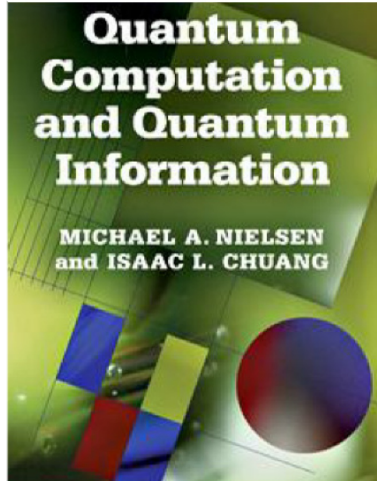




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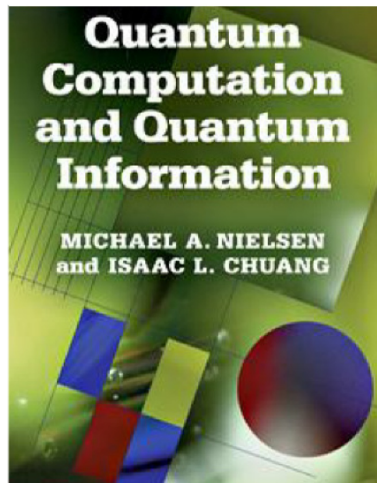
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Michael Nielsen
b. 1974
Research Fellow
at the *Astera Institute*
' <https://astera.org/>



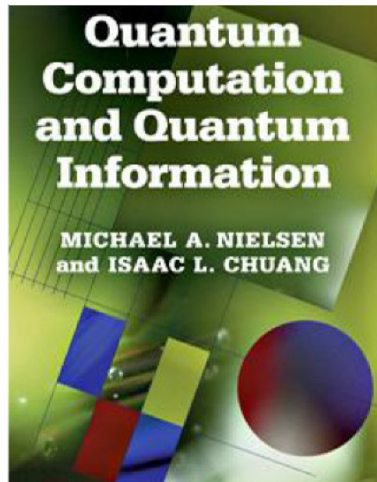
Isaac Chuang
Senior Associate Dean of
Digital Learning, and
Professor of Electrical
Engineering & Computer
Science,
and Professor of Physics, at
the *Massachusetts Institute
of Technology* (MIT)



Contents

Introduction to the Tenth Anniversary Edition	page xvii
Afterword to the Tenth Anniversary Edition	xix
Preface	xxi
Acknowledgements	xxvii
Nomenclature and notation	xxix
Part I Fundamental concepts	1
1 Introduction and overview	1
1.1 Global perspectives	1
1.1.1 History of quantum computation and quantum information	2
1.1.2 Future directions	12
1.2 Quantum bits	13
1.2.1 Multiple qubits	16
1.3 Quantum computation	17
1.3.1 Single qubit gates	17
1.3.2 Multiple qubit gates	20
1.3.3 Measurements in bases other than the computational basis	22
1.3.4 Quantum circuits	22
1.3.5 Qubit copying circuit?	24
1.3.6 Example: Bell states	25
1.3.7 Example: quantum teleportation	26
1.4 Quantum algorithms	28
1.4.1 Classical computations on a quantum computer	29
1.4.2 Quantum parallelism	30
1.4.3 Deutsch's algorithm	32
1.4.4 The Deutsch-Jozsa algorithm	34
1.4.5 Quantum algorithms summarized	36
1.5 Experimental quantum information processing	42
1.5.1 The Stern-Gerlach experiment	43
1.5.2 Prospects for practical quantum information processing	46
1.6 Quantum information	50
1.6.1 Quantum information theory: example problems	52
1.6.2 Quantum information in a wider context	58

2 Introduction to quantum mechanics	60	4.3 Controlled operations	177
2.1 Linear algebra	61	4.4 Measurement	185
2.1.1 Bases and linear independence	62	4.5 Universal quantum gates	188
2.1.2 Linear operators and matrices	63	4.5.1 Two-level unitary gates are universal	189
2.1.3 The Pauli matrices	65	4.5.2 Single qubit and CNOT gates are universal	191
2.1.4 Inner products	65	4.5.3 A discrete set of universal operations	194
2.1.5 Eigenvectors and eigenvalues	68	4.5.4 Approximating arbitrary unitary gates is generically hard	198
2.1.6 Adjoint and Hermitian operators	69	4.5.5 Quantum computational complexity	200
2.1.7 Tensor products	71	4.6 Summary of the quantum circuit model of computation	202
2.1.8 Operator functions	75	4.7 Simulation of quantum systems	204
2.1.9 The commutator and anti-commutator	76	4.7.1 Simulation in action	204
2.1.10 The polar and singular value decompositions	78	4.7.2 The quantum simulation algorithm	206
2.2 The postulates of quantum mechanics	80	4.7.3 An illustrative example	209
2.2.1 State space	80	4.7.4 Perspectives on quantum simulation	211
2.2.2 Evolution	81	5 The quantum Fourier transform and its applications	216
2.2.3 Quantum measurement	84	5.1 The quantum Fourier transform	217
2.2.4 Distinguishing quantum states	86	5.2 Phase estimation	221
2.2.5 Projective measurements	87	5.2.1 Performance and requirements	223
2.2.6 POVM measurements	90	5.3 Applications: order-finding and factoring	226
2.2.7 Phase	93	5.3.1 Application: order-finding	226
2.2.8 Composite systems	93	5.3.2 Application: factoring	232
2.2.9 Quantum mechanics: a global view	96	5.4 General applications of the quantum Fourier transform	234
2.3 Application: superdense coding	97	5.4.1 Period-finding	236
2.4 The density operator	98	5.4.2 Discrete logarithms	238
2.4.1 Ensembles of quantum states	99	5.4.3 The hidden subgroup problem	240
2.4.2 General properties of the density operator	101	5.4.4 Other quantum algorithms?	242
2.4.3 The reduced density operator	105	6 Quantum search algorithms	248
2.5 The Schmidt decomposition and purifications	109	6.1 The quantum search algorithm	248
2.6 EPR and the Bell inequality	111	6.1.1 The oracle	248
3 Introduction to computer science	120	6.1.2 The procedure	250
3.1 Models for computation	122	6.1.3 Geometric visualization	252
3.1.1 Turing machines	122	6.1.4 Performance	253
3.1.2 Circuits	129	6.2 Quantum search as a quantum simulation	255
3.2 The analysis of computational problems	135	6.3 Quantum counting	261
3.2.1 How to quantify computational resources	136	6.4 Speeding up the solution of NP-complete problems	263
3.2.2 Computational complexity	138	6.5 Quantum search of an unstructured database	265
3.2.3 Decision problems and the complexity classes P and NP	141	6.6 Optimality of the search algorithm	269
3.2.4 A plethora of complexity classes	150	6.7 Black box algorithm limits	271
3.2.5 Energy and computation	153	7 Quantum computers: physical realization	277
3.3 Perspectives on computer science	161	7.1 Guiding principles	277
Part II Quantum computation	171	7.2 Conditions for quantum computation	279
4 Quantum circuits	171	7.2.1 Representation of quantum information	279
4.1 Quantum algorithms	172	7.2.2 Performance of unitary transformations	281
4.2 Single qubit operations	174		



Contents

Introduction to the Tenth Anniversary Edition	page xvii
Afterword to the Tenth Anniversary Edition	xix
Preface	xxi
Acknowledgements	xxvii
Nomenclature and notation	xxix
Part I Fundamental concepts	1
1 Introduction and overview	1
1.1 Global perspectives	1
1.1.1 History of quantum computation and quantum information	2
1.1.2 Future directions	12
1.2 Quantum bits	13
1.2.1 Multiple qubits	16
1.3 Quantum computation	17
1.3.1 Single qubit gates	17
1.3.2 Multiple qubit gates	20
1.3.3 Measurements in bases other than the computational basis	22
1.3.4 Quantum circuits	22
1.3.5 Qubit copying circuit?	24
1.3.6 Example: Bell states	25
1.3.7 Example: quantum teleportation	26
1.4 Quantum algorithms	28
1.4.1 Classical computations on a quantum computer	29
1.4.2 Quantum parallelism	30
1.4.3 Deutsch's algorithm	32
1.4.4 The Deutsch–Jozsa algorithm	34
1.4.5 Quantum algorithms summarized	36
1.5 Experimental quantum information processing	42
1.5.1 The Stern–Gerlach experiment	43
1.5.2 Prospects for practical quantum information processing	46
1.6 Quantum information	50
1.6.1 Quantum information theory: example problems	52
1.6.2 Quantum information in a wider context	58

2 Introduction to quantum mechanics	60
2.1 Linear algebra	61
2.1.1 Bases and linear independence	62
2.1.2 Linear operators and matrices	63
2.1.3 The Pauli matrices	65
2.1.4 Inner products	65
2.1.5 Eigenvectors and eigenvalues	68
2.1.6 Adjoint and Hermitian operators	69
2.1.7 Tensor products	71
2.1.8 Operator functions	75
2.1.9 The commutator and anti-commutator	76
2.1.10 The polar and singular value decompositions	78
2.2 The postulates of quantum mechanics	80
2.2.1 State space	80
2.2.2 Evolution	81
2.2.3 Quantum measurement	84
2.2.4 Distinguishing quantum states	86
2.2.5 Projective measurements	87
2.2.6 POVM measurements	90
2.2.7 Phase	93
2.2.8 Composite systems	93
2.2.9 Quantum mechanics: a global view	96
2.3 Application: superdense coding	97
2.4 The density operator	98
2.4.1 Ensembles of quantum states	99
2.4.2 General properties of the density operator	101
2.4.3 The reduced density operator	105
2.5 The Schmidt decomposition and purifications	109
2.6 EPR and the Bell inequality	111
3 Introduction to computer science	120
3.1 Models for computation	122
3.1.1 Turing machines	122
3.1.2 Circuits	129
3.2 The analysis of computational problems	135
3.2.1 How to quantify computational resources	136
3.2.2 Computational complexity	138
3.2.3 Decision problems and the complexity classes P and NP	141
3.2.4 A plethora of complexity classes	150
3.2.5 Energy and computation	153
3.3 Perspectives on computer science	161
Part II Quantum computation	171
4 Quantum circuits	171
4.1 Quantum algorithms	172
4.2 Single qubit operations	174
4.3 Controlled operations	177
4.4 Measurement	185
4.5 Universal quantum gates	188
4.5.1 Two-level unitary gates are universal	189
4.5.2 Single qubit and CNOT gates are universal	191
4.5.3 A discrete set of universal operations	194
4.5.4 Approximating arbitrary unitary gates is generically hard	198
4.5.5 Quantum computational complexity	200
4.6 Summary of the quantum circuit model of computation	202
4.7 Simulation of quantum systems	204
4.7.1 Simulation in action	204
4.7.2 The quantum simulation algorithm	206
4.7.3 An illustrative example	209
4.7.4 Perspectives on quantum simulation	211
5 The quantum Fourier transform and its applications	216
5.1 The quantum Fourier transform	217
5.2 Phase estimation	221
5.2.1 Performance and requirements	223
5.3 Applications: order-finding and factoring	226
5.3.1 Application: order-finding	226
5.3.2 Application: factoring	232
5.4 General applications of the quantum Fourier transform	234
5.4.1 Period-finding	236
5.4.2 Discrete logarithms	238
5.4.3 The hidden subgroup problem	240
5.4.4 Other quantum algorithms?	242
6 Quantum search algorithms	248
6.1 The quantum search algorithm	248
6.1.1 The oracle	248
6.1.2 The procedure	250
6.1.3 Geometric visualization	252
6.1.4 Performance	253
6.2 Quantum search as a quantum simulation	255
6.3 Quantum counting	261
6.4 Speeding up the solution of NP-complete problems	263
6.5 Quantum search of an unstructured database	265
6.6 Optimality of the search algorithm	269
6.7 Black box algorithm limits	271
7 Quantum computers: physical realization	277
7.1 Guiding principles	277
7.2 Conditions for quantum computation	279
7.2.1 Representation of quantum information	279
7.2.2 Performance of unitary transformations	281

Contents			Contents			Contents			Contents		
	7.2.3 Preparation of fiducial initial states	281	8.4 Applications of quantum operations	386	11.2.3 Conditional entropy and mutual information	505	A2.2 Representations	612			
	7.2.4 Measurement of output result	282	8.4.1 Master equations	386	11.2.4 The data processing inequality	509	A2.2.1 Equivalence and reducibility	612			
7.3	Harmonic oscillator quantum computer	283	8.4.2 Quantum process tomography	389	11.3 Von Neumann entropy	510	A2.2.2 Orthogonality	613			
	7.3.1 Physical apparatus	283	8.5 Limitations of the quantum operations formalism	394	11.3.1 Quantum relative entropy	511	A2.2.3 The regular representation	614			
	7.3.2 The Hamiltonian	284			11.3.2 Basic properties of entropy	513	A2.3 Fourier transforms	615			
	7.3.3 Quantum computation	286	9 Distance measures for quantum information	399	11.3.3 Measurements and entropy	514					
	7.3.4 Drawbacks	286	9.1 Distance measures for classical information	399	11.3.4 Subadditivity	515	Appendix 3: The Solovay–Kitaev theorem	617			
7.4	Optical photon quantum computer	287	9.2 How close are two quantum states?	403	11.3.5 Concavity of the entropy	516					
	7.4.1 Physical apparatus	287	9.2.1 Trace distance	403	11.3.6 The entropy of a mixture of quantum states	518	Appendix 4: Number theory	625			
	7.4.2 Quantum computation	290	9.2.2 Fidelity	409	11.4 Strong subadditivity	519	A4.1 Fundamentals	625			
	7.4.3 Drawbacks	296	9.2.3 Relationships between distance measures	415	11.4.1 Proof of strong subadditivity	519	A4.2 Modular arithmetic and Euclid's algorithm	626			
7.5	Optical cavity quantum electrodynamics	297	9.3 How well does a quantum channel preserve information?	416	11.4.2 Strong subadditivity: elementary applications	522	A4.3 Reduction of factoring to order-finding	633			
	7.5.1 Physical apparatus	298					A4.4 Continued fractions	635			
	7.5.2 The Hamiltonian	300	10 Quantum error-correction	425	12 Quantum information theory	528	Appendix 5: Public key cryptography and the RSA cryptosystem	640			
	7.5.3 Single-photon single-atom absorption and refraction	303	10.1 Introduction	426	12.1 Distinguishing quantum states and the accessible information	529					
	7.5.4 Quantum computation	306	10.1.1 The three qubit bit flip code	427	12.1.1 The Holevo bound	531	Appendix 6: Proof of Lieb's theorem	645			
7.6	Ion traps	309	10.1.2 Three qubit phase flip code	430	12.1.2 Example applications of the Holevo bound	534					
	7.6.1 Physical apparatus	309	10.2 The Shor code	432	12.2 Data compression	536	Bibliography	649			
	7.6.2 The Hamiltonian	317	10.3 Theory of quantum error-correction	435	12.2.1 Shannon's noiseless channel coding theorem	537	Index	665			
	7.6.3 Quantum computation	319	10.3.1 Discretization of the errors	438	12.2.2 Schumacher's quantum noiseless channel coding theorem	542					
	7.6.4 Experiment	321	10.3.2 Independent error models	441	12.3 Classical information over noisy quantum channels	546					
7.7	Nuclear magnetic resonance	324	10.3.3 Degenerate codes	444	12.3.1 Communication over noisy classical channels	548					
	7.7.1 Physical apparatus	325	10.3.4 The quantum Hamming bound	444	12.3.2 Communication over noisy quantum channels	554					
	7.7.2 The Hamiltonian	326	10.4 Constructing quantum codes	445	12.4 Quantum information over noisy quantum channels	561					
	7.7.3 Quantum computation	331	10.4.1 Classical linear codes	445	12.4.1 Entropy exchange and the quantum Fano inequality	561					
	7.7.4 Experiment	336	10.4.2 Calderbank–Shor–Steane codes	450	12.4.2 The quantum data processing inequality	564					
7.8	Other implementation schemes	343	10.5 Stabilizer codes	453	12.4.3 Quantum Singleton bound	568					
			10.5.1 The stabilizer formalism	454	12.4.4 Quantum error-correction, refrigeration and Maxwell's demon	569					
			10.5.2 Unitary gates and the stabilizer formalism	459	12.5 Entanglement as a physical resource	571					
			10.5.3 Measurement in the stabilizer formalism	463	12.5.1 Transforming bipartite pure state entanglement	573					
			10.5.4 The Gottesman–Knill theorem	464	12.5.2 Entanglement distillation and dilution	578					
			10.5.5 Stabilizer code constructions	464	12.5.3 Entanglement distillation and quantum error-correction	580					
			10.5.6 Examples	467	12.6 Quantum cryptography	582					
			10.5.7 Standard form for a stabilizer code	470	12.6.1 Private key cryptography	582					
			10.5.8 Quantum circuits for encoding, decoding, and correction	472	12.6.2 Privacy amplification and information reconciliation	584					
			10.6 Fault-tolerant quantum computation	474	12.6.3 Quantum key distribution	586					
			10.6.1 Fault-tolerance: the big picture	475	12.6.4 Privacy and coherent information	592					
			10.6.2 Fault-tolerant quantum logic	482	12.6.5 The security of quantum key distribution	593					
			10.6.3 Fault-tolerant measurement	489							
			10.6.4 Elements of resilient quantum computation	493	Appendices	608					
					Appendix 1: Notes on basic probability theory	608					
Part III Quantum information	353		11 Entropy and information	500	Appendix 2: Group theory	610					
8 Quantum noise and quantum operations	353		11.1 Shannon entropy	500	A2.1 Basic definitions	610					
8.1 Classical noise and Markov processes	354		11.2 Basic properties of entropy	502	A2.1.1 Generators	611					
8.2 Quantum operations	356		11.2.1 The binary entropy	502	A2.1.2 Cyclic groups	611					
8.2.1 Overview	356		11.2.2 The relative entropy	504	A2.1.3 Cosets	612					
8.2.2 Environments and quantum operations	357										
8.2.3 Operator-sum representation	360										
8.2.4 Axiomatic approach to quantum operations	366										
8.3 Examples of quantum noise and quantum operations	373										
8.3.1 Trace and partial trace	374										
8.3.2 Geometric picture of single qubit quantum operations	374										
8.3.3 Bit flip and phase flip channels	376										
8.3.4 Depolarizing channel	378										
8.3.5 Amplitude damping	380										
8.3.6 Phase damping	383										

Contents			Contents			Contents			Contents		
	7.2.3 Preparation of fiducial initial states	281	8.4 Applications of quantum operations	386	11.2.3 Conditional entropy and mutual information	505	A2.2 Representations	612			
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7.3	Harmonic oscillator quantum computer	283	8.4.2 Quantum process tomography	389	11.3 Von Neumann entropy	510	A2.2.2 Orthogonality	613			
	7.3.1 Physical apparatus	283	8.5 Limitations of the quantum operations formalism	394	11.3.1 Quantum relative entropy	511	A2.2.3 The regular representation	614			
	7.3.2 The Hamiltonian	284			11.3.2 Basic properties of entropy	513	A2.3 Fourier transforms	615			
	7.3.3 Quantum computation	286	9 Distance measures for quantum information	399	11.3.3 Measurements and entropy	514					
	7.3.4 Drawbacks	286	9.1 Distance measures for classical information	399	11.3.4 Subadditivity	515	Appendix 3: The Solovay–Kitaev theorem	617			
7.4	Optical photon quantum computer	287	9.2 How close are two quantum states?	403	11.3.5 Concavity of the entropy	516					
	7.4.1 Physical apparatus	287	9.2.1 Trace distance	403	11.3.6 The entropy of a mixture of quantum states	518	Appendix 4: Number theory	625			
	7.4.2 Quantum computation	290	9.2.2 Fidelity	409	11.4 Strong subadditivity	519	A4.1 Fundamentals	625			
	7.4.3 Drawbacks	296	9.2.3 Relationships between distance measures	415	11.4.1 Proof of strong subadditivity	519	A4.2 Modular arithmetic and Euclid's algorithm	626			
7.5	Optical cavity quantum electrodynamics	297	9.3 How well does a quantum channel preserve information?	416	11.4.2 Strong subadditivity: elementary applications	522	A4.3 Reduction of factoring to order-finding	633			
	7.5.1 Physical apparatus	298					A4.4 Continued fractions	635			
	7.5.2 The Hamiltonian	300	10 Quantum error-correction	425	12 Quantum information theory	528	Appendix 5: Public key cryptography and the RSA cryptosystem	640			
	7.5.3 Single-photon single-atom absorption and refraction	303	10.1 Introduction	426	12.1 Distinguishing quantum states and the accessible information	529					
	7.5.4 Quantum computation	306	10.1.1 The three qubit bit flip code	427	12.1.1 The Holevo bound	531	Appendix 6: Proof of Lieb's theorem	645			
7.6	Ion traps	309	10.1.2 Three qubit phase flip code	430	12.1.2 Example applications of the Holevo bound	534					
	7.6.1 Physical apparatus	309	10.2 The Shor code	432	12.2 Data compression	536	Bibliography	649			
	7.6.2 The Hamiltonian	317	10.3 Theory of quantum error-correction	435	12.2.1 Shannon's noiseless channel coding theorem	537	Index	665			
	7.6.3 Quantum computation	319	10.3.1 Discretization of the errors	438	12.2.2 Schumacher's quantum noiseless channel coding theorem	542					
	7.6.4 Experiment	321	10.3.2 Independent error models	441	12.3 Classical information over noisy quantum channels	546					
7.7	Nuclear magnetic resonance	324	10.3.3 Degenerate codes	444	12.3.1 Communication over noisy classical channels	548					
	7.7.1 Physical apparatus	325	10.3.4 The quantum Hamming bound	444	12.3.2 Communication over noisy quantum channels	554					
	7.7.2 The Hamiltonian	326	10.4 Constructing quantum codes	445	12.4 Quantum information over noisy quantum channels	561					
	7.7.3 Quantum computation	331	10.4.1 Classical linear codes	445	12.4.1 Entropy exchange and the quantum Fano inequality	561					
	7.7.4 Experiment	336	10.4.2 Calderbank–Shor–Steane codes	450	12.4.2 The quantum data processing inequality	564					
7.8	Other implementation schemes	343	10.5 Stabilizer codes	453	12.4.3 Quantum Singleton bound	568					
			10.5.1 The stabilizer formalism	454	12.4.4 Quantum error-correction, refrigeration and Maxwell's demon	569					
			10.5.2 Unitary gates and the stabilizer formalism	459	12.5 Entanglement as a physical resource	571					
Part III Quantum information	353		10.5.3 Measurement in the stabilizer formalism	463	12.5.1 Transforming bipartite pure state entanglement	573					
			10.5.4 The Gottesman–Knill theorem	464	12.5.2 Entanglement distillation and dilution	578					
8 Quantum noise and quantum operations	353		10.5.5 Stabilizer code constructions	464	12.5.3 Entanglement distillation and quantum error-correction	580					
8.1	Classical noise and Markov processes	354	10.5.6 Examples	467	12.6 Quantum cryptography	582					
8.2	Quantum operations	356	10.5.7 Standard form for a stabilizer code	470	12.6.1 Private key cryptography	582					
	8.2.1 Overview	356	10.5.8 Quantum circuits for encoding, decoding, and correction	472	12.6.2 Privacy amplification and information reconciliation	584					
	8.2.2 Environments and quantum operations	357			12.6.3 Quantum key distribution	586					
	8.2.3 Operator-sum representation	360	10.6 Fault-tolerant quantum computation	474	12.6.4 Privacy and coherent information	592					
	8.2.4 Axiomatic approach to quantum operations	366	10.6.1 Fault-tolerance: the big picture	475	12.6.5 The security of quantum key distribution	593					
8.3	Examples of quantum noise and quantum operations	373	10.6.2 Fault-tolerant quantum logic	482							
	8.3.1 Trace and partial trace	374	10.6.3 Fault-tolerant measurement	489	Appendices	608					
	8.3.2 Geometric picture of single qubit quantum operations	374	10.6.4 Elements of resilient quantum computation	493	Appendix 1: Notes on basic probability theory	608					
	8.3.3 Bit flip and phase flip channels	376			Appendix 2: Group theory	610					
	8.3.4 Depolarizing channel	378	11 Entropy and information	500	A2.1 Basic definitions	610					
	8.3.5 Amplitude damping	380	11.1 Shannon entropy	500	A2.1.1 Generators	611					
	8.3.6 Phase damping	383	11.2 Basic properties of entropy	502	A2.1.2 Cyclic groups	611					
			11.2.1 The binary entropy	502	A2.1.3 Cosets	612					
			11.2.2 The relative entropy	504							

2.1 Linear algebra

2.1.1 Bases and linear independence	62
2.1.2 Linear operators and matrices	63
2.1.3 The Pauli matrices	65
2.1.4 Inner products	65
2.1.5 Eigenvectors and eigenvalues	68
2.1.6 Adjoints and Hermitian operators	69
2.1.7 Tensor products	71
2.1.8 Operator functions	75
2.1.9 The commutator and anti-commutator	76
2.1.10 The polar and singular value decompositions	78

2.1 Linear algebra

2.1 Linear algebra

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

(2.1)

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} z'_1 \\ \vdots \\ z'_n \end{bmatrix} \equiv \begin{bmatrix} z_1 + z'_1 \\ \vdots \\ z_n + z'_n \end{bmatrix},$$

(2.2)

$$z \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \equiv \begin{bmatrix} zz_1 \\ \vdots \\ zz_n \end{bmatrix},$$

(2.3)

2.1.1 Bases and linear independence	62
2.1.2 Linear operators and matrices	63
2.1.3 The Pauli matrices	65
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2.1.5 Eigenvectors and eigenvalues	68
2.1.6 Adjoints and Hermitian operators	69
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$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} z'_1 \\ \vdots \\ z'_n \end{bmatrix} \equiv \begin{bmatrix} z_1 + z'_1 \\ \vdots \\ z_n + z'_n \end{bmatrix},$$

$$z \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \equiv \begin{bmatrix} zz_1 \\ \vdots \\ zz_n \end{bmatrix},$$

(2.1)

(2.2)

(2.3)

Notation	Description
z^*	Complex conjugate of the complex number z . $(1 + i)^* = 1 - i$
$ \psi\rangle$	Vector. Also known as a <i>ket</i> .
$\langle\psi $	Vector dual to $ \psi\rangle$. Also known as a <i>bra</i> .
$\langle\varphi \psi\rangle$	Inner product between the vectors $ \varphi\rangle$ and $ \psi\rangle$.
$ \varphi\rangle \otimes \psi\rangle$	Tensor product of $ \varphi\rangle$ and $ \psi\rangle$.
$ \varphi\rangle \psi\rangle$	Abbreviated notation for tensor product of $ \varphi\rangle$ and $ \psi\rangle$.
A^*	Complex conjugate of the A matrix.
A^T	Transpose of the A matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the A matrix, $A^\dagger = (A^T)^*$.
$\langle\varphi A \psi\rangle$	Inner product between $ \varphi\rangle$ and $A \psi\rangle$. Equivalently, inner product between $A^\dagger \varphi\rangle$ and $ \psi\rangle$.

Figure 2.1. Summary of some standard quantum mechanical notation for notions from linear algebra. This style of notation is known as the *Dirac* notation.

$$|v_1\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |v_2\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (2.5)$$

since any vector

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (2.6)$$

$$((y_1, \dots, y_n), (z_1, \dots, z_n)) \equiv \sum_i y_i^* z_i = [y_1^* \dots y_n^*] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}. \quad (2.14)$$

$$|v_1\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |v_2\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (2.5)$$

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стр. 2 - 9

(стр. 4 ↓)

Бра-кет - формализъм. Напомняме: $\mathbb{C}^n$ се състои от вектор-стълбове $\equiv$ кет-вектори
за бра-вектори $\in (\mathbb{C}^n)^*$

$$\langle \Phi | \Psi \rangle = \Phi^* \Psi \quad (z_0 \Phi, \Psi \in \mathbb{C}^n)$$

$\langle \Phi |$ и $|\Psi\rangle$ са кет-вектори = вектор-стълбове
бра-вектори = вектор-редове

Конвенция: $|\Psi\rangle \equiv \Psi$ - операция идентитет,
 $\langle \Psi| \equiv \Psi^*$ - операция на ермитово спрягане.
Тоест, поставянето на скоби $|\bullet\rangle, \langle \bullet|$ са едноместни операции.

Примери:

$$1) \langle \Phi | |\Psi\rangle = \Phi^* \Psi = \langle \Phi | \Psi \rangle - \text{число} \quad (z_0 \Phi, \Psi \in \mathbb{C}^n)$$

правило 1: две вертикални линии стоящи една до друга могат да се слеят в една, както и обратно, всяка вертикална линия може да се раздвои

2.1.6 Adjoint and Hermitian operators

Suppose A is any linear operator on a Hilbert space, V . It turns out that there exists a unique linear operator $A^\dagger$ on V such that for all vectors $|v\rangle, |w\rangle \in V$,

$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle). \quad (2.32)$$

This linear operator is known as the *adjoint* or *Hermitian conjugate* of the operator A . From the definition it is easy to see that $(AB)^\dagger = B^\dagger A^\dagger$. By convention, if $|v\rangle$ is a vector, then we define $|v\rangle^\dagger \equiv \langle v|$. With this definition it is not difficult to see that $(A|v\rangle)^\dagger = \langle v|A^\dagger$.

$$\left(\sum_i a_i A_i \right)^\dagger = \sum_i a_i^* A_i^\dagger. \quad (2.33)$$

$$P \equiv \sum_{i=1}^k |i\rangle\langle i| \quad (2.35)$$

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стр. 2 - 9

(стр. 3, 8 ↓)

Ермитово свързване / Hermitian conjugation : $(m \times n)^* = (n \times m)$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}^* = \begin{pmatrix} \overline{a_{11}} & \dots & \overline{a_{m1}} \\ \vdots & & \vdots \\ \overline{a_{1n}} & \dots & \overline{a_{mn}} \end{pmatrix}, \quad A^* := \bar{A}^T = \overline{A^T}$$

$$\text{Основен закон : } (AB)^* = B^* A^* \Leftrightarrow \begin{cases} (AB)^T = B^T A^T \\ \overline{AB} = \bar{A} \bar{B} \end{cases}$$

$A^{**} = A$ - инволютивност / involution property

Извод : За всеки ортонормиран базис $f_1, \dots, f_n \in \mathbb{C}^n$ е в сила

(алгебрично) разлагане на 1
$\hat{1} = f_1\rangle\langle f_1 + \dots + f_n\rangle\langle f_n $

2.1.5 Eigenvectors and eigenvalues

An *eigenvector* of a linear operator A on a vector space is a non-zero vector $|v\rangle$ such that $A|v\rangle = v|v\rangle$, where v is a complex number known as the *eigenvalue* of A corresponding to $|v\rangle$. It will often be convenient to use the notation v both as a label for the eigenvector, and to represent the eigenvalue. We assume that you are familiar with the elementary properties of eigenvalues and eigenvectors – in particular, how to find them, via the characteristic equation. The *characteristic function* is defined to be $c(\lambda) \equiv \det |A - \lambda I|$,

A *diagonal representation* for an operator A on a vector space V is a representation $A = \sum_i \lambda_i |i\rangle\langle i|$, where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A , with corresponding eigenvalues λ_i . An operator is said to be *diagonalizable* if it has a diagonal representation. In the next section we will find a simple set of necessary and sufficient conditions for an operator on a Hilbert space to be diagonalizable.

2.1.5 Eigenvectors and eigenvalues

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слайд 9

Спектрална теорема Нека $A = A^*$ (самоспрегнат)

$A \in \text{Mat}_N(\mathbb{C})$, т.е., $A \in N \times N$, ермитова

Съществува ортонормиран базис на $\mathbb{C}^N$:
 $f_1, \dots, f_N \in \mathbb{C}^N$, от собствени вектори за A

$A f_j = \lambda_j f_j$, $\lambda_j \in \mathbb{R}$, $j = 1, \dots, N$ реални собствени стойности

При това:

$A = \lambda_1 |f_1\rangle\langle f_1| + \dots + \lambda_N |f_N\rangle\langle f_N|$
 $\hat{1} = |f_1\rangle\langle f_1| + \dots + |f_N\rangle\langle f_N|$
 упрощение
 - припокръпете двете страни върху $|f_j\rangle$
 и използвайте равенство върху базис

$A \in \mathcal{A}$ - нитомна алгебра

Съществуват двойки $(\alpha_1, Q_1), \dots, (\alpha_n, Q_n)$, които са единствени с точност до преназидане и са такива, че:

- $\alpha_1, \dots, \alpha_n$ са реални и са две по две различни
- $Q_1, \dots, Q_n$ са самоспрегнати идиempотенти
 $Q_j^* = Q_j = Q_j^2$

$$A = \alpha_1 Q_1 + \dots + \alpha_n Q_n$$

$$\hat{1} = Q_1 + \dots + Q_n, \quad Q_j Q_k = \begin{cases} Q_j & \text{при } j=k \\ 0 & \text{при } j \neq k \end{cases}$$

раздидване на $\hat{1}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad (2.29)$$

2.1.5 Eigenvectors and eigenvalues

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слайд 9

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$A f_j = \lambda_j f_j$, $\lambda_j \in \mathbb{R}$, $j = 1, \dots, N$ реални собствени стойности

При това:

$$A = \lambda_1 |f_1\rangle\langle f_1| + \dots + \lambda_N |f_N\rangle\langle f_N|$$

$$\hat{1} = |f_1\rangle\langle f_1| + \dots + |f_N\rangle\langle f_N|$$

упростение

-приполагаме двете страни върху $|f_j\rangle$
и използваме равенство върху базис

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- $Q_1, \dots, Q_n$ са самоспрегнати идиempотенти

$$Q_j^* = Q_j = Q_j^2$$

- $A = \alpha_1 Q_1 + \dots + \alpha_n Q_n$
- $\hat{1} = Q_1 + \dots + Q_n$, $Q_j Q_k = \begin{cases} Q_j & \text{при } j=k \\ 0 & \text{при } j \neq k \end{cases}$

разбиране на $\hat{1}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad (2.29)$$

Задача. Как се прилага теоремата за случая $A := Q$ при $Q^* = Q = Q^2$ самоспрегнат идиempотент

2.1 Linear algebra

68

Introduction to quantum mechanics

2.1.5 Eigenvectors and eigenvalues

An *eigenvector* of a linear operator A on a vector space is a non-zero vector $|v\rangle$ such that $A|v\rangle = v|v\rangle$, where v is a complex number known as the *eigenvalue* of A corresponding to $|v\rangle$. It will often be convenient to use the notation v both as a label for the eigenvector, and to represent the eigenvalue. We assume that you are familiar with the elementary properties of eigenvalues and eigenvectors – in particular, how to find them, via the characteristic equation. The *characteristic function* is defined to be $c(\lambda) \equiv \det |A - \lambda I|$,

61 - 79

Linear algebra

69

A *diagonal representation* for an operator A on a vector space V is a representation $A = \sum_i \lambda_i |i\rangle\langle i|$, where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A , with corresponding eigenvalues λ_i . An operator is said to be *diagonalizable* if it has a diagonal representation. In the next section we will find a simple set of necessary and sufficient conditions for an operator on a Hilbert space to be diagonalizable.

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слайд 9, 11 ↓

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Съществува ортонормиран базис на $\mathbb{C}^N$: $f_1, \dots, f_N \in \mathbb{C}^N$, от собствени вектори за A

$Af_j = \lambda_j f_j$, $\lambda_j \in \mathbb{R}$, $j = 1, \dots, N$ реални собствени стойности

При това:

$$\begin{aligned} A &= \lambda_1 |f_1\rangle\langle f_1| + \dots + \lambda_N |f_N\rangle\langle f_N| \\ \hat{I} &= |f_1\rangle\langle f_1| + \dots + |f_N\rangle\langle f_N| \end{aligned}$$

упростение
- приложете двете страни върху $|f_j\rangle$
и използвайте равенство върху базис

$A \in \mathcal{H}$ - хермитова алгебра

Съществуват двойки $(\alpha_1, Q_1), \dots, (\alpha_n, Q_n)$, които са единствени с точност до преназидане и са такива, че:

- $\alpha_1, \dots, \alpha_n$ са реални и са две по две различни
- $Q_1, \dots, Q_n$ са самоспрегнати идиempотенти $Q_j^2 = Q_j = Q_j^2$

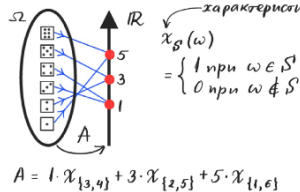
$$\begin{aligned} A &= \alpha_1 Q_1 + \dots + \alpha_n Q_n \\ \hat{I} &= Q_1 + \dots + Q_n, \quad Q_j Q_k = \begin{cases} Q_j & \text{при } j=k \\ 0 & \text{при } j \neq k \end{cases} \end{aligned}$$

разбиране на $\hat{I}$

Определение: $\text{Spec } A := \{\alpha_1, \dots, \alpha_n\}$ - спектр на наблюдаемата A

$\alpha_1, \dots, \alpha_n$ - спектрални стойности, $Q_1, \dots, Q_n$ - съответни спектрални ебидности

Интерпретация: • в класическа система, ако $A: \Omega \rightarrow \mathbb{R}$ (класически наблюдаема = функция)
тогава $\text{Spec } A = A(\Omega)$ (стойностите на A като функция) за $\alpha_j \in \text{Spec } A: Q_j = \chi_{S_j}$, $S_j := A^{-1}(\alpha_j)$



За $A = A^* \exists! \alpha_1 < \dots < \alpha_n, Q_1, \dots, Q_n$

- $Q_j^* = Q_j = Q_j^2$, $j = 1, \dots, n$
- $A = \alpha_1 Q_1 + \dots + \alpha_n Q_n$
- $\hat{I} = Q_1 + \dots + Q_n, \quad Q_j Q_k = \begin{cases} Q_j & \text{при } j=k \\ 0 & \text{при } j \neq k \end{cases}$

разбиране на $\hat{I}$

Kronecker product

$$A \otimes B \equiv \left[\overbrace{\begin{matrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{matrix}}^{nq} \right] \Bigg\} mp. \quad (2.50)$$

Exercise 2.28: Show that the transpose, complex conjugation, and adjoint operations distribute over the tensor product,

$$(A \otimes B)^* = A^* \otimes B^*; (A \otimes B)^T = A^T \otimes B^T; (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger. (2.53)$$

Kronecker product

$$A \otimes B \equiv \left[\overbrace{\begin{matrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{matrix}}^{nq} \right] \Bigg\} mp. \quad (2.50)$$

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$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$$

2.1 Linear algebra

61 - 79

74

Introduction to quantum mechanics

Kronecker product

$$A \otimes B \equiv \left[\begin{array}{cccc} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{array} \right] \quad mp. \quad (2.50)$$

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$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$$

Властниста: $i_1(A) =: A_{(1)}$

$$A \mapsto A \otimes \hat{1}$$

$$\mathcal{A} \xrightarrow{i_1} \mathcal{C} \xleftarrow{i_2} \mathcal{B}$$

$$\hat{1} \otimes B \leftarrow B$$

$$i_2(B) =: B_{(2)}$$

$$\begin{aligned} \Rightarrow A \otimes B &= \\ &= (A \cdot 1) \otimes (1 \cdot B) = (A \otimes 1)(1 \otimes B) \\ &= (1 \cdot A) \otimes (B \cdot 1) = (1 \otimes B)(A \otimes 1) \end{aligned}$$

$\Rightarrow$

$$A_{(1)} B_{(2)} = B_{(2)} A_{(1)} \equiv A \otimes B$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} \cdot 1 & a_{11} \cdot 0 & a_{12} \cdot 1 & a_{12} \cdot 0 \\ a_{11} \cdot 0 & a_{11} \cdot 1 & a_{12} \cdot 0 & a_{12} \cdot 1 \\ a_{21} \cdot 1 & a_{21} \cdot 0 & a_{22} \cdot 1 & a_{22} \cdot 0 \\ a_{21} \cdot 0 & a_{21} \cdot 1 & a_{22} \cdot 0 & a_{22} \cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \cdot 0 & a_{12} \cdot 0 & 0 & 0 \\ 0 & a_{11} \cdot 0 & 0 & a_{12} \cdot 0 \\ a_{21} \cdot 0 & a_{22} \cdot 0 & 0 & 0 \\ 0 & a_{21} \cdot 0 & 0 & a_{22} \cdot 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 \cdot b_{11} & 1 \cdot b_{12} & 0 \cdot b_{21} & 0 \cdot b_{22} \\ 1 \cdot b_{21} & 1 \cdot b_{22} & 0 \cdot b_{11} & 0 \cdot b_{12} \\ 0 \cdot b_{11} & 0 \cdot b_{12} & 1 \cdot b_{21} & 1 \cdot b_{22} \\ 0 \cdot b_{21} & 0 \cdot b_{22} & 1 \cdot b_{11} & 1 \cdot b_{12} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}$$

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слайд 15



2.1 Linear algebra

61 - 79

74

Introduction to quantum mechanics

Kronecker product

$$A \otimes B \equiv \left[\begin{array}{cccc} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{array} \right]_{mp}. \quad (2.50)$$

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$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$$

Властниста: $i_1(A) =: A_{(1)}$

$$A \mapsto A \otimes \hat{I}$$

$$\mathcal{A} \xrightarrow{i_1} \mathcal{C} \xleftarrow{i_2} \mathcal{B}$$

$$\hat{I} \otimes B \leftarrow \theta$$

$$i_2(B) =: B_{(2)}$$

$$\begin{aligned} \Rightarrow A \otimes B &= \\ &= (A \cdot I) \otimes (I \cdot B) = (A \otimes I)(I \otimes B) \\ &= (I \cdot A) \otimes (B \cdot I) = (I \otimes B)(A \otimes I) \end{aligned}$$

$$\Rightarrow A_{(1)} B_{(2)} = B_{(2)} A_{(1)} \equiv A \otimes B$$

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a_{11} \cdot 1 & a_{11} \cdot 0 & a_{12} \cdot 1 & a_{12} \cdot 0 \\ a_{11} \cdot 0 & a_{11} \cdot 1 & a_{12} \cdot 0 & a_{12} \cdot 1 \\ a_{21} \cdot 1 & a_{21} \cdot 0 & a_{22} \cdot 1 & a_{22} \cdot 0 \\ a_{21} \cdot 0 & a_{21} \cdot 1 & a_{22} \cdot 0 & a_{22} \cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} a_{11} \cdot 0 & a_{12} \cdot 0 \\ 0 & a_{11} \cdot 0 & a_{12} \cdot 0 \\ a_{21} \cdot 0 & a_{22} \cdot 0 \\ 0 & a_{21} \cdot 0 & a_{22} \cdot 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} &= \begin{pmatrix} 1 \cdot b_{11} & 1 \cdot b_{12} & 0 \cdot b_{11} & 0 \cdot b_{12} \\ 1 \cdot b_{21} & 1 \cdot b_{22} & 0 \cdot b_{21} & 0 \cdot b_{22} \\ 0 \cdot b_{11} & 0 \cdot b_{12} & 1 \cdot b_{11} & 1 \cdot b_{12} \\ 0 \cdot b_{21} & 0 \cdot b_{22} & 1 \cdot b_{21} & 1 \cdot b_{22} \end{pmatrix} \end{aligned}$$

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слайд 15



ВНИМАНИЕ: Всички, които се подготвят върху квантови алгоритми, непременно трябва да са запознати и с материала на този слайд както и с слайдове 45 - 63 на лекция 8.

$$\operatorname{tr}(A) \equiv \sum_i A_{ii}. \quad (2.59)$$

Exercise 2.37: (Cyclic property of the trace) If A and B are two linear operators show that

$$\operatorname{tr}(AB) = \operatorname{tr}(BA). \quad (2.62)$$

Exercise 2.38: (Linearity of the trace) If A and B are two linear operators, show that

$$\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B) \quad (2.63)$$

and if z is an arbitrary complex number show that

$$\operatorname{tr}(zA) = z\operatorname{tr}(A). \quad (2.64)$$

$$\operatorname{tr}(A) \equiv \sum_i A_{ii}. \quad (2.59)$$

Exercise 2.37: (Cyclic property of the trace) If A and B are two linear operators show that

$$\operatorname{tr}(AB) = \operatorname{tr}(BA). \quad (2.62)$$

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$$\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B) \quad (2.63)$$

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2.1 Linear algebra

61 - 79

Linear algebra

75

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Следа на $A = \begin{pmatrix} a_{11} & \dots & a_{1N} \\ \vdots & & \vdots \\ a_{N1} & \dots & a_{NN} \end{pmatrix}$ ← квадратна

↓ trace

$$\text{tr} A := a_{11} + a_{22} + \dots + a_{NN}$$

Основни свойства на следата

- $\text{tr}(AB \dots C) = \text{tr}(B \dots CA)$ ← цикличност
 - $\text{tr} \hat{1}_N = N$ — $N \times N$
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2.1 Linear algebra

2.1.1 Bases and linear independence	62
2.1.2 Linear operators and matrices	63
2.1.3 The Pauli matrices	65
2.1.4 Inner products	65
2.1.5 Eigenvectors and eigenvalues	68
2.1.6 Adjoints and Hermitian operators	69
2.1.7 Tensor products	71
2.1.8 Operator functions	75
2.1.9 The commutator and anti-commutator	76
2.1.10 The polar and singular value decompositions	78

2.2 The postulates of quantum mechanics

2.2.1 State space	80
2.2.2 Evolution	81
2.2.3 Quantum measurement	84
2.2.4 Distinguishing quantum states	86
2.2.5 Projective measurements	87
2.2.6 POVM measurements	90
2.2.7 Phase	93
2.2.8 Composite systems	93
2.2.9 Quantum mechanics: a global view	96
 2.4 The density operator	 98
2.4.1 Ensembles of quantum states	99
2.4.2 General properties of the density operator	101
2.4.3 The reduced density operator	105

2.2 The postulates of quantum mechanics

80 - 96

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.

80

Introduction to quantum mechanics

2.2 The postulates of quantum mechanics

2.2.1 State space	80
2.2.2 Evolution	81
2.2.3 Quantum measurement	84
2.2.4 Distinguishing quantum states	86
2.2.5 Projective measurements	87
2.2.6 POVM measurements	90
2.2.7 Phase	93
2.2.8 Composite systems	93
2.2.9 Quantum mechanics: a global view	96

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80

Introduction to quantum mechanics

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2.2.1 State space	80
2.2.2 Evolution	81
2.2.3 Quantum measurement	84
2.2.4 Distinguishing quantum states	86
2.2.5 Projective measurements	87
2.2.6 POVM measurements	90
2.2.7 Phase	93
2.2.8 Composite systems	93
2.2.9 Quantum mechanics: a global view	96

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Postulate 3: Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle, \quad (2.92)$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}. \quad (2.93)$$

The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I. \quad (2.94)$$

80

Introduction to quantum mechanics

2.2 The postulates of quantum mechanics

2.2.1 State space	80
2.2.2 Evolution	81
2.2.3 Quantum measurement	84
2.2.4 Distinguishing quantum states	86
2.2.5 Projective measurements	87
2.2.6 POVM measurements	90
2.2.7 Phase	93
2.2.8 Composite systems	93
2.2.9 Quantum mechanics: a global view	96

2.4 The density operator	98
2.4.1 Ensembles of quantum states	99
2.4.2 General properties of the density operator	101
2.4.3 The reduced density operator	105

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80 - 96

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Postulate 4: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n , and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

2.2.1 State space	80
2.2.2 Evolution	81
2.2.3 Quantum measurement	84
2.2.4 Distinguishing quantum states	86
2.2.5 Projective measurements	87
2.2.6 POVM measurements	90
2.2.7 Phase	93
2.2.8 Composite systems	93
2.2.9 Quantum mechanics: a global view	96

2.4 The density operator	98
2.4.1 Ensembles of quantum states	99
2.4.2 General properties of the density operator	101
2.4.3 The reduced density operator	105

2.4 The density operator

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *density operator*, which is a positive operator ρ with trace one, acting on the state space of the system. If a quantum system is in the state ρ_i with probability p_i , then the density operator for the system is $\sum_i p_i \rho_i$.

Postulate 2: The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2 ,

$$\rho' = U \rho U^\dagger. \quad (2.158)$$

Postulate 3: Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \text{tr}(M_m^\dagger M_m \rho), \quad (2.159)$$

and the state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho)}. \quad (2.160)$$

The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I. \quad (2.161)$$

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2.2.2 Evolution	81
2.2.3 Quantum measurement	84
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2.2.5 Projective measurements	87
2.2.6 POVM measurements	90
2.2.7 Phase	93
2.2.8 Composite systems	93
2.2.9 Quantum mechanics: a global view	96

2.4 The density operator	98
2.4.1 Ensembles of quantum states	99
2.4.2 General properties of the density operator	101
2.4.3 The reduced density operator	105

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1) Да се постигне аналогия, при която една квантова система да изглежда като класическа вероятностна (статистическа) система, в която не е фиксирано определено вероятностно разпределение, а отнапред е зададено само пространството на събитията (т.е., т.нар. "измеримо пространство" в теория на вероятностите). След това допускаме, че може да имаме различни вероятностни разпределения върху събитията и тях ги наричаме "състояния" на разглежданата статистическа система.

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- 2) Описаната постановка в точка 1), когато се вземе точно за класически вероятностни модели наричаме "класическа статистическа система". Искаме тя да бъде частен случай в нашите аксиоми.

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- 3) Системите изучавани, в квантовата физика трябва да бъдат друг частен случай на аксиомите на квантовата теория, но заедно с тях искаме да има и "хибридни" модели, които да съчетават и двата гранични случая: напълно класическия и чисто квантовия. Последното е полезно, когато трябва да се прави паралел или преход между класически и квантови понятия.