual $V^\#$ of V. We will often write ω in 'Sweedler's form' (but with indices (1) and (2) written as superscripts)

$$\omega = \sum_{s\in S} \omega_s^{(1)} \otimes \omega_s^{(2)}, \tag{5.84}$$

where S is a (finite) set of indices.

Suppose that N is even, $N = 2K$. By a *decomposition of* $\{1,\dots,N\}$ *into pairs* we mean a choice of subsets $\{i_1,j_1\}$, $\{i_2,j_2\}$, $\dots$, $\{i_K,j_K\}$ of $\{1,\dots,N\}$, $i_1 < j_1$, $i_2 < j_2,\dots, i_K < j_K$, such that $\{i_1,j_1,\dots,i_K,j_K\} = \{1,\dots,N\}$. For such a decomposition and indices $s_1,\dots,s_K \in S^{\times K}$ we denote

$$\varphi_k = \begin{cases} \omega_{s_r}^{(1)}, & \text{if } k = i_r \text{ and}\\ \omega_{s_r}^{(2)}, & \text{if } k = j_r,\end{cases}$$

for $1 \le k \le N$ and, finally,

$$\omega_{i_1,j_1}\omega_{i_2,j_2}\cdots\omega_{i_K,j_K} := \sum_{s_1,\ \ ,s_K} \varphi_1\otimes\cdots\otimes\varphi_N \in V^{\otimes N}.$$

EXAMPLE 5.71. There are three decompositions of $\{1,2,3,4\}$ into pairs:

$$\{1,2\},\{3,4\},\ \ \{1,3\},\{2,4\} \text{ and } \{1,4\},\{2,3\}.$$

The corresponding $\omega_{i_1,j_1}\omega_{i_2,j_2} \in V^{\otimes 4}$ are

$$\omega_{1,2}\omega_{3,4} = \omega\otimes\omega,\quad \omega_{1,3}\omega_{2,4} = \textstyle\sum_{i,j} \omega_i^{(1)}\otimes\omega_j^{(1)}\otimes\omega_i^{(2)}\otimes\omega_j^{(2)} \text{ and}$$
$$\omega_{1,4}\omega_{2,3} = \textstyle\sum_i \omega_i^{(1)}\otimes\omega\otimes\omega_i^{(2)}.$$

Observe that the notation itself already includes some symmetry, for example, $\omega_{1,2}\omega_{3,4} = \omega_{3,4}\omega_{1,2}$.

The following description of the $sp(2n)$-invariant subspace of $V^{\otimes N}$ follows from general principles of representation theory [**Wey97**].

PROPOSITION 5.72. *Suppose n is sufficiently large. Then*

$$(V^{\otimes N})^{sp(2n)} = \begin{cases} 0, & \textit{if } N \textit{ is odd, and}\\ Span_{\mathbb{Q}}(\omega_{i_1,j_1}\omega_{i_2,j_2}\cdots\omega_{i_K,j_K}), & \textit{if } N = 2K,\end{cases}$$

where $i_1,j_1,\dots,i_K,j_K$ runs over all decomposition of $\{1,\dots,N\}$ into pairs.

In the next couple of pages we sketch a PROOF OF THEOREM 5.69. The Chevalley-Eilenberg complex $CE_*(a_n)$ of the Lie algebra a_n can, according to Proposition 5.70, be written as

$$CE_*(a_n) = \bigoplus_{k\ge 2} \wedge^*((V^{\otimes k})^{\mathbb{Z}_k}) = \bigoplus_{k_2,k_3,\ \ \ge 0} \wedge^{k_2}\left((V^{\otimes 2})^{\mathbb{Z}_2}\right)\wedge\wedge^{k_3}\left((V^{\otimes 3})^{\mathbb{Z}_3}\right)\wedge\cdots \tag{5.85}$$

Observe next that each Lie algebra acts trivially on its homology via the adjoint representation. This statement is probably well known and can be easily verified

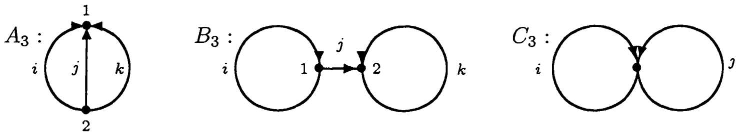


FIGURE 11. Graphs A_3, B_3 and C_3 with edges colored by summation indices.

directly. In particular, the subalgebra $sp(2n) \subset a_n$ acts trivially on a_n. Observe also that the action of $sp(2n)$ on $CE_*(a_n)$ is reductive.

It easily follows from the above observations that the $sp(2n)$-invariant subcomplex $CE_*(a_n)^{sp(2n)} \subset CE_*(a_n)$ is quasi-isomorphic to $CE_*(a_n)$. We will calculate the homology of a_n using this subcomplex. Observe that elements of $CE_*(a_n)^{sp(2n)}$ can be interpreted as elements of the space

$$J_*(a_n) := \bigoplus_{k_2,k_3,\ \geq 0} \left(V^{\otimes 2}\right)^{\otimes k_2} \wedge \left(V^{\otimes 3}\right)^{\otimes k_3} \wedge \cdots$$

enjoying a specific type of symmetry, namely the $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \cdots$-symmetry determining in (5.85) the subspace $CE_*(a_n)$ combined with the $sp(2n)$-symmetry of the subspace of invariants $CE_*(a_n)^{sp(2n)} \subset CE_*(a_n)$.

Let us show that, for n sufficiently large, ribbon graphs describe a basis of $CE_*(a_n)^{sp(2n)}$. To be more precise, let Γ be a ribbon graph, but this time Γ need not be connected and may also have binary vertices. At each vertex of valence m we imagine a copy of $V^{\otimes m}$. Let us now perform the 'state sum' by contracting along the edges of Γ using ω's. This state sum will be, by the antisymmetry of ω, an element of $J_*(a_n)$. Now we must 'average' this state sum over $\mathbb{Z}_m$ at each vertex of valence m. The result is an element ω_Γ of $CE_*(a_n)^{sp(2n)}$. We believe that the following example will make the construction clear.

EXAMPLE 5.73. Consider graph A_3 in Figure 10 and attach to its edges summation indices i, k, j as shown in Figure 11. There are three edges with indices i, k, j (in this cyclic order) coming to the vertex labeled 1, so the factor 'sitting' at this vertex is $\omega_i^{(2)}\omega_k^{(2)}\omega_j^{(2)}$. Similarly, there are three edges leaving the vertex labeled 2 with indices i, j, k (in this cyclic order), so the factor attached to this vertex is $\omega_i^{(1)}\omega_j^{(1)}\omega_k^{(1)}$. The resulting 'state sum' is $\omega_i^{(2)}\omega_k^{(2)}\omega_j^{(2)} \wedge \omega_i^{(1)}\omega_j^{(1)}\omega_k^{(1)} \in V^{\otimes 3} \wedge V^{\otimes 3} \subset J_2(a_n)$. So the $sp(2n)$-invariant element corresponding to A_3 is

$$\omega_{A_3} := \{\omega_i^{(2)}\omega_k^{(2)}\omega_j^{(2)}\}_{\mathbb{Z}_3} \wedge \{\omega_i^{(1)}\omega_j^{(1)}\omega_k^{(1)}\}_{\mathbb{Z}_3} \in \left(\wedge^2\left((V^{\otimes 3})^{\mathbb{Z}_3}\right)\right)^{sp(2n)},$$

where $\{-\}_{\mathbb{Z}_3}$ denotes the 'averaging' over $\mathbb{Z}_3$. Observe that this element has the requisite symmetry, it does not change sign if the orientation of the underlying graph changes and it is invariant with respect to cyclic permutations of labels of edges. Similarly, the invariant element corresponding to graph B_3 is

$$\omega_{B_3} := \{\omega_i^{(2)}\omega_i^{(1)}\omega_j^{(1)}\}_{\mathbb{Z}_3} \wedge \{\omega_k^{(1)}\omega_k^{(2)}\omega_j^{(2)}\}_{\mathbb{Z}_3} \in \left(\wedge^2\left((V^{\otimes 3})^{\mathbb{Z}_3}\right)\right)^{sp(2n)}$$

and the element corresponding to C_3 is

$$\omega_{C_2} := \{\omega_i^{(2)}\omega_i^{(1)}\}_{\mathbb{Z}_2} \wedge \{\omega_j^{(1)}\omega_j^{(2)}\}_{\mathbb{Z}_2} \in \left(\wedge^2\left((V^{\otimes 2})^{\mathbb{Z}_2}\right)\right)^{sp(2n)}.$$

By Proposition 5.72, if n is sufficiently large, ribbon graphs form a basis $\{\omega_\Gamma\}$ of $CE_*(a_n)^{sp(2n)}$. This is in particular true if we pass to the limit and calculate the homology of a_∞. So we identify $CE_*(a_\infty)^{sp(\infty)}$ with the vector space spanned by all (not necessarily connected) ribbon graphs modulo normalization (5.78). The Chevalley-Eilenberg differential translates to the differential of the graph complex (5 79).

The multiplication in $CE_*(a_\infty)^{sp(\infty)}$ is reflected by the disjoint union of graphs,

$$\omega_{\Gamma_1} \wedge \omega_{\Gamma_2} = \omega_{\Gamma_1 \sqcup \Gamma_2},$$

so *connected* graphs form the basis of the primitive subspace $Prim\ CE_*(a_\infty)^{sp(\infty)}$. We can decompose $Prim\ CE_*(a_\infty)^{sp(\infty)}$ as

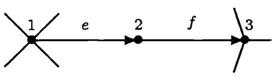
$$Prim\ CE_*(a_\infty)^{sp(\infty)} = C_1 \oplus C_2 \oplus C_3, \tag{5.86}$$

where

(i) C_1 is spanned by 'polygons,' i.e. connected graphs all of whose vertices are binary,
(ii) C_2 is spanned by graphs having at least one vertex of valence ≥ 3 and at least one binary vertex and
(iii) C_3 is spanned by graphs with no binary vertices, i.e. C_3 is, modulo grading, exactly the ribbon graph complex defined above.

Let us verify that decomposition (5.86) is compatible with the differentials. While it is obvious that $\partial(C_1) \subset C_1$ and $\partial(C_3) \subset C_3$, it is not so clear why $\partial(C_2) \subset C_2$.

If $\Gamma \in C_2$ has more than one binary vertex, then evidently $\partial(\Gamma) \in C_2$. If Γ has exactly one binary vertex, then it must contain (after an appropriate number of changes of labels of vertices and orientations of edges) a subgraph



There are two terms in $\partial(\Gamma)$ that have no binary vertex and thus might intersect with C_3, one given by contracting e, the second one by contracting f. Since clearly $\Gamma/e = -\Gamma/f$, these two terms cancel out, so all terms of $\partial(\Gamma)$ have at least one binary vertex, therefore $\partial(\Gamma) \subset C_2$. Similar arguments show that in fact C_2 *is acyclic.*

Let us turn our attention to C_1. It is generated by k-gons L_k, $k \geq 2$, with orientations indicated in Figure 12. The rotation of L_k by $\exp(2\pi i/k)$ clearly, for k even, reverses the orientation (see the octagon in Figure 12), so $L_k = 0$ in the graph complex for k even. The flip around the vertical axis reverses, for $k \equiv 1$ mod 4, the orientation (see the pentagon in Figure 12), so L_k is a nontrivial generator if and only if $k \equiv 3$ mod 4 (the triangle in Figure 12), therefore $H_*(C_1) = H_*(sp(\infty))$ as claimed.

As we have already observed, C_3 is the ribbon graph complex, but the gradings are different. If $x \in C_3$, then the Chevalley-Eilenberg degree $\deg_{CE}(x)$ and the degree $\deg_{RGC^{g,s}}(x)$ in the ribbon graph complex are related by

$$\deg_{CE}(x) = \deg_{RGC^{g,s}}(x) + 1 - 2g - s.$$

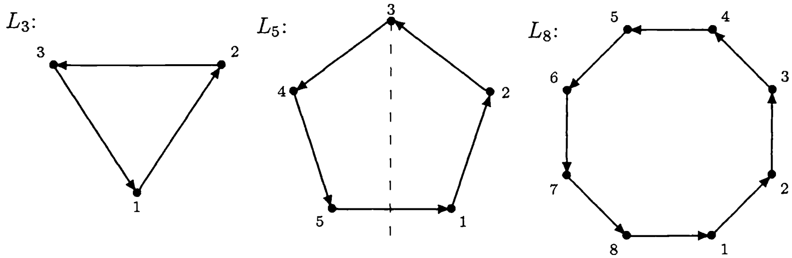


FIGURE 12. The triangle, pentagon and octagon.

Therefore

$$\begin{aligned} Prim\ H_k(a_\infty) &\cong H_k(Prim\ CE(a_\infty)^{sp(\infty)}) \cong H_k(C_1) \oplus H_k(C_2) \oplus H_k(C_3) \\ &\cong H_k(sp(\infty)) \oplus 0 \oplus \bigoplus_{\substack{s>0 \\ 2g-2+s>0}} H_{2g+s-1+k}(RGC_*^{g,s}; \mathbb{Q}). \end{aligned}$$

This finishes the proof. ■

GRAPHS AND AUTOMORPHISMS OF FREE GROUPS. We are going to consider another version of the graph complex, this time related to the classifying space of the group of outer automorphisms of a free group.

Let $Gr^{(n)}$ denote the set of isomorphism classes of connected graphs Γ (no ribbon structure assumed) with all vertices at least ternary and Euler characteristic $\chi(\Gamma) = 1 - n$. A *metric* on a graph $\Gamma \in Gr^{(n)}$ is defined exactly as before, that is, as a map from the set $edge(\Gamma)$ of edges to $\mathbb{R}_{>0}$. The set $Gr^{\mathrm{met}}_{(n)}$ of isomorphism classes of graphs from $Gr^{(n)}$ with a metric is a smooth orbifold of virtual dimension $3n - 3$. Therefore

$$H^{\mathrm{closed}}_{3n-3-*}(Gr^{\mathrm{met}}_{(n)}; \epsilon) \cong H^*(Gr^{\mathrm{met}}_{(n)}; \mathbb{Q}), \tag{5.87}$$

where the orientation sheaf ϵ is defined as in (5.76).

The orientation of $\Gamma \in Gr^{(n)}$ is given, as in Definition 5.64, by ordering the vertices of Γ and assigning an orientation to each edge of Γ. (Co)homology (5.87) can be calculated using the chain complex $GC_*^{(n)} = (GC_*^{(n)}, \partial)$ generated by all oriented graphs $\Gamma \in Gr^{(n)}$ modulo normalization (5.78). We observe the convention of [**Kon94, Kon93**] and assign to a generator $\Gamma \in GC_*^{(n)}$ degree $|Vert(\Gamma)|$. This grading differs from the grading of $RGC^{g,s}$ which was given by the number of edges. The differential is given by formula (5.79).

The following analog of Theorem 5.67 follows from the above remarks and from the fact that $Gr^{\mathrm{met}}_{(n)}$ is the rational classifying space of the group $OutFree(n)$ of outer automorphisms of the free group on n generators [**CV86**].

THEOREM 5.74. *The complex $GC_*^{(n)} = (GC_*^{(n)}, \partial)$ calculates the rational cohomology of the classifying space $B\,OutFree(n)$,*

$$H_{2n-2-*}(GC^{(n)}, \partial) \cong H^*(B\,OutFree(n); \mathbb{Q})$$

M. Kontsevich proved for the complex $GC_*^{(n)}$ a formula similar to (5.81), with the algebra $\mathfrak{a}_\infty$ replaced by $\mathfrak{c}_\infty$ defined as follows. Let $\mathfrak{c}_n$ be the Lie algebra of derivations of the polynomial algebra $\mathbb{Q}[p_1, \ldots, p_n, q_1, \ldots, q_n]$ that preserve the symplectic form $\sum_{1 \leq i \leq n} dp_i \wedge dq_i$ and the augmentation ideal. In fact, $\mathfrak{c}_n$ is the Lie algebra of Hamiltonian vector fields on the standard symplectic plane preserving the origin. Finally, define $\mathfrak{c}_\infty = \varinjlim \mathfrak{c}_n$.

THEOREM 5.75. *The primitive part Prim $H_*(\mathfrak{c}_\infty)$ of the Chevalley-Eilenberg cohomology of $\mathfrak{c}_\infty$ can be expressed as*

$$\mathit{Prim}\ H_k(\mathfrak{c}_\infty) = \mathit{Prim}\ H_k(sp(\infty)) \oplus \bigoplus_{n \geq 2} H_k(GC_*^{(n)}),$$

where Prim $H_k(sp(\infty))$ is described in (5.82).

The proof is similar to that of Theorem 5.69; see [**Kon93**, Theorem 1.1].

GRAPH COMPLEXES AS FEYNMAN TRANSFORMS. We show that the above graph complexes are special cases of the Feynman transform which we recalled in Section 5.4. This fact was first observed in [**GK98**] and then made more precise in [**Mar99b**]. In the rest of this section, $\mathbf{k}$ will be an arbitrary field of characteristic zero.

In Example 5.57 we recalled the cocycle $\mathfrak{Det}$ given by $\mathfrak{Det}(\Gamma) := Det(H_1(|\Gamma|; \mathbf{k}))$ and argued that each cyclic pseudo-operad $\mathcal{Q} = \{\mathcal{Q}(n)\}_{n \geq 2}$ can be considered as a $\mathfrak{Det}$-modular operad (denoted by the same symbol) with

$$\mathcal{Q}(g,n) = \begin{cases} \mathcal{Q}(n), & \text{for } g = 0,\ n \geq 2, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

We can thus form, for an arbitrary cyclic (pseudo)-operad $\mathcal{Q}$, its Feynman transform $\mathsf{F}_{\mathfrak{Det}}\mathcal{Q} = \{\mathsf{F}_{\mathfrak{Det}}\mathcal{Q}(g,n)\}_{(g,n) \in \mathfrak{S}}$. We show that the complex

$$\mathsf{F}_{\mathfrak{Det}}\mathcal{Q}(g,0) = (\mathsf{F}_{\mathfrak{Det}}\mathcal{Q}(g,0), \partial_{\mathsf{F}})$$

coincides, for $\mathcal{Q} = \mathcal{A}ss$ and $\mathcal{C}om$ (but considered as pseudo-operads, that is, with $\mathcal{C}om(1) = \mathcal{A}ss(1) = 0$) with the linear duals of the above mentioned graph complexes of M. Kontsevich.

Let us look more closely at $\mathsf{F}_{\mathfrak{Det}}\mathcal{Q}(g,0)$. As a graded vector space it is, by definition, isomorphic to

$$\begin{aligned} (5.88)\quad \mathsf{M}_{\mathfrak{K} \otimes \mathfrak{Det}^{-1}}(\mathcal{Q})(g,0) &\cong \bigoplus_{\gamma \in \{\mathbf{\Gamma}((g,0))\}} [\mathfrak{Det}^{-1}(\Gamma_\gamma) \otimes \mathfrak{K}(\Gamma_\gamma) \otimes \mathcal{Q}^\#((\Gamma_\gamma))]_{Aut(\Gamma_\gamma)} \\ &= \bigoplus_{\gamma \in \{\mathbf{\Gamma}((g,0))\}} [Det^{-1}(H_1(|\Gamma_\gamma|; \mathbf{k})) \otimes Det(edge(\Gamma_\gamma)) \otimes \mathcal{Q}^\#((\Gamma_\gamma))]_{Aut(\Gamma_\gamma)}, \end{aligned}$$

where $\{\mathbf{\Gamma}((g,0))\}$ is the set of isomorphism classes of all stable labeled genus g graphs with no legs and Γ_γ a representative of a class $\gamma \in \{\mathbf{\Gamma}((g,0))\}$; see Section 5.3 for the notation.

Because $\mathcal{Q}(g,n) = 0$ if $g > 0$ or $g = 0$ and $n = 1$, the summand indexed by Γ_γ may be nontrivial only if $g(v) = 0$ and $|edge(v)| \geq 3$ for all vertices v of Γ_γ. Therefore the summation is in fact indexed by the set $Gr^{(1-g)}$ of isomorphism classes of genus g graphs with all vertices at least ternary.

We will use the standard simplification and identify γ with its representative $\Gamma = \Gamma_\gamma$. By definition, $Det^{-1}(H_1(|\Gamma|; \mathbf{k})) \otimes Det(edge(\Gamma))$ is concentrated in degree

$$b_1(\Gamma) - |edge(\Gamma)| = 1 - |Vert(\Gamma)|.$$

So we may finally rewrite (5.88) as

$$\mathsf{F}_{\mathfrak{Det}}\mathcal{Q}(g,0)_k = \bigoplus_{\substack{\Gamma \in Gr^{(1-g)} \\ k = 1 - |Vert(\Gamma)|}} [Det^{-1}(H_1(|\Gamma|;\mathbf{k})) \otimes Det(edge(\Gamma)) \otimes \mathcal{Q}^{\#}((\Gamma))]_{Aut(\Gamma)}. \tag{5.89}$$

CASE $\mathcal{Q} = \mathcal{A}ss$. Let us see what happens for $\mathcal{Q} = \mathcal{A}ss$ in (5.89). We show first that for each finite set S with at least three elements, the space $\mathcal{A}ss((S))$ is the $\mathbf{k}$-linear space span of the cyclic orders of the set S. Recall (Example 5.27) that $\mathcal{A}ss(n) \cong \mathbf{k}[\Sigma_n]$, with the right Σ_n^+-action induced by the projection

$$\Sigma_n^+ \longrightarrow C_n^+ \backslash \Sigma_n^+ \cong \Sigma_n \tag{5.90}$$

of Σ_n^+ to the left classes of the action of the cyclic group C_n^+ of cyclic permutations of $\{0, \dots, n\}$ on Σ_n^+. From this we see that for each nonzero element

$$p \in \mathcal{A}ss((S)) = \left(\bigoplus_{f\ S \to ((n))} \mathbf{k}[\Sigma_n] \right)_{\Sigma_n^+} \tag{5.91}$$

there exists some indexing isomorphism $f : S \to ((n))$ such that the coordinate $p_f \in \mathbf{k}[\Sigma_n]_f$ of p is of the form $\alpha \cdot \mathbb{1}_n$, where $\mathbb{1}_n$ is the unit of Σ_n and $\alpha \in \mathbf{k}$. By (5.90), all maps f with this property differ by composition with an element of C_n^+; they thus determine the same cyclic order on S.

On the other hand, given $\alpha \in \mathbf{k}$ and a cyclic order of S represented by an isomorphism $f : S \to ((n))$, the equivalence class of $p_f := \mathbb{1}_n \in \mathbf{k}[\Sigma_n]_f$ in (5.91) does not depend on the choice of f and we may put $p := \alpha[p_f]$.

Thus $\mathcal{A}ss((\Gamma))$ is the linear span of cyclic orders of edges at each vertex of Γ, that is, the linear span of all ribbon structures on Γ and $\mathcal{A}ss^{\#}((\Gamma))$ is the dual of this space. The factor $Det^{-1}(H_1(|\Gamma|;\mathbf{k})) \otimes Det(edge(\Gamma))$ in (5.89) then expresses an orientation in the sense of Definition 5.64. We thus conclude that

$$(\mathsf{F}_{\mathfrak{Det}}\mathcal{A}ss(g,0)_{-k})^{\#} \cong RGC^{g,*}_{2g+*-1+k}, \ k \geq 0. \tag{5.92}$$

CASE $\mathcal{Q} = \mathcal{C}om$. Since $\mathcal{C}om((S)) \cong \mathbf{k}$ for each finite set S, the right-hand side of (5.89) is just the linear dual of the span of oriented (in the sense of Definition 5.64) graphs with no ribbon structure assumed. Recalling the definition of the graph complex $Gr_*^{(n)}$, we conclude that

$$(\mathsf{F}_{\mathfrak{Det}}\mathcal{C}om(g,0)_{-k})^{\#} \cong Gr_{1+k}^{(1-g)}, \ k \geq 0. \tag{5.93}$$

It is evident from the definitions that, under the identifications (5.92) and (5.93), the differentials of the graph complexes are duals to the differentials of the Feynman transform, at least up to signs. The sign issue is a tiresome but straightforward exercise which we leave to the reader.

REMARK 5.76. For $\mathcal{P} = \mathcal{L}ie$, the Feynman transform $\mathsf{F}_{\mathfrak{Det}}\mathcal{L}ie(g,0)$ can be described by the graph complex whose definition is indicated in [**Kon93**, §5]. We may imagine these 'Lie graphs' as oriented graphs whose vertices of arity $k+1$ are colored by $(k-1)!$ 'colors' representing a basis of $\mathcal{L}ie(k)$.

5.6. Application: moduli spaces of surfaces of arbitrary genera

As we already mentioned at the beginning of Section 5.3, modular operads were tailored to describe moduli spaces of surfaces of arbitrary genera. We illustrated this on the (unstable) modular Σ-module $\widehat{\mathcal{M}} = \{\widehat{\mathcal{M}}(g,n)\}_{g\geq 0, n\geq -1}$ of Riemann surfaces with parameterized holes and the obvious modular operad structure induced by sewing along the holes. This modular operad was studied by E. Getzler in [**Get94a, Get94b**] who called algebras for this operad *topological conformal field theories* with trivial ghost number anomaly.

The aim of this section is to discuss less straightforward examples related to Deligne-Knudsen-Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable genus g Riemann surfaces with n punctures. We show that these spaces form a modular operad $\overline{\mathcal{M}}$ in the category of complex projective varieties (Theorem 5.77) and then introduce three algebraic modular operads $\mathcal{G}rav$, varG and Grav related to $\overline{\mathcal{M}}$. Theorem 5.82 then says that the Feynman dual of the operad Grav is weakly equivalent to the operad $\mathcal{C}_*(\overline{\mathcal{M}})$ of de Rham currents on $\overline{\mathcal{M}}$.

DELIGNE-MUMFORD MODULI SPACES. Let us generalize the material of Section 4.2 to higher genera. A *stable curve* with n-marked points (also called punctures), $n \geq 0$, is a connected complex projective curve C whose only singularities are ordinary double points (nodal singularities), together with a 'marking' given by an embedding of the set $\{x_1,\ldots,x_n\}$ in the set of smooth points of C. The stability means that we assume that there are no infinitesimal automorphisms of C fixing the marked points and double points. This is the same as to say that each smooth component of C isomorphic to the complex projective space $\mathbb{CP}^1$ has at least three special points and that each smooth component isomorphic to the torus has at least one special point, where by a special point we mean either a double point or a singular point.

The *dual graph* $\Gamma = \Gamma(C)$ of a stable curve $C = (C, x_1, \ldots, x_n)$ is a labeled graph whose vertices are the components of C, edges are the nodes and its legs are the points $\{x_i\}_{1\leq i\leq n}$. An edge e_y corresponding to a nodal point y joins the vertices corresponding to the components intersecting at y. The vertex v_K corresponding to a branch K is labeled by the genus of the normalization of K. (See [**Har77**, page 23] for the normalization and recall that a curve is normal if and only if it is nonsingular.)

The construction of $\Gamma(C)$ from a curve C is visualized on Figure 13. Let us denote by $\overline{\mathcal{M}}_{g,n}$ the coarse moduli space [**Har77**, page 347] of curves $C = (C, x_1, \ldots, x_n)$ such that the dual graph $\Gamma(C)$ has genus g (see (5.28) for the definition of the genus of a labeled graph).

Observe that the genus of the graph $\Gamma(C)$ equals the arithmetic genus of the curve C, $p_a(C) = g(\Gamma(C))$. This follows from the relation between the arithmetic genus of C and of its normalization [**Har77**, p. 298]. For example, the curve in Figure 13 has (arithmetic) genus $g(A_1)+g(A_2)+g(A_3)+g(A_4)+g(A_5)+2$, where $g(-)$ denotes the genus of the normalization of the corresponding component. Thus $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of arithmetic genus g with n marked points.

Let us observe that, for a curve $C \in \overline{\mathcal{M}}_{0,n}$, the graph $\Gamma(C)$ must necessarily be a tree and all components of C must be smooth of genus 0, therefore $\overline{\mathcal{M}}_{0,n}$ coincides

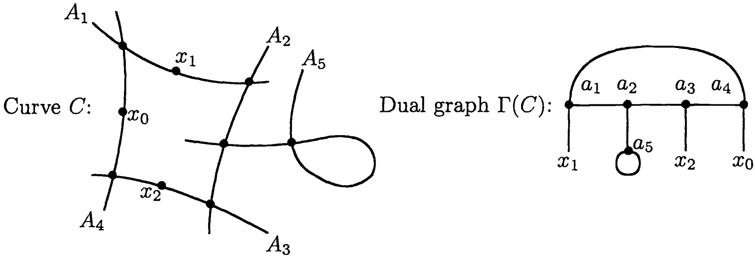


FIGURE 13. A stable curve and its dual graph. The curve C on the left has five components, A_1, A_2, A_3, A_4 and A_5, and three marked points x_0, x_1 and x_2. The dual graph $\Gamma(C)$ on the right has five vertices a_1, a_2, a_3, a_4 and a_5 corresponding to the components of the curve and three legs labeled by the marked points.

with the moduli space $\overline{\mathcal{M}}(n)$ of genus 0 stable curves with n labeled punctures that we discussed in Section 4.2. An immediate consequence of the stability is that dual graphs are stable as labeled graphs in the sense of Definition 5.38.

There is a subset $\mathcal{M}_{g,n}$ inside each $\overline{\mathcal{M}}_{g,n}$ consisting of curves C such that the dual graph $\Gamma(C)$ is the corolla with n legs, no edges and one vertex of genus g. By a result of P. Deligne, F.F. Knudsen and D. Mumford [**DM69, KM76, Knu83**], $\overline{\mathcal{M}}_{g,n}$ is a projective variety obtained by adjoining to $\mathcal{M}_{g,n}$ a divisor $D_{g,n}$ with normal crossings, though one must interpret this result with care, since $\overline{\mathcal{M}}_{g,n}$ is, for $g \geq 1$, not smooth. We shall discuss this subtlety later in the section.

The symmetric group Σ_n acts on $\overline{\mathcal{M}}_{g,n}$ by renumbering the marked points. Therefore, with $\overline{\mathcal{M}}(g,n) := \overline{\mathcal{M}}_{g,n+1}$,

$$\overline{\mathcal{M}} := \{\overline{\mathcal{M}}(g,n)\}_{g\geq 0, n\geq -1}$$

is a modular Σ-module in the category of projective varieties. There are clearly no stable curves of genus g with n punctures if $2(g-1)+n \leq 0$, so $\overline{\mathcal{M}}$ is a stable modular Σ-module in the sense of Definition 5.33, $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(g,n)\}_{(g,n)\in\mathfrak{S}}$.

Let us define the contraction along a stable graph $\Gamma \in \mathbf{\Gamma}(g,n)$

$$\alpha_\Gamma : \overline{\mathcal{M}}((\Gamma)) = \prod_{v\in \mathit{Vert}(\Gamma)} \overline{\mathcal{M}}((g(v), \mathit{Leg}(v))) \to \overline{\mathcal{M}}(g,n), \tag{5.94}$$

by gluing the marked points of curves from $\overline{\mathcal{M}}((g(v), \mathit{Leg}(v)))$, $v \in \mathit{Vert}(\Gamma)$, according to the graph Γ. To be more precise, let

$$\prod_{v\in \mathit{Vert}(\Gamma)} C_v, \quad \text{where } C_v \in \overline{\mathcal{M}}((g(v), \mathit{Leg}(v))),$$

be an element of $\overline{\mathcal{M}}((\Gamma))$. Let e be an edge of the graph Γ connecting vertices v_1 and v_2, $\mathrm{e} = \{y^e_{v_1}, y^e_{v_2}\}$, where $y^e_{v_i}$ is a marked point of the component C_{v_i}, $i = 1,2$, which is also the name of the corresponding flag of the graph Γ. The curve $\alpha_\Gamma(C)$ is then obtained by the identifications $y^e_{v_1} = y^e_{v_2}$ introducing a nodal singularity, for all $e \in \mathit{edge}(\Gamma)$. The procedure is the same as that described for the tree level in Section 4.2. The contraction maps are iterations of operations studied in [**Knu83**, page 190] and they are morphisms of projective varieties. The 'associativity' (5.42) of the contractions is clear immediately. Summing up the above observations, we formulate:

THEOREM 5.77. *The contraction maps (5.94) define on the modular* Σ*-module of coarse moduli spaces* $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(g,n)\}_{(g,n)\in\mathfrak{S}}$ *the structure of a modular operad in the category of complex projective varieties.*

Passing to homology, we obtain a modular operad $H_*(\overline{\mathcal{M}})$ in the category of graded vector spaces. An algebra over this operad is, by definition, a *cohomological field theory* in the sense of M. Kontsevich and Yu. Manin [**KM94**]; see also [**Get94b**].

The contraction maps are closed imbeddings [**Knu83**] and they describe the divisor $D(g,n) := D_{g,n+1}$ as follows. For a stable graph $\Gamma \in \mathbf{\Gamma}(g,n)$, denote $D_\Gamma := \mathrm{Im}(\alpha_\Gamma)$. Then the union

$$D(g,n) = \bigcup_\Gamma D_\Gamma, \tag{5.95}$$

over all (isomorphism classes of) stable graphs $\Gamma \in \mathbf{\Gamma}(g,n)$ with exactly one edge is a decomposition of $D(g,n)$ into smooth irreducible components. The same union as (5.95), but taken over graphs Γ with exactly two edges, gives the locus of double points of $D(g,n)$, etc. In fact, (5.95) describes a decomposition of $D(g,n)$ into *closed* strata; compare (4.10).

Let us turn out attention to open parts $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$. The modular Σ-module

$$\mathcal{M} = \{\mathcal{M}(g,n)\}_{(g,n)\in\mathfrak{S}} \text{ with } \mathcal{M}(g,n) := \mathcal{M}_{g,n+1}$$

is not a suboperad of $\overline{\mathcal{M}}$, because the images of the contraction maps (5.94) are contained in the divisor $D(g,n)$ while $D(g,n) \cap \mathcal{M}(g,n) = \emptyset$.

The aim of the rest of this section is to show, following again [**GK98**], that suitable 'chains' on $\mathcal{M}(g,n)$ nevertheless form a certain $\mathfrak{K}$-modular operad, where $\mathfrak{K}$ is the dualizing cocycle introduced in Example 5.52. By chains we mean elements of the topological dual

$$\mathcal{E}^*(\overline{\mathcal{M}}(g,n), \log D(g,n))' \tag{5.96}$$

of the de Rham complex $\mathcal{E}^*(\overline{\mathcal{M}}(g,n), \log D(g,n))$ of smooth differential forms on the variety $\overline{\mathcal{M}}(g,n)$ with logarithmic singularities on the divisor $D(g,n)$. Notice we used $(-)'$ to distinguish this topological dual from the algebraic one $(-)^\#$. The precise meaning of this complex is explained below. Its homology is, by [**HL71**, Theorem 2.3], isomorphic to the Čech homology of $\mathcal{M}(g,n)$ with real coefficients,

$$\check{H}_*(\mathcal{M}(g,n)) \cong H_*(\mathcal{E}^*(\overline{\mathcal{M}}(g,n), \log D(g,n))'),$$

so (5.96) indeed consists of a kind of 'chains' on $\mathcal{M}(g,n)$. Let us start with the necessary preliminary material.

SINGULARITIES AND RESIDUES. The theory of de Rham forms with logarithmic singularities stems from the following result of [**HL71**]. Let X be a smooth complex variety of dimension m and U an open (in the complex analytic topology) subset of X. Let ζ and θ be smooth differential forms with compact support on U of dimension $2m$ and $2m-1$, respectively. Let φ be an arbitrary holomorphic function on U. Then the limits

$$\lim_{\delta\to 0}\int_{|\varphi|>\delta}\frac{\zeta}{\varphi} \quad\text{and}\quad \lim_{\delta\to 0}\int_{|\varphi|=\delta}\frac{\theta}{\varphi} \tag{5.97}$$

exist. Let us point out that we do not assume the forms ζ and θ to be holomorphic.

Suppose that $D \subset X$ is a divisor. Let us denote by $\mathcal{E}^*_X(\log D)$ the sheaf on X generated, as a sheaf of algebras, by de Rham forms $\mathcal{E}^*_X$ on X and expressions $d\varphi/\varphi$, where φ is a holomorphic equation for D. A more formal way to introduce $\mathcal{E}^*_X(\log D)$ is the following. Let ι be the inclusion $\iota : X - D \hookrightarrow X$ and $\iota_*\mathcal{E}^*_{X-D}$ the direct image of the sheaf $\mathcal{E}^*_{X-D}$ of de Rham forms on $X - D$. The sections over an open set $U \subset X$ are, by definition, given as

$$\iota_*\mathcal{E}^*_{X-D}(U) = \mathcal{E}^*(U - (U \cap D)).$$

The restriction

$$r : \mathcal{E}^*(U) \to \mathcal{E}^*(U - (U \cap D))$$

is clearly monic and it identifies $\mathcal{E}^*_X$ with a subsheaf of $\iota_*\mathcal{E}^*_{X-D}$. Observe also that if φ is a holomorphic equation for D in U, then the 1-form $d\varphi/\varphi$ can be considered as an element of $\mathcal{E}^*(U-(U\cap D))$. We may then define $\mathcal{E}^*_X(\log D)$ as the subsheaf of $\iota_*\mathcal{E}^*_{X-D}$ generated as a sheaf of subalgebras by $\mathcal{E}^*_X$ and 1-forms $d\varphi/\varphi$, where φ is a local holomorphic equation for D. Let us define the *principal value* of a compactly supported form $\omega \in \mathcal{E}^{2m}(U, \log D)$ to be the integral

$$\mathrm{pv}\int_U \omega := \lim_{\delta\to 0}\int_{|\varphi|>\delta}\omega, \tag{5.98}$$

where φ is as above a holomorphic equation for D in U. The existence of this limit again follows from the results of [**HL71**]; it is more or less the first limit of (5.97). When $D \subset X$ is empty, $\mathcal{E}^*_X(\log D) = \mathcal{E}^*_X$ and the principal value is the ordinary integral over U.

Let us recall that, given an open subset U of a smooth manifold X, the space of smooth functions $\mathcal{C}^{(\infty)}(U)$ with the topology given by seminorms of the maxima of partial derivatives on compact subsets of U is a nuclear Fréchet linear topological space. The category NF of nuclear Fréchet spaces is a strict symmetric monoidal category, with the monoidal structure given by a certain completion $\widehat{\otimes}$ of the tensor product $\otimes$. A miraculous property of this topology is that, for any two open subsets $U_1 \subset X_1$ and $U_2 \subset X_2$, there exists a natural isomorphism

$$\mathcal{C}^{(\infty)}(U_1 \times U_2) \xrightarrow{\cong} \mathcal{C}^{(\infty)}(U_1)\widehat{\otimes}\, \mathcal{C}^{(\infty)}(U_2)$$

of nuclear Fréchet spaces; see [**Gro55**] or a bit easier [**Jar81**, page 500]. The above isomorphism of course induces an isomorphism

$$Z : \mathcal{C}^{(\infty)}_{X_1\times X_2} \to \mathcal{C}^{(\infty)}_{X_1}\widehat{\otimes}\, \mathcal{C}^{(\infty)}_{X_2} \tag{5.99}$$

of sheaves of smooth functions, where the tensor product on the right is an obvious exterior product induced by the $\widehat{\otimes}$-product of NF-spaces.

Let us return back to our case of de Rham forms with logarithmic singularities. Suppose that X_1, X_2 are smooth manifolds and $D_1 \subset X_1$, $D_2 \subset X_2$ divisors. Then

$$(D_1 \times X_1) \cup (X_1 \times D_2)$$

is a divisor in $X_1 \times X_2$ and the isomorphism (5.99) generalizes to

$$Z : \mathcal{E}^*_{X_1\times X_2}(\log D_1 \times X_1 \cup X_1 \times D_2) \xrightarrow{\cong} \mathcal{E}_{X_1}(\log D_1)\widehat{\otimes}\mathcal{E}_{X_2}(\log D_2). \tag{5.100}$$

Suppose for a moment that D has no crossings, i.e. it that is a disjoint union of nonsingular components. Let $j : D \hookrightarrow X$ be the inclusion and $j_*\mathcal{E}^*_D$ the direct image of the sheaf $\mathcal{E}^*_D$. This means, of course, that

$$j_*\mathcal{E}^*_D(U) = \mathcal{E}^*(D \cap U), \tag{5.101}$$

for any open subset $U \subset X$. A result of [**Del71**] says that there exists a degree -1 map of graded sheaves

$$Res : \mathcal{E}^*_X(\log D) \to j_*\mathcal{E}^{*-1}_D$$

such that for each open $U \subset X$ the induced map of sections $Res : \mathcal{E}^*(U, \log D) \to \mathcal{E}^{*-1}(D \cap U)$ satisfies

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{|\varphi| = \delta} \omega\alpha = \int_D Res(\omega)\alpha, \tag{5.102}$$

for each l-form $\omega \in \mathcal{E}^l(U, \log D)$, for each compactly supported form $\alpha \in \mathcal{E}^{2m-l-1}(U)$ and for each holomorphic equation φ of D in U.

EXAMPLE 5.78. Formula (5.102) is a deep generalization of the residue theorem from complex analysis. If $X = \mathbb{C}$ is the complex plane, $D = \{0\}$ the origin interpreted as the divisor given by the equation $z = 0$ and U an open neighborhood of the origin, then $\mathcal{E}^1(U, \log D)$ consists of expressions $\omega = \omega_1 + \omega_2 \frac{dz}{z}$, where $\omega_1 \in \mathcal{E}^1(U)$, $\omega_2 \in \mathcal{E}^0(U)$ are smooth forms. The residue $Res : \mathcal{E}^1(U) \to j_*\mathcal{E}^0(\{0\}) = \mathbb{C}$ is given by the classical formula $Res(\omega) := \omega_2(0)$. Equation (5.102) then translates to

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{S_\delta} \omega\alpha = Res(\omega)\alpha(0), \tag{5.103}$$

where S_δ is the circle around the origin with diameter δ. Observe that, for meromorphic ω and holomorphic α, this equation follows from the classical residue theorem.

The situation is more complicated when the divisor D has crossings. It is then necessary to consider the normalization $\pi : \widetilde{D} \to D$. The inverse image $\widehat{D}^2$ of the locus of double points of D is a divisor with normal crossings in $\widetilde{D}$. Let j be as in (5.101) the inclusion $D \hookrightarrow X$. The residue is then the map

$$Res : \mathcal{E}^*_X(\log D) \to (j\pi)_*\mathcal{E}^{*-1}_{\widetilde{D}}(\log \widehat{D}^2) \tag{5.104}$$

characterized as follows. For each open $U \subset X$ and for each l-form $\omega \in \mathcal{E}^l(U, \log D)$ its residue $Res(\omega) \in (j\pi)_*\mathcal{E}_{\widetilde{D}}(\log \widehat{D}^2)(U) = \mathcal{E}^{l-1}(\pi^{-1}(U), \log \widehat{D}^2)$ satisfies

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{|\varphi| = \delta} \omega\alpha = pv \int_{\widetilde{D}} Res(\omega)\pi^*\alpha,$$

for each compactly supported $\alpha \in \mathcal{E}^{2m-l-1}(U)$.

Let us analyze (5.104) in more detail. The normalization $\widetilde{D}$ is the disjoint union

$$\widetilde{D} = \bigsqcup_{1 \le i \le s} \widetilde{D}_i,$$

where each $\widetilde{D}_i$ is isomorphic to an irreducible component D_i of D. Let $\pi_i := \pi|_{\widetilde{D}_i}$, $D^2_i := D^2 \cap D_i$ and $\widehat{D}^2_i := \pi^{-1}(D^2_i) = \widehat{D}^2 \cap \widetilde{D}_i$. From the isomorphism

$$(j\pi)_*\mathcal{E}^*_{\widetilde{D}}(\log \widehat{D}^2) \cong \bigoplus_{1 \le i \le s} (j\pi_i)_*\mathcal{E}^*_{\widetilde{D}_i}(\log \widehat{D}^2_i)$$

and the isomorphism

$$(j\pi_i)_* \mathcal{E}^*_{\widetilde{D}_i}(\log \widehat{D}_i^2) \cong j_* \mathcal{E}^*_{D_i}(\log D_i^2),\ 1 \le i \le s,$$

induced by $\pi_i : \widetilde{D}_i \xrightarrow{\cong} D_i$, it follows that the residue (5.104) is given by an s-tuple of maps

$$Res : \mathcal{E}^*_X(\log D) \to j_* \mathcal{E}^{*-1}_{D_i}(\log D_i^2),\ 1 \le i \le s.$$

For sections over X these maps give

$$Res_i : \mathcal{E}^*(X, \log D) \to \mathcal{E}^{*-1}(D_i, \log D_i^2). \tag{5.105}$$

EXAMPLE 5.79. Let us illustrate the above material on $X := \mathbb{C}^2$ and $D := \{z_1 z_2 = 0\}$. The divisor D decomposes as $D = D_1 \cup D_2$ with $D_1 = \{z_1 = 0\}$, $D_2 = \{z_2 = 0\}$, and the locus of double points is $D^2 = D_1 \cap D_2 = \{(0,0)\}$. System (5.105) for this special case is

$$Res_i : \mathcal{E}^*(\mathbb{C}^2, \log D) \to \mathcal{E}^{*-1}(D_i, \log\{0\}),\ i = 1, 2.$$

Similarly, we have the residues for $(D_i, \{0\})$:

$$Res : \mathcal{E}^1(D_i, \log\{0\}) \to \mathcal{E}^0(\{0\}) = \mathbb{C},\ i = 1, 2.$$

The above maps form the diagram

$$\begin{tikzcd}
 & \mathcal{E}^1(D_1, \log\{0\}) \arrow[dr, "Res"] & \\
\mathcal{E}^2(\mathbb{C}^2, \log(D_1 \cup D_2)) \arrow[ur, "Res_1"] \arrow[dr, "Res_2"'] & & \mathbb{C} \\
 & \mathcal{E}^1(D_2, \log\{0\}) \arrow[ur, "Res"'] &
\end{tikzcd}$$

We claim the above diagram is *anticommutative*. This can be verified by a direct calculation, which we recommend as a nice exercise, or by the following argument.

The 'upper' composition U differs from the lower one L by the exchange $z_1 \leftrightarrow z_2$ or, formally, $U = L \circ T^*$, where $T : \mathbb{C}^2 \to \mathbb{C}^2$ is the flip $T(u \otimes v) := v \otimes u$. The anticommutativity now follows from

$$T^* \left(\frac{dz_1}{z_1} \frac{dz_2}{z_2} \right) = \frac{dz_2}{z_2} \frac{dz_1}{z_1} = -\frac{dz_1}{z_1} \frac{dz_2}{z_2}.$$

OPERADS $\mathcal{G}rav$, varG AND Grav. Let us ignore for the moment the unpleasant fact that $\overline{\mathcal{M}}_{g,n}$ is not smooth and apply the above machinery to the divisor $D(g,n)$ in $\overline{\mathcal{M}}(g,n)$.

As in (5.95), the components D_Γ of the divisor $D(g,n)$ are indexed by stable modular graphs Γ with exactly one edge. We also observed that $D^2(g,n)$ (the locus of double points) is the union (5.95) over graphs $\Gamma \in \mathbf{\Gamma}(g,n)$ with exactly two internal edges, therefore

$$D^2_\Gamma := D^2(g,n) \cap D_\Gamma = \bigcup_{\Delta < \Gamma} D_\Delta,$$

where the union runs over all graphs Δ, with exactly two edges. The relation $\Delta < \Gamma$ means that the graph Γ is a contraction of Δ.

System (5.105) in this case gives, for each $\Gamma \in \mathbf{\Gamma}(g,n)$ with exactly one edge, the residue

$$Res_\Gamma : \mathcal{E}^*(\overline{\mathcal{M}}(g,n), \log D(g,n)) \to \mathcal{E}^{*-1}(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma).$$

Let us denote by $\mathsf{varG} := \{\mathsf{varG}(g,n)\}_{(g,n)\in\mathfrak{S}}$ the modular Σ-module in the monoidal category `Chain(NF)` of chain complexes of Fréchet nuclear spaces with

$$\mathsf{varG}(g,n) := \mathcal{E}^*(\overline{\mathcal{M}}(g,n), \log D(g,n));$$

the meaning of this strange notation will become clear later. We claim that the isomorphism Z of (5.100) defines, for any $\Gamma \in \mathbf{\Gamma}(g,n)$ with exactly one edge, an isomorphism

$$Z_\Gamma : \mathcal{E}^*(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma) \xrightarrow{\cong} \mathsf{varG}((\Gamma)).$$

Indeed, if the unique edge of Γ joins two distinct vertices, then Γ is obtained from two corollae $*_1 \in \mathbf{\Gamma}(g_1,l)$, $*_2 \in \mathbf{\Gamma}(g_2,k)$, $g_1+g_2=g$, $k+l=n-1$, by joining a leg of $*_1$ with a leg of $*_2$. Then clearly $\overline{\mathcal{M}}((\Gamma)) \cong \overline{\mathcal{M}}((*_1)) \times \overline{\mathcal{M}}((*_2))$ and

$$D^2_\Gamma \cong (\overline{\mathcal{M}}((*_1)) \times D_{*_2}) \cup (D_{*_1} \times \overline{\mathcal{M}}((*_2))),$$

therefore

$$\mathcal{E}^*(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma) \cong \mathcal{E}^*(\overline{\mathcal{M}}((\xi_1)), \log D_{\xi_1}) \hat{\otimes} \mathcal{E}^*(\overline{\mathcal{M}}((\xi_2)), \log D_{\xi_2}) = \mathsf{varG}((\Gamma)).$$

The same argument applies when the edge of Γ is a loop around a single vertex. We may now define, for any graph Γ with exactly one edge, a degree -1 map

$$c_\Gamma : \mathsf{varG}(g,n) \to \mathsf{varG}((\Gamma)) \tag{5.106}$$

as the composition

$$\begin{aligned} \mathsf{varG}(g,n) &= \mathcal{E}^*(\overline{\mathcal{M}}(g,n), \log D(g,n)) \\ &\xrightarrow{Res_\Gamma} \mathcal{E}^{*-1}(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma) \xrightarrow{Z_\Gamma} \mathsf{varG}((\Gamma)). \end{aligned}$$

Before going further, recall that the strong topological dual $V \mapsto V'$ defines an equivalence of the opposite category $\mathtt{NF}^{\mathtt{op}}$ with the category `DF` of dual Fréchet spaces; see again [**Jar81**] or [**Gro55**]. Thus $\mathsf{Grav}(g,n) := \mathsf{varG}(g,n)'$ is a stable modular Σ-module in the monoidal category `Chain(DF)` of chain complexes of dual Fréchet spaces. We believe that the meaning of the name varG is clear now – varG is Grav written from right to left.

Let us recall (Example 5.52) that the dualizing cocycle $\mathfrak{K}$ was, for $\Gamma \in \mathbf{\Gamma}(g,n)$, defined by

$$\mathfrak{K}(\Gamma) = Det(edge(\Gamma)),$$

the top exterior power of $\mathrm{Span}_{\mathbf{k}}(edge(\Gamma))$ concentrated in degree $-\#(edge(\Gamma))$. For any $\Gamma \in \mathbf{\Gamma}(g,n)$ with exactly one edge the map (5.106) induces a degree 0 map

$$\alpha_\Gamma : \mathsf{Grav}((\Gamma)) \otimes \mathfrak{K}(\Gamma)^{-1} \to \mathsf{Grav}(g,n) \tag{5.107}$$

by

$$\alpha_\Gamma(\omega \otimes Det(\mathrm{e})) := 2\pi i\, c'_\Gamma(\omega),\ \omega \in \mathsf{varG}((\Gamma)),$$

where e is the unique edge of Γ. Iterating this map gives Grav a structure of a $\mathfrak{K}^{-1}$-modular operad, since the determinant in the definition of $\mathfrak{K}$ compensates for the anticommutativity of the diagram in Example 5.79.

The homology $\mathcal{G}rav(g,n)$ of $\mathsf{Grav}(g,n)$ forms a modular $\mathfrak{K}^{-1}$-operad in the category of finite dimensional vector spaces such that

$$\mathcal{G}rav(g,n) \cong H_*(\mathcal{M}(g,n)).$$

An 'elementary' construction of this structure at the 'tree level' (for genus zero) was given by T. Kimura, J. Stasheff and A.A. Voronov in [**KSV95**].

We owe the reader an explanation of why we may assume that $\overline{\mathcal{M}}_{g,n}$ is smooth. The magic trick is that of an *algebraic stack*. For our purposes, an algebraic stack is a groupoid (a small category in which all morphisms are invertible) $\mathcal{G}$ such that both the set of objects $Ob(\mathcal{G})$ and the set of morphisms $Mor(\mathcal{G})$ are smooth manifolds. We also assume that the target and source maps $s, t : Mor(\mathcal{G}) \longrightarrow Ob(\mathcal{G})$ are *étale*, which in our case means that they are local diffeomorphisms of smooth manifolds.

The definition of an algebraic stack is in fact more difficult. An intuitive idea of a stack is that of a scheme whose points admit nontrivial automorphisms. This is formalized by a certain category F fibered over the category of schemes, together with an *atlas*, which is, by definition, a scheme U together with a surjective étale (in an appropriate sense) map $U \longrightarrow F$. From these data one construct a *presentation* of the stack, which is a category with objects U and morphisms the fibered products $U \times_F U$. Our groupoid $\mathcal{G}$ is then the set of complex points of the presentation. We recommend [**Vis89**, Appendix] or the original paper [**DM69**] as an introduction to stacks.

A *coarse moduli space* $|\mathcal{G}|$ of an algebraic stack $\mathcal{G}$ is the quotient $Ob(\mathcal{G})/Mor(\mathcal{G})$. Notice that the coarse moduli space need not be smooth. A result of [**DM69**, Theorem 5.2] says that there exists an algebraic stack $\overline{\mathfrak{M}}_{g,n}$ such that the corresponding coarse moduli space is $\overline{\mathcal{M}}_{g,n}$.

The main trick is then to replace objects on $\overline{\mathcal{M}}_{g,n}$ with objects on $Ob(\overline{\mathfrak{M}}_{g,n})$ that are invariant under the 'action' of $Mor(\overline{\mathfrak{M}}_{g,n})$. For instance, a divisor D gives rise to a divisor in $Ob(\overline{\mathfrak{M}}_{g,n})$ such that $s^{-1}(D) = t^{-1}(D)$. In the same manner, a sheaf $\mathcal{S}$ gives rise to a sheaf $\mathcal{S}$ on $Mor(\overline{\mathfrak{M}}_{g,n})$ such that $s^*(\mathcal{S}) = t^*(\mathcal{S})$.

EXAMPLE 5.80. An example one has to have in mind is a smooth manifold Y with an action of a finite group G. These objects define a stack $\mathcal{G}$ with $Ob(\mathcal{G}) := Y$, $Mor(\mathcal{G}) := G \times Y$ and the source and target maps given by

$$s(g \times y) := y, \; t(g \times y) := gy, \text{ for } g \in G, \; y \in Y.$$

The coarse moduli space $|\mathcal{G}|$ is the standard quotient Y/G and $\mathrm{Mod}(\mathcal{G})$-invariant objects are G-invariant objects on Y. Our trick then reflects the principle that, in characteristic zero, there is a one-to-one correspondence between objects on Y/G and G-invariant objects on Y.

As observed in [**GK98**], all constructions above can be made for $Mor(\overline{\mathfrak{M}}_{g,n})$-invariant objects in $Ob(\overline{\mathfrak{M}}_{g,n})$. Let us summarize the results into the following theorem.

THEOREM 5.81. *The modular Σ-module* $\mathsf{Grav} = \{\mathsf{Grav}(g,n)\}_{(g,n)\in\mathfrak{S}}$ *is a $\mathfrak{K}^{-1}$-modular operad in the monoidal category* $\mathtt{Chain(DF)}$ *of chain complexes of dual Fréchet spaces.*

The $\mathfrak{K}^{-1}$-modular operad Grav of Theorem 5.81 is called the *gravity operad*. We close this section with a statement describing the Feynman transform of Grav which we give without proof. Let us recall that the Feynman transform $\mathsf{F}_{\mathfrak{D}}$ introduced in Section 5.4 was a functor from the category of $\mathfrak{D}$-modular operads in the monoidal category of finite dimensional vector spaces to the category of $\check{\mathfrak{D}}$-modular operads in the same category, where $\check{\mathfrak{D}} = \mathfrak{K} \otimes \mathfrak{D}^{-1}$. An obvious modification of $\mathsf{F}_{\mathfrak{D}}$ gives a functor $\mathsf{F}^{\mathrm{top}}_{\mathfrak{D}}$, the *topological* Feynman transform, from the category of $\mathfrak{D}$-modular operads in Chain(NF) to the category of $\check{\mathfrak{D}}$-modular operads in Chain(DF).

Recall that in Section 5.3 we introduced the invertible stable modular Σ-module $\mathfrak{p}$ with $\mathfrak{p}(g,n) = \downarrow^{6(g-1)-2n} \mathbf{k}$ and the property that $\mathfrak{D}_{\mathfrak{p}} = \mathfrak{K}^2$ (Proposition 5.54). From the last equation it easily follows that $\mathfrak{p}\mathsf{Grav} := \mathfrak{p}(\mathsf{Grav})$ (see (5.60) for the notation) is a $\mathfrak{K}$-modular operad in Chain(DF), thus $\mathsf{F}^{\mathrm{top}}_{\mathfrak{K}}(\mathfrak{p}\mathsf{Grav})$ is a modular operad in Chain(NF).

There is another example of a modular Chain(NF)-operad, namely the operad $\mathcal{C}_*(\overline{\mathcal{M}}) = \{\mathcal{C}_*(\overline{\mathcal{M}})(g,n)\}_{(g,n)\in\mathfrak{S}}$, where $\mathcal{C}_*(-)$ denotes the complex of the de Rham currents (see [**Fed69**] for an introduction to de Rham currents). The following theorem was proved in [**GK98**]:

THEOREM 5.82. *The operads* $\mathsf{F}^{\mathrm{top}}_{\mathfrak{K}}(\mathfrak{p}\mathsf{Grav})$ *and* $\mathcal{C}_*(\overline{\mathcal{M}})$ *are weakly equivalent modular operads in the monoidal category* Chain(NF).

In the above theorem, weak equivalence means as usual the existence of a chain of operadic maps that are homology isomorphisms.

5.7. Application: closed string field theory

In this section we analyze multilinear string products on the BRST complex of a combined conformal field theory of matter and ghosts. These products give rise to an object $V = (V, B, \{l^g_n\}_{g,n\geq 0})$ consisting of a graded vector space V, a nondegenerate bilinear form B and a system of multilinear maps $l^g_n : V^{\otimes n} \to V$. The 'tree level' specialization of this structure (see Example 5.86) is the L_∞-algebra constructed in [**KSV95**]; see also Section I.1.16.

The object $V = (V, B, \{l^g_n\}_{g,n\geq 0})$ is an example of a *loop homotopy Lie algebra*, introduced in [**Mar01b**] as a natural generalization of strongly homotopy Lie algebra. Recall that there are two equivalent descriptions of L_∞-algebras, the first one by coderivations of cofree nilpotent cocommutative coalgebras (Example 3.90), the second one by the dual dg operad of the operad $\mathcal{C}om$ for commutative algebras (see Remark 3.131).

A suitable generalization of these two types of description is available also for loop homotopy algebras. The coderivation type uses higher order coderivations and it is worked out in detail in [**Mar01b**]. This section is devoted to the 'operadic' one that describes loop homotopy algebras as algebras over the Feynman transform of the modular completion of the operad $\mathcal{C}om$. For physical motivations of objects discussed here, see [**Zwi93**] and Section I.1.16.

Let $\mathcal{H}$ be the Hilbert space of a combined conformal field theory of matter and ghosts and let $\mathcal{H}_{\mathrm{rel}} \subset \mathcal{H}$ be the subspace of elements annihilated by $b_0^- := b_0 - \bar{b}_0$ and $L_0^- := L_0 - \bar{L}_0$ (see, for example, [**KSV95**, Section 4] or Section I.1.16 of this book). Barton Zwiebach constructed in [**Zwi93**], for each 'genus' $g \geq 0$ and for each $n \geq 0$, multilinear *'string products'*

$$\mathcal{H}^{\otimes n}_{\mathrm{rel}} \ni B_1 \otimes \cdots \otimes B_n \longmapsto [B_1, \ldots, B_n]_g \in \mathcal{H}_{\mathrm{rel}}.$$

Here are the basic properties of these products. If $gh(-)$ denotes the ghost number, then [**Zwi93**, (4.8)]

$$gh([B_1,\dots,B_n]_g) = 3 - 2n + \sum_{i=1}^{n} gh(B_i).$$

The string products are graded commutative [**Zwi93**, (4.4)]:

$$[B_1,\dots,B_i,B_{i+1},\dots,B_n]_g = (-1)^{B_iB_{i+1}}[B_1,\dots,B_{i+1},B_i,\dots,B_n]_g. \tag{5.108}$$

Here we used the notation

$$(-1)^{B_iB_{i+1}} := (-1)^{gh(B_i)gh(B_{i+1})}.$$

For $n = 0$ and $g \geq 0$, $[\,.\,]_g \in \mathcal{H}_{\rm rel}$ is just a constant and the products are constructed in such a way that $[\,.\,]_0 = 0$ [**Zwi93**, (4.6)]. The linear operation $[B]_0 =: QB$ is identified with the BRST differential of the theory. These products satisfy, for all n, g, the *main identity* [**Zwi93**, (4.13)]

$$\begin{aligned} 0 = {} & \sum \sigma(i_l, j_k)\, [B_{i_1},\dots,B_{i_l},[B_{j_1},\dots,B_{j_k}]_{g_2}]_{g_1} \\ & + \frac{1}{2}\sum_s (-1)^{\Phi_s}[\Phi_s,\Phi^s,B_1,\dots,B_n]_{g-1}. \end{aligned} \tag{5.109}$$

Here the first sum runs over all $g_1 + g_2 = g$, $k + l = n$ and all (k,l)-unshuffles $i_1 < \cdots < i_l$, $j_1 < \cdots < j_k$ of $\{1,\dots,n\}$ (see (2.5) for the terminology). The sign $\sigma(i_l, j_k)$ is picked up from rearranging the sequence $(Q, B_1,\dots,B_n)$ into the order $(B_{i_1},\dots,B_{i_l},Q,B_{j_1},\dots,B_{j_k})$, where Q is the BRST differential of the theory, $\deg(Q) = 1$. In the second sum, $\{\Phi_s\}$ is a basis of $\mathcal{H}_{\rm rel}$ and $\{\Phi^s\} \subset \mathcal{H}_{\rm rel}$ its dual basis in the sense that

$$(-1)^{\Phi_r}\langle \Phi_r, \Phi^s\rangle = \delta_r^s \text{ (Kronecker delta)}, \tag{5.110}$$

where $\langle -,-\rangle$ denotes the bilinear inner product on $\mathcal{H}$ [**Zwi93**, (2.44)]. Let us remark that, in the original formulation of [**Zwi93**], $\{\Phi_s\}$ was a basis of the whole $\mathcal{H}$, but the sum in (5.109) was restricted to $\mathcal{H}_{\rm rel}$. The product satisfies [**Zwi93**, (2.62)]:

$$\langle A, B\rangle = (-1)^{(A+1)(B+1)}\langle B, A\rangle \tag{5.111}$$

and, by definition [**Zwi93**, (2.44)], it is nontrivial only for elements whose ghost numbers add up to five:

$$\text{if } \langle A, B\rangle \neq 0, \text{ then } gh(A) + gh(B) = 5. \tag{5.112}$$

The above two conditions in fact imply that $\langle A, B\rangle = \langle B, A\rangle$. Moreover, the product $\langle -,-\rangle$ is Q-invariant [**Zwi93**, 2.63]:

$$\langle QA, B\rangle = (-1)^A\langle A, QB\rangle. \tag{5.113}$$

Conditions (5.111) and (5.112) together with $gh(\Phi^s) \equiv gh(\Phi_s) + 1$ $(mod\ 2)$ imply that the element $\Phi := (-1)^{\Phi_s}\Phi_s \otimes \Phi^s \in \mathcal{H}_{\rm rel}^{\otimes 2}$ is symmetric in the sense that

$$(-1)^{\Phi_s}\Phi_s \otimes \Phi^s = (-1)^{\Phi_s}\Phi^s \otimes \Phi_s = -(-1)^{\Phi^s}\Phi^s \otimes \Phi_s. \tag{5.114}$$

We use, in the previous formula as well as at many places in the rest of the section, the Einstein convention of summing over repeated indices.

Let us prove (5.114). Since the form $\langle -,-\rangle$ is nondegenerate, to show that two elements of $\mathcal{H}_{\mathrm{rel}}^{\otimes 2}$ coincide, it is enough to prove that they are mapped by the map

$$\langle \Phi_r, -\rangle\langle \Phi_t, -\rangle : \mathcal{H}_{\mathrm{rel}} \otimes \mathcal{H}_{\mathrm{rel}} \longrightarrow \mathbf{k}$$

to the same element, for any indices r, t. Thus, to prove the first equation of (5.114), it is enough to show that

$$(-1)^{\Phi_s}\langle \Phi_r, \Phi_s\rangle\langle \Phi_t, \Phi^s\rangle = (-1)^{\Phi_s}\langle \Phi_r, \Phi^s\rangle\langle \Phi_t, \Phi_s\rangle.$$

But, by (5.110), both sides of the above equation are equal to $\langle \Phi_r, \Phi_t\rangle$. This proves the first equation of (5.114). The second one follows from $gh(\Phi^s) \equiv gh(\Phi_s) + 1$ (*mod* 2).

The last important property of string products is that the element

$$\Phi_s \otimes [\Phi^s, B_1, \ldots, B_{n-1}]_g \in \mathcal{H}_{\mathrm{rel}}^{\otimes 2} \tag{5.115}$$

is *antisymmetric*. This property is not explicitly stated in [**Zwi93**], but it follows from the fact that the string products are defined with the aid of the *multilinear string functions* [**Zwi93**, (7.72)]

$$\mathcal{H}_{\mathrm{rel}}^{\otimes(n+1)} \ni B_0 \otimes \cdots \otimes B_n \longmapsto \{B_0, \ldots, B_n\}_g \in \mathbb{C}.$$

The details can be found in [**Mar01b**].

Let us rewrite the axioms of string products into a more usual and convenient formalism so that it becomes manifest that they generalize strongly homotopy Lie algebras recalled in Example 3.133. Recall that, for a graded vector space U, the *suspension* (respectively the *desuspension*) of U is denoted $\uparrow U$ (respectively $\downarrow U$) and defined to be the graded vector space with $(\uparrow U)_p := U_{p-1}$ (respectively $(\downarrow U)_p := U_{p+1}$).

For a graded vector space U, let its *reflection* $\mathbf{r}(U)$ be the graded vector space defined by $\mathbf{r}(U)_p := U^{-p}$. There is an obvious natural map $\mathbf{r} : U \longrightarrow \mathbf{r}(U)$. Observe that $\mathbf{r}^2 = \mathbb{1}$, $\mathbf{r} \circ \uparrow = \downarrow \circ\, \mathbf{r}$ and $\mathbf{r} \circ \downarrow = \uparrow \circ\, \mathbf{r}$.

Take now $V := \mathbf{r}(\downarrow \mathcal{H}_{\mathrm{rel}})$. Define, for each $g \geq 0$ and $n \geq 0$, multilinear maps $l_n^g : V^{\otimes n} \longrightarrow V$ by

$$l_n^g(v_1, \ldots, v_n) := (-1)^{(n-1)v_1 + (n-2)v_2 + \quad + v_{n-1}} \downarrow [\uparrow \mathbf{r}(v_1), \ldots, \uparrow \mathbf{r}(v_n)]_g$$

for $v_1, \ldots, v_n \in V^{\otimes n}$. Define also the bilinear form $B : V \otimes V \longrightarrow \mathbb{C}$ by

$$B(u, v) := \langle \uparrow \mathbf{r}(u), \uparrow \mathbf{r}(v)\rangle. \tag{5.116}$$

Finally, let $h_s := (-1)^{\Phi_s}\mathbf{r}(\downarrow \Phi_s)$ and $h^s := \mathbf{r}(\downarrow \Phi^s)$, which means that

$$h_s \otimes h^s := (-1)^{\Phi_s}\mathbf{r}(\downarrow \Phi_s) \otimes \mathbf{r}(\downarrow \Phi^s).$$

The following definition abstracts the essentials of the above structure as can be verified by a technical, but absolutely straightforward, calculation.

DEFINITION 5.83. A *loop homotopy Lie algebra* is a triple $V = (V, B, \{l_n^g\})$ consisting of

(i) a $\mathbb{Z}$-graded vector space V, $V_* = \bigoplus V_i$,
(ii) a graded symmetric nondegenerate bilinear degree $+3$ form $B : V \otimes V \longrightarrow \mathbf{k}$ and
(iii) a set $\{l_n^g\}_{n,g \geq 0}$ of degree $n-2$ multilinear graded antisymmetric operations $l_n^g : V^{\otimes n} \longrightarrow V$.

These data are supposed to satisfy the following two axioms.

(A1) For any $n, g \geq 0$ and $v_1, \ldots, v_n \in V$, the following 'main identity'

$$(5.117)\quad 0 = \sum_{\substack{k+l=n+1 \\ g_1+g_2=g \\ k\geq 1}} \sum_{\sigma\in unsh(l,n-l)} \chi(\sigma)(-1)^{l(k-1)} l_k^{g_1}(l_l^{g_2}(v_{\sigma(1)},\ldots, v_{\sigma(l)}), v_{\sigma(l+1)},\ldots, v_{\sigma(n)}) + \frac{1}{2}\sum_s (-1)^{h_s+n} l_{n+2}^{g-1}(h_s, h^s, v_1, \ldots, v_n)$$

holds. In the second sum, $\{h_s\}$ and $\{h^s\}$ are bases of the vector space V dual to each other in the sense that

$$(5.118)\qquad B(h^s, h_t) = \delta_t^s.$$

(A2) The element

$$(5.119)\qquad (-1)^{(n+1)h_s} h_s \otimes l_n^g(h^s, v_2, \ldots, v_n) \in V\otimes V$$

is symmetric, for all $g \geq 0, n \geq 0$ and $v_2, \ldots, v_n \in V$.

We are going to show how 'abstract' loop homotopy Lie algebras are related to a certain construction over the operad $\mathcal{C}om$ for commutative algebras. All algebraic objects in the rest of this section are defined over a field $\mathbf{k}$ of characteristic zero.

REMARK 5.84. To give a reasonable meaning to the 'basis $\{h_s\}$ of V,' we must suppose either that V is finite dimensional or at least of finite type or that it has a suitable topology, as in the case of string products. We will always tacitly assume that assumptions of this form have been made. In the 'main identity' for $g = 0$ we put, by definition, $l_n^{-1} = 0$.

Because $\deg(h_s) + \deg(h^s) = -3$, we have that $\deg(h_s)\deg(h^s)$ is even. The graded symmetry of B then implies that, in addition to (5.118), also $B(h_s, h^t) = \delta_s^t$. The element $h = h_s\otimes h^s$ is easily seen to be symmetric, $h_s\otimes h^s = (-1)^{h_s h^s} h^s\otimes h_s = h^s\otimes h_s$.

For $n = 0$ axiom (5.109) gives

$$0 = \sum_{g_1+g_2=g} l_1^{g_1}(l_0^{g_2}(.)) + \frac{1}{2}\sum_s (-1)^{h_s} l_2^{g-1}(h_s, h^s),$$

while for $n = 1$ it gives

$$(5.120)\qquad 0= \sum_{g_1+g_2=g} (l_1^{g_1}(l_1^{g_2}(v)) + l_2^{g_1}(l_0^{g_2}(.), v)) - \frac{1}{2}\sum_s (-1)^{h_s} l_3^{g-1}(h_s, h^s, v),$$

for all $v \in V$. From this moment on, we will assume that $l_0^g = 0$, for all $g \geq 0$, that is, the theory has 'no constants.' This assumption is not really necessary, but it will considerably simplify our exposition.

EXERCISE 5.85. Let us denote $\partial := l_1^0$. Equation (5.120) implies that $\partial^2 = 0$ (recall our assumption $l_0^g = 0$ for $g \geq 0$!). Thus ∂ is a degree -1 differential on the space V. The symmetry of $h_s\otimes\partial(\tilde{h}_s)$ (axiom (A2) with $n = 1$ and $g = 1$) is equivalent to the ∂-invariance of the form B, that is, for $u, v \in V$, $B(\partial u, v) + (-1)^u B(u, \partial v) = 0$.

EXAMPLE 5.86. Let us discuss the 'tree level' ($g = 0$) specialization of the above structure, when the only nontrivial l^g_n's are $l_n := l^0_n$, $n \geq 1$. The main identity (5.117) for $g = 0$ reduces to

$$0 = \sum_{k+l=n+1} \sum_{\sigma \in unsh(l,n-l)} \chi(\sigma)(-1)^{l(k-1)} l_k(l_l(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)}).$$

We immediately recognize the above as the defining equation for strongly homotopy Lie algebras; see Example 3.133.

For $g = 1$, the main identity gives (after forgetting the overall factor $\frac{(-1)^n}{2}$)

$$0 = \sum_s (-1)^{h_s} l_{n+2}(h_s, h^s, v_1, \ldots, v_n). \tag{5.121}$$

Axiom (A2) says that the elements

$$(-1)^{(n+1)h_s} h_s \otimes l_n(h^s, v_1, \ldots, v_n) \tag{5.122}$$

are symmetric. Thus the tree level loop homotopy Lie algebra is a strongly homotopy Lie algebra $(V, \{l_n\})$ which acquires an additional structure given by a bilinear form B such that the element $h = h_s \otimes h^s$, uniquely determined by B, satisfies (5.121).

We see that the 'tree level' specialization is a richer structure than just a strongly homotopy Lie algebra as it is usually understood. A proper name for such a structure would be a *cyclic* strongly homotopy Lie algebra; compare [**Kon94, PS95**].

OPERADIC INTERPRETATION. We show that loop homotopy algebras (without 'constants' see (Remark 5.84)) are algebras over a certain modular operad, constructed from the cyclic operad $\mathcal{C}om$ for commutative algebras (Theorem 5.92 and Theorem 5.93). The constants can be easily incorporated by applying the same construction to the operad $\mathcal{UC}om$ for *unital* commutative associative algebras.

Let us consider the following harmless 'symmetric version' of Example 5.56.

EXAMPLE 5.87. Let $W = (W, H)$ be a graded vector space with a nondegenerate degree -1 symmetric bilinear form H. Define the modular Σ-module $\mathcal{E}nd^{\mathfrak{K}}_W$ by

$$\mathcal{E}nd^{\mathfrak{K}}_W((g, S)) := W^{\otimes S},$$

for $g \geq 0$ and a finite set S. For $\Gamma \in \mathbf{\Gamma}((g, S))$, the $\mathfrak{K}$-modular 'composition map'

$$\alpha^{\mathfrak{K}}_\Gamma : \mathcal{E}nd^{\mathfrak{K}}_W((\Gamma)) \otimes \mathfrak{K}(\Gamma) \to \mathcal{E}nd^{\mathfrak{K}}_W((g, S))$$

is defined as in Example 5.56. This means that we choose labels s_e, t_e such that $e = \{s_e, t_e\}$ for each edge $e \in edge(\Gamma)$ and define $\alpha^{\mathfrak{K}}_\Gamma$ to be the composition:

$$\begin{aligned} \mathcal{E}nd^{\mathfrak{K}}_W((\Gamma)) \otimes \mathfrak{K}(\Gamma) &\cong W^{\otimes Flag(\Gamma)} \otimes Det(edge(\Gamma)) \\ &\cong W^{\otimes S} \otimes \bigotimes_{e \in edge(\Gamma)} \left(W^{\otimes \{s_e, t_e\}} \otimes Span(\downarrow e) \right) \\ &\cong W^{\otimes S} \otimes \bigotimes_{e \in edge(\Gamma)} \left(W_{s_e} \otimes W_{t_e} \otimes Span(\downarrow e) \right) \\ &\xrightarrow{\mathbb{1} \otimes \bigotimes_e H_e} W^{\otimes S} \otimes \mathbf{k}^{\otimes edge(\Gamma)} \cong \mathcal{E}nd^{\mathfrak{K}}_W(g, S), \end{aligned} \tag{5.123}$$

where H_e is the map that sends $u \otimes v \otimes \downarrow e \in W_{s_e} \otimes W_{t_e} \otimes Span(\downarrow e)$ to $H(u,v) \in \mathbf{k}$. As in Example 5.56, the symmetry of H implies that the definition of $\alpha_\Gamma^{\mathfrak{K}}$ does not depend on the choice of labels. It is easy to verify that $\{\alpha_\Gamma^{\mathfrak{K}} | \Gamma \in \mathbf{\Gamma}((g,S))\}$ induces on $\mathcal{E}nd_W^{\mathfrak{K}}$ the structure of a $\mathfrak{K}$-modular operad.

By definition, a loop homotopy Lie algebra is a structure that lives on a graded vector space $V = (V, B)$ with a nondegenerate degree $+3$ bilinear symmetric form B. Then $W = (W, H)$ with $W := \uparrow^2 V$ and the form H defined by $H(u,v) := B(\downarrow^2 u, \downarrow^2 v)$, $u, v \in W$, provide the data as in Example 5.87 and we may consider the modular $\mathfrak{K}$-operad $\mathcal{E}nd_{\uparrow^2 V}^{\mathfrak{K}}$.

Let us consider the functor $U : \mathtt{MOp} \to \mathtt{Op}^+$ from the category of modular operads to the category of cyclic operads given, for any finite set S, by $U(\mathcal{A})((S)) := \mathcal{A}(0,S)$. We are going to describe a left adjoint $\mathrm{Mod} : \mathtt{Op}^+ \to \mathtt{MOp}$ to U.

Let $\underline{\mathbf{\Gamma}}((g,S))$ be the subcategory of the category $\mathbf{\Gamma}((g,S))$ whose objects are graphs $\Gamma \in \mathbf{\Gamma}((g,S))$ such that $g(v) = 0$ for each vertex $v \in Vert(\Gamma)$. A morphism $f : \Gamma_0 \to \Gamma_1 \in \mathbf{\Gamma}((g,S))$ is a morphism in $\underline{\mathbf{\Gamma}}((g,S))$ if and only if f has a factorization (5.54) such that each π_{e_i} is a contraction along an edge joining two *distinct* vertices. Another, equivalent, way to characterize morphisms $f : \Gamma_0 \to \Gamma_1$ of $\underline{\mathbf{\Gamma}}((g,S))$ is that the graph $f^{-1}(v)$ is a tree, for any $v \in Vert(\Gamma_1)$.

For a cyclic operad $\mathcal{P}$ and a graph $\gamma \in \underline{\mathbf{\Gamma}}((g,S))$, consider the graded vector space

$$\mathcal{P}((\Gamma)) = \bigotimes_{v \in Vert(\Gamma)} \mathcal{P}((Leg(v))).$$

For each $f : \Gamma_0 \to \Gamma_1 \in \underline{\mathbf{\Gamma}}((g,S))$ define a morphism $\mathcal{P}((f)) : \mathcal{P}((\Gamma_0)) \to \mathcal{P}((\Gamma_1))$ by

$$\begin{aligned}\mathcal{P}((\Gamma_0)) &= \bigotimes_{u \in Vert(\Gamma_0)} \mathcal{P}((Leg(u))) \cong \bigotimes_{v \in Vert(\Gamma_1)} \mathcal{P}((f^{-1}(v))) \\ &\xrightarrow{\otimes_v p_{f^{-1}(v)}} \bigotimes_{v \in Vert(\Gamma_1)} \mathcal{P}((Leg(v))) = \mathcal{P}((\Gamma_1)),\end{aligned}$$

where $p_f^{-1}(v)$ is the $\mathcal{P}$-composition along the graph $f^{-1}(v)$. This composition exists because $f^{-1}(v)$ is a tree, by assumption. The definition of the map $\mathcal{P}((f))$ is formally the same as that of $A(\psi)$ in (1.45), except the graphs Γ_1 and Γ_2 need not be trees. We have the following analog of Theorem 5.42.

PROPOSITION 5.88. *The correspondence $\Gamma \mapsto \mathcal{P}((\Gamma))$, $f \mapsto p(f)$, defines, for each $g \geq 0$ and each finite set S, a functor from the category $\underline{\mathbf{\Gamma}}((g,S))$ to the category of graded vector spaces.*

By the above proposition, it makes sense to put, for any $\mathcal{P} \in \mathtt{Op}^+$,

$$\mathrm{Mod}(\mathcal{P})((g,S)) := \operatorname*{colim}_{\Gamma \in \underline{\mathbf{\Gamma}}((g,S))} \mathcal{P}((\Gamma)).$$

Observe the analogy to the definition of Γ_K in Section 1.11 where the colimit (1.65) is taken over the compositions that are already defined for a K-collection, whereas here the colimit is over the operadic 'compositions' already defined for a cyclic operad.

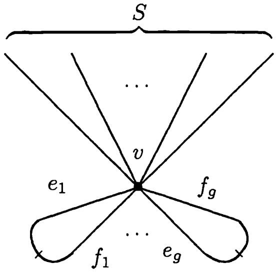


FIGURE 14. The graph $\Upsilon_{(\!(g,S)\!)}$.

The modular operad structure on the modular Σ-module $\mathrm{Mod}(\mathcal{P})$ can be defined as follows. Consider the category $\underline{\mathbf{\Gamma}}_1((g,S))$ whose objects are graph morphisms $\Gamma_0 \xrightarrow{f} \Gamma_1$, where $\Gamma_0 \in \underline{\mathbf{\Gamma}}((g,S))$, $\Gamma_1 \in \mathbf{\Gamma}((g,S))$, and morphisms are commutative diagrams

$$\begin{tikzcd} \Gamma_0 \arrow[r, "f"] \arrow[d, "u_0"'] & \Gamma_1 \arrow[d, "u_1"] \\ \Gamma_0' \arrow[r, "f'"] & \Gamma_1' \end{tikzcd}$$

where u_0 is a morphism in $\underline{\mathbf{\Gamma}}((g,S))$ and u_1 is an isomorphism. We may show, as in the proof of Proposition 5.39, that

$$\mathbb{M}(\mathrm{Mod}(\mathcal{P})) \cong \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \underline{\mathbf{\Gamma}}_1((g,S))} \mathcal{P}((\Gamma_0)).$$

The functor $\underline{\mathbf{\Gamma}}_1((g,S)) \to \underline{\mathbf{\Gamma}}((g,S))$ that sends $[\Gamma_0 \xrightarrow{f} \Gamma_1]$ to Γ_0 defines a map $\mathbb{M}(\mathrm{Mod}(\mathcal{P})) \to \mathrm{Mod}(\mathcal{P})$ which induces a modular operad structure on $\mathrm{Mod}(\mathcal{P})$.

DEFINITION 5.89. The modular operad $\mathrm{Mod}(\mathcal{P})$ is called the *modular operadic completion* of the cyclic operad $\mathcal{P}$.

PROPOSITION 5.90. *The functor* $\mathrm{Mod} : \mathtt{Op}^+ \to \mathtt{MOp}$ *is a left adjoint to the forgetful functor* $U : \mathtt{MOp} \to \mathtt{Op}^+$.

Since we will not need the proposition, we omit its proof. There is no easy way to express $\mathrm{Mod}(\mathcal{P})$ in terms of $\mathcal{P}$. The category $\underline{\mathbf{\Gamma}}((g,S))$ has, for $g > 0$, no terminal object, though there exists an important distinguished graph $\Upsilon_{(\!(g,S)\!)}$ described as follows.

The graph $\Upsilon_{(\!(g,S)\!)}$ has one vertex v of genus 0, the set of legs $Leg(\Upsilon_{(\!(g,S)\!)}) := S$, the set of internal flags $IFlag(\Upsilon_{(\!(g,S)\!)}) := \{e_1, f_1, \dots, e_g, f_g\}$ and the involution σ acting on $IFlag(\Upsilon_{(\!(g,S)\!)})$ by $\sigma(e_i) = f_i$, for $1 \leq i \leq g$. The geometric realization of the graph $\Upsilon_{(\!(g,S)\!)}$ is depicted in Figure 14.

The most important property of the graph $\Upsilon_{(\!(g,S)\!)}$ is that

(5.124) for each $\Gamma \in \underline{\mathbf{\Gamma}}((g,S))$, there exists a map $f : \Gamma \to \Upsilon_{(\!(g,S)\!)} \in \underline{\mathbf{\Gamma}}((g,S))$,

but the map f need not be unique. In fact, the automorphism group of the graph $\Upsilon_{(\!(g,S)\!)}$ is quite large, namely $Aut(\Upsilon_{(\!(g,S)\!)}) \cong \Sigma_g \rtimes (\mathbb{Z}_2)^{\times g}$, the semidirect product of the symmetric group Σ_g and the g-fold cartesian power of $\mathbb{Z}_2$, with Σ_g acting on $(\mathbb{Z}_2)^{\times g}$ by permuting the factors. The graph $\Upsilon_{(g,n)}$ can still, however, be used to define a map $\rho : \mathcal{P}(n+2g) \to \mathrm{Mod}(\mathcal{P})(g,n)$, for any $g \geq 0$ and $n \geq 1$, as follows.

The order $(0, 1, \dots, n, e_1, f_1, \dots, e_g, f_g)$ of $Flag(\Upsilon_{(g,n)})$ defines an isomorphism $P(n+2g) \cong \mathcal{P}((\Upsilon_{(g,n)}))$, which then composes with the projection to yield the canonical map

$$\rho : P(n+2g) \xrightarrow{\cong} \mathcal{P}((\Upsilon_{(g,n)})) \longrightarrow \operatorname*{colim}_{\Gamma \in \underline{\mathbf{\Gamma}}((g,S))} \mathcal{P}((\Gamma)) = \mathrm{Mod}(\mathcal{P})(g,n). \tag{5.125}$$

The map ρ is always an epimorphism, as easily follows from (5.124), but, in general, it need not be an isomorphism (the map ρ cannot distinguish between elements which differ only by the action of $Aut(\Upsilon_{(g,n)})$).

EXAMPLE 5.91. Since $Com(n) = \mathbf{k}$ for $n \geq 1$, the vector space $Com((\Gamma))$ is canonically isomorphic to $\mathbf{k}$ and, for each map $f : \Gamma_0 \to \Gamma_1 \in \underline{\mathbf{\Gamma}}((g,S))$, the induced map $Com((f)) : Com((\Gamma_0)) \to Com((\Gamma_1))$ is, under this identification, the identity,

$$Com((f)) : Com((\Gamma_0)) \cong \mathbf{k} \xrightarrow{\mathbb{1}} \mathbf{k} \cong Com((\Gamma_1)).$$

The map $\rho : Com(n+2g) \to \mathrm{Mod}(Com)(g,n)$ of (5.125) is in this case easily seen to be an isomorphism, therefore

$$\mathrm{Mod}(Com)(g,n) \cong \mathbf{k}, \text{ for each } g \geq 0,\ n \geq 1, \tag{5.126}$$

with the trivial action of the symmetric group Σ_n^+. To determine the modular operad structure, observe that, for each $\Gamma \in \mathbf{\Gamma}((g,S))$, also $\mathrm{Mod}(Com)((\Gamma))$ is canonically isomorphic to $\mathbf{k}$. The structure map is, under this identification, the isomorphism

$$\alpha_\Gamma : \mathrm{Mod}(Com)((\Gamma)) \cong \mathbf{k} \xrightarrow{\mathbb{1}} \mathbf{k} \cong \mathrm{Mod}(Com)((g,S)).$$

Let us recall that in Section 5.4 we described the Feynman transform of a modular operad as an analog of the cobar complex of an ordinary operad. We are going to describe explicitly the Feynman transform $\mathsf{F} = \mathsf{F}(\mathrm{Mod}(Com)) = (\mathsf{F}(\mathrm{Mod}(Com)), \partial_{\mathsf{F}})$ of the modular operad $\mathrm{Mod}(Com)$. Ignoring the differential, it is, by definition, the free $\mathfrak{K}$-modular operad $\mathbb{M}_{\mathfrak{K}}((\mathrm{Mod}(Com))^{\#})$ on the linear dual of the modular Σ-module $\mathrm{Mod}(Com)$; see (5.57) for the definition of free twisted modular operads and Example 5.52 for the definition of $\mathfrak{K}$.

By (5.126), $\mathrm{Mod}(Com)(g,n) \cong \mathbf{k}$, for all $g \geq 0$, $n \geq 1$. Let

$$\omega_n^g \in (\mathrm{Mod}(Com))^{\#}(g,n)$$

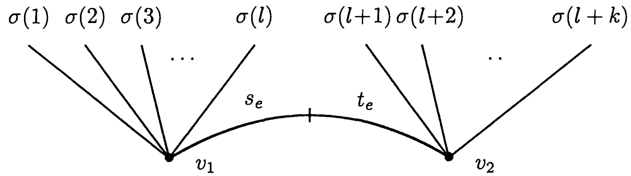
be the dual of $1 \in \mathbf{k} \cong \mathrm{Mod}(Com)(g,n)$. Therefore, as a modular Σ-module,

$$\mathsf{F}(\mathrm{Mod}(Com)) = \mathbb{M}_{\mathfrak{K}}\Big(\{\omega_n^g \mid g \geq 0,\ n \geq 1\}\Big). \tag{5.127}$$

Since the action of the symmetric group Σ_n^+ on $\mathrm{Mod}(Com)^{\#}(g,n)$ is trivial, the vector space $\mathrm{Mod}(Com)^{\#}(g,S)$ is *canonically* isomorphic to $\mathrm{Mod}(Com)^{\#}(g,n)$ for *any* finite set S with $card(S) = n+1$, thus ω_n^g determines the element (denoted by the same symbol) $\omega_n^g \in \mathrm{Mod}(Com)^{\#}((g,S))$. Therefore each graph $\Gamma \in \mathbf{\Gamma}((g,S))$ determines the *canonical element*

$$\gamma := \bigotimes_{v \in Vert(\Gamma)} \omega^{g(v)}_{|Flag(v)|-1} \in \bigotimes_{v \in Vert(\Gamma)} \mathrm{Mod}(Com)^{\#}((g(v), Flag(v))) = \mathrm{Mod}(Com)^{\#}((\Gamma)).$$

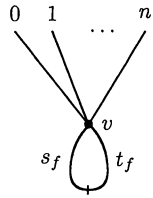
For $g_1, g_2 \geq 0$ with $g_1 + g_2 = g$, $k, l \geq 1$ with $k + l = n + 1$ and $\sigma \in unsh(k, l)$, let $\Phi^{g_1,g_2}_{k,l}(\sigma) \in \mathbf{\Gamma}(g, n)$ be the graph



with $g(v_1) = g_1$ and $g(v_2) = g_2$. Let

$$\phi^{g_1,g_2}_{k,l}(\sigma) \in \mathrm{Mod}(Com)^{\#}((\Phi^{g_1,g_2}_{k,l}(\sigma)))$$

be the canonical element. In a similar manner, let $\Psi^g_n \in \mathbf{\Gamma}(g, n)$ be the graph



where $g(v) = g - 1$ and let $\psi^g_n \in \mathrm{Mod}(Com)^{\#}((\Psi^g_n))$ be the corresponding canonical element. It is immediate from the definition of the differential ∂_{F} as given in Section 5.4 that

$$\partial_{\mathsf{F}}(\omega^g_n) = \sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g,\ \sigma\in unsh(k,l)}} \phi^{g_1,g_2}_{k,l}(\sigma) \otimes \uparrow e + \psi^g_n \otimes \uparrow f, \tag{5.128}$$

where $e := \{s_e, t_e\} \in edge(\Phi^{g_1,g_2}_{k,l}(\sigma))$ and $f := \{s_f, t_f\} \in edge(\Psi^g_n)$. The obvious symmetry

$$\Phi^{g_1,g_2}_{k,l}(\sigma) \longleftrightarrow \Phi^{g_2,g_1}_{l,k}(\sigma'),$$

where σ' is, for $\sigma \in unsh(k, l)$, the 'opposite unshuffle,'

$$\sigma' := (\sigma(k+1), \dots, \sigma(k+l), \sigma(1), \dots, \sigma(k)),$$

enables one to rewrite (5.128) as

$$\partial_{\mathsf{F}}(\omega^g_n) = \sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g,\ \sigma\in unsh_0(k,l)}} 2\phi^{g_1,g_2}_{k,l}(\sigma) \otimes \uparrow e + \psi^g_n \otimes \uparrow f, \tag{5.129}$$

where $unsh_0(k, l) := \{\sigma \in unsh(k, l) |\ \sigma(1) = 1\}$.

Let us recall the following notation. For $u_1 \otimes \cdots \otimes u_r \in V^{\otimes r}$ and $v_1 \otimes \cdots \otimes v_s \in V^{\otimes s}$, let $Sh(u_1 \otimes \cdots \otimes u_r | v_1 \otimes \cdots \otimes v_s)$ denote the *shuffle product* of $u_1 \otimes \cdots \otimes u_k$ with $v_1 \otimes \cdots \otimes v_l$ in $TV = \bigoplus_{n \geq 0} V^{\otimes n}$, i.e. the sum of all shuffles of $u_1 \otimes \cdots \otimes u_r$ and $v_1 \otimes \cdots \otimes v_s$; see (3.98). A general element $x \in V^{\otimes k}$ is of the form

$$x = \sum x_{i_1} \otimes \cdots \otimes x_{i_k} \text{ where } x_{i_1}, \dots, x_{i_k} \in V,$$

where the summation runs over an appropriate set of indices. We usually omit the summation symbol and write simply

$$x = x_{i_1} \otimes \cdots \otimes x_{i_k}.$$

Let $V = (V, B)$ be a graded vector space with a nondegenerate degree $+3$ symmetric bilinear form and let $\{h_s\}$, $\{h^s\}$ be the bases of V as in Definition 5.83.

THEOREM 5.92. *A $\mathfrak{K}$-modular $\mathsf{F}(\mathrm{Mod}(\mathcal{C}om))$-algebra structure on the double suspension $\uparrow^2 V = (\uparrow^2 V, B)$ is given by a sequence $\{\xi^g_n \mid g \geq 0, n \geq 1\}$, where $\xi^g_n = x^g_{i_0} \otimes \cdots \otimes x^g_{i_n} \in V^{\otimes n+1}$ is a degree $-2(n+1)$ symmetric element of $V^{\otimes n+1}$ such that, for each $n \geq 1$ and $g \geq 0$, the following 'main equation' is satisfied:*

$$(5.130)\quad 0 = \sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g}} (-1)^{x^{g_1}_{i_k}} B(x^{g_1}_{i_k}, x^{g_2}_{j_0}) x^{g_1}_{i_0} \otimes Sh(x^{g_1}_{i_1} \otimes \cdots \otimes x^{g_1}_{i_{k-1}} | x^{g_2}_{j_1} \otimes \cdots \otimes x^{g_2}_{j_l})$$

$$- \quad \frac{1}{2} B(x^{g-1}_{i_{n+1}}, x^{g-1}_{i_{n+2}}) x^{g-1}_{i_0} \otimes \cdots \otimes x^{g-1}_{i_n}.$$

In the above identity for $g = 0$ we put, by definition, $\xi^{g-1}_{n+2} = x^{g-1}_{i_0} \otimes \cdots \otimes x^{g-1}_{i_{n+1}} := 0$.

PROOF. An $\mathsf{F}(\mathrm{Mod}(\mathcal{C}om))$-algebra structure on $(\uparrow^2 V, B)$ is an $\mathfrak{K}$-modular differential operad map

$$a : (\mathsf{F}(\mathrm{Mod}(\mathcal{C}om)), \partial_{\mathsf{F}}) \longrightarrow (\mathcal{E}nd^{\mathfrak{K}}_{\uparrow^2 V}, \partial = 0).$$

Description (5.127) shows that such a map a is given by its values $\xi^g_n := a(\omega^g_n)$ on the generators. Moreover, the map a commutes with the differentials, so the equation $a(\partial_{\mathsf{F}}(\omega^g_n)) = 0$ must be satisfied. This is, by (5.129), the same as

$$(5.131)\qquad a\Big(\sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g,\ \sigma\in unsh_0(k,l)}} 2\phi^{g_1,g_2}_{k,l}(\sigma) \otimes \uparrow e + \psi^g_n \otimes \uparrow f \Big) = 0.$$

The 'main identity' (5.130) is then (5.131) expressed in terms of the structure operations of the endomorphism operad $\mathcal{E}nd^{\mathfrak{K}}_{\uparrow^2 V}$. ■

Let us introduce our basic tool which converts the structures of Theorem 5.92 into loop homotopy algebras, the isomorphism $\Xi : V^{\otimes n+1} \longrightarrow Hom(V^{\otimes n}, V)$, where V is a graded vector space of finite type. The nondegenerate bilinear form B identifies the space V with its dual, $V \cong V^{\#}$, thus $V^{\otimes n+1} \cong V^{\#^{\otimes n}} \otimes V$, while, as always, $V^{\#^{\otimes n}} \otimes V \cong Hom(V^{\otimes n}, V)$. Thus the existence of Ξ is not a surprise. The subtle issue is a proper choice of signs, which is provided by the formula

$$(5.132)\quad \Xi(x_0 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n)$$
$$:= (-1)^{nx_0+(n-1)x_1+\cdots+x_{n-1}} x_0 B(x_1, v_1) B(x_2, v_2) \cdots B(x_n, v_n),$$

for $x_0 \otimes \cdots \otimes x_n \in V^{\otimes n+1}$ and $v_1, \ldots, v_n \in V$. The map Ξ is clearly a degree $3n$ isomorphism of $V^{\otimes n+1}$ and $Hom(V^{\otimes n}, V)$. Observe that $\Xi(x_0 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) \neq 0$ only if $\deg(x_i) = \deg(v_i) + 1\ (mod\ 2)$, $1 \leq i \leq n$.

THEOREM 5.93. *Let us define, for $g \geq 0$, $n \geq 1$ and $v_1, \ldots, v_n \in V$,*

$$l^g_n(v_1, \ldots, v_n) := (-1)^{\frac{n(n+1)}{2} + n(v_1+\cdots+v_n)} \Xi(\xi^g_n)(v_1, \ldots, v_n).$$

Then $\{\xi^g_n\} \mapsto \{l^g_n\}$ is a one-to-one correspondence between the structures of Theorem 5.92 and of loop homotopy algebras in the sense of Definition 5.83.

The proof of Theorem 5.93, based on the properties of the map Ξ, is mostly a tedious exercise in getting the signs right. It is given in the following appendix.

5.7.1. Appendix. The first step in the proof of Theorem 5.93 is the following lemma, which describes the behavior of Ξ under the symmetric group action.

LEMMA 5.94. *Let $x_0 \otimes \cdots \otimes x_n \in V^{\otimes n+1}$ and $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. Then, for any $\sigma \in \Sigma_n$,*

$$\begin{aligned} \text{(5.133)} \qquad & \epsilon(\sigma)\Xi(x_0 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)})(v_1, \ldots, v_n) \\ &= \chi(\sigma)\Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n)(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}), \end{aligned}$$

where $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ is the Koszul sign (3.96) of the permutation σ and $\chi(\sigma) = \chi(\sigma; v_1, \ldots, v_n) := \mathrm{sgn}(\sigma)\epsilon(\sigma; v_1, \ldots, v_n)$ (3.99). Moreover,

$$\begin{aligned} \text{(5.134)} \quad & \Xi(x_1 \otimes x_0 \otimes x_2 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) \\ &= (-1)^{x_0 + x_1} h_s \otimes B\Big(\Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n)(h^s, v_2, \ldots, v_n), v_1\Big). \end{aligned}$$

PROOF. Let us prove (5.133). Since the symmetric group is generated by transpositions, it is clearly enough to show that

$$\begin{aligned} & (-1)^{x_i x_{i+1}} \Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) \\ &\quad = -(-1)^{v_i v_{i+1}} \Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n)(v_1, \ldots, v_{i+1}, v_i, \ldots, v_n), \end{aligned}$$

for all $1 \le i < n$. By (5.132), the left-hand side of the above equation is

$$x_0 B(x_1, v_1) \cdots B(x_{i+1}, v_i) B(x_i, v_{i+1}) \cdots B(x_n, v_n)$$

with the sign

$$(-1)^{x_i x_{i+1} + n x_0 + \cdots + (n-i) x_{i+1} + (n-i-1) x_i + \cdots + x_{n-1}}$$

while the right-hand side is almost the same:

$$x_0 B(x_1, v_1) \cdots B(x_i, v_{i+1}) B(x_{i+1}, v_i) \cdots B(x_n, v_n)$$

with the sign

$$-(-1)^{v_i v_{i+1} + n x_0 + \cdots + (n-i) x_i + (n-i-1) x_{i+1} + \cdots + x_{n-1}}.$$

Modulo signs, the above two expressions are the same, while their signs differ by

$$-(-1)^{v_i v_{i+1} + x_i x_{i+1} + x_i + x_{i+1}}$$

which is 1, because $v_i = x_{i+1} + 1$ (*mod* 2) and $v_{i+1} = x_i + 1$ (*mod* 2), if we assume the expressions in (5.133) to be nontrivial.

Equation (5.134) can be proved in the same straightforward manner. ■

COROLLARY 5.95. *The map Ξ induces an isomorphism between the space of symmetric elements $x = x_{i_0} \otimes \cdots \otimes x_{i_n} \in V^{\otimes n+1}$ and the space of graded antisymmetric maps $k : V^{\otimes n} \to V$ with the property that the element*

$$\text{(5.135)} \qquad (-1)^{h^s} h_s \otimes k(h^s, v_2, \ldots, v_n) \in V^{\otimes 2}$$

is graded symmetric for any $v_2, \ldots, v_n \in V^{\otimes n-1}$.

PROOF. Let $k \in Hom(V^{\otimes n}, V)$ correspond to $x = x_{i_0} \otimes \cdots \otimes x_{i_n} \in V^{\otimes n+1}$,

$$k(v_1, \ldots, v_n) := \Xi(x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \ldots, v_n), \quad \text{for } v_1, \ldots, v_n \in V.$$

The symmetry of x implies that

$$x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_n} = \epsilon(\sigma) x_{i_0} \otimes x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(n)}}$$

for any $\sigma \in \Sigma_n$. This means that

$$\begin{aligned} k(v_1, \dots, v_n) &= \Xi(x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_n})(v_1, \dots, v_n) \\ &= \epsilon(\sigma)\Xi(x_{i_0} \otimes x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(n)}})(v_1, \dots, v_n) \quad \text{(by (5.133))} \\ &= \chi(\sigma)\Xi(x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_n})(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}) \\ &= \chi(\sigma)k(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}), \end{aligned}$$

which is the antisymmetry of k.

To prove the second part of the corollary, observe that the symmetry of $x = x_0 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}$ in the first two factors,

$$x_{i_1} \otimes x_{i_0} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} = (-1)^{x_{i_0} x_{i_1}} x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n},$$

implies that

$$\begin{aligned} &\Xi(x_{i_1} \otimes x_{i_0} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \dots, v_n) \\ &\quad = (-1)^{x_{i_0} x_{i_1}} \Xi(x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \dots, v_n) \\ &\quad = (-1)^{x_{i_0}(v_1+1)} \Xi(x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \dots, v_n) \\ &\quad = (-1)^{(\deg(k)+v_1+\cdots+v_n)(v_1+1)} k(v_1, \dots, v_n). \end{aligned}$$

Notice also that

$$\begin{aligned} (-1)^{x_{i_0}+x_{i_1}} &\Xi(x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(h^s, v_2, \dots, v_n) \\ &= (-1)^{\deg(k)+v_2+\cdots+v_n+1} k(h^s, v_2, \dots, v_n), \end{aligned}$$

because $x_{i_0} = \deg(k) + h^s + v_2 + \cdots + v_n$ (*mod* 2) and $x_{i_1} = h^s + 1$ (*mod* 2) when the two sides are not zero. Thus (5.134) can be written as

$$\begin{aligned} &(-1)^{(\deg(k)+v_1+\cdots+v_n)(v_1+1)} k(v_1, \dots, v_n) \\ &\quad = (-1)^{\deg(k)+v_2+\cdots+v_n+1} h_s \otimes B\Big(k(h^s, v_2, \dots, v_n), v_1\Big). \end{aligned} \tag{5.136}$$

The left-hand side of (5.136) can be obviously expressed as

$$\begin{aligned} &(-1)^{(\deg(k)+v_1+\cdots+v_n)(v_1+1)} k(v_1, \dots, v_n) \\ &\quad = (-1)^{(\deg(k)+v_1+\cdots+v_n)h_s} k(h^s, v_2, \dots, v_n) B(h_s, v_1). \end{aligned} \tag{5.137}$$

Because the form B is nondegenerate, equations (5.136) and (5.137) imply that

$$\begin{aligned} &(-1)^{(\deg(k)+h^s+v_2+\cdots+v_n)h_s} k(h^s, v_2, \dots, v_n) \otimes h_s \\ &\quad = (-1)^{\deg(k)+v_2+\cdots+v_n+1} h_s \otimes k(h^s, v_2, \dots, v_n). \end{aligned} \tag{5.138}$$

On the other hand, the symmetry of (5.135) means by definition that

$$\begin{aligned} &(-1)^{h^s} h_s \otimes k(h^s, v_2, \dots, v_n) \\ &\quad = (-1)^{h^s + (\deg(k)+h^s+v_2+\cdots+v_n)h_s} k(h^s, v_2, \dots, v_n) \otimes h_s \end{aligned}$$

which gives, after multiplying by the overall sign factor $(-1)^{\deg(k)+v_2+\cdots+v_n}$,

$$\begin{aligned} &(-1)^{\deg(k)+h^s+v_2+\cdots+v_n} h_s \otimes k(h^s, v_2, \dots, v_n) \\ &\quad = (-1)^{(\deg(k)+h^s+v_2+\cdots+v_n)h^s} k(h^s, v_2, \dots, v_n) \otimes h_s. \end{aligned} \tag{5.139}$$

We finish the proof by observing that (5.138) is mapped to (5.139) by the endomorphism $\varphi : V \otimes V \to V \otimes V$ defined by $\varphi(u \otimes v) := (-1)^u u \otimes v$, for $u, v \in V$. ■

LEMMA 5.96. *For any* $u_0 \otimes \cdots \otimes u_k \in V^{\otimes k+1}$ *and* $v_0 \otimes \cdots \otimes v_l \in V^{\otimes l+1}$,

$$(5.140)\quad (-1)^{u_0+\ \ +u_{k-1}} B(u_k, v_0)\Xi(u_0 \otimes u_1 \otimes \cdots \otimes u_{k-1} \otimes v_1 \otimes \cdots \otimes v_l) \\ = (-1)^{l+l(u_0+\ \ +u_k)}\Xi(u_0 \otimes \cdots \otimes u_k) \circ_k \Xi(v_0 \otimes \cdots \otimes v_l).$$

PROOF. The proof is a direct verification. The left-hand side of (5.140) equals, by definition,

$$(5.141)\quad (-1)^{u_0+\ \ +u_{k-1}+(l+k-1)u_0+(l+k-2)u_1+\ \ +lu_{k-1}+(l-1)v_1+\ \ +v_{l-1}} u_0 B(u_1, -)\cdots \\ \cdots B(u_k, -)B(u_k, v_0)B(v_1, -)\cdots B(v_l, -)$$

while the right-hand side of (5.140) is

$$(5.142)\quad (-1)^{l+l(u_0+\ \ +u_k)+ku_0+(k-1)u_1+\ \ +u_{k-1}+lv_0+(l-1)v_0+\ \ +v_{l-1}} u_0 B(u_1, -)\cdots \\ \cdots B(u_k, -)B(u_k, v_0)B(v_1, -)\cdot\ \ \cdot B(v_l, -).$$

A straightforward calculation shows that the signs of (5.141) and (5.142) differ by $(-1)^{l+l(u_k+v_0)}$ which is $+1$, since $u_k + v_0 = 1$ $(mod\ 2)$. ■

LEMMA 5.97. *For any* $x_0 \otimes \cdots \otimes x_{n+2} \in V^{\otimes n+3}$ *and* $v_1, \ldots, v_n \in V$,

$$(5.143)\quad B(x_{n+1}, x_{n+2})\Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) \\ = (-1)^{x_{n+1}}\Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \otimes x_{n+2})(v_1, \ldots, v_n, h_s, h^s).$$

PROOF. The left-hand side of (5.143) equals

$$(5.144)\quad (-1)^{nx_0+(n-1)x_1+\ \ +x_{n-1}} x_0 B(x_1, v_1)\cdots B(x_n, v_n)B(x_{n+1}, x_{n+2}),$$

while the right-hand side of (5.143) is

$$(5.145)\quad (-1)^{x_{n+1}+(n+2)x_0+(n+1)x_1+\ \ +2x_n+x_{n+1}} x_0 B(x_1, v_1)\cdots \\ \cdots B(x_n, v_n)B(x_{n+1}, h_s)B(x_{n+2}, h^s).$$

It is immediate to see that the signs of (5.144) and (5.145) agree. The proof is finished by observing that $B(x_{n+1}, h_s)B(x_{n+2}, h^s) = B(x_{n+1}, x_{n+2})$. ■

PROOF OF THEOREM 5.93. For $n \geq 1$, $g \geq 0$ and $v_1, \ .., v_n \in V$, let

$$k_n^g(v_1, \ldots, v_n) := \Xi(x^g_{i_0} \otimes \cdots \otimes x^g_{i_n})(v_1, \ldots, v_n).$$

Since $\deg(x^g_{i_0} \otimes \cdots \otimes x^g_{i_n}) = -2(n+1)$ by assumption, $\deg(k_n^g) = n-2$.

Let us apply the map $\Xi : V^{\otimes n+1} \to Hom(V^{\otimes n}, V)$ to the main equation (5.130). We claim that the result is

$$(5.146)\quad 0 = \sum_{\substack{k+l=n+1\\ g_1+g_2=g}} \sum_{\sigma \in unsh(n-l,l)} (-1)^l \chi(\sigma) k_k^{g_1}(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)}, k_l^{g_2}(v_{\sigma(k)}, \ldots, v_{\sigma(n)})) \\ + \frac{1}{2}(-1)^{h_s} k_{n+2}^{g-1}(v_1, \ldots, v_n, h_s, h^s).$$

Indeed, the first term of (5.130) gives

$$
\begin{aligned}
&(-1)^{x^{g_1}_{i_k}} B(x^{g_1}_{i_k}, x^{g_2}_{j_0})\Xi\Big(x^{g_1}_{i_0}\, Sh(x^{g_1}_{i_1} \otimes \cdots \otimes x^{g_1}_{i_{k-1}} | x^{g_2}_{j_1} \otimes \cdots \otimes x^{g_2}_{j_l}) \Big)(v_1, \ldots, v_n) \\
&\quad = (\text{by } (5.133)) \\
&\qquad \sum \chi(\sigma)(-1)^{x^{g_1}_{i_k}} B(x^{g_1}_{i_k}, x^{g_2}_{j_0})\Xi(x^{g_1}_{i_0} \otimes \cdots \otimes x^{g_1}_{i_{k-1}} \otimes x^{g_2}_{j_1} \otimes \cdots \otimes x^{g_2}_{j_l})(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \\
&\quad = (\text{by } (5.140)) \\
&\qquad \sum \chi(\sigma)(-1)^l \Big\{ \Xi(x^{g_1}_{i_0} \otimes \cdots \otimes x^{g_1}_{i_{k-1}}) \circ_k \Xi(x^{g_2}_{j_1} \otimes \cdots \otimes x^{g_2}_{j_l}) \Big\}(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \\
&\quad = \sum \chi(\sigma)(-1)^l (k^{g_1}_k \circ_k k^{g_2}_l)(v_{\sigma(1)}, \ldots, v_{\sigma(n)}),
\end{aligned}
$$

which is the first term of (5.146). For brevity, we did not specify the ranges of summations which are the same as in (5.146). We also used the obvious identity $x^{g_1}_{i_0} + \cdots + x^{g_1}_{i_k} = 0 \ (mod\ 2)$. The second term of (5.130) gives

$$
\begin{aligned}
-\frac{1}{2}B(x^{g-1}_{i_{n+1}}, x^{g-1}_{i_{n+2}})\Xi(x^{g-1}_{i_0} \otimes \cdots \otimes x^{g-1}_{i_n})(v_1, \ldots, v_n) \quad (\text{by } (5.143)) \\
= \quad -\frac{1}{2}(-1)^{x^{g-1}_{i_{n+1}}}\Xi(x^{g-1}_{i_0} \otimes \cdots \otimes x^{g-1}_{i_{n+2}})(v_1, \ldots, v_n, h_s, h^s) \\
= \quad \frac{1}{2}(-1)^{h_s} k^{g-1}_{n+2}(v_1, \ldots, v_n, h_s, h^s),
\end{aligned}
$$

since $x^{g-1}_{i_{n+1}} = h_s + 1 \ (mod\ 2)$. The operations k^g_n are, by Corollary 5.95, antisymmetric, so we can rewrite (5.146) as

$$
\begin{aligned}
0 = \sum (-1)^{k(l+1)+l(v_{\sigma(l+1)}+ \cdots +v_{\sigma(n)})}\chi(\sigma) k^{g_1}_k (k^{g_2}_l(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)}) \\
- \quad \frac{1}{2}(-1)^{h_s+v_1+ \cdots +v_n} k^{g-1}_{n+2}(h_s, h^s, v_1, \ldots, v_n)
\end{aligned}
$$

where the summation is the same as in (5.146). The substitution

$$
\overline{k}^g_n(v_1, \ldots, v_n) := (-1)^{n(v_1+ \cdots +v_n)} k^g_n(v_1, \ldots, v_n)
$$

converts the above equation to

$$
\begin{aligned}
0 = \sum (-1)^{k+(n+1)(v_1+ \cdots +v_n)}\chi(\sigma) \overline{k}^{g_1}_k (\overline{k}^{g_2}_l(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)}) \\
- \quad \frac{1}{2}(-1)^{h_s+(n+1)(v_1+ \cdots +v_n)+n} \overline{k}^{g-1}_{n+2}(h_s, h^s, v_1, \ldots, v_n).
\end{aligned}
$$

Multiplying by the overall sign $(-1)^{(n+1)(v_1+\cdots+v_n)}$, this can be further simplified to

$$
\begin{aligned}
(5.147) \qquad 0 = \sum (-1)^k \chi(\sigma) \overline{k}^{g_1}_k (\overline{k}^{g_2}_l(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)}) \\
- \quad \frac{1}{2}(-1)^{h_s+n} \overline{k}^{g-1}_{n+2}(h_s, h^s, v_1, \ldots, v_n).
\end{aligned}
$$

Finally, the substitution

$$
\begin{aligned}
(5.148) \qquad l^g_n(v_1, \ldots, v_n) \quad &:= \quad (-1)^{\frac{n(n+1)}{2}} \overline{k}^g_n(v_1, \ldots, v_n) \\
&= \quad (-1)^{\frac{n(n+1)}{2}+n(v_1+ \cdots +v_n)} k^g_n(v_1, \ldots, v_n)
\end{aligned}
$$

converts (5.147) to

$$\sum (-1)^{l(k-1)}\chi(\sigma) l_k^{g_1}(l_l^{g_2}(v_{\sigma(1)},\dots,v_{\sigma(l)}),v_{\sigma(l+1)},\dots,v_{\sigma(n)}) + \frac{1}{2}(-1)^{h_s+n} l_{n+2}^{g-1}(h_s,h^s,v_1,\dots,v_n),$$

which is, quite miraculously, (5.117). It remains to show that the element

$$(-1)^{(n+1)h_s} h_s \otimes l_n^g(h^s,v_2,\dots,v_n) \in V\otimes V,$$

where l_n^g is given by (5.148), is symmetric. In terms of the operations k_n^g, this is the same as

$$(-1)^{(n+1)h_s+\frac{n(n+1)}{2}+n(h^s+v_2+\cdots+v_n)} h_s\otimes k_n^g(h^s,v_2,\dots,v_n) = (-1)^{h_s+\frac{n(n+1)}{2}+n(1+v_2+\cdots+v_n)} h_s\otimes k_n^g(h^s,v_2,\dots,v_n). \tag{5.149}$$

On the other hand, the element

$$(-1)^{h_s} h_s\otimes k_n^g(h^s,v_2,\dots,v_n),$$

which is symmetric by Corollary 5.95, differs from (5.149) only by an overall sign factor $(-1)^{\frac{n(n+1)}{2}+n(1+v_2+\cdots+v_n)}$, thus (5.72) is symmetric, too. ■

Epilog

We have tried to provide a reasonably complete survey of operads and their applications, although with our own viewpoints and emphases. As we were completing the present text, a search to FIND *operad* on the arXiv produced a list of 54 articles and a search for *operad* ANYWHERE in the listings on MathSciNet produced abstracts of 203 articles. Of these, 15 were posted to the arXiv in the past year and 31 have publication dates listed on MathSciNet of 2000 or 2001. Many of these do appear in the previous chapters (and we will not cite them here again); rather than add the rest to our bibliography, which would become out of date during the publication process, we recommend that the reader access those lists directly.

Several of the most recent, however, deserve further attention here; we group them loosely by topic.

GEOMETRY: Operads are playing an increasing role in geometry, in particular, symplectic [**Xu99, Gin01**] and also algebraic [**Kap98, Man99**]. Ginzburg [**Gin01**] in fact generalizes to a 'noncommutative geometry' for an algebra over an arbitrary cyclic quadratic Koszul operad. Recent results of Salvatore [**Sal99, Sal01**] on topology of configuration spaces and their completions use the presence of operadic structures on these spaces.

HOMOTOPY THEORY: Several papers devoted to homotopy invariant algebraic structures, homotopy theory of algebras over operads and homotopy theory in the category of operads appeared recently; see [**Smi99, Til00, Hin01, Smi01**] or the proceedings [**MMW99**]. Also some classical topics received new attention; see [**May97, Ber99**].

PHYSICS: There continues to be strong interaction between operads and mathematical physics, and especially topological field theories and string theory.

In [**KVZ97**], Kimura, Voronov and Zuckerman study the role of homotopy Gerstenhaber algebras and Tillmann [**Til99**] teases out the operad algebra structure, particularly the BV-algebra, of TCFTs. Chas and Sullivan [**CS99**] had a breakthough development of an appropriate algebra of closed strings by inventing the algebra of cacti; see also Voronov [**Vor01**]. This algebraic structure has been developed further as homotopy theory by R. Cohen and J.D.S. Jones [**CJ01**].

HIGHER CATEGORY THEORY: The pentagon and hexagons of monoidal category coherence appeared also in the associahedra and the multiplihedra for A_∞-spaces and, of course, also in A_∞-categories. The higher dimensional cells appear in 'higher category theory,' manifestly, for example, in Gordon, Power and Street [**GPS95**]. Perhaps the most interesting cross-fertilization occurs in the work of Baez and Dolan on higher category theory [**BD98**] using their consideration of *opeotopes*; compare also the 'globular' approach of Batanin [**Bat98a, Bat98b**].

Conversely, operads themselves are examined from a categorical point of view by Beke [**Bek99**].

POLYTOPES: Those early 'hedra' mentioned above continue to occur in a variety of contexts as do their progeny.' Devadoss [**Dev99**] tessellates the moduli space and produces a *mosaic operad*. See also [**LR01**].

ALGEBRAS AND GROUPS: Several generalizations of classical algebras, such as dialgebras of Loday and his 'school' [**Lod95, Lod01**] and various k-ary algebras [**Gne95b, Gne95a**] appeared recently; see also [**GW00**]. Formal groups and cogroups over operads are studied in [**Fre98b, Fre98a**]. Morita equivalence for modules over operads is studied by Kapranov and Manin in [**KM01**].

NONZERO CHARACTERISTIC: In discussing dg operads, we have worked primarily in characteristic 0. For non-Σ operads, this is unimportant but is crucial for many major operads, e.g the $\mathcal{L}ie$-$\mathcal{C}om$ Koszul duality. Historically, in nonoperadic language, this is reflected in the difference between Harrison and André-Quillen cohomology for commutative algebras. As we saw in Section II.3.8, Harrison cohomology can be generalized straightforwardly for algebras over any Koszul quadratic operad. André and Quillen make essential use of simplicial techniques. For algebras over general operads, the simplicial point of view is utilized in [**GH99, Fre97, Fre00**].

Our selection of articles mentioned above necessarily depended on our personal inclinations and taste and certainly forms only a small portion of the rapidly growing literature on operads. We thus apologize to all whose work we omitted referencing here.

Bar Ilan, Chapel Hill and Praha, November 28, 2001

Bibliography

[Ada] J F. Adams, *Private communication*

[Ada56] J F Adams, *On the cobar construction*, Proc Nat. Acad. Sci USA **42** (1956), 409–412

[AK00] R. Akbarpour and M. Khalkhali, *Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras*, Preprint math.KT/0011248, November 2000

[APB97] J A Azcárraga and J C. Pérez Bueno, *Higher-order simple Lie algebras*, Comm Math Phys. **184** (1997), 669–681.

[Arn83] V I. Arnol'd, *Remarks on the perturbation theory for problems of Mathieu type*, Russian Math. Surveys **38** (1983), no 4, 215–233

[AS94] S Axelrod and I M Singer, *Chern-Simons perturbation theory II*, J Differential Geom **39** (1994), 173–213.

[Bal94] D. Balavoine, *Déformations des algèbres de Leibniz*, C R Acad. Sci Paris Sér. I Math **319** (1994), 783–788

[Bal96] D. Balavoine, *Deformations de structure algebraiques et operades*, Thesis, Université de Montpellier II, June 1996

[Bal98] D Balavoine, *Homology and cohomology with coefficients, of an algebra over a quadratic operad*, J Pure Appl Algebra **132** (1998), 221–258

[Bar66] M Barr, *Cohomology in tensored categories*, Conference on Categorical Algebra (M Kelly and S Eilenberg, eds.), Springer-Verlag, Berlin, Heidelberg, New York, 1966, pp 344–354

[Bar71] M.G Barratt, *A free group functor for stable homotopy*, Algebraic topology (Proc Sympos Pure Math., Vol XXII, Univ. Wisconsin, Madison, Wis , 1970), Amer Math Soc., Providence, RI, 1971, pp 31–35.

[Bat98a] M.A Batanin, *Computads for finitary monads on globular sets*, Higher category theory, Contemporary Math., vol. 230, Amer Math Soc., Providence, RI, 1998

[Bat98b] M A. Batanin, *Homotopy coherent category theory and A_∞-structures in monoidal categories*, J Pure Appl Algebra **123** (1998), no 1-3, 67–103

[Bau81] H -J Baues, *The double bar and cobar constructions*, Compositio Math **43** (1981), 331–341

BC64] R L Bishop and R J Crittenden, *Geometry of manifolds*, Applied Mathematics, vol XV, Academic Press, New York-London, 1964

BD98] J C Baez and J. Dolan, *Higher-dimensional algebra III· n-categories and the algebra of opetopes*, Adv in Math **2** (1998), no 135, 145–206

Bec69] J. Beck, *On H-spaces and infinite loop spaces*, Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash , 1968, Vol 3), Springer, Berlin, 1969, pp 139–153

Bek99] T Beke, *Operads from the viewpoint of categorical algebra*, Higher Homotopy Structures in Topology and Mathematical Physics (J McCleary, ed), Contemporary Math , vol 227, Amer Math. Soc , 1999, pp. 29–47

Ber96] C Berger, *Opérades cellulaires et espaces de lacets itérés*, Ann Inst. Fourier (Grenoble) **46** (1996), no 4, 1125–1157

Ber97] C Berger, *Combinatorial models for real configuration spaces and E_n-operads*, Operads Proceedings of Renaissance Conferences (J L. Loday, J D Stasheff, and A.A Voronov, eds), Contemporary Math , vol. 202, Amer Math Soc., 1997

Ber99] C Berger, *Double loop spaces, braided monoidal categories and algebraic 3-type of space*, Higher Homotopy Structures in Topology and Mathematical Physics (J McCleary, ed.), Contemporary Math , vol 227, Amer Math Soc , 1999, pp 49–66

[BFSV] C Baltenau, Z. Fiedorowicz, R Schwanzl, and R M Vogt, *Iterated monoidal categories*, Preprint

[BG92] A Beilinson and V Ginzburg, *Infinitesimal structure of moduli spaces of G-bundles*, Internat Math. Res Notices (1992), 63–74

[BGS88] A. Beilinson, V Ginzburg, and V. Schechtman, *Koszul duality*, J Geom Phys **5** (1988), 317–350

[BJT97] H -J. Baues, M. Jibladze, and A. Tonks, *Cohomology of monoids in monoidal categories*, Operads: Proceedings of Renaissance Conferences (J.L Loday, J.D Stasheff, and A A Voronov, eds.), Contemporary Math , vol 202, Amer. Math Soc , 1997, pp 137–165

[Blo85] R E. Block, *Commutative Hopf algebras, Lie coalgebras, and divided powers*, J. Algebra **96** (1985), 275–306

[Bor53] A Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts*, Ann of Math. **57** (1953), 115–207

[Bot88] R Bott, *Morse theory indomitable*, Inst. Hautes Études Sci Publ Math (1988), no. 68, 99–114 (1989).

[Bou] N Bourbaki, *Algèbre*, Chapitre X. Algèbre Homologique. Mason, 1980.

[Bou92] A.K. Bousfield, *The simplicial homotopy theory of iterated loop spaces*, Unpublished, 1992

[BP72] M G Barratt and S. Priddy, *On the homology of non-connected monoids and their associated groups*, Comment. Math. Helv. **47** (1972), 1–14

[Bre67] G.E. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967

[Bro60] W Browder, *Homology operations and loop spaces*, Illinois J. Math **4** (1960), no. 3, 347–357

[BT94] R Bott and C. Taubes, *On the self-linking of knots*, J Math Phys **35** (1994), no. 10, 5247–5287

[BV68] J.M Boardman and R.M Vogt, *Homotopy-everything H-spaces*, Bull. Amer Math Soc **74** (1968), no 6, 1117–1122

[BV73] J M Boardman and R.M Vogt, *Homotopy invariant algebraic structures on topological spaces*, Springer-Verlag, 1973

[BV81] I.A Batalin and G.S Vilkovisky, *Gauge algebra and quantization*, Phys Lett. B **102** (1981), 27–31.

[Car] P. Cartier, *Homologie cycliques rapport sur des travaux récents de Connes, Karoubi, Loday, Quillen...*, Séminaire Bourbaki 621, 1983–84

[CF99] A. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, Preprint `math.QA/9902090`, February 1999

[Cha00] F. Chapoton, *Un théorème de Cartier-Milnor-Moore-Quillen pour les bigèbres dendriformes et les algèbres braces*, Preprint `math.QA/0005253`, May 2000.

[Cha01] F. Chapoton, *Opérades différentielles graduées sur les simplexes et les permutoèdres*, Preprint `math.QA/0102172`, February 2001.

[Che56] C Chevalley, *Review of Lois des groupes et analyseurs*, Math Reviews, vol. 17, Amer Math. Soc , 1956.

[CJ01] R L Cohen and J.D.S Jones, *A homotopy theoretic realization of string topology*, preprint `math.GT/0107187`, July 2001.

[CK00] A Connes and D Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem I. The Hopf algebra structure of graphs and the main theorem*, Comm Math. Phys. **210** (2000), no. 1, 249–273

[CL00] F. Chapoton and M Livernet, *Pre-Lie algebras and the rooted trees operad*, Preprint `math.QA/0002069`, February 2000.

[CMT78] F.R Cohen, J.P. May, and L R. Taylor, *Splitting of certain spaces CX*, Math. Proc. Cambridge Philos Soc **84** (1978), no 3, 465–496.

[Coh76] F.R. Cohen, *The homology of C_{n+1}-spaces, $n \geq 0$*, The homology of iterated loop spaces (F.R Cohen, T.J. Lada, and J.P. May, eds), Lecture Notes in Math., vol 533, Springer-Verlag, 1976

[Coh88] F.R Cohen, *Artin's braid groups, classical homotopy theory, and sundry other curiosities*, Braids, AMS-IMS-SIAM Jt. Summer Res Conf., Santa Cruz, Calif 1986, Contemp Math., vol. 78, 1988, pp 167–206

[Con83] A Connes, *Non commutative differential geometry*, Prépublications I H.E.S., March 1983.

[CS99] M Chas and D Sullivan, *Strng topology*, preprint `math.GT/9911159`, November 1999

[Cun00] J Cuntz, *Cyclic theory and the bivarıant Chern-Connes character*, Preprint SFB 478, University of Munster, December 2000

[CV86] M Culler and K Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent Math **84** (1986), 91–119

[Dei00] P. Deidel, *Vanishing cycles and mutation*, Preprint `math.SG/0007115`, July 2000

[Del71] P. Deligne, *Théorıe de Hodge II*, Inst Hautes Études Sci. Publ Math **40** (1971), 5–58

[Del72] P. Deligne, *Resumé des premiérs exposés de A Grothendieck*, Groupes de Monodromie en Géometrie Algébrique, vol. 288, Springer-Verlag, Berlin, 1972, pp 1–24

[Del93] P. Deligne, *A letter to Stasheff, Gerstenhaber, May, Schechtman and Drınfeld*, Unpublished, 1993

[Dev99] S L. Devadoss, *Tessellations of moduli spaces and the mosaic operad*, Homotopy Invariant Algebraic Structures (J P Meyer, J Morava, and W S Wilson, eds), Contemporary Math , vol 239, 1999, pp 91–114

[Dev01] S L. Devadoss, *A space of cyclohedra*, Preprint `math.QA/0102166`, February 2001

[DL] E Dyer and R.K Lashof, *Homology of iterated loop spaces*, Unpublished preprint containing material not in the published version of the same title.

[DL59] A Dold and R K Lashof, *Prıncipal quasi-fibrations and fibre homotopy equivalence of bundles.*, Illinois J Math. **3** (1959), 285–305

[DL62] E Dyer and R K Lashof, *Homology of iterated loop spaces*, Amer J. Math **84** (1962), no 1, 35–88

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, vol. 36, Inst. Hautes Études Sci Publ Math., no 75, I.H.E S , 1969

[DMGS75] J. Deligne, P Morgan, P Griffiths, and D. Sullivan, *The real homotopy type of Kahler manifolds*, Invent Math **29** (1975), 245–254

[DR00] S L Devadoss and R C Read, *Cellular structures determined by polygons and trees*, Preprint `math.CO/0008145`, August 2000

[Eli00] Y Eliashberg, *Introduction to symplectic field theory*, Preprint `math.SG/0010059`, October 2000.

[EM65] S. Eilenberg and J.C Moore, *Adjoint functors and trıples*, Illinois J Math **9** (1965), 381–389.

[EML53] S Eilenberg and S. Mac Lane, *On the groups $H(\pi, n)$ I*, Ann of Math (2) **58** (1953), 55–106.

[ENS01] J.B Etnyre, L L. Ng, and J.M. Sabloff, *Invarıants of legendrıan knots and coherent orıentations*, Preprint `math.SG/0101145`, January 2001

[Fed69] H. Federer, *Geometrıc measure theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1969

[FHT95] Y Félix, S. Halperin, and J-C Thomas, *Differential graded algebras in topology*, Handbook of algebraic topology (I.M James, ed), North-Holland, Amsterdam, 1995, pp. 829–865.

[FM94] W. Fulton and R MacPherson, *A compactification of configuration spaces*, Ann of Math. **139** (1994), 183–225

[FM97] T F Fox and M. Markl, *Distrıbutive laws, bialgebras, and cohomology*, Operads Proceedings of Renaissance Conferences (J.L. Loday, J D Stasheff, and A A Voronov, eds.), Contemporary Math., vol. 202, Amer. Math Soc., 1997, pp 167–205

[Fox93] T.F. Fox, *The construction of cofree coalgebras*, J Pure Appl Algebra **84** (1993), no 2, 191–198

[Fre97] B. Fresse, *Opérations de Cartan pour les algèbres simpliciales sur une opérade*, C R Acad Sci. Paris Sér I Math. **325** (1997), no 3, 247–252

[Fre98a] B Fresse, *Cogroups in algebras over an operad are free algebras*, Comment Math Helv. **73** (1998), no 4, 637–676

[Fre98b] B. Fresse, *Lie theory of formal groups over an operad*, J Algebra **202** (1998), 455–511

[Fre00] B Fresse, *On the homotopy of simplicial algebras over an operad*, Trans Amer Math Soc **352** (2000), no 9, 4113–4141

[FS97] K. Fukaya and P. Seidel, *Floer homology, A_∞-categorıes and topological field theory*, Geometry and Physics (J Andersen, J Dupont, H. Petersen, and A. Swan, eds.), Lecture Notes in Pure and Applied Mathematics, vol. 184, Marcel-Dekker, 1997, pp 9–32

[Ger63] M Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math **78** (1963), no 2, 267–288.

[Ger64] M Gerstenhaber, *On the deformation of rings and algebras*, Ann of Math. **79,1** (1964), 59–104

[Get93] E Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, Quantum deformations of algebras and their representations (A Joseph and S Shnider, eds.), Bar Ilan University, 1993, Israel Mathematical Conference Proceedings, pp. 65–79

[Get94a] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm Math. Phys **159** (1994), 265–285.

[Get94b] E. Getzler, *Two-dimensional topological gravity and equivariant cohomology*, Comm Math Phys. **163** (1994), no. 3, 473–489

[GH99] P G Goerss and M.J. Hopkins, *André-Quillen (co)-homology for simplicial algebras over simplicial operads*, Homotopy theory in the Swiss Alps, Contemporary Math , vol 265, Amer. Math Soc., Providence, RI, 1999, pp 41–85

[Gin95] V. Ginzburg, *Resolution of diagonals and moduli space*, The Moduli Space of Curves (R. Dijkgraaf, C Faber, and G van der Geer, eds.), Progress in Mathematics, vol 129, Birkhauser, 1995, pp. 231–266

[Gin01] V Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math Res Lett **8** (2001), no 3, 377–400

[GJ94] E Getzler and J D.S Jones, *Operads, homotopy algebra, and iterated integrals for double loop spaces*, Preprint hep-th/9403055, March 1994

[GK94] V Ginzburg and M.M. Kapranov, *Koszul duality for operads*, Duke Math. J **76** (1994), no. 1, 203–272

[GK95] E Getzler and M M. Kapranov, *Cyclic operads and cyclic homology*, Geometry, Topology and Physics for Raoul Bott (S.-T Yau, ed.), Conf. Proc Lect. Notes. Geom. Topol , vol 4, International Press, 1995, pp 167–201.

[GK98] E. Getzler and M M Kapranov, *Modular operads*, Compositio Math **110** (1998), no. 1, 65–126

[Gne95a] A V. Gnedbaye, *Les algèbres k-aires et leurs opérades*, C R. Acad. Sci. Paris Sér I Math. **321** (1995), no 2, 147–152.

[Gne95b] A V Gnedbaye, *Sur l'homologie des algèbres de Leibniz, opérades des algèbres k-aires*, preprint, Université Louis Pasteur, France, 1995.

[Gne96] A V. Gnedbaye, *Opérades des algèbres $(k+1)$-aires*, Operads Proceedings of Renaissance Conferences (J L Loday, J.D Stasheff, and A.A Voronov, eds), Contemporary Math., vol. 202, Amer Math. Soc., 1996, pp 83–114

[GPS95] R Gordon, A.J. Power, and R Street, *Coherence for tricategories*, Mem Amer Math. Soc , vol. 558, AMS, 1995

[Gro55] A. Grothendieck, *Produits tensoriels et produits topologiques*, Mem Amer Math Soc , vol 16, AMS, 1955

[Gug82] V K.A.M Gugenheim, *On a perturbation theory for the homology of a loop space*, J Pure Appl. Algebra **25** (1982), 197–207

[GV95] M Gerstenhaber and A A Voronov, *Higher operations on the Hochschild complex*, Functional Anal Appl. **29** (1995), no. 1, 1–5

[GW00] A V. Gnedbaye and M Wambst, *Jordan triples and operads*, J Algebra **231** (2000), no 2, 744–757

[Hai84] M. Haiman, *Constructing the associahedron*, Unpublished manuscript, 11 pages, 1984

[Hal83] S. Halperin, *Lectures on minimal models*, Memoirs Soc. Math. France, Nouv. Ser. 9-10, vol 261, Soc. Mat France, 1983

[Har77] R Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol 52, Springer-Verlag, 1977

[Har88] J Harer, *The cohomology of the moduli space of curves*, Theory of Moduli, Lect. 3rd Sess. Cent. Int Mat Estivo, Montecatini Terme, Italy 1985, Lecture Notes in Mathematics, vol 1337, Springer-Verlag, 1988, pp 138–221

[Hin97] V. Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **10** (1997), no 25, 3291–3323

[Hin01] V Hinich, *Virtual operad algebras and realization of homotopy types*, J. Pure Appl. Algebra **159** (2001), no 2-3, 173–185

[HL71] M. Herrera and D Lieberman, *Residues and principal values on complex spaces*, Math. Ann. **194** (1971), 259–294

[HL93] Y -Z Huang and J Lepowsky, *Vertex operator algebras and operads*, The Gelfand Mathematics Seminars, 1990–1992, Birkhauser, Boston, 1993, pp 145–161

[Hop42] H Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment Math Helv **14** (1942), 257–309

[HP91] P J. Hilton and J Pedersen, *Catalan numbers, their generalization, and their uses*, Math Intelligencer **13** (1991), no 2, 64–75.

[HS71] P J Hilton and U. Stammbach, *A course in homological algebra*, Graduate texts in Mathematics, vol 4, Springer-Verlag, 1971

[HS87] V A. Hinich and V V. Schechtman, *On homotopy limit of homotopy algebras*, Lecture Notes in Mathematics **1289** (1987), 240–264

[HS93] V.A. Hinich and V V Schechtman, *Homotopy Lie algebras*, Advances in Soviet Mathem **16** (1993), no. 2, 1–28

[Hua94] Y -Z. Huang, *Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras*, Comm Math. Phys **164** (1994), 145–161

[Hua97] Y.-Z Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Birkhauser Boston Inc , Boston, MA, 1997

[HW95] P. Hanlon and M L. Wachs, *On Lie k-algebras*, Adv in Math. **113** (1995), 206–236

[Jam55] I M. James, *Reduced product spaces*, Ann. of Math. **62** (1955), 170–197

[Jar81] H. Jarchov, *Locally convex spaces*, Teubner, Stuttgart, 1981.

[Joy81] A. Joyal, *Une théorie combinatoire des séries formelle*, Adv in Math **42** (1981), 1–82.

[KA56] T Kudo and S Araki, *Topology of H_n-spaces and H-squaring operations*, Mem. Fac Sci. Kyūsyū Univ. Ser A. **10** (1956), 85–120

[Kad80] T V Kadeishvili, *On the homology theory of fibre spaces*, Russian Math Surveys **35** (1980), no. 3, 231–238

[Kad85] T V Kadeishvili, *O kategorii differentialnych koalgebr i kategorii $A(\infty)$-algebr*, Trudy Tbiliss Mat. Inst Razmadze Akad. Nauk Gruzin. SSR **77** (1985), 50–70, in Russian

[Kad88] T V Kadeishvili, *$A(\infty)$-algebra structure in cohomology and the rational homotopy type*, Preprint 37, Forschungsschwerpunkt Geometrie, Universität Heidelberg, Mathematisches Institut, 1988

[Kap93] M Kapranov, *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation*, J Pure Appl. Algebra **85** (1993), 119–142

[Kap98] M Kapranov, *Operads in algebraic geometry*, Doc Math. (1998), 277–286, Extra Volume ICM

[Kel57] J L Kelley, *General topology*, D Van Nostrand, 1957

[Kel64] G M. Kelly, *On Mac Lane's conditions for coherence of natural associativities, commutativities, etc.*, J. Algebra **1** (1964), 397–402

[Kly74] A A Klyatchko, *Lie elements in the tensor algebra*, Siberian Math. J **15** (1974), 914–920

[KM76] F F. Knudsen and D Mumford, *The projectivity of the moduli space of stable curves I Preliminaries on "det" and "Div"*, Math Scand **39** (1976), 19–55

[KM93] I. Kříž and J P May, *Operads, algebras, and modules, I: Definitions and examples*, Preprint, University of Chicago, 1993.

[KM94] M Kontsevich and Y Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Preprint `hep-th/9402147`, February 1994

[KM95] I. Kříz and J P May, *Operads, algebras, modules and motives*, Astérisque, vol 233, Société Mathématique de France, 1995

[KM01] M. Kapranov and Y Manin, *Modules and Morita theorem for operads*, American Journal of Mathematics **123** (2001), no 5, 811–838.

[Knu83] F F Knudsen, *The projectivity of the moduli space of stable curves, II The stacks $M_{g,n}$*, Math Scand. **52** (1983), 161–199

[Kon93] M Kontsevich, *Formal (non)commutative symplectic geometry*, The Gel'fand mathematics seminars 1990–1992, Birkhauser, 1993.

[Kon94] M Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics II, Progr Math , vol 120, Birkhauser, Basel, 1994

[Kon97] M Kontsevich, *Deformation quantization of Poisson manifolds, I*, Preprint `q-alg/9709040`, September 1997

[Kon99] M Kontsevich, *Operads and motives in deformation quantization*, Preprint math.QA/9904055, April 1999

[KS00] M Kontsevich and Y Soibelman, *Deformations of algebras over operads and Deligne's conjecture*, Preprint math.QA/0001151, January 2000

[KSV95] T. Kimura, J.D. Stasheff, and A A Voronov, *On operad structures of moduli spaces and string theory*, Comm Math Phys **171** (1995), 1–25

[KVZ97] T Kimura, A A Voronov, and G J Zuckerman, *Homotopy Gerstenhaber algebras and topological field theory*, Operads Proceedings of Renaissance Conferences (J D Stasheff, J -L Loday and A.A. Voronov, eds.), Contemporary Mathematics, vol 202, 1997, pp 305–333

[Lad76] T Lada, *Strong homotopy algebras over monads*, The Homology of Iterated Loop Spaces (F.R Cohen, T. Lada, and P May, eds), Lecture Notes in Mathematics, vol 533, Springer, 1976, pp 399–479.

[Law63] F.W Lawvere, *Functorial semantics of algebraic theories*, Proc Nat Acad Sci U S.A **50** (1963), 869–872.

[Laz55] M Lazard, *Lois des groupes et analyseurs*, Ann Sci Ecole Norm Sup , III. Sér **72** (1955), no 3, 299–400.

[Laz59] M Lazard, *Lois des groupes et analyseurs*, Séminaire Bourbaki, 7e année 1954/1955, vol 109, Soc. Math Francé, 1959, pp 1–15

[Lee85] C. Lee, *Some notes on triangulating polytopes*, 3. Kolloquium uber Diskrete Geometrie, Institut fur Mathematik, Universitàt Salzburg, 1985, pp. 173–181

[Lee89] C. Lee, *The associahedron and triangulations of the n-gon*, European J. Combin **10** (1989), 551–560

[Leh77] D. Lehmann, *Théorie homotopique des formes différentielles*, Astérisque, vol 54, Soc Math France, 1977.

[LFCG01] J -L Loday, A. Frabetti, F Chapoton, and F Goichot, *Dialgebras and related operads*, Lecture Notes in Mathematics, vol 1763, Springer-Verlag, 2001

[LM95] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm Algebra **23** (1995), no. 6, 2147–2161.

[Lod93] J -L. Loday, *Une version non commutative des algébres de Liebniz*, L'Ens. Math. **39** (1993), 269–293

[Lod95] J -L. Loday, *Algèbres ayant deux opérations associatives (digèbres)*, C. R Acad Sci Paris Sér I Math. **321** (1995), no. 2, 141–146.

[Lod01] J.-L Loday, *Dialgebras*, preprint math.QA/0102053, February 2001.

[LQ84] J -L. Loday and D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv **59** (1984), 565–591.

[LR01] J -L. Loday and M. O Ronco, *Une dualité entre simplexes standards et polytopes de Stasheff*, C R. Acad Sci Paris Sér I Math **333** (2001), no. 2, 81–86

[LS93] T. Lada and J D Stasheff, *Introduction to sh Lie algebras for physicists*, Internat J Theoret. Phys. **32** (1993), no 7, 1087–1103.

[Mac63a] S Mac Lane, *Homology*, Springer-Verlag, 1963.

[Mac63b] S Mac Lane, *Natural associativity and commutativity*, Rice Univ. Stud **49** (1963), no 1, 28–46

[Mac65] S Mac Lane, *Categorical algebra*, Bull. Amer Math Soc **71** (1965), 40–106

[Mac71] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, 1971

[Mal74] P. Malraison, *Homotopy associative categories*, Lecture Notes in Math., vol. 428, Springer-Verlag, 1974, pp. 108–131

[Man99] Y I Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, Amer Math. Soc. Colloquium Publications, vol. 47, Amer Math. Soc., Providence, RI, 1999

[Man01] M A Mandell, *E_∞ algebras and p-adic homotopy theory*, Topology **40** (2001), no. 1, 43–94

[Mar92] M Markl, *A cohomology theory for $A(m)$-algebras and applications*, J Pure Appl Algebra **83** (1992), 141–175

[Mar96a] M Markl, *Cotangent cohomology of a category and deformations*, J Pure Appl Algebra **113** (1996), 195–218

[Mar96b] M Markl, *Distributive laws and Koszulness*, Ann. Inst. Fourier (Grenoble) **46** (1996), no 4, 307–323.

[Mar96c] M Markl, *Models for operads*, Comm Algebra **24** (1996), no 4, 1471–1500

[Mar99a] M Markl, *A compactification of the real configuration space as an operadic completion*, J Algebra **215** (1999), 185–204.

[Mar99b] M Markl, *Cyclic operads and homology of graph complexes*, Proceedings of the 18th Winter School "Geometry and physics", Srní, Czech Republic, January 10-17, 1998, Supplem ai Rend Circ Matem Palermo, Ser II, vol 59, 1999, pp 161–170

[Mar99c] M Markl, *Homotopy algebras are homotopy algebras*, Preprint `math.AT/9907138`, July 1999

[Mar99d] M Markl, *Simplex, associahedron, and cyclohedron*, Higher Homotopy Structures in Topology and Mathematical Physics (J. McCleary, ed), Contemporary Math , vol 227, Amer Math Soc , 1999, pp 235–265

[Mar00] M Markl, *Homotopy algebras via resolutions of operads*, Proceedings of the 19th Winter School "Geometry and physics", Srní, Czech Republic, January 9-15, 1999, Supplem ai Rend Circ. Matem Palermo, Ser. II, vol 63, 2000, pp 157–164

[Mar01a] M Markl, *Ideal perturbation lemma*, Comm Algebra **29** (2001), no 11, 5209–5232

[Mar01b] M Markl, *Loop homotopy algebras in closed string field theory*, Comm Math Phys **221** (2001), 367–384

[Mas58] W.S Massey, *Some higher order cohomology operations*, Symposium internacional de topología algebraica, 1958, pp. 145–154

[Mas69] W.S Massey, *Higher order linking numbers*, Conf Algebr Topol., Univ Ill Chicago Circle, 1969, pp 174–205

[Mas98] W S Massey, *Higher order linking numbers*, J. Knot Theory Ramifications **7** (1998), no 3, 393–414

[Mat97] O. Mathieu, *Homologies associated with Poisson structures*, Deformation theory and symplectic geometry (Ascona, 1996), Kluwer Acad Publ., Dordrecht, 1997, pp 77–199

[May69] J P May, *Matrix Massey products*, J. Algebra **12** (1969), 533–568.

[May70] J P. May, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and its Applications (F P Peterson, ed), Lectures Notes on Mathematics, vol 168, Springer Verlag, 1970, pp 153–231.

[May72] J P May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, vol 271, Springer-Verlag, New York, 1972

[May97] J P May, *Operadic tensor products and smash products*, Operads· Proceedings of Renaissance Conferences (J L Loday, J.D Stasheff, and A A. Voronov, eds.), Contemporary Math , vol 202, Amer Math Soc , 1997, pp. 287–303

[May98] J.P May, *Brave new worlds in stable homotopy theory*, Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), Contemporary Math., vol. 220, Amer Math Soc., Providence, RI, 1998, pp. 193–212

[Mil56] J. Milnor, *Construction of universal bundles. II*, Ann of Math (2) **63** (1956), 430–436

[Mil66] R.J Milgram, *Iterated loop spaces*, Ann of Math. **84** (1966), 386–403

[Mil67] R.J Milgram, *The bar construction and abelian H-spaces*, Illinois J Math **11** (1967), 242–250

[Mil74] R.J. Milgram, *Unstable homotopy from the stable point of view*, Lecture Notes in Mathematics, vol 36, Springer-Verlag, 1974

[MMW99] J.P Meyer, J Morava, and W.S. Wilson (eds), *Homotopy invariant algebraic structures A conference in honor of J M Boardman*, Contemporary Math., vol 239, Amer Math Soc , Providence, RI, 1999

[Moo71] J C Moore, *Differential homological algebra*, Actes Congres Intern Math **1** (1971), 335–339

[MP98] M. Mulase and M Penkava, *Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over $\bar{Q}$*, Preprint `math-ph/9811024`, November 1998

[MS99] J E. McClure and J H Smith, *A solution of Deligne's conjecture*, Preprint `math.QA/9910126`, October 1999

[MS01] M. Markl and S Shnider, *Coherence constraints for operads, categories and algebras*, Proceedings of the 20th Winter School "Geometry and Physics", Srní, Czech Republic, January 15-22, 2000, Supplem ai Rend. Circ Matem. Palermo, Ser II, vol 66, 2001, pp 29–57.

[MT68] R.E Mosher and M C Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, New York, Evanston and London, 1968

[MU57] W S Massey and H Uehara, *The Jacobi identity for Whitehead products*, Princeton Univ Press, 1957

[NT97] R Nest and B Tsygan, *The Fukaya type categories for associative algebras*, Deformation theory and symplectic geometry (Ascona, 1996), Math. Phys Stud , vol. 20, Kluwer Acad Publ., Dordrecht, 1997

[Pen87] R.C Penner, *The decorated Teichmüller space of punctured surfaces*, Comm Math Phys **113** (1987), 299–339

[Pen96] M Penkava, *Infinity algebras and the cohomology of graph complexes*, Preprint q-alg/9601018, January 1996.

[Pir94] T. Pirashvili, *On Leibniz homology*, Ann. Inst. Fourier (Grenoble) **44** (1994), no 2, 401–411.

[Pol98] A. Polishchuk, *Massey and Fukaya products on elliptic curves*, Preprint math.AG/9803017, March 1998.

[Pri70] S.B Priddy, *Koszul resolutions*, Trans. Amer Math Soc **152** (1970), 39–60

[Pri71] S B. Priddy, *On $\Omega^\infty S^\infty$ and the infinite symmetric group*, Proceedings of Symposia in Pure Mathematics, Madison, Wisconsin, 1970 (A Liulevicius, ed.), vol 12, 1971, pp. 217–220

[PS95] M Penkava and A. Schwarz, *A_∞ algebras and the cohomology of moduli spaces*, Trans Amer. Math. Soc **169** (1995), 91–107

[Qui67] D Quillen, *Homotopical algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, 1967

[Qui69] D. Quillen, *Rational homotopy theory*, Ann of Math **90** (1969), 205–295.

[Qui70] D Quillen, *Cohomology of groups*, Actes Congrès International Math **2** (1970), 47–51

[Ree58] R Ree, *Lie elements and an algebra associated with shuffles*, Ann of Math **68** (1958), 210–220

[Ret93] V.S. Retakh, *Lie-Massey brackets and n-homotopy multiplicative maps of differential graded Lie algebras*, J. Pure Appl Algebra **89** (1993), 217–229.

[Sal99] P. Salvatore, *Configuration spaces with summable labels*, preprint math.AT/9907073, July 1999

[Sal01] P. Salvatore, *Homotopy type of Euclidean configuration spaces*, Proceedings of the 20th Winter School "Geometry and Physics", Srní, Czech Republic, January 15-22, 2000, Supplem. ai Rend Circ. Matem. Palermo, Ser. II, vol. 66, 2001, pp. 161–164.

[Sat56] I Satake, *On a generalization of the notion of manifold*, Proc. Nat Acad Sci U S A. **42** (1956), 359–363

[Sch93] M Schwarz, *Morse homology*, Progress in Mathematics, vol 111, Birkhäuser Verlag, 1993

[SE62] N E Steenrod and D B A. Epstein, *Cohomology operations*, Princeton University Press, 1962

[Seg73] G Segal, *Configuration-spaces and iterated loop-spaces*, Invent Math. **21** (1973), 213–221

[Seg74] G Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312

[Sem] *Séminaire Bourbaki, vol.3, Années 1954/55-1955/5*, Société Mathématique de France, Paris

[Ser65] J -P Serre, *Lie algebras and Lie groups*, Benjamin, 1965

[Smi82] V.A Smirnov, *On the cochain complex of topological spaces*, Math USSR Sbornik **43** (1982), 133–144

[Smi85] V.A. Smirnov, *Homotopy theory of coalgebras*, Izv Akad Nauk SSSR, Ser Mat **49** (1985), no. 6, 1302–1321, In Russian.

[Smi92] V A Smirnov, *Secondary operations in the homology of the operad E*, Izv Ross. Akad. Nauk, Ser Mat **56** (1992), 449–468.

[Smi94] J R. Smith, *Iterating the cobar construction*, Mem. Amer Math Soc., vol 109, Amer. Math. Soc., 1994

[Smi99] V A. Smirnov, *A_∞-simplicial objects and A_∞-topological groups*, Mat Zametki **66** (1999), no 6, 913–919

[Smi01] V A. Smirnov, *Simplicial and operad methods in algebraic topology*, Translations of Mathematical Monographs, vol. 198, Amer. Math. Soc., Providence, RI, 2001.

[Spa66] E.H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966

[SS84] M. Schlessinger and J D. Stasheff, *Deformation theory and rational homotopy type*, Preprint, 1984

[SS85] M Schlessinger and J D Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl Algebra **38** (1985), 313–322

[SS94] S Shnider and S Sternberg, *Quantum groups from coalgebras to Drinfel'd algebras A guided tour*, International Press, Cambridge, Mass , 1994, second edition 1997

[Sta61] J D Stasheff, *On the homotopy associativity of H-spaces*, Dissertation, Princeton University, 1961

[Sta63a] J D Stasheff, *Homotopy associativity of H-spaces, I*, Trans Amer Math Soc **108** (1963), 275–292.

[Sta63b] J D Stasheff, *Homotopy associativity of H-spaces, II*, Trans Amer Math Soc **108** (1963), 293–312.

[Sta70] J D Stasheff, *H-spaces from a homotopy point of view*, Lecture Notes in Math., vol 161, Springer-Verlag, 1970

[Sta83] J D Stasheff, *Rational Poincaré duality spaces*, Illinois J Math **27** (1983), 104–109

[Sta95] J D Stasheff, *The cycloedra for compactification of configuration spaces of the circle*, Preprint, 1995

[Sta97] J D. Stasheff, *From operads to 'physically' inspired theories*, Operads Proceedings of Renaissance Conferences (J -L. Loday, J.D Stasheff, and A.A Voronov, eds), Contemporary Math., vol. 202, 1997, pp 53–81

[Str84] K Strebel, *Quadratic differentials*, Springer-Verlag, Berlin, 1984

[SU00] S Saneblidze and R Umble, *A diagonal on the associahedra*, Preprint, March 2000

[Sug57a] M Sugawara, *On a condition that a space is an h-space*, Math J Okayama Univ **6** (1957), 109–129

[Sug57b] M Sugawara, *On a condition that a space is group-like*, Math J Okayama Univ **7** (1957), 123–149

[Sug61] M. Sugawara, *On the homotopy-commutativity of groups and loop spaces*, Mem Coll Sci Univ. Kyoto, Ser A Math **33** (1960/61), 257–269

[Sul77] D Sullivan, *Infinitesimal computations in topology*, Inst Hautes Études Sci Publ Math **47** (1977), 269–331

[SV92a] D Sahoo and M C Valsakumar, *Nambu mechanics and its quantization*, Phys. Rev A **46** (1992), 4410–4412

[SV92b] R. Schwanzl and R. Vogt, *Homotopy homomorphisms and the hammock localization*, Bol. Soc. Mat Mexicana (2) **37** (1992), no 1 -2 , 431–448.

[SVO99] S Shnider and D.H Van Osdol, *Operads as abstract algebras, and the Koszul property*, J. Pure Appl. Algebra **1-3** (1999), no 143, 381–407

[SW01] P Salvatore and N. Wahl, *Framed discs operads and the equivariant recognition principle*, Preprint `math.AT/0106242`, July 2001

[Tak94] L Takhtajan, *On foundation of the generalized Nambu mechanics*, Comm Math Phys **160** (1994), 295–315

[Tam98a] D E Tamarkin, *Another proof of M Kontsevich formality theorem*, Preprint `math.QA/9803025`, March 1998

[Tam98b] D E Tamarkin, *Formality of chain operad of small squares*, Preprint `math.QA/9809164`, September 1998

[Tan83] D Tanré, *Homotopie rationnelle Modèles de Chen, Quillen, Sullivan*, Lect Notes in Math., vol. 1025, Springer-Verlag, 1983

Til99] U. Tillmann, *Vanishing of the Batalin-Vilkovisky algebra structure for TCFTs*, Comm. Math Phys **205** (1999), no. 2, 283–286

Til00] U. Tillmann, *Higher genus surface operad detects infinite loop spaces*, Math Ann **317** (2000), no. 3, 613–628

Tsy83] B.L Tsygan, *Homology of matrix algebras over rings and Hochschild homology*, Uspekhi Mat. Nauk **38** (1983), 217–218, in Russian

TT00] D E Tamarkin and B. Tsygan, *Noncommutative differential calculus, homotopy BV algebras and formality conjectures*, Preprint `math.KT/0002116`, February 2000

Uly99] A Ulyanov, *Polydiagonal compactification of configuration spaces*, preprint `math.AG/9904049`, April 1999

Vis89] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math **97** (1989), 613–670

[Vog] R.M. Vogt, *Cofibrant operads and universal E_∞ operads*, Preprint E99-005, 81–89, available at http //www.mathematik.uni-bielefeld de/sfb343/preprints/index99.html, submitted to Ann. Inst. Fourier

[Vog98] R.M Vogt, *My time as Mike Boardman's student and our work on infinite loop spaces*, Preprint, 1998

[Vor94] A A Voronov, *Topological field theories, string backgrounds and homotopy algebras*, Proceedings of the XXIInd International Conference on Differential Geometric Methods in Theoretical Physics, Ixtapa-Zihuatanejo, Mexico (J Keller and Z Oziewicz, eds.), Advances in Applied Clifford Algebras, vol 4 (S1), 1994, pp 167–178

[Vor98] A.A Voronov, *The Swiss cheese operad of open-closed string field theory*, Technical report, MIT, preprint `math/98`, 1998.

[Vor99a] A A Voronov, *Homotopy Gerstenhaber algebras*, Preprint `math.QA/9908040`, August 1999

[Vor99b] A.A Voronov, *The Swiss-cheese operad*, Homotopy Invariant Algebraic Structures (J.P Meyer, J Morava, and W S. Wilson, eds), Contemporary Math , vol 239, 1999, pp 365–373

[Vor01] A.A. Voronov, *Notes on universal algebra*, preprint `math.QA/0111009`, November 2001

[VV98] A. Vinogradov and M. Vinogradov, *On multiple generalizations of Lie algebras and Poisson manifolds*, Secondary calculus and cohomological physics, Moscow, August 24-31, 1997, Contemp Math., vol 219, Amer. Math Soc., Providence, RI, 1998

[Wah01] N Wahl, *Ribbon braid and related operads*, Ph.D. thesis, Oxford, 2001

[Wei] C Weibel, *e-letter to Stasheff*

[Wei94] C Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol 38, Cambridge University Press, 1994

[Wel80] R O Wells, *Differential analysis on complex manifolds*, Graduate Texts in Mathematics, vol. 65, Springer-Verlag, 1980

[Wey97] H. Weyl, *The classical groups Their invariants and representations. Fifteenth printing*, Princeton University Press, 1997

[Whi98] A.N. Whitehead, *A treatise on universal algebra*, Cambridge, 1898

[WZ94] M Wolf and B Zwiebach, *The plumbing of minimal area surfaces*, J. Geom. Phys **15** (1994), 23–56

[Xu99] P. Xu, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, Comm Math. Phys **200** (1999), no 3, 545–560.

[Zwi93] B Zwiebach, *Closed string field theory· Quantum action and the Batalin-Vilkovisky master equation*, Nuclear Phys. B **390** (1993), 33–152

Glossary of notations

$\mathsf{a}(v)$	number of inputs (arity) of a vertex v of a rooted tree, 83, 217
Aff	group of affine transformations of the complex plane $\mathbb{C}$, 213
$\mathit{Aff}(V)$	group of affine transformations of a vector space V, 220
$\mathcal{A}ss$	operad for associative algebras, 20, 44, 168
$\underline{\mathcal{A}ss}$	non-Σ operad for associative algebras, 14, 45
$\mathcal{A}ss_\infty$	operad for A_∞-algebras, 20
$\underline{\mathcal{A}ss}_\infty$	non-Σ operad for A_∞-algebras, 13, 26, 106
$\mathcal{B}(\mathcal{P})$	bar construction on an operad $\mathcal{P}$, 20
$\mathcal{B}race$	operad for brace algebras, 31
$\mathcal{BV}$	operad for Batalin-Vilkovisky algebras, 206, 257
BY	classifying space of an A_∞-space Y, 11
$\mathcal{B}_k$	operad for k-braid algebras, 27
B_∞	Baues operad, 30
$CH_*(A;A)$	Hochschild chains of A with coefficients in itself, 259
$CH^*(A;A)$	Hochschild cochains of A with coefficients in itself, 29, 104
$\mathcal{C}mon$	operad for commutative associative monoids, 111
$\mathcal{C}o\mathcal{E}nd_X$	coendomorphism operad, 43
$\mathtt{Col}$	category of collections, 85
$\mathtt{Col}_K$	category of K-collections, 85
$\mathcal{C}om$	operad for commutative associative algebras, 20, 44, 168
$\mathcal{C}om_\infty$	operad for C_∞-algebras, 20
$\mathsf{Con}(\mathbb{R}^k,n)$	configuration space of n distinct labeled points in $\mathbb{R}^k$, 26, 100
$\mathsf{Con}(M,n)$	configuration space of n distinct labeled points in a manifold M, 234
$\overline{\mathsf{Con}}(M,n)$	compactification of the configuration space $\mathsf{Con}(M,n)$, 234, 239
$\mathsf{Con}(V,n)$	configuration space of n distinct labeled points in a vector space V, 220
$\mathcal{C}_1$	little intervals operad, 95
$\mathcal{C}_2$	little squares operad, 95
$\mathcal{C}_k$	little k-cubes operad, 12, 94, 203
$\mathbf{C}(\mathcal{P})$	cobar complex of an operad $\mathcal{P}$, 20, 121, 125
$\mathbb{CP}^1$	complex projective line, 207
$C_{\mathcal{P}}^{-*}(V)$	$\mathcal{P}$-algebra chain complex with trivial coefficients, 173
$C_{\mathcal{P}}^{*}(V;V)$	$\mathcal{P}$-algebra cochain complex with coefficients in itself, 177

$\check{\mathfrak{D}}$	dual of a cocycle $\mathfrak{D}$, 284
$det(S)$	determinant of a finite set S, 124
$Det(S)$	Determinant of a finite set S, 283
$det(T)$	determinant of a tree T, 124
$Det(T)$	Determinant of a tree T, 128
$Det(V)$	Determinant of a vector space V, 283
$\mathtt{dgVec}$	category of differential graded vector spaces, 38, 121
$\mathtt{dgOp}$	category of differential graded operads, 133
$\mathtt{dg\Psi Op}$	category of differential graded pseudo-operads, 126
D^k, B^k	standard unit k-dimensional disk (ball), 204
$\mathcal{D}_k$	little k-disks operad, 12, 203
$\mathbf{D}(\mathcal{P})$	dual dg operad of an operad $\mathcal{P}$, 20, 121, 127
$D\mathcal{P}$	Σ-module of decomposables of an operad $\mathcal{P}$, 187
e_2	operad for Gerstenhaber algebras, 30, 176
$Edg(T)$	set of all edges of a tree T minus the root edge, 51, 128
$Edge(T)$	set of all edges of a tree T including the root edge, 51
$edge(T)$	set of internal edges of a tree T, 51, 217, 123
$edge(\Gamma)$	set of edges of a graph Γ, 269
$edge(v)$	set of edges incident with a vertex v, 250
$\mathcal{E}nd_X$	endomorphism operad, 6, 43, 254
$\mathcal{E}\mathcal{Z}$	Eilenberg-Zilber operad, 105
$\mathcal{E}tree(n)$	set of of isomorphism classes of n-trees with a special vertex, 234
$E^{\#}$	linear dual of a Σ-module, 141
$E^{\vee}$	Czech dual of a Σ-module E, 142
$\mathfrak{f}\mathsf{Con}(M,n)$	framed configuration space of n distinct labeled points in a manifold M, 210, 234
$\mathfrak{f}\overline{\mathsf{Con}}(M,n)$	compactification of the space $\mathfrak{f}\mathsf{Con}(M,n)$, 234
$\mathfrak{f}\mathcal{D}_k$	framed little k-disks operad, 13, 203
$\mathsf{F}_{\mathfrak{D}}(\mathcal{A})$	Feynman transform of a $\mathfrak{D}$-modular operad $\mathcal{A}$, 287
$Flag(\Gamma)$	set of flags (half-edges) of a graph Γ, 269
$\mathcal{F}_{\mathcal{P}}(-)$	free $\mathcal{P}$-algebra functor, 47
$\mathcal{F}^c_{\mathcal{P}}(-)$	cofree nilpotent $\mathcal{P}$-coalgebra functor, 166
$\overset{\circ}{\mathsf{F}}_V(n)$	moduli space $\overset{\circ}{\mathsf{F}}_V(n) := \mathsf{Con}(V,n)/\mathit{Aff}(V)$, 220
$\mathsf{F}_V(n)$	compactification of the moduli space $\overset{\circ}{\mathsf{F}}_V(n)$, 225
$\overset{\circ}{\mathsf{F}}_V$	Σ-module $\{\overset{\circ}{\mathsf{F}}_V(n)\}_{n\geq 1}$, 220
F_V	operad $\{\mathsf{F}_V(n)\}_{n\geq 1}$, 225
$\overset{\circ}{\mathsf{F}}_k(n)$	simplified notation for the space $\overset{\circ}{\mathsf{F}}_{\mathbb{R}^k}(n)$, 220
$\mathsf{F}_k(n)$	simplified notation for the space $\mathsf{F}_k(n)$, 27, 225
$\overset{\circ}{\mathsf{F}}_k$	simplified notation for the space $\overset{\circ}{\mathsf{F}}_{\mathbb{R}^k}$, 220
F_k	simplified notation for the operad $\mathsf{F}_{\mathbb{R}^k}$, 27, 226
G_∞	minimal model of the operad e_2, 30
$GC^{(n)}_*$	graph complex, 301
$Gr^{(n)}$	set of connected graphs with Euler characteristic $1-n$, 301

$Gr_{(n)}^{\mathrm{met}}$	set of graphs $\Gamma \in Gr^{(n)}$ with a metric, 301
Grav	gravity operad, 311
gVec	category of graded vector spaces, 38
$g(\Gamma)$	genus of a labeled graph Γ, 270
$\mathcal{H} = (\mathcal{H}, Q)$	BRST complex, 24, 208
$Harr_*(A, M)$	Harrison homology of A with coefficients in M, 266
$\mathcal{HG}$	McClure-Smith operad, 30, 31
$HH^*(A; A)$	Hochschild cohomology of A with coefficients in itself, 29
$HH_*(A; A)$	Hochschild homology of A with coefficients in itself, 259
$\mathcal{H}_{\mathrm{rel}} = (\mathcal{H}_{\mathrm{rel}}, Q)$	relative BRST complex, 26
$hS(-\|-)$	skew-symmetric shuffle product, 197
$H_*^\lambda(A)$	cyclic homology of an associative algebra A, 259
$H_{\mathrm{DR}}^*(A)$	de Rham cohomology of an algebra A, 260
$H_{\mathcal{P}}^*(V; V)$	operadic cohomology of a $\mathcal{P}$-algebra V with coefficients in itself, 177
$IFlag(\Gamma)$	set of internal flags of a graph Γ, 285
$In(v)$	set of incoming edges of a vertex v of a rooted tree, 51
$Iso\,(\mathcal{D})$	category of isomorphisms of a category $\mathcal{D}$, 72, 271
JX	James' reduced product, 97
$\mathbf{K}(\mathcal{P}^!)$	Koszul complex of an operad $\mathcal{P}$, 121, 145
$\mathbf{K}(\mathcal{P}^!)^\#$	dual Koszul complex of an operad $\mathcal{P}$, 146
K_n	Stasheff's associahedron, 9, 56
$Leaf(T)$	set of leaves of a tree T, 51
$Leg(\Gamma)$	set of legs (half-edges) of a graph Γ, 269
$Leg(v)$	set of legs (half-edges) adjacent to a vertex v, 269
$\mathcal{L}eib$	operad for Leibniz algebras, 257
$\mathcal{L}i$	linear isometries operad, 106, 107
$\mathtt{LI}$	category of real inner-product spaces, 107
$\mathcal{L}ie$	operad for Lie algebras, 20, 50
$\mathcal{L}ie_\infty$	operad for L_∞-algebras, 18
$L(T)$	number of leaves of a tree T, 128
$\mathcal{M}_{0,n}$	moduli space of n distinct labeled points in $\mathbb{CP}^1$, 212
$\overline{\mathcal{M}}(n)$	compactification of the moduli space $\mathcal{M}_{0,n+1}$, 213, 305
$\mathcal{M}_{g,n}$	moduli space of genus g Riemann surfaces with n distinct labeled points, 292, 305
$\overline{\mathcal{M}}(g,n)$, $\overline{\mathcal{M}}_{g,n+1}$	compactification of the moduli space $\mathcal{M}_{g,n+1}$, 304
$\overline{\mathcal{M}}$	configuration pseudo-operad $\{\overline{\mathcal{M}}(n)\}_{n\geq 2}$, 25, 216; modular operad $\{\overline{\mathcal{M}}(g,n)\}_{(g,n)\in\mathfrak{S}}$, 306
$\widehat{\mathcal{M}}_0(n)$	moduli space of Riemann spheres with $n+1$ labeled parametrized holes, 23, 207, 267
$\widehat{\mathcal{M}}_0$	operad $\{\widehat{\mathcal{M}}_0(n)\}_{n\geq 1}$, 23, 207, 247
$\widehat{\mathcal{M}}(g,n)$	moduli space of genus g Riemann surfaces with $n+1$ labeled parametrized holes, 267, 304
$\widehat{\mathcal{M}}$	unstable modular operad $\{\widehat{\mathcal{M}}(g,n)\}_{g\geq 0, n\geq -1}$, 267

$\mathfrak{M}_{g,s}$	moduli space of genus g surfaces with s distinct unlabeled points, 292
$\mathfrak{M}_{g,s}^{\mathrm{dec}}$	moduli space of genus g surfaces with s distinct unlabeled decorated points, 292
$\mathbb{M}(-)$	triple for modular operads, 271
$\mathbb{M}_{\mathfrak{D}}(-)$	triple for $\mathfrak{D}$-twisted modular operads, 281
$MC_{g,s}$	mapping class group of a genus g surface with s distinct labeled points, 292
$Met(T)$	space of metrics on a tree T, 109, 221, 235
$\mathtt{MMod}$	category of stable modular Σ-modules, 268
$\mathtt{Mod}_{\mathbf{k}}$	category of $\mathbf{k}$-modules, 38
$\mathrm{Mod}(\mathcal{P})$	modular completion of an operad $\mathcal{P}$, 317
$\mathtt{Mod}_{\mathcal{P}}$	category of $\mathcal{P}$-modules, 139
$\mathcal{M}on$	operad for associative topological monoids, 28, 99
$\underline{\mathcal{M}on}$	non-Σ operad for associative topological monoids, 112
$\mathtt{Mon}(\mathcal{C})$	category of $\mathcal{C}$-monoids, 40
MX	rectification of a $W\mathcal{P}$-space X, 116
$\mathcal{N}(n)$	moduli space of Riemann spheres with $n+1$ marked decorated points, 25
$\underline{\mathcal{N}}(n)$	compactification of the space $\mathcal{N}(n)$, 25
$\underline{\mathcal{N}}$	operad $\{\underline{\mathcal{N}}(n)\}_{n\geq 1}$, 23, 25
$\mathtt{Nerve}_\bullet(\mathcal{C})$	nerve of a category $\mathcal{C}$, 272
$\mathbf{N}(\mathcal{P})$	categorial cobar complex of an operad $\mathcal{P}$, 121, 149
$\mathtt{Op}$	category of operads, 42
$Ord(X)$	set of orderings of X, 62
$\mathcal{P} = \langle E; R\rangle$	presentation of an operad $\mathcal{P}$, 139
$\mathcal{P}^{!}$	quadratic dual of a quadratic operad $\mathcal{P}$, 21
$\mathcal{P}^{+}$	pseudo-operad associated to an operad $\mathcal{P}$, 110; augmentation ideal of an augmented operad $\mathcal{P}$, 187
$\mathcal{P}^{\perp}$	operad $(s\mathcal{P}^{!})^{\#}$, 262
$\mathcal{P}\rtimes G$	semidirect product of an operad $\mathcal{P}$ with a group G, 204
$\mathcal{P}\langle-\rangle$	free $\mathcal{P}$-module functor, 139
$\mathcal{P}_{\mathtt{B}\to\mathtt{W}}$	colored operad for homomorphisms of $\mathcal{P}$-spaces, 115
P_n	permutahedron, 97
$\mathcal{P}oiss$	operad for Poisson algebras, 176, 256, 261
$Q\mathcal{P}$	Σ-module of indecomposables of an operad $\mathcal{P}$, 188
R	Smith operad, 105
$RGr_{g,s}$	set of ribbon graphs Γ such that $Surf(\Gamma)$ has s holes and genus g, 292
$RGr_{g,s}^{\mathrm{met}}$	set of ribbon graphs $\Gamma \in RGr_{g,s}$ with a metric, 292
$RGC_{*}^{g,s}$	ribbon graph complex, 295
$\mathcal{R}mtree(n)$	space of isomorphism classes of reduced rooted metric n-trees, 109, 221
$\mathcal{R}mtree$	pseudo-operad of reduced rooted metric trees, 110

$\mathcal{R}m\underline{tree}(n)$	space of isomorphism classes of reduced rooted metric planar n-trees, 112
$\mathcal{R}m\underline{tree}$	non-Σ pseudo-operad of reduced rooted planar metric trees, 112
$\mathcal{R}tree(n)$	set of isomorphism classes of reduced rooted n-trees, 55
$\mathcal{R}tree$	pseudo-operad of reduced rooted trees, 55
$\mathcal{R}\underline{tree}(n)$	set of isomorphism classes of reduced planar rooted n-trees, 55
$\mathcal{R}\underline{tree}$	non-Σ operad of reduced planar rooted trees, 55, 109
$\mathtt{Rtree}$	category of reduced rooted n-trees, 87, 124
$rv(T)$	vertex attached to the root of a rooted tree T, 221, 235
$\mathfrak{s}A$	operadic suspension of a Σ-module A, 127, 258
$\mathfrak{s}^{-1}A$	operadic desuspension of a Σ-module A, 127
$\mathfrak{s}\mathcal{E}$	modular suspension of a modular Σ-module $\mathcal{E}$, 283
$\mathtt{Set}$	category of sets, 38
$\mathtt{Set}_f$	category of finite sets, 40
$\mathtt{Set}_f\text{-}\mathtt{Mod}$	category of $\mathtt{Set}_f$-modules, 40
sgn_n	signum representation of the symmetric group Σ_n, 264
$Sh(-\|-)$	shuffle product, 196
SS_n^p	set of surjection sequences, 152
$Surj[j,n]$	set of surjections $f : [n] \twoheadrightarrow [j]$, 150
$\mathtt{Surj}_f$	category of surjections of finite sets, 60
$Surf(\Gamma)$	surface associated to a ribbon graph Γ, 292
$\mathcal{T}_{g,s}$	Teichmüller space of genus g surfaces with s distinct labeled points, 291
$\mathcal{T}_{g,s}^{\mathrm{dec}}$	decorated Teichmüller space of genus g surfaces with s distinct labeled points, 291
$T_{[f]}$	tree corresponding to a surjection f, 151
$\mathcal{T}ree(n)$	set of isomorphism classes of rooted n-trees, 8
$\mathcal{T}ree$	operad of rooted trees, 8, 54
$\underline{\mathcal{T}ree}(n)$	set of isomorphism classes of planar rooted n-trees, 8
$\underline{\mathcal{T}ree}$	non-Σ operad of planar rooted trees, 8, 54
$\mathcal{T}ree(X)$	set of isomorphism classes of X-labeled trees, 53
$\mathtt{Tree}$	category of labeled rooted trees, 52
$\mathtt{Tree}_n$	category of labeled rooted n-trees, 52
$\mathtt{Tree}_X$	category of X-labeled rooted trees, 52
$\mathtt{Tree}_n^+$	category of unrooted trees with legs labeled by $\{0,\ldots,n\}$, 250
$\mathcal{UA}ss$	operad for associative algebras with unit, 267
$\mathcal{UC}om$	operad for commutative associative algebras with unit, 267, 316
$unsh(-,\ldots,-)$	set of unshuffles, 99
$\mathcal{UP}oiss$	operad for Poisson algebras with unit, 267
$\mathtt{Vec}$	category of vector spaces, 38
$Vert(T)$	set of (internal) vertices of a tree T, 51, 250
$\overline{Vert}(T)$	set of all (including external) vertices of a tree T, 51

$Vert(\Gamma)$	set of vertices of a graph Γ, 269
$V^{\#} = (V^{\#}, d^{\#})$	linear dual of a dg complex, 121
$W(\mathcal{P})$	W-construction on an operad $\mathcal{P}$, 28, 111
$\underline{W}(\underline{\mathcal{P}})$	non-Σ W-construction on a non-Σ operad $\underline{\mathcal{P}}$, 111,
W_n	cyclohedron, 241
$\mathfrak{Z}(h)$	zero set of a metric $h \in Met(T)$, 221
$\mathbb{Z}_n$	cyclic group $\mathbb{Z}/n\mathbb{Z}$

- - - - -

$*_n$	n-star, 250
$*_{g,n}$	modular n-corolla of genus g, 271.
$\mathrm{c}(n)$	n-corolla, 251
$\mathbb{1}$	trivial operad, 42, 187
$[n]$	set $\{1, \ldots, n\}$, 40
$\lvert\Gamma\rvert$	geometric realization of a graph Γ, 269
$\lvert T\rvert$	number of internal edges of a tree T, 123
Λ	determinant operad, 128
$\square$	box product on Σ-`Mod`, 68
$\square_K$	box product on $\mathtt{Col}_K$, 86
$\circ_i$	composition (Gerstenhaber) product, 6
$\Gamma(C)$	dual graph of a stable curve C, 304
$\mathbf{\Gamma}(\!(g, S)\!)$	category of stable S-labeled graphs of genus g, 271
$\Gamma_+(-)$	free cyclic operad functor, 251
$\Gamma(-)$	free operad functor, 82
$\Gamma_K(-)$	free K-operad functor, 88
$\Psi(-)$	free pseudo-operad functor, 81
$\Psi_+(-)$	free cyclic pseudo-operad functor, 251
$\Lambda(\mathcal{P}, -)$	universal bilinear form, 261
Ω^*_A	module of differentials of an algebra A, 260
$\Omega^k X$	k-fold loop space of a topological space X, 93
$\Omega^*_{\mathrm{DR}}(M)$	dg algebra of de Rham forms, 29
$\Psi\mathtt{Op}$	category of pseudo-operads, 67
Σ-`Mod`	category of Σ-modules, 40
Σ^+-`Mod`	category of cyclic Σ-modules, 249
Σ_n	permutation group of $\{1, \ldots, n\}$, 40
Σ^+_n	permutation group of $\{0, \ldots, n\}$, 264
Σ	symmetric groupoid, 40
$\uparrow U$	suspension of a graded vector space U, 314
$\downarrow U$	desuspension of a graded vector space U, 314
$\mathbf{r}(U)$	reflection of a graded vector space U, 314
τ_n	cycle $(0, \ldots, n) \in \Sigma^+_n$, 247
$\epsilon(\sigma)$	Koszul sign, 195
$\chi(\sigma)$	skew-symmetric Koszul sign, 196

Index