Реализация на дискретните Фурие трансформации с квантови мрежи

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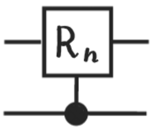
Използват се следните базисни гейтове :

-  $He_j = 2^{-1/2}\left(e_0 + (-1)^j e_1\right) = 2^{-1/2} \sum_{k=0,1} \underbrace{\exp\left(2\pi i \frac{jk}{2}\right)}_{e^{2\pi i \frac{jk}{2}}} e_k$

  $2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

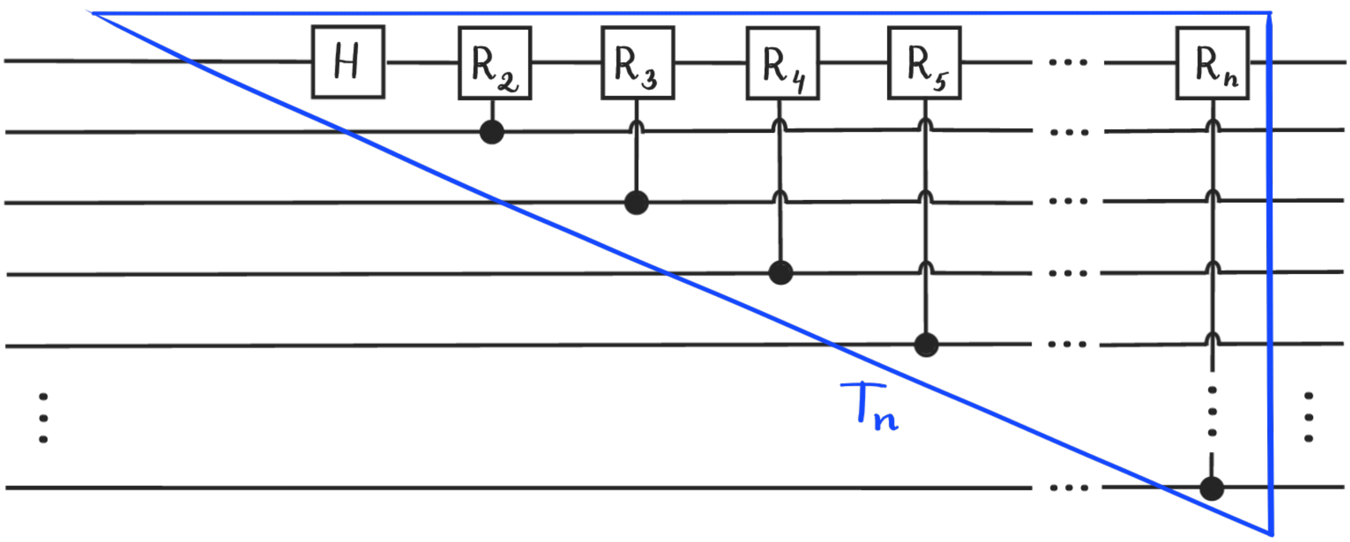
  където $j,k \in \{0,1\}$

-  $R_n \, e_j \otimes e_k = \exp\left(2\pi i \frac{jk}{2^n}\right) e_j \otimes e_k$

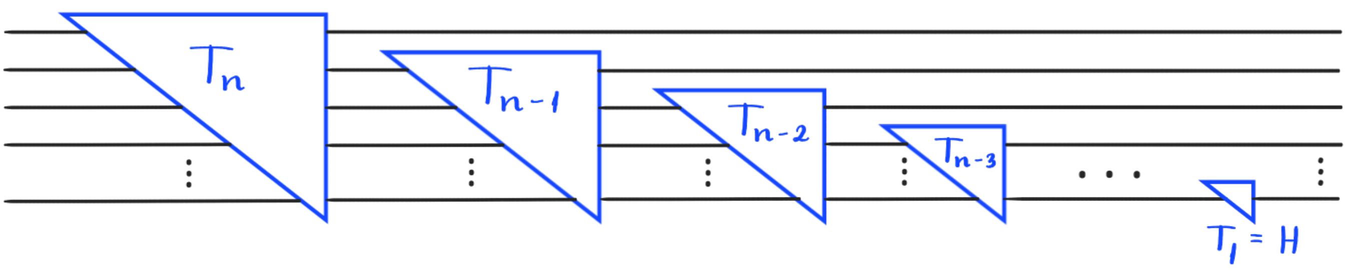
   } алтернативно означение (тип "условна операция")

  за $n = 2, 3, \dots$

Градивните блокове на веригата са триъгълниците



Цялата композиция е :



1. Как действа T_n: $T_n(e_{j_1}\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n})$

$$\overset{?}{=} 2^{-1/2}\Big(e_0+\exp\big(2\pi i\,[0.j_1j_2j_3j_4j_5\cdots j_n]_2\big)e_1\Big)\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n}$$

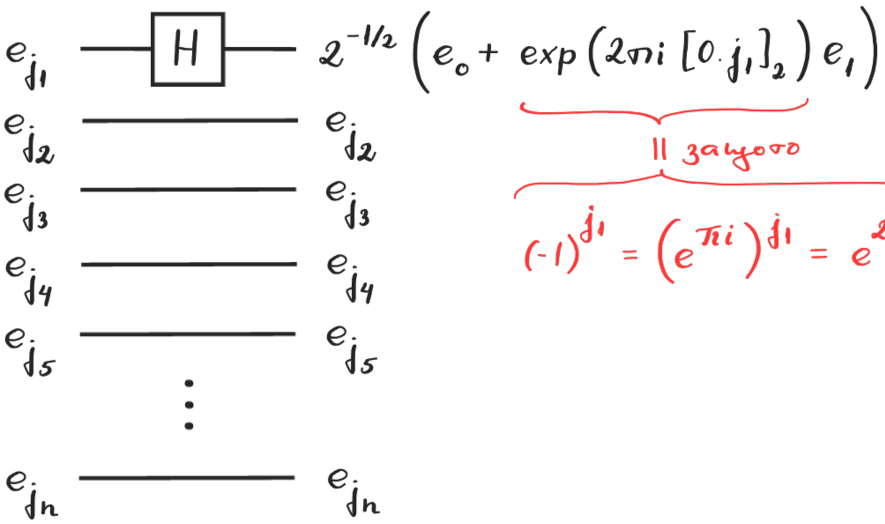
където $[0.j_1j_2j_3j_4j_5\cdots j_n]_2$ е двоичната дроб:

$$[0.j_1j_2j_3j_4j_5\cdots j_n]_2 := j_1 2^{-1}+j_2 2^{-2}+j_3 2^{-3}+j_4 2^{-4}+j_5 2^{-5}+\cdots+j_n 2^{-n}.$$

Доказателство:

$$T_n(e_{j_1}\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n}) = ?$$

Стъпка 1:



$\underbrace{\exp(2\pi i\,[0.j_1]_2)}_{\parallel \text{ защото}}$

$$(-1)^{j_1} = \left(e^{\pi i}\right)^{j_1} = e^{2\pi i \frac{j_1}{2}}$$

$[0.j_1]_2$

$$e_{j_1}\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n}$$

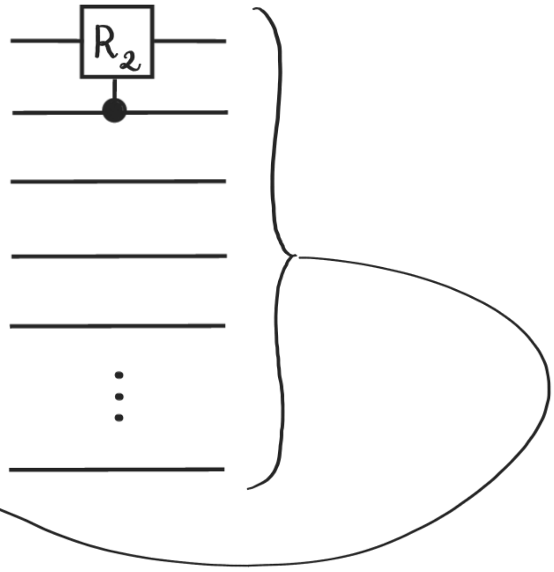
$$\mapsto He_{j_1}\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n}$$

$$= 2^{-1/2}\Big(e_0+\exp\big(2\pi i\,[0.j_1]_2\big)e_1\Big)\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n}$$

$$= 2^{-1/2}\, e_0\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n}$$

$$+\ 2^{-1/2}\exp\big(2\pi i\,[0.j_1]_2\big)e_1\otimes e_{j_2}\otimes e_{j_3}\otimes e_{j_4}\otimes e_{j_5}\otimes\cdots\otimes e_{j_n}$$

Стъпка 2 :



$$2^{-1/2}\, \underbrace{e_0 \otimes e_{j_2}}_{\downarrow R_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

$$\boxed{e^{2\pi i \frac{0\cdot j_2}{2^2}}}\; e_0 \otimes e_{j_2}$$

$\to 1$

$$+\, 2^{-1/2} \exp\left(2\pi i\, [0.j_1]_2\right)\, \underbrace{e_1 \otimes e_{j_2}}_{\downarrow R_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

$$\boxed{e^{2\pi i \frac{1\cdot j_2}{2^2}}}\; e_1 \otimes e_{j_2}$$

$\to \exp\left(2\pi i\, [0.0j_2]_2\right)$

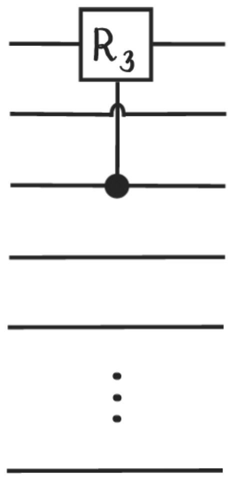
$$\exp\left(2\pi i\, [0.j_1]_2 + 2\pi i\, [0.0j_2]_2\right) = \exp\left(2\pi i\, [0.j_1j_2]_2\right)$$

$\Rightarrow$ резултата след стъпка 2 е :

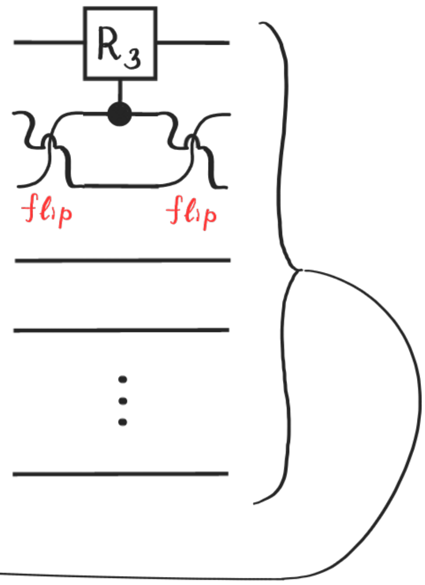
$$2^{-1/2}\, e_0 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

$$+\, 2^{-1/2} \exp\left(2\pi i\, [0.j_1j_2]_2\right)\, e_1 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

Стъпка 3:



- това е все едно -



$$2^{-1/2}\, e_0 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

R_3

$\boxed{e^{2\pi i \frac{0\cdot j_3}{2^3}}} \to 1$

$e_0 \otimes e_{j_2} \otimes e_{j_3}$

$$+\, 2^{-1/2} \underbrace{\exp\left(2\pi i\, [0.j_1 j_2]_2\right)}\, e_1 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

R_3

$\boxed{e^{2\pi i \frac{1\cdot j_3}{2^3}}} \to \underbrace{\exp\left(2\pi i\, [0.00 j_3]_2\right)}$

$e_0 \otimes e_{j_2} \otimes e_{j_3}$

$$\exp\left(2\pi i\, [0.j_1 j_2]_2 + 2\pi i\, [0.00 j_3]_2\right) = \exp\left(2\pi i\, [0.j_1 j_2 j_3]_2\right)$$

$\Rightarrow$ резултата след стъпка 3 е:

$$2^{-1/2}\, e_0 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

$$+\, 2^{-1/2} \exp\left(2\pi i\, [0.j_1 j_2 j_3]_2\right) e_1 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

Аналогично, след събика и резултата е:

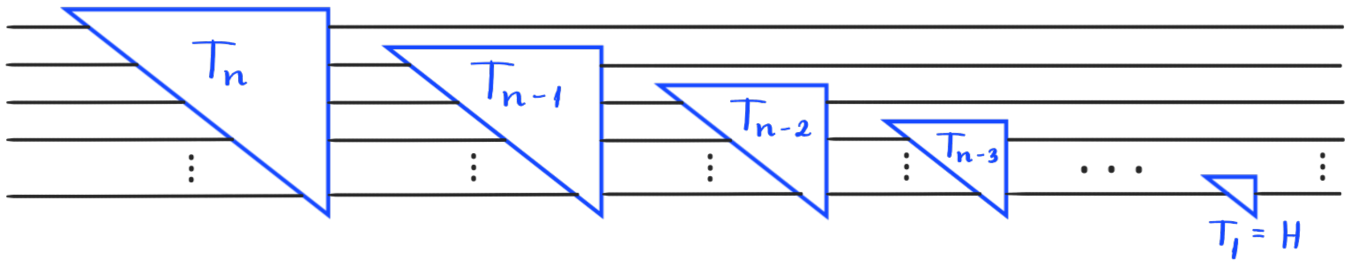
$$2^{-1/2}\, e_0 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

$$+\; 2^{-1/2} \exp\left(2\pi i\, [0.j_1 j_2 j_3 j_4 j_5 \cdots j_n]_2\right) e_1 \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

$$= 2^{-1/2}\left(e_0 + \exp\left(2\pi i\, [0.j_1 j_2 j_3 j_4 j_5 \cdots j_n]_2\right) e_1\right) \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4} \otimes e_{j_5} \otimes \cdots \otimes e_{j_n}$$

което и твърдяхме

2. Изчисляване на композицията



$$\underbrace{e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes \cdots \otimes e_{j_n}}$$

$$\Big\downarrow\, T_n$$

$$2^{-1/2}\left(e_0 + \exp\left(2\pi i\, [0.j_1 j_2 j_3 \cdots j_n]_2\right) e_1\right) \otimes \underbrace{e_{j_2} \otimes e_{j_3} \otimes \cdots \otimes e_{j_n}}$$

$$\Big\downarrow\, T_{n-1}$$

$$2^{-1/2}\left(e_0 + \exp\left(2\pi i\, [0.j_1 j_2 j_3 \cdots j_n]_2\right) e_1\right) \otimes 2^{-1/2}\left(e_0 + \exp\left(2\pi i\, [0.j_2 j_3 \cdots j_n]_2\right) e_1\right) \otimes e_{j_3} \otimes \cdots \otimes e_{j_n}$$

По индукция, резултата е:

$$e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes \cdots \otimes e_{j_n}$$

$$\longmapsto 2^{-n/2} \bigotimes_{r=1}^{n} \left(e_0 + \exp\left(2\pi i \, [0.j_r \cdots j_n]_2\right) e_1 \right) = \, ?$$

не е комутативно – реда е важен!

Забелязваме, че: $N = 2^n \Rightarrow 2^{-n/2} = \frac{1}{\sqrt{N}}$

$$[j_1 \cdots j_{r-1} . j_r \cdots j_n]_2 = \frac{[j_1 \cdots j_n]_2}{2^{n-r+1}}$$ и тогава

$$\exp\left(2\pi i \, [0.j_r \cdots j_n]_2\right) = \exp\left(2\pi i \left([j_1 \cdots j_{r-1}]_2 + [0.j_r \cdots j_n]_2 \right)\right) =$$

$$= \exp\left(\frac{2\pi i}{N} [j_1 \cdots j_n]_2 \cdot 2^{r-1} \right)$$

Следователно:

$$? = \frac{1}{\sqrt{N}} \bigotimes_{r=1}^{n} \left(e_0 + \exp\left(\frac{2\pi i}{N} [j_1 \cdots j_n]_2 \cdot 2^{r-1} \right) e_1 \right)$$

$$= \frac{1}{\sqrt{N}} \bigotimes_{r=1}^{n} \sum_{k_r = 0}^{1} \exp\left(\frac{2\pi i}{N} [j_1 \cdots j_n]_2 \cdot 2^{r-1} k_r \right) e_{k_r}$$

$$= \frac{1}{\sqrt{N}} \sum_{k_1, \ldots, k_n = 0}^{1} \exp\left(\frac{2\pi i}{N} [j_1 \cdots j_n]_2 \cdot \underbrace{\left(2^{n-1} k_n + \cdots + 2^{1-1} k_1 \right)}_{[k_n \cdots k_1]_2} \right) \underbrace{e_{k_1} \otimes \cdots \otimes e_{k_n}}_{|[k_1 \cdots k_n]_2\rangle}$$

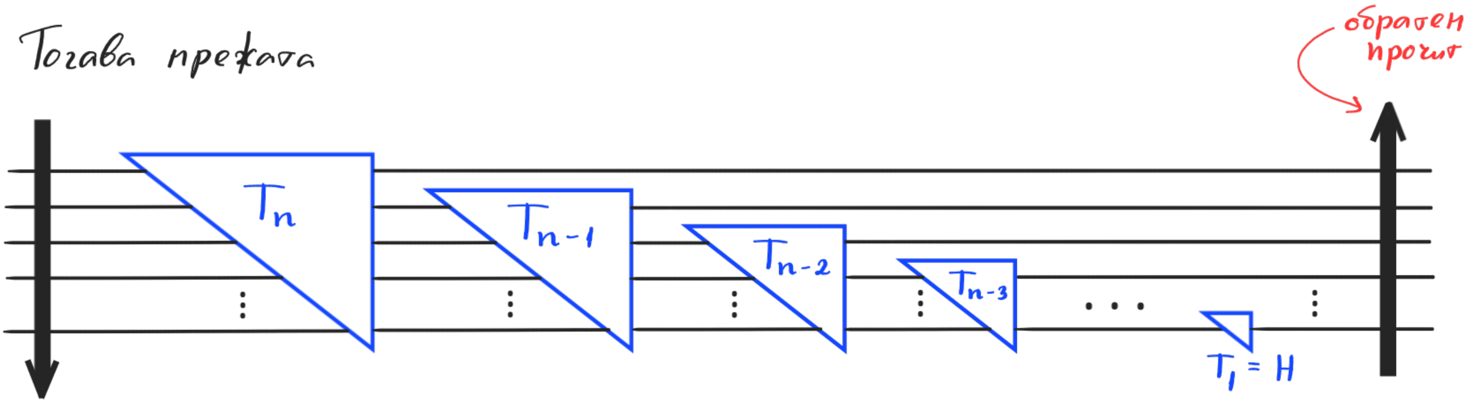
Крайни изводи :

Нека за $J, K = 0, \dots, N-1$ $(N = 2^n)$ имаме двоични разлагания

$J = [j_1 \cdots j_n]_2$ и $K = [k_1 \cdots k_n]_2$ и нека

$|J\rangle = |[j_1 \cdots j_n]_2\rangle = e_{j_1} \otimes \cdots \otimes e_{j_n}$ е стандартния базис на $\mathbb{C}^N$ изразен с тензорно произведение.

Тогава прежата



извършва дискретната Фурие трансформация

$$|J\rangle \longmapsto \frac{1}{\sqrt{N}} \sum_{K=0}^{N-1} \exp\left(\frac{2\pi i}{N} JK\right) |K\rangle$$