Algebraic approach to the variational calculus

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1. Basic calculus in differential geometry

Let $M \subseteq \mathbb{R}^D$ be an open subset with coordinates $x = (x^\mu) \in M$. Introduce the basic vector fields on M:

$$\partial_{x^\mu} \equiv \frac{\partial}{\partial x^\mu}$$

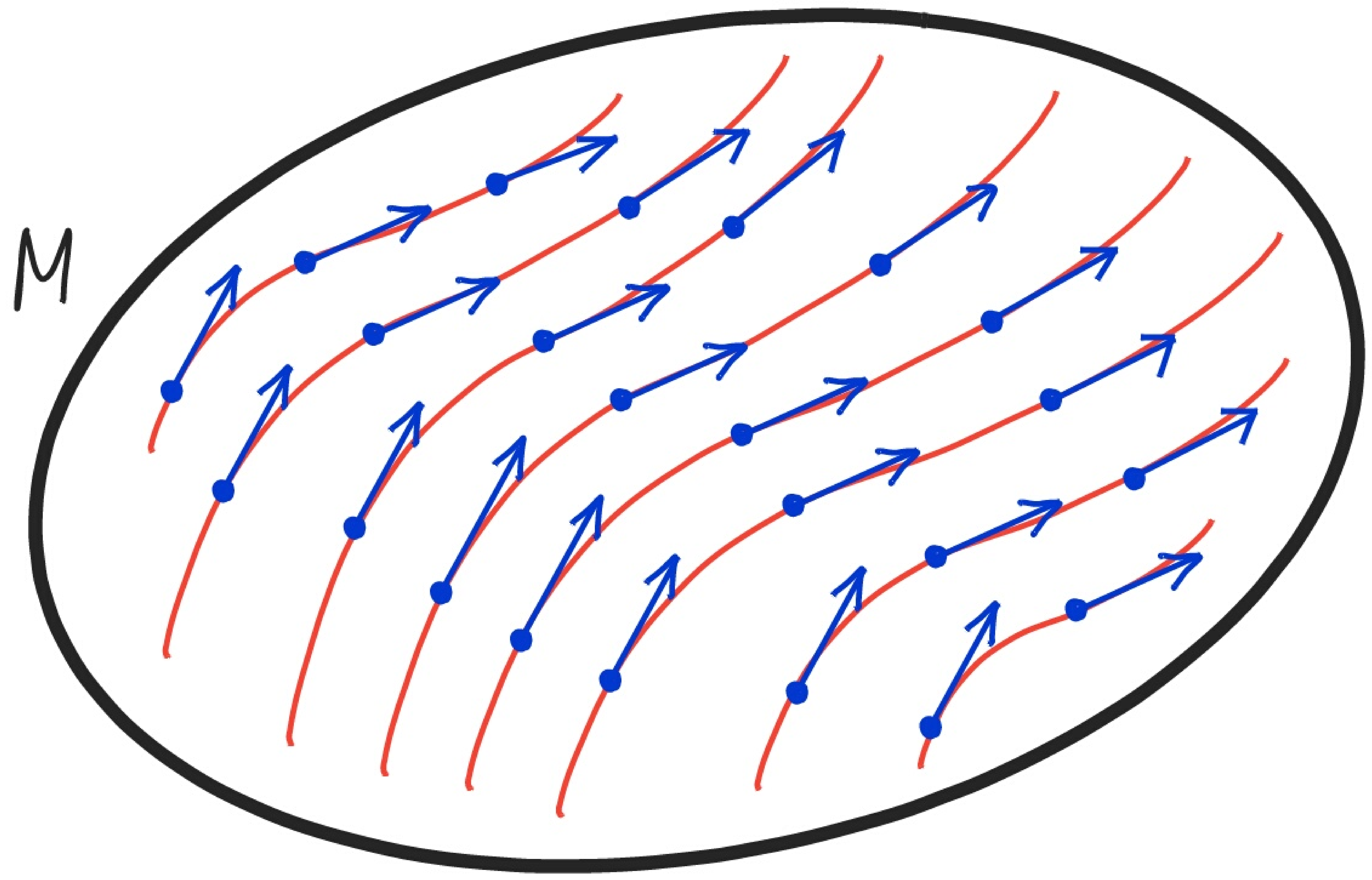
and the basic 1-forms dx^μ.

A general vector field on M takes the form $X^\mu(x)\, \partial_{x^\mu}$ and it generates a (local) flow on M:

$$x = (x^\mu) \longmapsto \left(\mathcal{F}^\mu_{(\varepsilon)}(x) \right) = \mathcal{F}_{(\varepsilon)}(x)$$

$$\mathcal{F}^\mu_{(\varepsilon)}(x) \Big|_{\varepsilon = 0} = x^\mu$$

Hence, $X^\mu(x) = \dfrac{\partial}{\partial \varepsilon} \mathcal{F}^\mu_{(\varepsilon)}(x) \Big|_{\varepsilon = 0}$



The primary meaning of a vector field $X = X^\mu(x)\,\partial_{x^\mu}$ is as a linear operator

$$X : C^\infty(M) \to C^\infty(M)$$

$$f \mapsto X(f) := X^\mu \partial_{x^\mu} f$$

on the vector space $C^\infty(M)$ of all smooth functions $f : M \to \mathbb{R}$, which are derivations in the sense that

$$X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2)$$

(X satisfies the Leibniz rule)

Note that conversely, if X is a linear operator on $C^\infty(M)$, which satisfies the Leibniz rule then X is uniquely representable in the form $X = X^\mu(x)\,\partial_{x^\mu}$, as the coeficient functions $X^\mu(x)$ can be determined as $X^\mu(x) = X$ acting on x^μ.

As an application of the last result one observes that if $X_1 = X_1^\mu(x)\,\partial_{x^\mu}$ and $X_2 = X_2^\mu(x)\,\partial_{x^\mu}$ are two vector fields on M then their comutator $X = [X_1, X_2]$, as operators acting on $C^\infty(M)$, is a derivation again and hence, is a vector field. One obtains

$$X = X^\mu(x)\,\partial_{x^\mu} \text{ for}$$

$$X^\mu(x) = X_1^\nu(x)\,\partial_{x^\nu} X_2^\mu(x) - X_2^\nu(x)\,\partial_{x^\nu} X_1^\mu(x)$$

A general (external) r-form on M has the form

$$\gamma = \Gamma_{\mu_1 \dots \mu_r}(x)\, dx^{\mu_1} \dots dx^{\mu_r}$$

(note, $dx^\mu dx^\nu \equiv dx^\mu \wedge dx^\nu$
$= - dx^\nu dx^\mu$)

Here, $\Gamma_{\mu_1 \dots \mu_r}(x) \in C^\infty(M)$

$\gamma \in \Omega^r(M)$ - the space of all r-forms on M.

The de Rham (exterior) differential is

$$d\gamma = \partial_{x^\mu} \Gamma_{\mu_1 \dots \mu_r}(x)\, dx^\mu dx^{\mu_1} \dots dx^{\mu_r}$$

and $d\gamma \in \Omega^{r+1}(M)$, $d^2\gamma = 0$.

Having a (local) flow

$$\mathcal{F}_{(\varepsilon)} : (x^\mu) \longmapsto \left(\mathcal{F}^\mu_{(\varepsilon)}(x) \right)$$

one has an induced pullback action on forms:

$$\mathcal{F}^*_{(\varepsilon)} g(x) := \frac{\partial \mathcal{F}^{\nu_1}_{(\varepsilon)}(x)}{\partial x^{\mu_1}} \cdots \frac{\partial \mathcal{F}^{\nu_r}_{(\varepsilon)}(x)}{\partial x^{\mu_r}}$$

$$\times \Gamma_{\nu_1 \dots \nu_r}\left(\mathcal{F}_{(\varepsilon)}(x)\right) dx^{\mu_1} \dots dx^{\mu_r}$$

It induces the Lie derivative $L_X g$ of g along the vector field

$$X := X^{\mu}(x)\, \partial_{x^{\mu}}$$

that generates the flow $\mathcal{F}_{(\varepsilon)}$, by the formula

$$L_X g := \frac{\partial}{\partial \varepsilon} \mathcal{F}^*_{(\varepsilon)} g \Big|_{\varepsilon = 0}$$

One calculates the explicit form of

$$L_X g = \Gamma'_{\mu_1 \dots \mu_r}(x)\, dx^{\mu_1} \dots dx^{\mu_r}$$

where $\Gamma'_{\mu_1 \dots \mu_r}(x) =$

$$= X^{\mu}(x)\, \partial_{x^{\mu}} \Gamma_{\mu_1 \dots \mu_r}(x)$$

$$+ \sum_{j=1}^{r} \left(\partial_{x^{\mu_j}} X^{\nu_j}(x)\right) \Gamma_{\mu_1 \dots \nu_j \dots \mu_r}(x)$$

Recall that $\Omega^\bullet(M) := \bigoplus_{r=0}^{D} \Omega^r(M)$

$\left(\text{where } \Omega^0(M) := C^\infty(M)\right)$ is a graded commutative algera, i.e.,

if $\gamma_1 \in \Omega^{r_1}(M)$ and $\gamma_2 \in \Omega^{r_2}(M)$

then $\gamma_1\gamma_2 \equiv \gamma_1 \wedge \gamma_2 \in \Omega^{r_1+r_2}(M)$

and $\gamma_1\gamma_2 = (-1)^{r_1 r_2}\, \gamma_2\gamma_1$

One checks that the Lie derivatives

$$L_X : \Omega^\bullet(M) \to \Omega^\bullet(M) \text{ are}$$

derivations of the algebra $\Omega^\bullet(M)$ of degree 0. The latter means that

a) $L_X : \Omega^r(M) \to \Omega^r(M)$ (preserves the degrees).

b) $L_X(\gamma_1\gamma_2) = L_X(\gamma_1)\gamma_2 + \gamma_1 L_X(\gamma_2)$. (Leibniz rule).

In particular, on $\Omega^0(M) = C^\infty(M)$, L_X acts as a vector field, i.e., $L_X(f) \equiv X(f)$.

In general, a linear operator $A: \Omega^\bullet(M) \to \Omega^\bullet(M)$ is said to be a derivation of degree $p \in \mathbb{Z}$ iff:

a) $A: \Omega^r(M) \to \Omega^{r+p}(M)$
(A shifts the degree by p).

b) $$A(\gamma_1 \gamma_2) = A(\gamma_1)\gamma_2 + (-1)^{pr_1} \gamma_1 A(\gamma_2).$$

(the graded Leibniz rule), where $\gamma_j \in \Omega^{r_j}(M)$ $(j = 1, 2)$.

An example of derivation of degree 1 is the exterior differential:

$$d(\gamma_1 \gamma_2) = d(\gamma_1)\gamma_2 + (-1)^{r_1} \gamma_1 d(\gamma_2)$$

Another example is the "iota-operation"

$$\iota_X : \Omega^r(M) \to \Omega^{r-1}(M)$$

defined for a vector field X, which is a derivation of order -1:

$$\iota_X(\gamma_1 \gamma_2) = \iota_X(\gamma_1)\gamma_2 + $$
$$+ (-1)^{r_1} \gamma_1 \iota_X(\gamma_2).$$

This operation is defined by

$$\iota_X \gamma = \Gamma''_{\mu_1 \dots \mu_{r-1}} dx^{\mu_1} \dots dx^{\mu_{r-1}},$$

$$\Gamma''_{\mu_1 \dots \mu_{r-1}}(x)$$
$$:= \sum_{j=1}^{r} X^\nu(x) \Gamma_{\mu_1 \dots \nu \dots \mu_r}(x)$$

It follows that

$$\iota_{fX}\gamma = f \iota_X$$

$$L_X = d\iota_X + \iota_X d$$ (Cartan's formula)

The Cartan formula is easily proven by the following two observations:

1) If A_1 and A_2 are two derivations of $\Omega^\bullet(M)$ of degrees p_1 and p_2, respectively, then their graded commutator, defined by:

$$[A_1, A_2] := A_1 A_2 - (-1)^{p_1 p_2} A_2 A_1$$

is a derivation of $\Omega^\bullet(M)$ of degree $p_1 + p_2$.

2) $\Omega^\bullet(M)$ is generated, as an algebra by $\Omega^0(M) = C^\infty(M)$ and all dx^μ. Therefore, to prove an identity between two derivations it is enough to check it for funtions $f \in C^\infty(M)$ and all dx^μ.

By the Cartan formula it follows also that

$$L_X d = d L_X$$

since

$$L_X d = (d\imath_X + \imath_X d)d = d\imath_X d = d L_X$$

In fact, one can prove that d commutes with any (local) flow $\mathcal{F}_{(\varepsilon)}$:

$$d\mathcal{F}^*_{(\varepsilon)} = \mathcal{F}^*_{(\varepsilon)} d$$

Note also that if N is a manifold with (local) cordinates $y = (y^a)$ and

$$\mathcal{F} : N \longrightarrow M : (y^a) \mapsto (x^\mu = \mathcal{F}^\mu(y))$$

is a smooth map then it induces a pullback map:

note: reverse order!

$$\mathcal{F}^* : \Omega^r(M) \to \Omega^r(N) \quad (\forall r)$$

defined in the same way as for flows:

$$(\mathcal{F}^*\gamma)(y) := \frac{\partial \mathcal{F}^{\mu_1}(y)}{\partial y^{a_1}} \cdots \frac{\partial \mathcal{F}^{\mu_r}(y)}{\partial y^{a_r}}$$

$$\times \Gamma_{\mu_1 \cdots \mu_r}(\mathcal{F}(y))\, dy^{a_1} \cdots dy^{a_r}$$

$$\left(\text{iff } \gamma = \Gamma_{\mu_1 \cdots \mu_r}(x)\, dx^{\mu_1} \cdots dx^{\mu_r}\right)$$

One has:

$$\mathcal{F}^*(\gamma_1 \gamma_2) = \mathcal{F}^*(\gamma_1)\, \mathcal{F}^*(\gamma_2)$$

$$(\mathcal{F}_1 \circ \mathcal{F}_2)^* = \mathcal{F}_2^* \circ \mathcal{F}_1^*$$

$$d \circ \mathcal{F}^* = \mathcal{F}^* \circ d$$

In particular, for a submanifold

$$j : N \hookrightarrow M$$

$j^*\gamma =: \gamma\big|_N$ is called a restriction of γ on N.

Integration

Volume forms on M are called the exterior forms $\Lambda \in \Omega^D(M)$ of a top degree $D = \dim(M)$.

Then $\Lambda = \mathcal{L}(x)\, dx^1 \dots dx^D$

and we define

$$\int_M \Lambda := \int_M \mathcal{L}(x)\, dx^1 \dots dx^D$$

(note: this definition relies on an orientation on M provided by the order $dx^1 \dots dx^D$).

More generally, if γ is an exterior form on M of degree $\dim N$, for an oriented submanifold $j : N \hookrightarrow M$, then $j^* \gamma$ is a volume form on N and we set:

$$\int_N \gamma := \int_N j^* \gamma$$

Stokes' theorem $\quad \int_N d\gamma = \int_{\partial N} \gamma$

Useful formulas

$$L_X f = X(f)$$

$$L_X(\gamma_1 \gamma_2) = L_X(\gamma_1)\gamma_2 + \gamma_1 L_X(\gamma_2).$$

$$L_X d = d L_X$$

$$L_X = d\iota_X + \iota_X d$$

$$\Longrightarrow \quad L_{fX} = f L_X + (df)\,\iota_X$$

$$[L_X, L_Y] = L_{[X,Y]}$$

$$d = \partial_{x^\mu} \otimes dx^\mu :$$

$$\underbrace{C^\infty(M) \otimes \Lambda^\bullet[dx^1, \dots, dx^D]}_{\Omega^\bullet(M)} \circlearrowleft$$

$$d = L_{\partial_{x^\mu}} dx^\mu \equiv L_{\partial_{x^\mu}} \circ dx^\mu$$

2. Basic calculus on bundles

Let $M \subseteq \mathbb{R}^D$ and $F \subseteq \mathbb{R}^N$ be open subsets, set $E = M \times F$ and consider the trivial bundle $\xi = (E, p : E \to M, M)$ with coordinates $x = (x^\mu) \in M$ and $u = (u^A) \in F$. Introduce the basic vector fields on E:

$$\partial_{x^\mu} \equiv \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \partial_{u^A} \equiv \frac{\partial}{\partial u^A}$$

and the basic 1-forms dx^μ, du^A.

A general vector field on E takes the form:

$$Z = X^\mu(x,u)\,\partial_{x^\mu} + Y^A(x,u)\,\partial_{u^A}$$

and it generates a (local) flow on E

$$(x,u) \longmapsto \left(\mathcal{F}_{(\varepsilon)}(x,u),\, G_{(\varepsilon)}(x,u)\right)$$

Hence,

$$X^\mu(x,u) = \frac{\partial}{\partial \varepsilon} \mathcal{F}^\mu_{(\varepsilon)}(x,u)\Big|_{\varepsilon=0}$$

$$Y^A(x,u) = \frac{\partial}{\partial \varepsilon} G^A_{(\varepsilon)}(x,u)\Big|_{\varepsilon=0}.$$

The flow

$$(x,u) \longmapsto \left(\mathcal{F}_{(\varepsilon)}(x,u),\, G_{(\varepsilon)}(x,u) \right)$$

is called vertical iff $\mathcal{F}^{\mu}_{(\varepsilon)}(x,u) \equiv x^{\mu}$

and hence, iff $X^{\mu}(x,u) = 0$.

Then, the generating vector field

$Z = Y^{A}(x,u)\, \partial_{u^A}$ is also called

vertical

The flow

$$(x,u) \longmapsto \left(\mathcal{F}_{(\varepsilon)}(x,u),\, G_{(\varepsilon)}(x,u) \right)$$

is called vertically consistent iff

$\mathcal{F}_{(\varepsilon)}$ depends only on x, and hence,

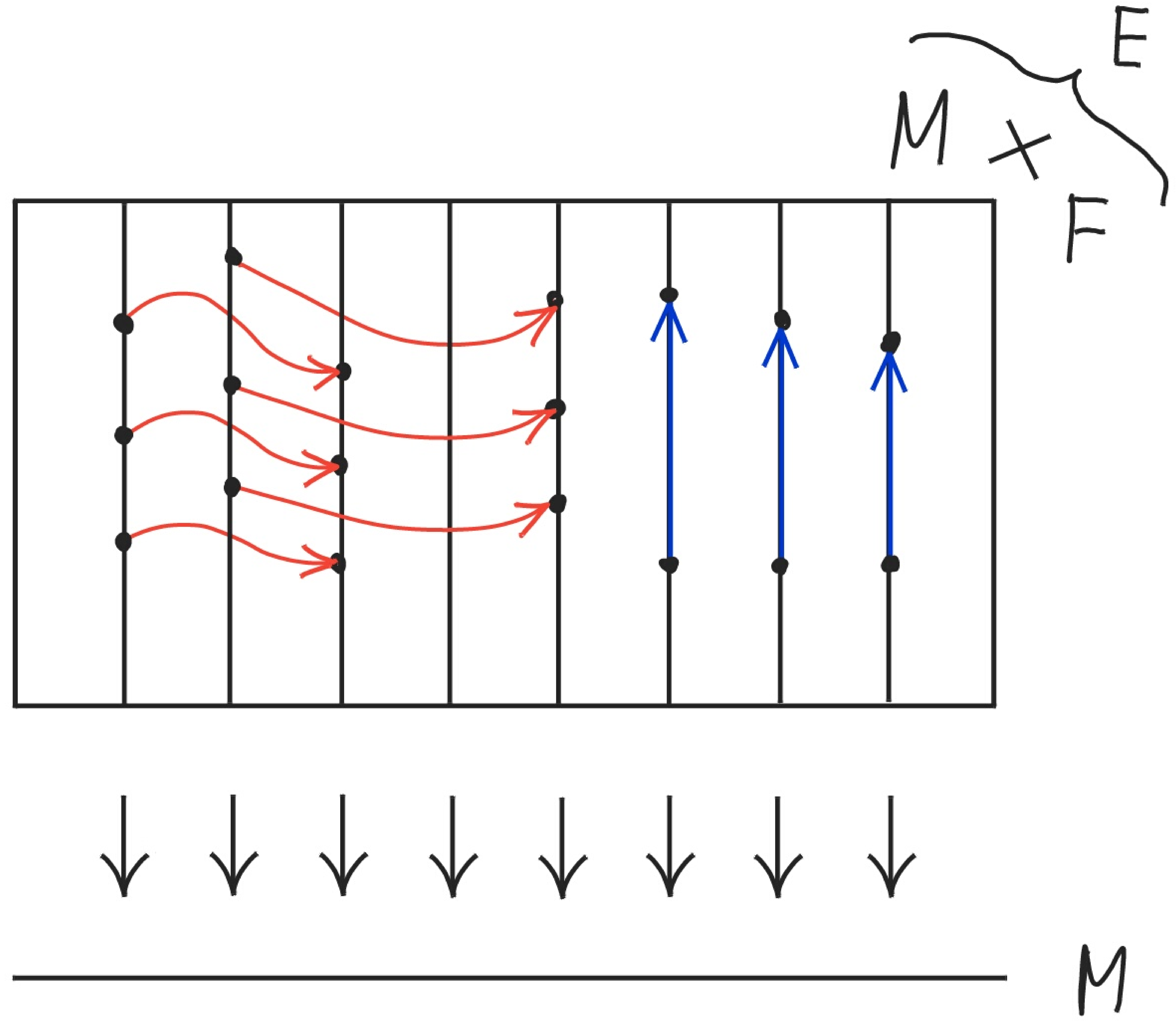
iff X^{μ} depend only on x.

Then, the generating vector field

$Z = X^{\mu}(x)\, \partial_{x^{\mu}} + Y^{A}(x,u)\, \partial_{u^A}$

is also called vertically consistent

Note: vertical consitency of a map (flow) means that fibers are not teared



Note also that every vertical flow, or, a vector field is also a vertically consistent.

The compositions of maps (flows) and the commutators preserve these categories.

Note that every vertically consistent map $\mathcal{H}: E \to E$ induces a unique map $\mathcal{F}: M \to M$, which makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{H}} & E \\ p\downarrow & & \downarrow p \\ M & \xrightarrow{\mathcal{F}} & M \end{array}$$

commutative and vice versa, the commutativity of the above diagram esures that the map $\mathcal{H}$ is vertically consistent.

Sometimes $\mathcal{H}$ is called a *lift* of the map $\mathcal{F}$ to E.

Example: the *tangential lift* of $\mathcal{F}$ on the tangential bundle $E = T(M)$ (in this case: $T(M) \cong M \times \mathbb{R}^D$):

$$((x^\mu), (u^\nu)) \mapsto$$

$$\left((\mathcal{F}^\mu(x)), \left(\frac{\partial \mathcal{F}^\nu(x)}{\partial x^\rho} u^\rho \right) \right)$$

The tangential lift of the map $\mathcal{F} : M \longrightarrow M$ is denoted by $\mathcal{F}^{\tau} : T(M) \longrightarrow T(M)$.

If $\mathcal{F}_{(\varepsilon)} : M \longrightarrow M$ is a (local) flow generated by the vector field

$$X = X^{\mu}(x)\, \partial_{x^{\mu}} \,,$$

$$X^{\mu}(x) = \frac{\partial}{\partial \varepsilon} \mathcal{F}^{\mu}_{(\varepsilon)}(x) \Big|_{\varepsilon = 0}$$

then the tangential lift $\mathcal{F}^{\tau}_{(\varepsilon)}$ is a flow on $T(M)$ generated by: $X^{(\tau)} :=$

$$X^{\mu}(x)\, \partial_{x^{\mu}} + \partial_{x^{\rho}} X^{\nu}(x)\, u^{\rho}\, \partial_{u^{\nu}}$$

Another examle is the lift of X to $T(M)$ provided by a paralel transport

$$X^{(\nabla)} := X^{\mu}(x)\, \partial_{x^{\mu}} + \Gamma^{\nu}_{\rho\sigma}(x)\, X^{\rho}(x)\, u^{\sigma}\, \partial_{u^{\nu}}$$

Note: $[X_1^{(\tau)}, X_2^{(\tau)}] = [X_1, X_2]^{(\tau)}$

$$[X_1^{(\nabla)}, X_2^{(\nabla)}] = [X_1, X_2]^{(\nabla)} + R(X_1, X_2)^{\nu}\, \partial_{u^{\nu}}$$

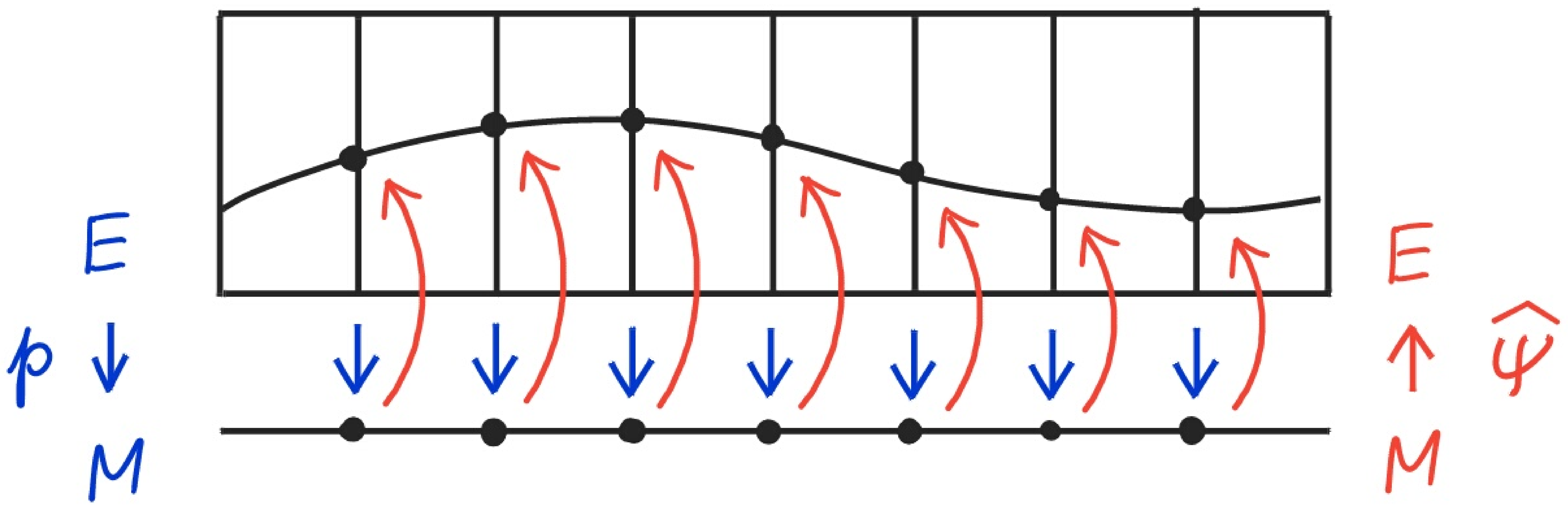
Sections: note that every "field" $\varphi : M \to F$ (i.e., a smooth map $M \to F$) can be equivalently seen as a <u>section</u> of the trivial bundle $\xi = (E, p : E \to M, M)$, $E = M \times F$, i.e., as a smooth map

$$\begin{array}{rcl} \hat{\varphi} : M & \longrightarrow & E = M \times F \\ \in & & \overbrace{\quad\quad\quad}^{\in} \\ x & \longmapsto & (x, \varphi(x)) \end{array}$$

such that $p \circ \hat{\varphi} = \mathrm{Id}_M$. Thus

$$\underbrace{C^\infty(M, F)}_{\substack{\text{the set of all smooth} \\ \text{maps } M \to F}} \cong \underbrace{C^\infty(\xi)}_{\substack{\text{the set of all} \\ \text{sections of } \xi}}$$

In fact, $\hat{\varphi}$ maps M to the graph of φ



2. Fields' transformations

a) Fields' transformations under vertically consistent flows.

Assume we have a (local) vertically consistent flow on E:

$$(x,u) \longmapsto \big(\mathcal{F}(x), G(x,u)\big)$$

$\big(\mathcal{F} = \mathcal{F}_{(\varepsilon)}, G = G_{(\varepsilon)}\big)$. Under this flow the fields (or, the multicomponet field) $u = \Psi(x)$ are transformed to new fields $u' = \Psi'(x')$ according to the rule

$$\Psi'\big(\underbrace{\mathcal{F}(x)}_{x'}\big) = G\big(x, \Psi(x)\big)$$

b) Generalized fields' transformations

Assume we have a (local) flow on E:

$$(x,u) \longmapsto \big(\mathcal{F}(x,u), G(x,u)\big)$$

$\big(\mathcal{F} = \mathcal{F}_{(\varepsilon)}, G = G_{(\varepsilon)}\big)$. Together with a field $u = \varphi(x)$, this induces:

$$\boxed{\begin{aligned} x' &= \mathcal{F}(x,u) \\ u' &= G(x,u) \\ u &= \varphi(x) \end{aligned}}$$

$$\boxed{u' = \varphi'(x')}$$

the transformed fields

$$\boxed{\begin{aligned} x' &= \mathcal{F}^{(\varphi)}(x) \\ u' &= G^{(\varphi)}(x', u') \\ \text{for } u' &= \varphi'(x') \end{aligned}}$$

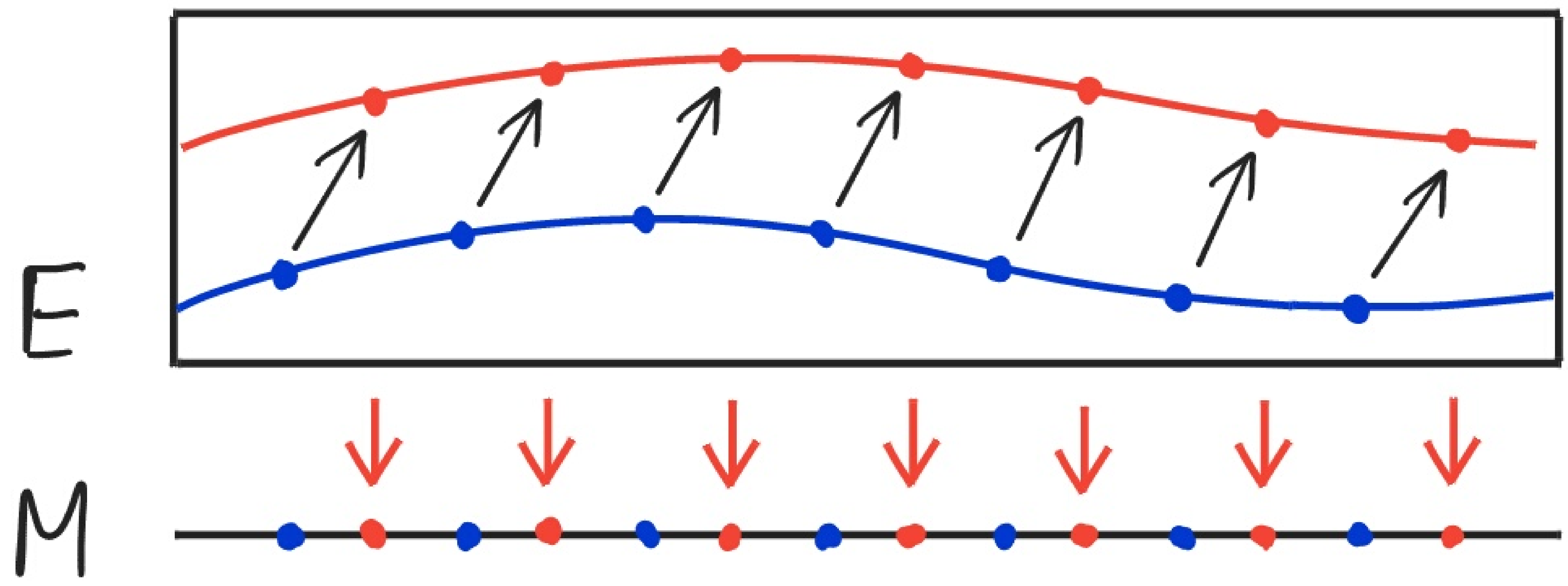
"effective" vertically consistent transformation

Here: $\varphi'\big(\mathcal{F}(x,\varphi(x))\big) = G(x,\varphi(x))$

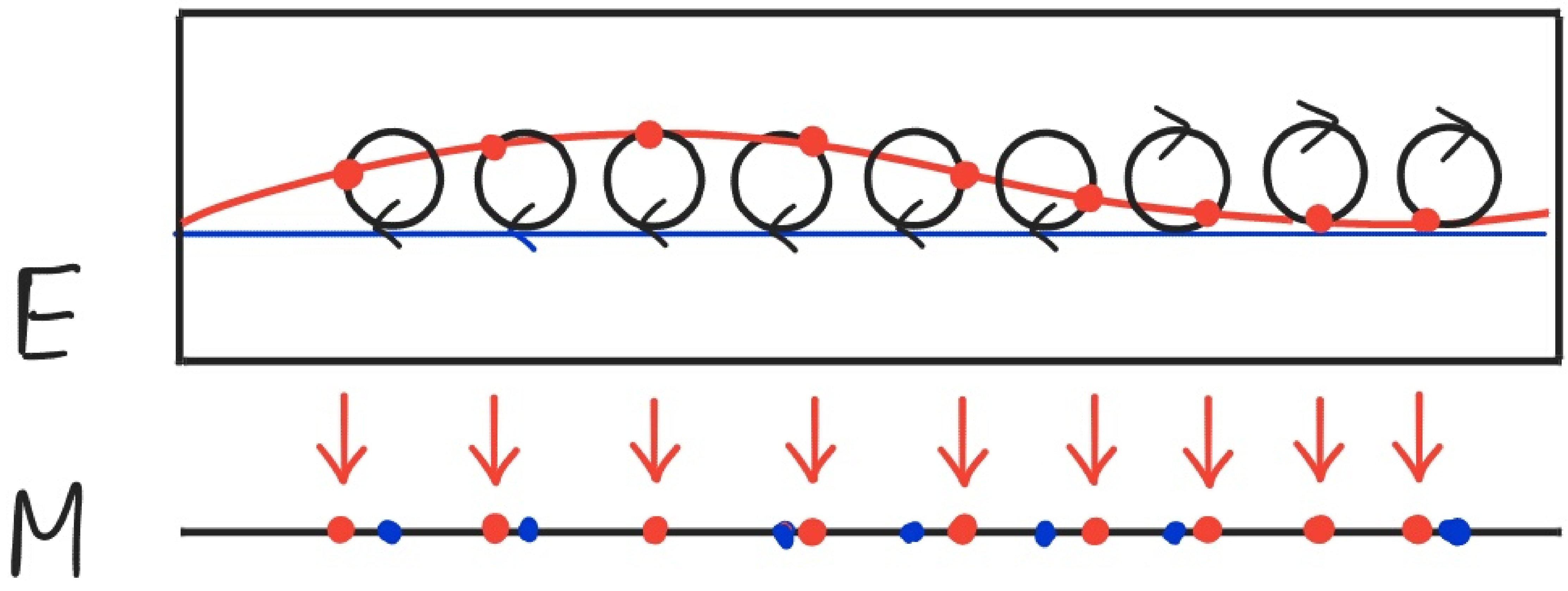
$$\mathcal{F}^{(\varphi)}(x) = \mathcal{F}(x,\varphi(x))$$

$$G^{(\varphi)}(x') = \varphi'(x')$$

a)



b)



Let us differentiate in ε at $\varepsilon = 0$.

(Note: $\mathcal{F} = \mathcal{F}_{(\varepsilon)}$, $G = G_{(\varepsilon)}$, $\psi' = \psi'_{(\varepsilon)}$, $\mathcal{F}^{(\psi)} = \mathcal{F}^{(\psi)}_{(\varepsilon)}$ and $G^{(\psi)} = G^{(\psi)}_{(\varepsilon)}$ but ψ do not depend on ε.)

And let, as before,

$$\frac{\partial}{\partial \varepsilon} \mathcal{F}^{\mu}_{(\varepsilon)}(x,u)\Big|_{\varepsilon=0} =: X^{\mu}(x,u),$$

$$\frac{\partial}{\partial \varepsilon} G^{A}_{(\varepsilon)}(x,u)\Big|_{\varepsilon=0} =: Y^{A}(x,u).$$

Then using $\mathcal{F}^{\mu}_{(\varepsilon)}(x,u)\Big|_{\varepsilon=0} = x^{\mu}$ $G^{A}_{(\varepsilon)}(x,u)\Big|_{\varepsilon=0} = u^{A}$ one gets:

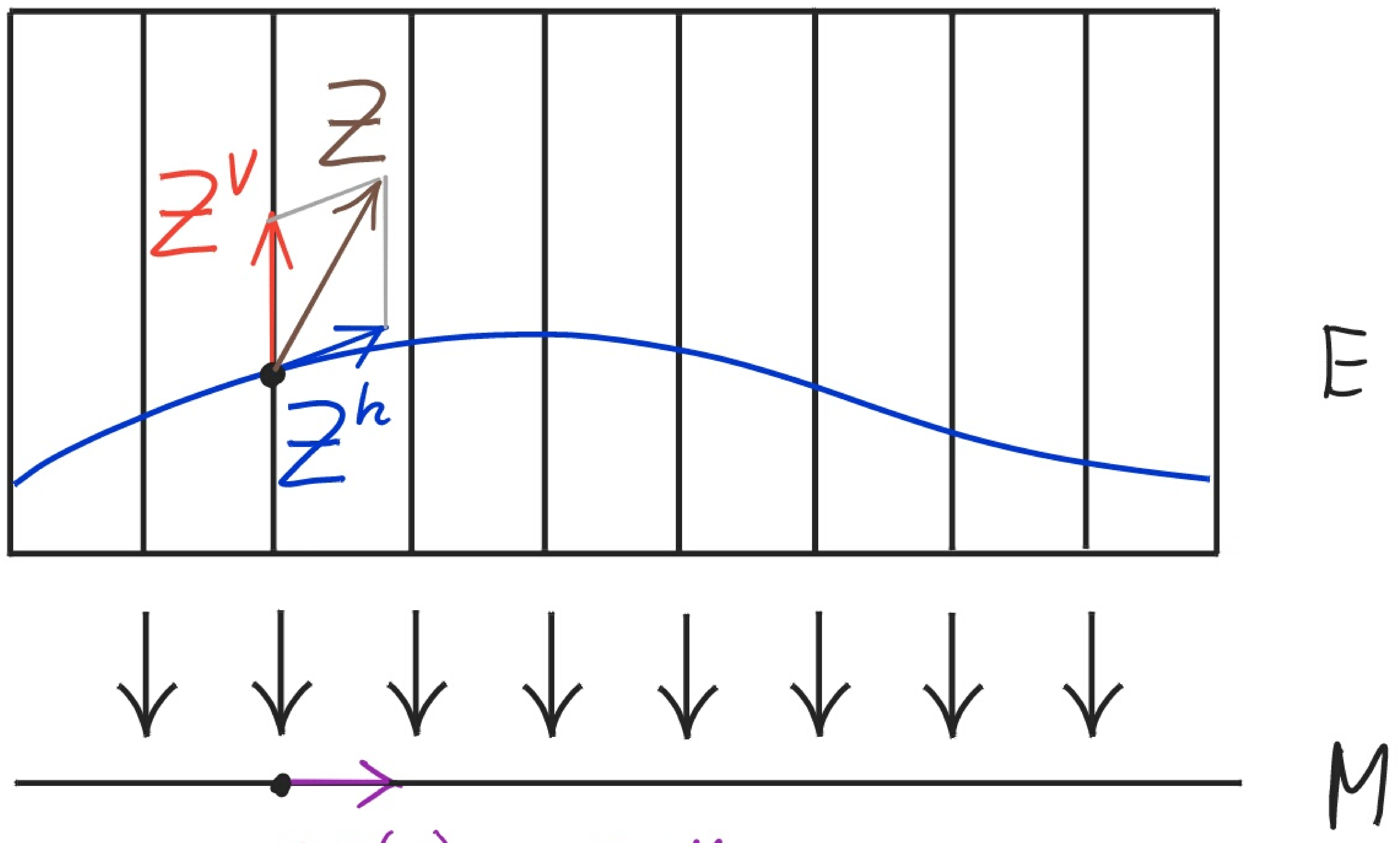
$$Z = X^\mu(x,u)\,\partial_{x^\mu} + Y^A(x,u)\,\partial_{u^A}$$

- the vector field on E generating the flow.

$$\frac{\partial}{\partial\varepsilon}\left(\mathcal{F}^{(\psi)}_{(\varepsilon)}\right)^\mu\Bigg|_{\varepsilon=0} = X^\mu(x,\psi(x))$$

$$\frac{\partial}{\partial\varepsilon}\left(G^{(\psi)}_{(\varepsilon)}\right)^A\Bigg|_{\varepsilon=0} = Y^A(x,\psi(x)) - X^\mu(x,\psi(x))\,\partial_{x^\mu}\psi^A(x)$$

The above result has the following geometric interpretation.



$$X^{(\psi)} = X^\mu(x,\psi(x))\,\partial_{x^\mu}$$

$$Z^V \equiv Z^{V,\psi} := \frac{d}{d\varepsilon}\left(G^{(\psi)}_{(\varepsilon)}\right)^A \partial_{u^A}$$

$$Z^h \equiv Z^{h,\psi} := Z - Z^{V,\psi}$$

where Z^V

$$= \left(Y^A(x,u) - X^\mu(x,u)\,\partial_{x^\mu}\Psi(x)^A\right)\partial_{u^A}$$

$$= \left(Y^A(x,u) - X^\mu(x,u)\,u^A_\mu\right)\partial_{u^A}$$

$$Z^h = X^\mu(x,u)\,\partial_{x^\mu}$$

$$+ X^\nu(x,u)\,\partial_{x^\nu}\Psi(x)^A\,\partial_{u^A}$$

$$= X^\mu(x,u)\left(\partial_{x^\mu} + u^A_\mu\,\partial_{u^A}\right)$$

defined only over the field's graph, i.e., for $(x,u) = (x,\Psi(x))$

$$u^A_\mu = \partial_{x^\mu}\Psi(x)^A$$

Note that Z^h is tangential to the graph of Ψ. To prove this one first observes that a vector field Z is tangential to the graph iff $Z^V = 0$, i.e., iff $Y^A(x,\Psi(x)) = X^\mu(x,\Psi(x))\,\partial_{x^\mu}\Psi^A$

Next, the necessary and sufficient condition for the latter is

$$\iota_Z d^v u^A = 0 \qquad (\forall A)$$

where $d^v u^A := du^A - u^A_\mu dx^\mu$

$$(\text{for } u^A_\mu = \partial_{x^\mu} \psi^A)$$

is the so called vertical differential of u^A. But then, one checks that

$$\iota_{Z^h} d^v u^A = 0.$$

Hence, Z^h is always tangential to the graph of ψ.

Note that in this section u^A_μ is not an independent coordinate but a function in x. Later, we shall introduce the so called jet bundles, where u^A_μ, togther with u, will become an independent variable.

c) Fields' variations

A fields' transformation generated by a vertical flow on E

$$(x, u) \longmapsto (x, G(x, u))$$

$(G = G_{(\varepsilon)})$ is called a fields' variation.

Under every fields' variation we obtain a one parameter family of fields φ_ε

$\varphi_\varepsilon(x) = \varphi'(x)$, such that

$\varphi_{\varepsilon=0}(x) = \varphi(x)$ - the initial field.

Note that for every fields $u = \varphi(x)$ and and for every one parameter family $\varphi_\varepsilon(x)$ such that $\varphi_{\varepsilon=0}(x) = \varphi(x)$ there exists a fields' transformation that generates the one parameter family $\varphi_\varepsilon(x)$.

Hence, with a slight abuse of the notation we shall call every one parameter family $\varphi_\varepsilon(x)$, such that $\varphi_{\varepsilon=0}(x) = \varphi(x)$, a fields' variation of $\varphi(x)$.

3. Multiindices

a) Symmetric multiindex

$$\underline{\mu} = (\mu_1, \ldots, \mu_n) = (\mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)})$$

$$|\underline{\mu}| := n \ , \quad \underline{\mu}\nu = \nu\underline{\mu} := (\mu_1, \ldots, \mu_n, \nu)$$

$$\partial_{\underline{\mu}} \psi(x) = \partial_{x^{\mu_1}} \cdots \partial_{x^{\mu_n}} \psi(x)$$

$$\Rightarrow \quad \partial_{x^\nu} \partial_{\underline{\mu}} \psi(x) = \partial_{\nu\underline{\mu}} \psi(x)$$

$$x^{\underline{\mu}} := x^{\mu_1} \ldots x^{\mu_n} \quad \Rightarrow \quad x^\nu x^{\underline{\mu}} = x^{\nu\underline{\mu}}$$

Taylor's formula in symmetric multiind.

$$\psi(x) = \sum_{\underline{\mu}} \frac{1}{|\underline{\mu}|!} \partial_{\underline{\mu}} \psi(x_0)(x-x_0)^{\underline{\mu}}$$

$$= \psi(x_0) + \sum_{\dot{\underline{\mu}}} \frac{1}{|\dot{\underline{\mu}}|!} \partial_{\dot{\underline{\mu}}} \psi(x_0)(x-x_0)^{\dot{\underline{\mu}}}$$

where $\dot{\underline{\mu}}$ stands for a multiindex with $|\dot{\underline{\mu}}| \geq 1$.

We also will use the Einstein's convention:

$$x^{\underline{\mu}} \partial_{\underline{\mu}} \psi := \sum_{\underline{\mu}} x^{\underline{\mu}} \partial_{\underline{\mu}} \psi$$

$$x^{\dot{\underline{\mu}}} \partial_{\dot{\underline{\mu}}} \psi := \sum_{\dot{\underline{\mu}}} x^{\dot{\underline{\mu}}} \partial_{\dot{\underline{\mu}}} \psi$$

b) Nonsymmetric multiindices

$$\underline{r} = (r_1, \dots, r_D) \in \{0, 1, 2, \dots\}^{\times D}$$

$$|\underline{r}| := r_1 + \dots + r_D \,, \quad \underline{e}_\mu := (0, \dots, \overset{\mu}{1}, \dots, 0)$$

$$\partial_{\underline{r}} \psi(x) \equiv (\partial_x)^{\underline{r}} \psi(x)$$

$$:= (\partial_{x^1})^{r_1} \cdots (\partial_{x^D})^{r_D} \psi(x)$$

$$\partial_{x^\mu} \partial_{\underline{r}} \psi(x) = \partial_{\underline{r} + \underline{e}_\mu} \psi(x)$$

$$x^{\underline{r}} := (x^1)^{r_1} \cdots (x^D)^{r_D}, \quad x^{\underline{r}} x^{\underline{s}} = x^{\underline{r} + \underline{s}}$$

$$\psi(x) = \sum_{\underline{r}} \frac{1}{\underline{r}!} \partial_{\underline{r}} \psi(x_0) (x - x_0)^{\underline{r}}$$

$$= \psi(x_0) + \sum_{\dot{\underline{r}}} \frac{1}{\dot{\underline{r}}!} \partial_{\dot{\underline{r}}} \psi(x_0) (x - x_0)^{\dot{\underline{r}}}$$

where $\dot{\underline{r}}$ stands for a multiindex with $|\dot{\underline{r}}| \geq 1$.

and $\underline{r}! := r_1! \cdots r_D!$

$$x^{\underline{r}} \partial_{\underline{r}} \psi := \sum_{\underline{r}} x^{\underline{r}} \partial_{\underline{r}} \psi$$

$$x^{\dot{\underline{r}}} \partial_{\dot{\underline{r}}} \psi := \sum_{\dot{\underline{r}}} x^{\dot{\underline{r}}} \partial_{\dot{\underline{r}}} \psi$$

c) The correspondence.

$$\underline{r} = (r_1, \dots, r_D) \equiv \underline{\mu} = (\mu_1, \dots, \mu_n)$$

$$\Updownarrow \text{ iff}$$

$$\underline{\mu} = (\underbrace{1, \dots, 1}_{r_1}, \dots, \underbrace{D, \dots, D}_{r_D})$$

$$\Updownarrow \text{ iff}$$

$$r_j = \#\{\mu_j = 1\} \text{ for } j = 1, \dots, D$$

d) Negative (or, impossible) multiindices.

The nonsymmetric index $\underline{r} = (r_1, \dots, r_D)$ is called negative (or, impossible) if all r_j are integers and some of them are negative. We set that every turm in a sum, which contains a negative multiindex, equals to zero. For example:

$$\partial_{\underline{r}} x^{\underline{s}} = \frac{\underline{s}!}{(\underline{s}-\underline{r})!} x^{\underline{s}-\underline{r}}$$

which is zero iff $\underline{s}-\underline{r}$ is negative.

However, one needs some precaution:

$x^{\underline{r}} x^{\underline{s}} \neq x^{\underline{r}+\underline{s}}$ if $\underline{r}$ is negative but $\underline{s}$ and $\underline{r}+\underline{s}$ are nonnegative (according to the previous convention)

It is convenient to set

$\underline{r} = (r_1, \dots, r_D) \geq \underline{0}$ iff $r_j \geq 0$ $(\forall j)$

and $\underline{r} \geq \underline{s}$ iff $\underline{r} - \underline{s} \geq 0$

Thus, $\underline{r}$ is nonnegative iff $\underline{r} \geq \underline{0}$,

r is negative iff $\underline{r} \ngeq \underline{0}$.

For a symmetric multiindex $\underline{\mu} = (\mu_1, \dots, \mu_n)$ and $\nu = 1, \dots, D$ we set

$\underline{\mu} \setminus \nu$ to be the symmetric multiindex corresponding to $\underline{r} - \underline{e}_\nu$ if $\underline{\mu} \equiv \underline{r}$.

This $\underline{\mu} \setminus \nu$ is negative iff $\underline{r} - \underline{e}_\nu$ is negative. The latter is also equivalent to the fact that ν does not appear in the list of $\underline{\mu} = (\mu_1, \dots, \mu_n)$. Otherwise, $\underline{\mu} \setminus \nu$ is obtained by removing one entry of ν in the list of $\underline{\mu}$.

e) Multiindex Newton's binomial formula and the binomial Leibniz rule

$$(x+y)^{\underline{r}} = \sum_{\underline{0} \leqslant \underline{s} \leqslant \underline{r}} \frac{\underline{r}!}{\underline{s}!(\underline{r}-\underline{s})!} x^{\underline{s}} y^{\underline{r}-\underline{s}}$$

Let $\mathcal{A}$ be an associative algebra and $A_1, \dots, A_D$ be a set of mutually commuting derivations on $\mathcal{A}$, i.e.

$$A_j(ab) = A_j(a)\, b + a\, A_j(b) \quad \forall a, b \in \mathcal{A}$$

$$A_j A_k = A_k A_j \quad \forall j, k = 1, \dots, D.$$

Set: $\underline{A}^{\underline{r}} := A_1^{r_1} \cdots A_D^{r_D}$. Then:

$$\underline{A}^{\underline{r}}(ab) = \sum_{\underline{0} \leqslant \underline{s} \leqslant \underline{r}} \frac{\underline{r}!}{\underline{s}!(\underline{r}-\underline{s})!} \times \underline{A}^{\underline{s}}(a)\, \underline{A}^{\underline{r}-\underline{s}}(b)$$

$$a\, \underline{A}^{\underline{r}}(b) = \sum_{\underline{0} \leqslant \underline{s} \leqslant \underline{r}} \frac{\underline{r}!}{\underline{s}!(\underline{r}-\underline{s})!} \times (-1)^{|\underline{s}|+1}\, \underline{A}^{\underline{s}}\left(\underline{A}^{\underline{r}-\underline{s}}(a)\, b\right)$$

f) Combinatorics

$$D_n := \#\{\underline{\mu} \mid |\underline{\mu}| = n\} = \binom{D+n}{D}$$

with $D_1 = D$

To prove this note that

$$\sum_{n=0}^{\infty} D_n t^n = \sum_{\underline{r}} x^{\underline{r}} t^{|\underline{r}|} \Big|_{x^1 = \cdots = x^D = 1}$$

$$= \prod_{\mu=1}^{D} \left(1 - x^\mu t\right)^{-1} \Big|_{x^1 = \cdots = x^D = 1}$$

$$= (1-t)^{-D} = \sum_{n=0}^{\infty} (-1)^n \binom{-n}{D} t^n$$

$$\Longrightarrow \quad D_n = (-1)^n \binom{-n}{D} = \binom{D+n}{D}$$

4. Jet bundles

Recall, $E = M \times F$

with $\underset{\text{open}}{M \subseteq} \mathbb{R}^D$, $\underset{\text{open}}{F \subseteq} \mathbb{R}^N$.

Set $E^{(n)} := M \times F \times F^{(1)} \times \dots \times F^{(n)}$.

where $F^{(k)} := \mathbb{R}^{ND_k}$ and we denote the coordinates in $F^{(k)}$ by $(u^A_{\underline{\mu}})$ where $A = 1, \dots, N$ and $\underline{\mu}$ is a symmetric multi-index with $|\underline{\mu}| = k$. Equivalently, one can work with nonsymmetric multiindices

$$u^A_{\underline{\mu}} \equiv u^A_{\underline{r}} \quad \text{iff} \quad \underline{\mu} \equiv \underline{r}.$$

Thus, the coordinates on $E^{(n)}$ are

$$(x, u^{(n)}), \text{ where } \quad x = (x^\mu)_{\mu=1}^D$$

$$u^{(n)} = \left(u^A_{\underline{\mu}}\right)_{A=1,\dots,N;\ |\underline{\mu}| \leq n}$$

with the convention $u^A = u^A_{\underline{\mu}}$ for the empty multiindex $\underline{\mu}$ (i.e., for $|\underline{\mu}| = 0$).

There are natural projections

$$p_n : E^{(n)} \twoheadrightarrow E^{(n-1)}$$

$$(x, u^{(n)}) \mapsto (x, u^{(n-1)})$$

$$\begin{tikzcd}
E & E^{(1)} \arrow[l, two heads, "p_1"'] \arrow[d] & E^{(2)} \arrow[l, two heads, "p_2"'] \arrow[d] & \cdots \arrow[l, two heads] \\
M \arrow[u, leftarrow, two heads, "p"] \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & \cdots
\end{tikzcd}$$

The pullback's of functions along p_n,

i.e. $p_n^* : C^\infty(E^{(n-1)}) \hookrightarrow C^\infty(E^{(n)})$

$$f \longmapsto f \circ p_n$$

induces inclusions:

$$C^\infty(M) \subseteq C^\infty(E) \subseteq C^\infty(E^{(1)}) \subseteq \cdots$$

and we set $C^\infty(E^{(\infty)}) := \bigcup_{n=1}^{\infty} C^\infty(E^{(n)})$

Thus, a function $\in C^\infty(E^{(\infty)})$ is a function on x and finitely many u^A_μ

5. Jet prolongations

(струйни продължения / повдигания)

a) Jet prolongations of sections

If we have a section of

$$\xi = (E, p: E \to M, M)$$

$$\begin{array}{cccc} \hat{\varphi}^{(0)}: & M & \longrightarrow & E = M \times F \\ & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\ & x & \longmapsto & \overbrace{(x, \varphi(x))} \end{array}$$

then we obtain a section

$$\begin{array}{cccc} \hat{\varphi}^{(n)}: & M & \longrightarrow & E^{(n)} \\ & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\ & x & \longmapsto & \overbrace{(x, u^{(n)} = \varphi^{(n)}(x))} \end{array}$$

where $u^{(n)} = \varphi^{(n)}(x)$ stands for

$$u^A_{\underline{\mu}} = \partial_{\underline{\mu}} \varphi^A(x)$$

for all $\underline{\mu}$ with $|\underline{\mu}| \leq n$

Thus, while the coordinate u^A serves for the values of the field's component $\varphi^A(x)$ $u^A_{\underline{\mu}}$ serves for the value of its partial derivative.

b) Jet prolongations of flows / maps and vector fields

Assume we have a (local) flow on E:

$E \longrightarrow E : (x,u) \longmapsto (\mathcal{F}(x,u), G(x,u))$

$(\mathcal{F} = \mathcal{F}_{(\varepsilon)},\ G = G_{(\varepsilon)})$. We look for a (local) flow on $E^{(n)}$

$(x, u^{(n)}) \longmapsto (\mathcal{F}(x,u), G^{(n)}(x, u^{(n)}))$

with the properties

(pr_1) $p_n(\mathcal{F}(x,u), G^{(n)}(x,u^{(n)}))$

$$= (\mathcal{F}(x,u), G^{(n-1)}(x, u^{(n-1)}))$$

(recall, $p_n : E^{(n)} \twoheadrightarrow E^{(n-1)}$).

(pr_2) If a field $u = \varphi(x)$ is transformed to a field $u' = \varphi'(x')$ under the rule

$$\varphi'(\mathcal{F}(x, \varphi(x))) = G(x, \varphi(x))$$

Then $\varphi'^{(n)}(\mathcal{F}(x,\varphi(x))) =$

$$= G^{(n)}(x, \varphi^{(n)}(x))$$

<u>Theorem</u> $\exists!$ (local) flow $E^{(n)} \to E^{(n)}$

$$(x, u^{(n)}) \longmapsto \left(\mathcal{F}(x,u), G^{(n)}(x, u^{(n)})\right)$$

$\left(\mathcal{F} = \mathcal{F}_{(\varepsilon)}, G = G^{(n)}_{(\varepsilon)}\right)$ for $\forall\, n = 1, 2, \ldots$

satisfying the conditions (pr_1) and (pr_2) for every field $u = \varphi(x)$

Proof. We use an induction in $n = 0, 1, 2, \ldots$ (for $n=0$ we take the starting map $E \to E$)

Assume we have constructed

$$G^{(n)}(x, u^{(n)})$$

$$= \left(G^A_{\underline{\mu}}(x, u^{(n)})\right)_{A=1,\ldots,N;\ |\underline{\mu}| \leq n}$$

so that $\partial_{\underline{\mu}} \varphi'^{A}\left(\mathcal{F}(x, \varphi(x))\right) =$

$$= G^A_{\underline{\mu}}\left(x, \varphi^{(n)}(x)\right)$$

We take a derivative ∂_{x^ν} on both sides and obtain

$$\partial_{\rho\underline{\mu}} \varphi'^{A}\left(\mathcal{F}(x,\varphi(x))\right)\left[\left(\partial_{x^\nu}\mathcal{F}^\rho\right)(x, \varphi(x)) + \left(\partial_{u^B}\mathcal{F}^\rho\right)(x, \varphi(x))\, \partial_{x^\nu}\varphi^B(x)\right]$$

$$= \left(\partial_{x^\nu} G^A_{\underline{\mu}}\right)(x, \psi^{(n)}(x))$$

$$+ \left(\partial_{u^B_{\underline{\sigma}}} G^A_{\underline{\mu}}\right)(x, \psi^{(n)}(x))\, \partial_{x^{\nu\underline{\sigma}}} \psi^B(x)$$

Let us try to use the desired identity

$$\partial_{\rho\underline{\mu}} \psi'^A\left(\mathcal{F}(x, \psi(x))\right)$$

$$= G^A_{\rho\underline{\mu}}(x, \psi^{(n+1)}(x))$$

at order n+1 for the construction of $G^{(n+1)}$

$$\boxed{\begin{aligned} & G^A_{\rho\underline{\mu}}(x, u^{(n+1)}) \Big[\left(\partial_{x^\nu} \mathcal{F}^\rho\right)(x, u) \\ & \qquad + \left(\partial_{u^B} \mathcal{F}^\rho\right)(x, u)\, u^B_\nu \Big] \\ & = \left(\partial_{x^\nu} G^A_{\underline{\mu}}\right)(x, u^{(n)}) \\ & \qquad + \left(\partial_{u^B_{\underline{\sigma}}} G^A_{\underline{\mu}}\right)(x, u^{(n)})\, u^B_{\nu\underline{\sigma}} \,\Big] \end{aligned}}$$

where we have replaced everywhere $\partial_{x^{\underline{\sigma}}} \psi^B$ by $u^B_{\underline{\sigma}}$ for all $B = 1, \dots, N$ and $|\underline{\sigma}| \leq n+1$ (beause for every $u^{(n+1)}$ and a fixed x there always exists a field ψ such that $u^{(n+1)} = \psi^{(n+1)}(x)$).

Now, in order to find $G^{A}_{\rho\underline{\mu}}(x, u^{(n+1)})$ in a unique way (and hence, to complete the proof) one needs to invert the matrix

$$\tilde{\Xi}^{\rho}_{\nu} = \left(\partial_{x^\nu}\mathcal{F}^{\rho}\right)(x,u) + \left(\partial_{u^B}\mathcal{F}^{\rho}\right)(x,u)\,u^B_\nu$$

But this is always possible for a sufficiently small flow parameter ε since

$$\left(\mathcal{F}_{(\varepsilon)}(x,u),\, G_{(\varepsilon)}(x,u)\right)\Big|_{\varepsilon=0} = (x,u)$$

$\square$

Let us calculate the vector field

$$\begin{aligned} Z^{(n)} &= X^{\mu}(x,u)\,\partial_{x^\mu} \\ &+ \sum_{k=0}^{n}\sum_{|\underline{\sigma}|=k} \tilde{Y}^{A}_{\underline{\sigma}}(x, u^{(k)})\,\partial_{u^A_{\underline{\sigma}}} \end{aligned}$$

(we use an explicit notation for the sum in $\underline{\sigma}$ because of the restriction $|\underline{\sigma}| \leqslant n$).

that generates the flow

$$\left(\mathcal{F}_{(\varepsilon)}(x,u),\, G_{(\varepsilon)}(x,u^{(n)})\right)$$

where

$$\widetilde{Y}^A_{\underline{\mu}}\left(x, u^{(k)}\right)$$

$$= \frac{\partial}{\partial \varepsilon}\left(\left(G_{(\varepsilon)}\right)^A_{\underline{\mu}}\left(x, u^{(n)}\right)\right)\Big|_{\varepsilon=0}$$

Differentiating in ε the equation in box at $\varepsilon = 0$ we obtain the following recursive identity:

$$\widetilde{Y}^A_{\underline{\mu}\nu}\left(x, u^{(n+1)}\right) + u^A_{\underline{\mu}\rho}\, \partial_{x^\nu} X^\rho(x,u) + u^A_{\underline{\mu}\rho}\, u^B_\nu\, \partial_{u^B} X^\rho(x,u)$$

$$= \partial_{x^\nu} \widetilde{Y}^A_{\underline{\mu}}\left(x, u^{(n)}\right) + u^B_{\nu\underline{\sigma}}\, \partial_{u^B_{\underline{\sigma}}} \widetilde{Y}^A_{\underline{\mu}}\left(x, u^{(n)}\right)$$

If one formally introduces.

$$\mathcal{D}_{x^\nu} = \partial_{x^\nu} + u^B_{\nu\underline{\sigma}}\, \partial_{u^B_{\underline{\sigma}}}$$

then the above relation becomes:

$$\widetilde{Y}^A_{\underline{\mu}\nu}\left(x, u^{(n+1)}\right) = \mathcal{D}_{x^\nu} \widetilde{Y}^A_{\underline{\mu}}\left(x, u^{(n)}\right) - u^A_{\underline{\mu}\rho}\, \mathcal{D}_{x^\nu} X^\rho(x,u)$$

6. Total derivatives and generalized vector fields.

a) A *generalized vector field* is a formal series of a form

$$\widetilde{Z} = \widetilde{X}^{\mu}\,\partial_{x^{\mu}} + \widetilde{Y}^{A}_{\underline{\sigma}}\,\partial_{u^{A}_{\underline{\sigma}}}\,,\quad \text{where}$$

all $\widetilde{X}^{\mu}$ and $\widetilde{Y}^{A}_{\underline{\sigma}}$ belong to $C^{\infty}(E^{(\infty)})$ i.e., each $\widetilde{X}^{\mu}$ and $\widetilde{Y}^{A}_{\underline{\sigma}}$ depend on x and finitely many $u^{B}_{\underline{\nu}}$.

The above introduced

$$\mathcal{D}_{x^{\nu}} = \partial_{x^{\nu}} + u_{\nu\underline{\sigma}}\,\partial_{u^{B}_{\underline{\sigma}}}$$

is our first example of a generalized vector field. It is called a total derivative (with respect to x^{ν}).

Note that although $\widetilde{Z}$ contains an infinite sum in $\underline{\sigma}$ it has a well dinied action:

$$\widetilde{Z}(f) := \widetilde{X}^{\mu}\,\partial_{x^{\mu}} f + \widetilde{Y}^{A}_{\underline{\sigma}}\,\partial_{u^{A}_{\underline{\sigma}}} f$$

for every $f \in C^\infty(E^{(\infty)})$ since f depend on finetely many $u^B_{\underline{\nu}}$. However, if

$f \in C^\infty(E^{(n)})$ $\left(\subseteq C^\infty(E^{(\infty)})\right)$

then $\tilde{Z}(f) \in C^\infty(E^{(m)})$ and it is possible that $m > n$.

An example is provided by

b) The total derivatives $\mathcal{D}_{x^\nu}$ defined as:

$$f \in C^\infty(E^{(n)}) \Rightarrow \mathcal{D}_{x^\nu}(f) \in C^\infty(E^{(n+1)})$$

In fact, $\mathcal{D}_{x^\nu}(f)$ has the following meaning

$$\partial_{x^\nu}\left(f(x, \varphi(x))\right) = \left(\mathcal{D}_{x^\nu} f\right)(x, \varphi^{(n)}(x))$$

which justifies the name total derivative

Another usefull formulas are

$$\mathcal{D}_{x^\nu} u^A_{\underline{\mu}} = u^A_{\nu\underline{\mu}} \equiv u^A_{\underline{\mu}\nu}$$

$$u^A_{\underline{\mu}} = \mathcal{D}_{\underline{\mu}} u^A$$

if we set $\mathcal{D}_{\underline{\mu}} := \mathcal{D}_{x^{\mu_1}} \cdots \mathcal{D}_{x^{\mu_n}}$

Not that every generalized vector field $\widetilde{Z}$ acts as a derivation of $C^\infty(E^{(\infty)})$

i.e., $\widetilde{Z}(f_1 f_2) = \widetilde{Z}(f_1) f_2 + f_1 \widetilde{Z}(f_2)$

Furthermore, if

$$\widetilde{Z}_j = \widetilde{X}^\mu_j \partial_{x^\mu} + \widetilde{Y}^A_{j,\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}} \qquad (j = 1, 2)$$

are two generalized vector fields then their commutator on functions

$$\widetilde{Z}(f) := \widetilde{Z}_1 \widetilde{Z}_2(f) - \widetilde{Z}_2 \widetilde{Z}_1(f)$$

is again a derivation provided by a unique generalized vector field

$$\widetilde{Z} = \widetilde{X}^\mu \partial_{x^\mu} + \widetilde{Y}^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}, \quad \text{where}$$

$$\widetilde{X}^\mu = \widetilde{Z}_1(\widetilde{X}^\mu_2) - \widetilde{Z}_2(\widetilde{X}^\mu_1)$$

$$\widetilde{Y}^A_{\underline{\sigma}} = \widetilde{Z}_1(\widetilde{Y}^A_{2,\underline{\sigma}}) - \widetilde{Z}_2(\widetilde{Y}^A_{1,\underline{\sigma}})$$

This defines the commutator of two generalized vector fields

$$\widetilde{Z} = [\widetilde{Z}_1, \widetilde{Z}_2]$$

As an example, one can calculate

$$\left[\mathcal{D}_{x^\alpha}, \mathcal{D}_{x^\beta}\right] = 0$$

Indeed, if

$$\left[\mathcal{D}_{x^\alpha}, \mathcal{D}_{x^\beta}\right] = \tilde{X}^\mu \partial_{x^\mu} + \tilde{Y}^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}$$

then $\tilde{X}^\mu = \mathcal{D}_{x^\alpha}(\delta^\mu_\beta) - \mathcal{D}_{x^\beta}(\delta^\mu_\alpha) = 0$

$$\tilde{Y}^A_{\underline{\sigma}} = \underbrace{\mathcal{D}_{x^\alpha}(u^A_{\beta\underline{\sigma}})}_{u^A_{\alpha\beta\underline{\sigma}}} - \underbrace{\mathcal{D}_{x^\beta}(u^A_{\alpha\underline{\sigma}})}_{u^A_{\beta\alpha\underline{\sigma}}} = 0$$

<u>Corollary</u> The component $Y^A_{\underline{\sigma}}(x, u^{(k)})$ of the prolongated vector field $Z^{(n)}$ (recursively determined at the end of the previous section):

$$\begin{aligned} &\tilde{Y}^A_{\underline{\sigma}}(x, u^{(k)}) \\ &= \mathcal{D}_{\underline{\sigma}}\left(Y^A(x,u) - u^A_\mu X^\mu(x,u)\right) \\ &+ u^A_{\underline{\sigma}\mu} X^\mu(x,u) \qquad (|\underline{\sigma}| = k) \end{aligned}$$

Proof We use the recursive identity obtained at the end of the previous section and follow an induction in k. For $k=1$ this formula gives directly the result:

$$
\begin{aligned}
&Y_\nu^A(x,u^{(1)}) \\
&= \mathcal{D}_{x^\nu} Y^A(x,u) - u^A_\rho\, \mathcal{D}_{x^\nu} X^\rho(x,u) \\
&= \mathcal{D}_{x^\nu}\left(Y^A(x,u) - u^A_\mu X^\mu(x,u)\right) \\
&\quad + u^A_{\nu\mu} X^\mu(x,u)
\end{aligned}
$$

Then, by induction: $Y^A_{\underline{\sigma}\nu}(x,u^{(k+1)})$

$$
\begin{aligned}
&= \mathcal{D}_{x^\nu} Y^A_{\underline{\mu}}(x,u^{(k)}) \\
&\qquad\qquad - u^A_{\underline{\sigma}\rho}\, \mathcal{D}_{x^\nu} X^\rho(x,u)
\end{aligned}
$$

$\mathcal{D}_{\nu\underline{\sigma}}$

$$
\begin{aligned}
&= \boxed{\mathcal{D}_{x^\nu}\mathcal{D}_{\underline{\sigma}}}\left(Y^A(x,u) - u^A_\mu X^\mu(x,u)\right) \\
&\quad + \boxed{\mathcal{D}_{x^\nu}\left(u^A_{\underline{\sigma}\mu} X^\mu(x,u)\right) - u^A_{\underline{\sigma}\rho}\, \mathcal{D}_{x^\nu} X^\rho(x,u)}
\end{aligned}
$$

$u^A_{\underline{\sigma}\nu\mu} X^\mu$

$\square$

c) "Temperete" generalized vector fields and flows

A generalized vector field

$$\tilde{Z} = \tilde{X}^\mu \partial_{x^\mu} + \tilde{Y}^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}$$

is called tempered iff $\exists\, n_0 \in \mathbb{N}$ s.t.

$$\tilde{Z}\left(C^\infty(E^{(n)})\right) \subseteq C^\infty(E^{(n)}) \quad \forall n \geq n_0$$

i.e., iff $\tilde{Y}^A_{\underline{\sigma}} \in C^\infty(E^{(n)})$ for $n = |\underline{\sigma}|$ when $n \geq n_0$

Thus, every tempered generalized vector field $\tilde{Z}$ can be restricted to an ordinary vector field on each $E^{(n)}$, when $n \geq n_0$, and hence, induces a local flow on $E^{(n)}$.

Note: $\mathcal{D}_{x^\mu}$ are not tempered

7. Vertical-horizontal decomposition

Let us call the generalized vector fields $\mathcal{D}_{x^\nu}$ - basic horizontal fields and the vector fields $\partial_{u^A_{\underline{\sigma}}}$ - basic vertical vector fields. Since

$$\partial_{x^\nu} = \mathcal{D}_{x^\nu} - u^B_{\nu\underline{\sigma}}\, \partial_{u^B_{\underline{\sigma}}}$$

one can expand a generalized vector field in both ways:

$$\begin{aligned} \tilde{Z} &= \tilde{X}^\mu\, \partial_{x^\mu} + \tilde{Y}^A_{\underline{\sigma}}\, \partial_{u^A_{\underline{\sigma}}} \\ &= \tilde{X}^\mu\, \mathcal{D}_{x^\mu} + \tilde{V}^A_{\underline{\sigma}}\, \partial_{u^A_{\underline{\sigma}}} \end{aligned}$$

where all $X^\mu, Y^A_{\underline{\sigma}}, \tilde{V}^A_{\underline{\sigma}} \in C^\infty(E^{(\infty)})$

and $\tilde{Y}^A_{\underline{\sigma}} = \tilde{V}^A_{\underline{\sigma}} + u^A_{\underline{\sigma}\mu}\, \tilde{X}^\mu$

The second equality above is called also a horizontal-vertical decomposition of $\tilde{Z}$:

$$\tilde{Z} = \tilde{Z}^{(hor)} + \tilde{Z}^{(ver)}$$ where:

$\widetilde{Z}^{(hor)} = X^\mu \mathcal{D}_{x^\mu}$ is called <u>horizontal part</u> of $\widetilde{Z}$ and

$\widetilde{Z}^{(ver)} = \widetilde{V}^A_{\underline{\sigma}} \, \partial_{u^A_{\underline{\sigma}}}$ is called <u>vertical part</u> of $\widetilde{Z}$ and

As an important example of such a decomposition let us consider the decomposition of the so called infinite jet prolongation of a vector field

$$Z = X^\mu(x,u)\, \partial_{x^\mu} + Y^A(x,u)\, \partial_{u^A}$$

defined as:

$$\overset{\infty}{Z} \equiv Z^{(\infty)} := X^\mu(x,u)\, \partial_{x^\mu} + \widetilde{Y}^A_{\underline{\sigma}}\, \partial_{u^A_{\underline{\sigma}}}$$

with $\widetilde{Y}^A_{\underline{\sigma}}$ calculated in the last corrolary of the previous section. Then we obtain:

$$\overset{\infty}{Z} = \underbrace{X^\mu(x,u)\, \mathcal{D}_{x^\mu}}_{\overset{\infty}{Z}^{(hor)}} + \underbrace{\left(\mathcal{D}_{\underline{\sigma}} \widetilde{V}^A\right) \partial_{u^A_{\underline{\sigma}}}}_{\overset{\infty}{Z}^{(ver)}}$$

where $\widetilde{V}^A := Y^A(x,u) - u^A_\mu X^\mu(x,u)$

is called *characteristic* of Z

(note: here $\widetilde{V}^A \in C^\infty(E^{(1)})$

Note: $\overset{\infty}{Z} \equiv Z^{(\infty)}$ is a tempered generalized vector field

In fact, the infinite jet prolongation $Z^{(\infty)}$ keeps invariant each subspace $C^\infty(E^{(n)}) \left(\subseteq C^\infty(E^{(\infty)})\right)$, i.e.,

$$Z^{(\infty)}\left(C^\infty(E^{(n)})\right) \subseteq C^\infty(E^{(n)})$$

and in fact,

$$Z^{(\infty)}\Big|_{C^\infty(E^{(n)})} = Z^{(n)}$$

However, neither $\overset{\infty}{Z}{}^{(hor)}$ nor $\overset{\infty}{Z}{}^{(ver)}$ keeps any of $C^\infty(E^{(n)})$ invariant. One has instead

$$\overset{\infty}{Z}{}^{(hor)}\left(C^\infty(E^{(n)})\right) \subseteq C^\infty(E^{(n+1)})$$

$$\overset{\infty}{Z}{}^{(ver)}\left(C^\infty(E^{(n)})\right) \subseteq C^\infty(E^{(n+1)})$$

__Theorem__ Consider a vertical generalized vector field

$$\widetilde{V} := \widetilde{V}^A_{\underline{\sigma}} \, \partial_{u^A_{\underline{\sigma}}}$$

Then the necessary and sufficient condition to have

$$\left[\mathcal{D}_{x^\mu}, \widetilde{V}\right] = 0 \quad (\forall \mu = 1, \dots, D)$$

is that

$$\widetilde{V}^A_{\underline{\sigma}} = \mathcal{D}_{\underline{\sigma}} \widetilde{V}^A \qquad (\forall A, \underline{\sigma})$$

for some $\widetilde{V}^A \in C^\infty(E^{(\infty)})$

The proof follow by the observation that the identities $\left[\mathcal{D}_{x^\mu}, \widetilde{V}\right] = 0$ are equivalent to the recursive identities

$$\widetilde{V}^A_{\underline{\sigma}\mu} = \mathcal{D}_{x^\mu}\left(\widetilde{V}^A_{\underline{\sigma}}\right) \qquad \square$$

Vertical generalized vector fields satisfying the conditions of this theorem will be called horizontally constant

In fact, the vertical generalized vector fields, which are horizontally constant (as defined above), are also called *evolutionary vector fields*

The reason for this terminology is the following. According to the previous Theorem such an evolutionary vector field is determined by the coefficients

$$\tilde{V}^A = \tilde{V}^A(x, u^{(n)}) \quad , \quad A = 1, \dots, N$$

The latter system determines a system of "evolutionary" P.D.E.

$$\frac{\partial}{\partial t} \varphi^A(x;t) = \tilde{V}^A\left(x, \varphi^{(n)}(x;t)\right)$$

Furthermore, since $\varphi^A_{\underline{\mu}}(x;t)$

$= \partial_{\underline{\mu}} \varphi^A(x;t)$ and $\tilde{V}^A_{\underline{\mu}} = \mathcal{D}_{\underline{\mu}} \tilde{V}^A$

Then:

$$\frac{\partial}{\partial t} \varphi^A_{\underline{\mu}}(x;t) = \tilde{V}^A_{\underline{\mu}}\left(x, \varphi^{(n+m)}(x;t)\right)$$

$$(m = |\underline{\mu}|)$$

In this way, $\varphi^A(x;t)$ can be regarded as a "flow" (of course, in a generalized sense) for the evolutionary vector field $\widetilde{V} := \widetilde{V}^A_{\underline{\sigma}}\, \partial_{u^A_{\underline{\sigma}}}$

Another consequence of the last theorem is

<u>Corollary</u> The commutor of two evolutionary vector fields is again an evolutionary vector field.

<u>Indeed</u>, if $\left[\mathcal{D}_{x^\mu}, \widetilde{V}\right] = 0$

$$= \left[\mathcal{D}_{x^\mu}, \widetilde{W}\right]$$

then by the Jacobi identity

$$\left[\mathcal{D}_{x^\mu}, \left[\widetilde{V}, \widetilde{W}\right]\right] = 0.$$

8. The variational bicomplex

a) Differential forms on $E^{(\infty)}$

Let us introduce the graded commutative algebra $\Omega^\bullet(E^{(\infty)})$, which similarly to $C^\infty(E^{(\infty)})$ is an inductive limit:

$$\Omega^\bullet(E^{(\infty)}) = \bigcup_{n=1}^{\infty} \Omega^\bullet(E^{(n)})$$

induced by the diagram

$$\begin{tikzcd} E \arrow[d, twoheadrightarrow, "p"'] & E^{(1)} \arrow[l, twoheadrightarrow, "p_1"'] \arrow[d, twoheadrightarrow] & E^{(2)} \arrow[l, twoheadrightarrow, "p_2"'] \arrow[d, twoheadrightarrow] & \cdots \arrow[l, twoheadrightarrow] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & \cdots \end{tikzcd}$$

(i.e., $\Omega^\bullet(E^{(n)})$ is embedded trough the pullback into $\Omega^\bullet(E^{(n+1)})$:

$$\Omega^\bullet(E^{(n)}) \xhookrightarrow[p_n^*]{} \Omega^\bullet(E^{(n+1)})\Big)$$

From a practical point of view $\Omega^\bullet(E^{(\infty)})$ is generated, as an algebra, by the elements of $C^\infty(E^{(\infty)})$ and the 1-forms

$$dx^\mu, \; du^A_{\underline{\sigma}}$$

More precisely,

$$\Omega^\bullet(E^{(\infty)})$$
$$\cong C^\infty(E^{(\infty)}) \otimes \Lambda^\bullet[dx^\mu, du^A_{\underline{\sigma}} : \forall \mu, \underline{\sigma}, A]$$

Under the above decomposition

$$d = \partial_{x^\mu} \otimes dx^\mu + \partial_{u^A_{\underline{\sigma}}} \otimes du^A_{\underline{\sigma}}$$

is the de Rham differential

$$d : \Omega^\bullet(E^{(\infty)}) \longrightarrow \Omega^{\bullet+1}(E^{(\infty)})$$

which is, as before, a graded derivation of $\Omega^\bullet(E^{(\infty)})$ of degree 1, with

$$d^2 = 0$$

b) The horizontal differential

We shall define in this section a new graded derivation of $\Omega^\bullet(E^{(\infty)})$ of degree 1

$$d^h : \Omega^\bullet(E^{(\infty)}) \longrightarrow \Omega^{\bullet+1}(E^{(\infty)})$$

of degree 1 with the following properties $(d^h\text{-}2)$ and $(d^h\text{-}3)$

$(d^h\text{-}1)$ $d^h f = \mathcal{D}_{x^\mu} f \, dx^\mu$

$(d^h\text{-}2)$ $(d^h)^2 = 0$

$(d^h\text{-}3)$ $d^h d = -d\, d^h$

<u>Theorem</u> The above conditions uniquely determine the derivation d^h

Proof. Let us show that d^h is uniquely determined on all the generators of the algebra $\Omega^\bullet(E^{(\infty)})$. It remains to calculate it on $du^A_{\underline{\sigma}}$. We have

$$d^h(du^A_{\underline{\sigma}}) = -d\, d^h u^A_{\underline{\sigma}}$$
$$= -d\left(\mathcal{D}_{x^\mu} u^A_{\underline{\sigma}} \, dx^\mu\right) = -d\left(u^A_{\underline{\sigma}\mu} \, dx^\mu\right)$$
$$= -du^A_{\underline{\sigma}\mu} \, dx^\mu$$

In this way, d^h is extended to a unique graded derivation of $\Omega^\bullet(E^{(\infty)})$ of degree 1

Still, it remains to confirm the properties $(d^h\text{-}2)$ and $(d^h\text{-}3)$. However, they are of the form of vanishing graded commutators, i.e.:

$$[d^h, d^h] \equiv 2(d^h)^2 = 0$$

$$[d^h, d] = d^h d + d d^h = 0$$

But these graded commutators are again graded derivations. Hence, it is enough to check they vanish on the generators of the algebra $\Omega^\bullet(E^{(\infty)})$, i.e., on the elements $f \in C^\infty(E^{(\infty)})$ and on all dx^μ and du^A_σ. The latter is done by a strightforward computation.

c) The vertical differential

It is definen as $d^v := d - d^h$

Since $[d^h, d^h] \equiv 2(d^h)^2 = 0$ $= [d^h, d] = [d, d] \equiv 2d^2$ it follows that

$$(d^v)^2 = 0 \text{ and}$$

$$d^v d^h = -d^h d^v, \quad d^v d = -d d^v$$

Let us calculate

$$d^v u^A_{\underline{\sigma}} \equiv d^v(u^A_{\underline{\sigma}}) = du^A_{\underline{\sigma}} - d^h(u^A_{\underline{\sigma}})$$

$$= du^A_{\underline{\sigma}} - u^A_{\underline{\sigma}\mu}\, dx^\mu$$

$$d^v f = df - d^h f$$

$$= \partial_{x^\mu} f\, dx^\mu + \partial_{u^A_{\underline{\sigma}}} f\, du^A_{\underline{\sigma}}$$

$$- \left(\partial_{x^\mu} + u^A_{\underline{\sigma}\mu}\, \partial_{u^A_{\underline{\sigma}}}\right) f\, dx^\mu$$

$$= \partial_{u^A_{\underline{\sigma}}} f \left(du^A_{\underline{\sigma}} - u^A_{\underline{\sigma}\mu}\, dx^\mu\right)$$

$$= \partial_{u^A_{\underline{\sigma}}} f\, d^v u^A_{\underline{\sigma}}$$

In particular, $d^v x^\mu = 0$

$$d^v du^A_{\underline{\sigma}} = -d\, d^v u^A_{\underline{\sigma}}$$

$$= -d\left(du^A_{\underline{\sigma}} - u^A_{\underline{\sigma}\mu}\, dx^\mu\right)$$

$$= du^A_{\underline{\sigma}\mu}\, dx^\mu$$

$$d^v dx^\mu = -d\, d^h x^\mu = -d\, dx^\mu = 0$$

Note that $\Omega^\bullet(E^{(\infty)})$ can be equally generated by $C^\infty(E^{(\infty)})$ and all dx^μ and $d^v u^A_{\underline{\sigma}}$, i.e.,

$$
\begin{aligned}
&\Omega^\bullet(E^{(\infty)}) \\
&\cong C^\infty(E^{(\infty)}) \otimes \Lambda^\bullet[dx^\mu, du^A_{\underline{\sigma}} : \forall \mu, \underline{\sigma}, A] \\
&\cong C^\infty(E^{(\infty)}) \otimes \Lambda^\bullet[dx^\mu, d^v u^A_{\underline{\sigma}} : \forall \mu, \underline{\sigma}, A]
\end{aligned}
$$

Then, under the isomorphism on the second row one has:

$$d^v = \partial_{u^A_{\underline{\sigma}}} \otimes d^v u^A_{\underline{\sigma}}$$

However, d^h does not act in a simple way in this representation as:

$$
\begin{aligned}
d^h d^v u^A_{\underline{\sigma}} &= - d^v d^h u^A_{\underline{\sigma}} \\
&= - d^v (u^A_{\underline{\sigma}\mu} dx^\mu) = - du^A_{\underline{\sigma}\mu} dx^\mu \\
&= dx^\mu d^v u^A_{\underline{\sigma}\mu} \\
&= dx^\mu du^A_{\underline{\sigma}\mu} = d^h du^A_{\underline{\sigma}}
\end{aligned}
$$

Thus, d^h has a nonzero action on $du^A_{\underline{\sigma}}$ as well as on $d^v u^A_{\underline{\sigma}}$

d) The horizonal-vertical bigrading

Let us set $\Omega^{(p,q)}(E^{(n)})$ = the span of all forms on $E^{(n)}$ of the form

$$f\, dx^{\mu_1} \cdots dx^{\mu_p}\, d^v u^{A_1}_{\underline{\sigma}_1} \cdots d^v u^{A_q}_{\underline{\sigma}_q}$$

with $|\underline{\sigma}_1|, \dots, |\underline{\sigma}_q| \leqslant n$. Also,

$$\Omega^{(p,q)}(E^{(\infty)}) := \bigcup_{n=1}^{\infty} \Omega^{(p,q)}(E^{(n)})$$

= the span of all forms on $E^{(\infty)}$ of the form $f\, dx^{\mu_1} \cdots dx^{\mu_p}\, d^v u^{A_1}_{\underline{\sigma}_1} \cdots d^v u^{A_q}_{\underline{\sigma}_q}$ (with no further restrictions).

Note:

$$d^h : \Omega^{(p,q)}(E^{(n)}) \longrightarrow \Omega^{(p+1,q)}(E^{(n+1)})$$
$$d^v : \Omega^{(p,q)}(E^{(n)}) \longrightarrow \Omega^{(p,q+1)}(E^{(n+1)})$$

e) Useful formulas

We define for every generalized vector field $\tilde{Z} = \tilde{X}^\mu \partial_{x^\mu} + \tilde{Y}^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}$

the graded derivation

$$\imath_{\tilde{Z}} : \Omega^\bullet(E^{(\infty)}) \to \Omega^{\bullet-1}(E^{(\infty)})$$

of degree -1 by:

$\imath_{\tilde{Z}} f = 0 \quad (\forall f \in C^\infty(E^{(\infty)}))$,

$\imath_{\tilde{Z}} dx^\mu = \tilde{X}^\mu$ and $\imath_{\tilde{Z}} du^A_{\underline{\sigma}} = \tilde{Y}^A_{\underline{\sigma}}$

We also set

$$L_{\tilde{Z}} := d\imath_{\tilde{Z}} + \imath_{\tilde{Z}} d$$

which is a derivation of $\Omega^\bullet(E^{(\infty)})$ of degree 0.

<u>Theorem A.</u> $d^v \imath_{\mathcal{D}_{x^\mu}} = -\imath_{\mathcal{D}_{x^\mu}} d^v \quad (\forall \mu)$

<u>Proof</u> We have that

$$\imath_{\mathcal{D}_{x^\mu}} d^v u^A_{\underline{\sigma}} = \imath_{\mathcal{D}_{x^\mu}} \left(du^A_{\underline{\sigma}} - u^A_{\underline{\sigma}\mu} dx^\mu \right)$$
$$= u^A_{\underline{\sigma}\mu} - u^A_{\underline{\sigma}\mu} = 0.$$

It follows that

$$d^{v}\iota_{\mathcal{D}_{x^\mu}} f = 0 = -\iota_{\mathcal{D}_{x^\mu}} d^{v} f \qquad (\forall f \in C^{\infty}(E^{(\infty)})),$$

$$d^{v}\iota_{\mathcal{D}_{x^\mu}} (dx^{\nu}) = 0 = -\iota_{\mathcal{D}_{x^\mu}} d^{v}(dx^{\nu})$$

$$d^{v}\iota_{\mathcal{D}_{x^\mu}} (d^{v}u^{A}_{\underline{\sigma}}) = 0 = -\iota_{\mathcal{D}_{x^\mu}} d^{v}(d^{v}u^{A}_{\underline{\sigma}})$$

Hence, the graded derivation $d^{v}\iota_{\mathcal{D}_{x^\mu}} + \iota_{\mathcal{D}_{x^\mu}} d^{v}$ of $\Omega^{\bullet}(E^{(\infty)})$ (of degree 0) is equal to 0. $\square$

<u>Corollary A</u> $d^{v} L_{\mathcal{D}_{x^\mu}} = -L_{\mathcal{D}_{x^\mu}} d^{v} \quad (\forall \mu)$

This is since $L_{\mathcal{D}_{x^\mu}} = [d, \iota_{\mathcal{D}_{x^\mu}}]$ and $[d^{v}, \iota_{\mathcal{D}_{x^\mu}}] = 0 = [d^{v}, d]$ (graded commutators), and then by the graded Jacobi identity one gets:

$$[d^{v}, L_{\mathcal{D}_{x^\mu}}] = [d^{v}, [d, \iota_{\mathcal{D}_{x^\mu}}]]$$

$$= [[d^{v}, d], \iota_{\mathcal{D}_{x^\mu}}] - [d, [d^{v}, \iota_{\mathcal{D}_{x^\mu}}]]$$

$$= 0 \qquad \square$$

__Theorem B.__ Let $\tilde{V} := \tilde{V}^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}$

be a vertical generalized vector field.

The following conditions are equivalent

(a) $\tilde{V}$ is horizontally constant, i.e.,

$$\left[\mathcal{D}_{x^\mu}, \tilde{V}\right] = 0 \quad (\forall \mu = 1, \ldots, D)$$

(or, $\tilde{V}$ is an evolutionary vector field).

(b) $d^h \imath_{\tilde{V}} = - \imath_{\tilde{V}} d^h$

(c) $d^h L_{\tilde{V}} = L_{\tilde{V}} d^h$

Proof Recall that $\tilde{V}$ is horizontally constant iff $\mathcal{D}_{x^\mu}\left(\tilde{V}^A_{\underline{\sigma}}\right) - \tilde{V}^A_{\underline{\sigma}\mu} = 0$ for all $A, \mu, \underline{\sigma}$ (acording to the theorem in Sect. 7).

To prove that (a) $\Rightarrow$ (b) note that both terms $d^h \imath_{\tilde{V}}$ and $\imath_{\tilde{V}} d^h$ vanish on $f \in C^\infty(E^{(\infty)})$ as well as on all dx^μ

Vanishing of $[d^h, \imath_{\tilde{V}}]$ on $du^A_{\underline{\sigma}}$ follows from the first identity below:

$$\left(d^h \iota_{\tilde V} + \iota_{\tilde V} d^h\right)\left(du^A_{\underline{\sigma}}\right)$$

$$= \left(\mathcal{D}_{x^\mu}\left(\tilde V^A_{\underline{\sigma}}\right) - \tilde V^A_{\underline{\sigma}\mu}\right) dx^\mu$$

$$= \left(d^h L_{\tilde V} - L_{\tilde V} d^h\right)\left(u^A_{\underline{\sigma}}\right)$$

It also implies (b) => (a) and the second identity above implies that (c) => (a).

Finally, (b) => (c) follows from the the graded Jacobi identity, as in the previous corollary. □

Let us introduce for a vertical generalized vector field $\tilde V := \tilde V^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}$ its vertical Lie derivative

$$L^v_{\tilde V} := \left[d^v, \iota_{\tilde V}\right] \equiv d^v \iota_{\tilde V} + \iota_{\tilde V} d^v$$

<u>Corollary B</u> If $\tilde V := \tilde V^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}$ is horizontally constant then

$$L^v_{\tilde V} \equiv L_{\tilde V}$$

To prove this recall that $d = d^h + d^v$ and $L_{\tilde{V}} := [d, \imath_{\tilde{V}}]$. On the other hand, $[d^h, \imath_{\tilde{V}}] = 0$ (by Theorem B).

Note that $\quad L^v_{\partial_{u^A_\sigma}} d^v u^B_\rho$

$$= \left(d^v \imath_{\partial_{u^A_\sigma}} + \imath_{\partial_{u^A_\sigma}} d^v\right) d^v u^B_\rho$$

$$= d^v \imath_{\partial_{u^A_\sigma}} d^v u^B_\rho = d^v\left(\delta^B_A \delta^\sigma_\rho\right) = 0$$

However, $L_{\partial_{u^A_\sigma}} d^v u^B_\rho \neq 0$ in some cases

Also, $\quad L^v_{\partial_{u^A_\sigma}} dx^\mu =$

$$= \left(d^v \imath_{\partial_{u^A_\sigma}} + \imath_{\partial_{u^A_\sigma}} d^v\right) dx^\mu$$

$$= \imath_{\partial_{u^A_\sigma}} d^v dx^\mu = 0.$$

$$L^v_{\partial_{u^A_\sigma}} f = \partial_{u^A_\sigma} f.$$

It follows that

$$d^v = L^v_{\partial_{u^A_{\underline{\sigma}}}} \circ d^v u^A_{\underline{\sigma}} = d^v u^A_{\underline{\sigma}} \circ L^v_{\partial_{u^A_{\underline{\sigma}}}}$$

where $d^v u^A_{\underline{\sigma}}$ are considered as operators of multiplication by $d^v u^A_{\underline{\sigma}}$ on $\Omega^\bullet(E^{(\infty)})$

Indeed, one first verifies that the right hand sides are graded derivations of $\Omega^\bullet(E^{(\infty)})$ of degree 1 and then evaluates both sides on the generators $f \in C^\infty(E^{(\infty)})$, dx^μ and $d^v u^A_{\underline{\sigma}}$.

Note that on an element $\gamma \in \Omega^\bullet(E^{(\infty)})$ the above identity reads as

$$\begin{aligned} d^v \gamma &= L^v_{\partial_{u^A_{\underline{\sigma}}}}\left(\left(d^v u^A_{\underline{\sigma}}\right)\gamma\right) \\ &= \left(d^v u^A_{\underline{\sigma}}\right) L^v_{\partial_{u^A_{\underline{\sigma}}}} \gamma \end{aligned}$$

f) Coordinate independence

Note that the exterior differential on $\Omega^\bullet(E)$, as well as, on $\Omega^\bullet(E^{(\infty)})$ is invariant under any any diffeomorphism on E (or, a local flow, or, a smooth map), i.e., if $\mathcal{H} = (\mathcal{F}, G) : E \to E$ is a diffeomorphism (or, a local flow, or, a smooth map), then

$$\mathcal{H}^* \circ d = d \circ \mathcal{H}^*$$

(this means, in particular, an independence of d from any coordinate transformation).

However, d^h and d^v are not invariant under arbitrary maps $\mathcal{H} : E \to E$, as they keep track of the vertical structure of E (i.e, the structure of a fiber bundle) The maps $\mathcal{H} : E \to E$ that respect the the later structure are the vertically consistent maps.

One needs however some precaution
Since d^h and d^v are defined on the infinite jet spaces, i.e., on $\Omega^\bullet(E^{(\infty)})$ one needs to extend $\mathcal{H}^*$ on $\Omega^\bullet(E^{(\infty)})$ according to its prolongations.

$$\mathcal{H}^* f = f \circ \mathcal{H}^{(n)} \quad \text{for } f \in C^\infty(E^{(n)})$$

i.e., $\mathcal{H}^* f\,(x, u^{(n)})$
$$= f\left(\mathcal{F}(x), G^{(n)}(x, u^{(n)})\right)$$

where $G^{(n)}$ is the n-th prolongation, as calculated in Sect. 5.b.

$$\mathcal{H}^* dx^\mu = d\mathcal{F}^\mu(x)$$

$$\mathcal{H}^* du^A_{\underline{\sigma}} = dG^A_{\underline{\sigma}}(x, u^{(s)}) \quad \text{for } s = |\underline{\sigma}|$$

<u>Theorem C</u>. Let $\mathcal{H} = (\mathcal{F}, G) : E \to E$ be a vertically consistent map (or, a local flow). Then:

$$\mathcal{H}^* \circ d^h = d^h \circ \mathcal{H}^*$$

$$\mathcal{H}^* \circ d^v = d^v \circ \mathcal{H}^*$$

Proof. It is enough to prove one of the above identities as $\mathcal{H}^*$ always commutes with d and $d = d^h + d^v$.

Note next that
$$A := \mathcal{H}^* \circ d^h$$
$$B := d^h \circ \mathcal{H}^*$$
are graded $\mathcal{H}^*$-derivations in the sense

$$A(\alpha\beta) = A(\alpha)\,\mathcal{H}^*(\beta) + (-1)^{|\alpha|}\,\mathcal{H}^*(\alpha)\,A(\beta)$$

$$B(\alpha\beta) = B(\alpha)\,\mathcal{H}^*(\beta) + (-1)^{|\alpha|}\,\mathcal{H}^*(\alpha)\,B(\beta)$$

Hence, in order to prove that $A = B$ one needs to check this only on the generators of the algebra $\Omega^\bullet(E^{(\infty)})$, i.e., on f and df for $f \in C^\infty(E^{(\infty)})$. Furthermore, it is even enough to check for $f \in C^\infty(E^{(\infty)})$ as then:

$$\mathcal{H}^* d^h df \overset{(-)}{=} -d\,\mathcal{H}^* d^h f$$

$$\overset{(-)}{=} -d\,d^h \mathcal{H}^* f = d^h \mathcal{H}^* df$$

Now, $\mathcal{H}^* d^h f = \mathcal{H}^*(\mathcal{D}_{x^\mu} f \, dx^\mu)$

$$= \mathcal{H}^*(\mathcal{D}_{x^\mu} f)\,\mathcal{H}^*(dx^\mu)$$

$$= (\mathcal{D}_{x^\mu} f)\left(\mathcal{F}(x), G^{(n)}(x, u^{(n)})\right)$$

$$\times \frac{\partial \mathcal{F}^\mu(x)}{\partial x^\nu}\, dx^\nu$$, and on the other hand

$$d^h \mathcal{H}^* f = \mathcal{D}_{x^\nu}(\mathcal{H}^* f)\, dx^\nu$$

$$= \mathcal{D}_{x^\nu}\left(f\left(\mathcal{F}(x), G^{(n)}(x, u^{(n)})\right)\right) dx^\nu$$

Hence, the proof is completed by noticing the chane rule for the total derivatives

$$\mathcal{D}_{x^\nu}\left(f\left(\mathcal{F}(x), G^{(n)}(x, u^{(n)})\right)\right)$$

$$= \frac{\partial \mathcal{F}^\mu(x)}{\partial x^\nu}\,(\mathcal{D}_{x^\mu} f)\left(\mathcal{F}(x), G^{(n)}(x, u^{(n)})\right)$$

$\square$

Similarly to the previous theore one proves

<u>Corollary</u> For every section

$$\begin{array}{ccc} \hat{\varphi} : M & \to & E = M \times F \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \mapsto & \overbrace{(x, \varphi(x))} \end{array}$$

of $\xi = (E, p : E \to M, M)$ $(E = M \times F)$ one has

$$\left(\hat{\varphi}^{(\infty)}\right)^* \circ d^h = d \circ \left(\hat{\varphi}^{(\infty)}\right)^*$$

The latter identity has a simple geometric meaning: as $\hat{\varphi}^{(n)}$ embeds M into $E^{(n)}$ as the graph of the prolongated fields $\varphi^{(n)}$ then any horizontal differential $d^h \gamma$ is restricted to the graph as the ordinary differential on M. Hence, the vertical differential has always a zero restriction to each graph.

9. The main identity of variational calculus

a) The main identity

Let us denote $\Omega_0^{(p,q)}(E^{(n)})$ = the span of all forms on $E^{(n)}$ of the form

$$f\, dx^{\mu_1} \cdots dx^{\mu_p} \underbrace{d^v u^{A_1} \cdots d^v u^{A_q}}_{\text{no jet multiindices!}}$$

In particular, $\Omega_0^{(p,0)}(E^{(n)}) \equiv \Omega^{(p,0)}(E^{(n)})$

<u>Theorem</u> There exists a unique linear map

note: a top degree

$$\delta : \Omega_0^{(D,q)}(E^{(\infty)}) \longrightarrow \Omega_0^{(D,q+1)}(E^{(\infty)})$$

defined for every $q = 0, 1, \ldots$, with the property that for $\forall \gamma \in \Omega_0^{(D,q)}(E^{(\infty)})$

$$\boxed{d\gamma \equiv d^v \gamma = \delta \gamma + d^h \mathcal{N} \gamma}$$

where $\mathcal{N}$ is a linear map:

$$\mathcal{N} : \Omega_0^{(D,q)}(E^{(\infty)}) \longrightarrow \Omega_{\cancel{0}}^{(D-1,q+1)}(E^{(\infty)})$$

In fact,

$$\delta : \Omega_0^{(D,q)}(E^{(n)}) \longrightarrow \Omega_0^{(D,q+1)}(E^{(2n)})$$

Proof The niqueness follows by the following

Lemma Let $\gamma_0, \gamma_0' \in \Omega_0^{(p,q+1)}(E^{(\infty)})$ be such that

$$\gamma_0 + d^h \gamma_1 = \gamma_0' + d^h \gamma_1'$$

Then $\gamma_0 = \gamma_0'$

To prove the lemma we need to show that if $\gamma_0 - \gamma_0' \in \Omega_0^{(p,q+1)}(E^{(\infty)})$ and

$$\gamma_0 - \gamma_0' = d^h(\gamma_1' - \gamma_1)$$

then $\gamma_0 - \gamma_0' = 0$. Indeed, we observe that if $d^h(\gamma_1' - \gamma_1) \neq 0$ then it will necessarily contain a term with $d^v u^A_{\underline{\sigma}}$ for $|\underline{\sigma}| \geq 1$ and there is no way to create only $d^v u^A$ since

$$d^h d^v u^A_{\underline{\rho}} = dx^\mu d^v u^A_{\underline{\rho}\mu}$$

We continue with the existence in the theorem and we start with the identity for

$$\gamma \in \Omega_0^{(D,q)}(E^{(\infty)})$$

$$d^{v}\gamma = (d^{v}u^{A}_{\underline{\sigma}})\, L^{v}_{\partial_{u^{A}_{\underline{\sigma}}}}\gamma$$

$$= (L_{\mathcal{D}})_{\underline{\sigma}}(d^{v}u^{A})\, L^{v}_{\partial_{u^{A}_{\underline{\sigma}}}}\gamma$$

where $(L_{\mathcal{D}})_{\underline{\sigma}} := L_{\mathcal{D}_{x^{\sigma_1}}}\cdots L_{\mathcal{D}_{x^{\sigma_k}}}$

we shall show that

$$\delta\gamma = (d^{v}u^{A})\Big((-L_{\mathcal{D}})_{\underline{\sigma}}\, L^{v}_{\partial_{u^{A}_{\underline{\sigma}}}}\gamma\Big)$$

where $(-L_{\mathcal{D}})_{\underline{\sigma}} := (-L_{\mathcal{D}_{x^{\sigma_1}}})\cdots(-L_{\mathcal{D}_{x^{\sigma_k}}})$

To this end we pass to nonsymmetric multiindices $\underline{r} = \underline{\sigma}$ and use the formula for "integration by parts" of Sect.

$$d^{v}\gamma = \sum_{\underline{r}}(L_{\mathcal{D}})_{\underline{r}}(d^{v}u^{A})\, L^{v}_{\partial_{u^{A}_{\underline{r}}}}\gamma$$

where $(L_{\mathcal{D}})_{\underline{r}} := (L_{\mathcal{D}_{x^1}})^{r_1}\cdots(L_{\mathcal{D}_{x^D}})^{r_D}$

and we passed to an explicit notation for the summation in $\underline{r}$ and continue to keep the convention for summation for the repeating index A. Then:

$$d^v \mathscr{L} = \sum_{\underline{r}} (L_{\mathcal{D}})_{\underline{r}} \, (d^v u^A) \, L^v_{\partial_{u^A_{\underline{r}}}} \mathscr{L}$$

$$= \sum_{\underline{r} \geqslant 0} \; \sum_{0 \leqslant \underline{s} \leqslant \underline{r}} \binom{\underline{r}}{\underline{s}}$$

$$\times (L_{\mathcal{D}})_{\underline{s}} \Big((d^v u^A) (-L_{\mathcal{D}})_{\underline{r}-\underline{s}} \, L^v_{\partial_{u^A_{\underline{r}}}} \mathscr{L} \Big)$$

$$= \sum_{\underline{s} \geqslant 0} \; \sum_{\underline{t} \geqslant 0} \binom{\underline{s}+\underline{t}}{\underline{s}}$$

$$\times (L_{\mathcal{D}})_{\underline{s}} \Big((d^v u^A) (-L_{\mathcal{D}})_{\underline{t}} \, L^v_{\partial_{u^A_{\underline{s}+\underline{t}}}} \mathscr{L} \Big)$$

The leading term in the above sum, at $\underline{s} = 0$, gives the expression for the δ-operator.

$$\delta \mathscr{L} = (d^v u^A) (-L_{\mathcal{D}})_{\underline{t}} \, L^v_{\partial_{u^A_{\underline{t}}}} \mathscr{L}$$

(where $\sum_{\underline{t} \geqslant 0}$ is assumed).

It remains to prove tha

$$d^v \gamma - \delta \gamma$$

$$= \sum_{\underline{s} \gneq 0} \sum_{\underline{t} \geq 0} \binom{\underline{s}+\underline{t}}{\underline{s}}$$

$$\times (L_{\mathcal{D}})_{\underline{s}} \left((d^v u^A) (-L_{\mathcal{D}})_{\underline{t}} L^v_{\partial u^A_{\underline{s}+\underline{t}}} \gamma \right)$$

has a form $d^h \gamma'$ for some $\gamma' = N\gamma$ linearly depending on γ.

This is because

1) The expression after the sums in the above equations, as well as, the expression after $(L_{\mathcal{D}})_{\underline{s}}$ belong to $\Omega^{(D, q+1)}(E^{(\infty)})$. This follows by the fact that $L_{\mathcal{D}_{x^\mu}}$ as well as $L^v_{\partial u^A_{\underline{r}}}$ do not decrease the horizontal degree (cf. the calculations in Sect. 8).

2) Since on each term in the above sum one has applied a product $(L_{\mathcal{D}})_{\underline{s}}$ that includes at least one $L_{\mathcal{D}_{x^\mu}}$ (as $\underline{s} \gneq 0$)

from one hand, and on the other hand,

$$L_{\mathcal{D}_{x^\mu}} = d\,\iota_{\mathcal{D}_{x^\mu}} + \iota_{\mathcal{D}_{x^\mu}}\,d$$
$$= d^h\,\iota_{\mathcal{D}_{x^\mu}} + \iota_{\mathcal{D}_{x^\mu}}\,d^h$$

(because of Theorem A of Sect. 8 and $d = d^h + d^v$), then by 1) it follows that only the term $d^h \iota_{\mathcal{D}_{x^\mu}}$ will survive (as $\iota_{\mathcal{D}_{x^\mu}} d^h$ will act on a form with a maximal horizontal degree).

Thus, the expression for $d^v\gamma - \delta\gamma$ is indeed of a form $d^h\gamma'$.

Using the above observations one can also find an expression for γ', which linearly depends on γ.

b) The variatianal differential and the pre-Noether current.

We call the operators

$$\delta : \Omega_0^{(D,\bullet)}(E^{(\infty)}) \longrightarrow \Omega_0^{(D,\bullet+1)}(E^{(\infty)})$$

and

$$\mathcal{N} : \Omega_0^{(D,q)}(E^{(\infty)}) \longrightarrow \Omega_{\cancel{0}}^{(D-1,q+1)}(E^{(\infty)})$$

constructed by the above theorem, variational differential and pre-Noether current. Since δ is uniquely fixed by the "main identity of the variational calculus",

$$d\gamma \equiv d^v\gamma = \delta\gamma + d^h \mathcal{N}\gamma$$

it follows that:

<u>Corollary</u> a) $\delta^2 = 0$

b) If $\mathcal{H} = (\mathcal{F}, G) : E \to E$ is a vertically consitent map (flow) then $\mathcal{H}^* \circ \delta = \delta \circ \mathcal{H}^*$

Proof a) Let $\gamma_1 = d\gamma$. We have

$$0 = d\gamma_1 = \delta\gamma_1 + d^h\mathcal{N}\gamma_1$$

$$\gamma_1 = \delta\gamma + d^h\mathcal{N}\gamma$$

Hence

$$\begin{aligned}
0 &= d\delta\gamma + dd^h\mathcal{N}\gamma \\
&= \delta^2\gamma + d^h\mathcal{N}\delta\gamma - d^h d\mathcal{N}\gamma \\
&= \delta^2\gamma + d^h\gamma_2
\end{aligned}$$

By the uniqueness: $\delta^2\gamma = 0$

b)

$$\begin{cases} d\mathcal{H}^*\gamma = \delta\mathcal{H}^*\gamma + d^h\gamma' \\ \mathcal{H}^*d\gamma = \mathcal{H}^*\delta\gamma + \underbrace{\mathcal{H}^*d^h}_{d^h\mathcal{H}^*}\gamma'' \end{cases}$$

$$\Rightarrow \underbrace{\mathcal{H}^*\delta\gamma}_{\in \, \Omega_0^{(D,0)}(E^{(\infty)})} - \underbrace{\delta\mathcal{H}^*\gamma}_{\in \, \Omega_0^{(D,0)}(E^{(\infty)})} = \underbrace{d^h\gamma'''}_{=\,0}$$

by the lemma

Why?

by construction

Why $\mathcal{H}^*$ maps $\Omega_0^{(D,0)}(E^{(\infty)})$ to itself?

- because: $\mathcal{H}^* dx^\mu =$

$$\frac{\partial \mathcal{F}^\mu}{\partial x^\nu} dx^\nu + \underbrace{\frac{\partial \mathcal{F}^\mu}{\partial u^A}}_{0} du^A$$

by the vertical consistency

Hence $\mathcal{H}^*$ does not decrease the horizontal degree - but in $\Omega_0^{(D,0)}(E^{(\infty)})$ it is maximal. $\square$

<u>Applycations</u> 1)

10. The variational principle

a) Local actions

let: $\Lambda \in \Omega_0^{(D,0)}(E^{(n)})$, i.e.,

$$\Lambda = \mathcal{L}(x, u^{(n)}) \underbrace{d^D x}_{dx^1 \cdots dx^D}$$

and for $\varphi : M \to F$ - a field, and
$U \subseteq M$ - a bound (precompact) open set

let: $$S_U\{\varphi\} = \int_U (\hat{\varphi}^{(n)})^* \Lambda$$

where we remind that

$$\hat{\varphi}^{(n)} : M \to E^{(n)}$$
$$x \mapsto (x, \underbrace{\varphi^{(n)}(x)}_{(\varphi(x),\, \partial_{x^\mu}\varphi(x),\, \ldots,\, \partial_{x^{\mu_1}} \cdots \partial_{x^{\mu_n}} \varphi(x))})$$

and hence:

$$(\hat{\varphi}^{(n)})^* \Lambda = \mathcal{L}(x, \varphi^{(n)}(x))\, d^D x$$

Here: S is called local action
Λ is called Lagrangean form.

Note: the bounded (i.e., precompact) set U is needed to ensure that the action is finite

b) The variational problem

Having a local action

$$S_U\{\psi\} = \int_U \left(\widehat{\psi}^{(n)}\right)^* \Lambda$$

we look for fields $\psi : M \to F$, which make the "action <u>stationary</u>" in the following sense

For every

- bound (or, precompact) open set $U \subseteq M$
- and for every fields' variation $\psi_{(\varepsilon)}$ of ψ (i.e, $\psi_{(\varepsilon=0)}(x) = \psi(x)$, cf Sect. 2c), which vanish in a neighbourhood of the bouundary ∂U, i.e., such that $\exists\, O$ - a neighbourhood of ∂U for which:

$$\psi_{(\varepsilon)}(x) = \psi(x) \quad \text{for } \forall\, x \in O.$$

we riquire that:

$$\frac{d}{d\varepsilon} S_U\{\psi_\varepsilon\}\Big|_{\varepsilon=0} = 0$$

c) The Euler-Lagrange equations

<u>Theorem</u> Let

$$S_U\{\psi\} = \int_U \left(\widehat{\varphi}^{(n)}\right)^* \Lambda$$

be a local action determined by Λ

Then the fields $\psi : M \to F$ make the action stationary if they satisfy

$$\delta\Lambda\left(x, \psi^{(2n)}(x)\right) = 0$$

These are the Euler-Lagrange equations.

<u>Proof</u>. As we explained in Sect. 2c, there exists a vertical flow on E:

$$\mathcal{H}_{(\varepsilon)} : (x,u) \mapsto \left(x, G_{(\varepsilon)}(x,u)\right)$$

such that $\widehat{\varphi}_{(\varepsilon)} = \mathcal{H}_{(\varepsilon)} \circ \widehat{\varphi}$. Then:

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S_U\{\psi_{(\varepsilon)}\}$$

$$= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_U \left(\hat{\varphi}^{(n)}\right)^* \mathcal{H}^*_{(\varepsilon)} \Lambda$$

$$= \int_U \left(\hat{\varphi}^{(n)}\right)^* \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{H}^*_{(\varepsilon)} \Lambda$$

$$= \int_U \left(\hat{\varphi}^{(n)}\right)^* L_{\tilde{V}} \Lambda$$

where $\tilde{V} = \tilde{V}^A_{\underline{\sigma}}(x, u^{(n)})\, \partial_{u^A_{\underline{\sigma}}}$ is the prolongation of the vertical field

$$V = V^A(x, u)\, \partial_{u^A}$$

that generates the flow $\mathcal{H}_{(\varepsilon)}$. Then:

$$L_{\tilde{V}} \Lambda = \iota_{\tilde{V}} d\Lambda + d \underbrace{\iota_{\tilde{V}} \Lambda}_{0}$$

$$= \iota_{\tilde{V}} \left(\delta\Lambda + d^h \Lambda_1\right)$$

$$= \iota_V \delta\Lambda + d^h \iota_{\tilde{V}} \Lambda_1$$

$$0 = \int_U \left(\widehat{\varphi}^{(n)}\right)^* \iota_V \, \delta\Lambda$$

$$+ \int_U \left(\widehat{\varphi}^{(n)}\right)^* d^h \iota_{\widetilde{V}} \Lambda_1$$

$$= \int_U V^A(x, \varphi(x))$$

$$\times (\delta\Lambda)_A \left(x, \varphi^{(2n)}(x)\right)$$

$$+ \int_U d\left(\widehat{\varphi}^{(n)}\right)^* \iota_{\widetilde{V}} \Lambda_1$$

where $\delta\Lambda = \underbrace{(\delta\Lambda)_A}_{\in \, \Omega_0^{(D,0)}(E^{(n)})} du^A$

The second integral vanishes by the Stokes theorem as $\widetilde{V}$ vanishes in a neighbourhood of U. Thus, $(\delta\Lambda)_A \left(x, \varphi^{(2n)}(x)\right) = 0$

as all their integrals against a compactly supported functions on M vanish. $\square$

11. The first Noether's theorem

a) The stronget notion of invariance of a local action and its infinitesimal expression

The most popular version of the first Noether's theorem is formulated under the most restricted conditions. Let

$$\mathcal{H}_{(\varepsilon)} : (x, u) \mapsto \left(\mathcal{F}_{(\varepsilon)}(x), G_{(\varepsilon)}(x, u) \right)$$

be a vertically consistent flow generated by a vector field (cf. Sect. 2)

$$Z = X^{\mu}(x)\, \partial_{x^{\mu}} + Y^{A}(x, u)\, \partial_{u^{A}}$$

Let $\varphi : M \to F$ be a field (system) and let $\varphi_{(\varepsilon)}$ be transformed fields.

We say that a local action

$$S_{U}\{\varphi\} = \int_{U} \left(\hat{\varphi}^{(n)} \right)^{*} \Lambda$$

is invariant under the flow $\mathcal{H}_{(\varepsilon)}$ iff

$$S_{U_{\varepsilon}}\{\varphi_{(\varepsilon)}\} = \text{const}(\varepsilon) \text{ for every}$$

- $U \subseteq M$ - bound open subset,
- $\varphi : M \to F$ - fields on M; where

$$U_\varepsilon := \mathcal{F}_{(\varepsilon)}(U)$$

Example Let $M = \mathbb{R} = F$

$$\Lambda = (u_1^2 - u^2)\, dx \ , \text{ i.e.,}$$

$$\psi^*\Lambda = \left((\partial_x \psi(x))^2 - \psi(x)^2\right) dx$$

Then for $U = (a, b)$

$$S_U\{\psi\} = \int_a^b \left((\partial_x \psi(x))^2 - \psi(x)^2\right) dx$$

Now if $Z = \partial_x$, then

$$\mathcal{F}_{(\varepsilon)}(x) = x + \varepsilon \ , \ G_{(\varepsilon)}(x, u) = u \ ,$$

$$\psi_{(\varepsilon)}(x) = \psi(x - \varepsilon) \ ,$$

$$U_\varepsilon = (a + \varepsilon \ , \ b + \varepsilon) \ ,$$

and the invariance condition reads

$$\int_a^b \mathcal{L}(x)\, dx = \int_{a+\varepsilon}^{b+\varepsilon} \mathcal{L}(x - \varepsilon)\, dx$$

where $\mathcal{L}(x) := (\partial_x \psi(x))^2 - \psi(x)^2$

Note that sinse

$$\widehat{\varphi}_{(\varepsilon)} = \mathcal{H}_{(\varepsilon)} \circ \widehat{\varphi} \circ \mathcal{F}_{(\varepsilon)}^{-1}$$

it follows then that

$$\mathcal{S}_{U_\varepsilon}\{\Psi_{(\varepsilon)}\}$$

$$= \int_{U_\varepsilon} \left(\mathcal{F}_{(\varepsilon)}^{-1}\right)^* \left(\widehat{\varphi}^{(n)}\right)^* \left(\mathcal{H}_{(\varepsilon)}^{(n)}\right)^* \Lambda$$

these compesate each other

$$= \int_{U} \left(\widehat{\varphi}^{(n)}\right)^* \left(\mathcal{H}_{(\varepsilon)}^{(n)}\right)^* \Lambda$$

Hence, the above invariance condition reads

$$0 = \frac{d}{d\varepsilon} \mathcal{S}_U\{\Psi_{(\varepsilon)}\} = \int_U \left(\widehat{\varphi}_{(\varepsilon)}^{(n)}\right)^* L_{\tilde{Z}} \Lambda$$

where $\tilde{Z}$ is the jet prolongation of Z. Since the latter equality is true for all U and Ψ we finally get the following equivalent infinitesimal condition

$$L_{\tilde{Z}} \Lambda = 0$$

b) The Noether theorem in the weakest form

<u>Theorem</u> Assume that the local action

$$S_U\{\varphi\} = \int_U \left(\hat{\varphi}^{(n)}\right)^* \Lambda$$

is invariant under a flow generated by a vertically consistent vector field on E

$$Z = X^\mu(x)\,\partial_{x^\mu} + Y^A(x,u)\,\partial_{u^A}$$

Then, for every solution $\varphi : M \to F$ of the Euler-Lagrange equations we have a closed $(D-1)$-form on M:

$$j = \left(\hat{\varphi}^{(2n-1)}\right)^* J$$

where $J \in \Omega^{(D-1,0)}\left(E^{(2n-1)}\right)$ is defined by

$$J := \iota_{\tilde{Z}^v} N - \iota_{\tilde{Z}^h} \Lambda$$

and $\tilde{Z}$ is the jet prolongation of Z with its horizontal-vertical decomposition

$$\tilde{Z} = \tilde{Z}^h + \tilde{Z}^v$$

<u>Lemma</u> The infinitesimal invariance law

$$L_{\tilde{Z}} \Lambda = 0 \text{ is equivalent to}$$

$$L_{\tilde{Z}^v} \Lambda = - d^h \iota_{\tilde{Z}^h} \Lambda$$

Proof $L_{\tilde{Z}} = L_{\tilde{Z}^h} + L_{\tilde{Z}^v}$ and

Next, since $\tilde{Z}^h = X^\mu(x) \mathcal{D}_{x^\mu}$ then

$$L_{\tilde{Z}^h} = X^\mu(x) L_{\mathcal{D}_{x^\mu}} + \left(d X^\mu(x)\right) \iota_{\mathcal{D}_{x^\mu}}$$

$$= X^\mu(x) \left(d \iota_{\mathcal{D}_{x^\mu}} + \iota_{\mathcal{D}_{x^\mu}} d\right)$$

$$+ \left(d^h X^\mu(x)\right) \iota_{\mathcal{D}_{x^\mu}}$$

$$= X^\mu(x) \left(d^h \iota_{\mathcal{D}_{x^\mu}} + \iota_{\mathcal{D}_{x^\mu}} d^h\right)$$

$$+ \left(d^h X^\mu(x)\right) \iota_{\mathcal{D}_{x^\mu}}$$

$$= d^h \iota_{\tilde{Z}^h} + \iota_{\tilde{Z}^h} d^h$$

Then, since $d^h \Lambda = 0$ we obtain

$$L_{\tilde{Z}} \Lambda = L_{\tilde{Z}^v} \Lambda + d^h \iota_{\tilde{Z}^h} \Lambda$$

We continue with the proof of the theorem.

$$0 = L_{\tilde{Z}^v} \Lambda + d^h \iota_{\tilde{Z}^h} \Lambda$$

$$= \underbrace{d \iota_{\tilde{Z}^v} \Lambda}_{0} + \iota_{\tilde{Z}^v} d\Lambda + d^h \iota_{\tilde{Z}^h} \Lambda$$

since Λ has 0 vertical degree

$$= \iota_{\tilde{Z}^v} \delta\Lambda + \underbrace{\iota_{\tilde{Z}^v}}_{(-)} d^h \mathcal{N}\Lambda + d^h \iota_{\tilde{Z}^h} \Lambda$$

since $\tilde{Z}^v$ is vertically constant (cf. Sect. 8e)

Thus, $d^h \mathcal{J} = \iota_{\tilde{Z}^v} \delta\Lambda$ and hence,

for every fields $\varphi : M \to F$

$$d\left(\hat{\varphi}^{(2n-1)}\right)^* \mathcal{J} \equiv \left(\hat{\varphi}^{(2n)}\right)^* d^h \mathcal{J}$$

$$= \left(\hat{\varphi}^{(2n)}\right)^* \iota_{\tilde{Z}^v} \delta\Lambda$$

$$= \tilde{V}^A\left(x, \varphi^{(1)}(x)\right) \underbrace{(\delta\Lambda)_A\left(x, \varphi^{(2n)}(x)\right)}_{\text{Euler-Lagrange eqs.}}$$

where

$\tilde{V}^A := Y^A(x,u) - u^A_\mu X^\mu(x)$ is the characteristics of Z (cf. Sect. 7) $\square$

c) Generalizations

The presented proof of the Noether's theorem suggests an immediate generalization. Suppose we have an evolutionary vector field $\tilde{V} := \tilde{V}^A_{\underline{\sigma}} \partial_{u^A_{\underline{\sigma}}}$, i.e., such that

$$\tilde{V}^A_{\underline{\sigma}} = \mathcal{D}_{\underline{\sigma}}(\tilde{V}^A) \quad \text{(cf. Sect. 7)}$$

We call $\tilde{V}$ a genaralized symmetry field of the local action

$$S_U\{\varphi\} = \int_U (\hat{\varphi}^{(n)})^* \Lambda$$

determined by Λ iff

$$L_{\tilde{V}} \Lambda = d^h \gamma$$

for some $\gamma \in \Omega^{(D-1,0)}(E^{(\infty)})$

We shall also call the pair

$$(\tilde{V}, \gamma)$$

a symmetry pair for Λ.

Theorem Let $(\tilde{V}, \gamma)$ be a symmetry pair for Λ then for every solution of the Euler-Lagrange equations,

$$\varphi : M \to F,$$

we have a closed $(D-1)$-form on M:

$$j = \left(\hat{\varphi}^{(\infty)}\right)^* \mathcal{J}$$

where $\mathcal{J} \in \Omega^{(D-1,0)}\left(E^{(\infty)}\right)$ is defined by

$$\mathcal{J} := \iota_{\tilde{V}} \mathcal{N} + \gamma$$

Proof. We start again by the symmetry condition

$$\begin{aligned} 0 &= L_{\tilde{V}} \Lambda - d^h \gamma \\ &= \underbrace{d\,\iota_{\tilde{V}} \Lambda}_{0} + \iota_{\tilde{V}} d\Lambda - d^h \gamma \end{aligned}$$

since Λ has 0 vertical degree

$$= \iota_{\tilde{V}}\, \delta\Lambda + \underbrace{\iota_{\tilde{V}}}_{(-)} d^h \mathcal{N}\Lambda - d^h \gamma$$

since $\tilde{Z}^v$ is vertically constant (cf. Sect. 8e)

and proceed as before. $\square$

d) The symmetry Lie algebra

<u>Theorem</u> Let $(\tilde V, \gamma)$ and $(\tilde V', \gamma')$ be two symmetry pairs for Λ then

$$\left([\tilde V, \tilde V'] \,,\, L_{\tilde V}\gamma' - L_{\tilde V'}\gamma \right)$$

is again a symmetry pair of Λ.

The proof is straightforward

$$L_{[\tilde V, \tilde V']}\Lambda = L_{\tilde V} \underbrace{L_{\tilde V'}\Lambda}_{d^h \gamma'} - L_{\tilde V'} \underbrace{L_{\tilde V}\Lambda}_{d^h \gamma}$$

$$= d^h\left(L_{\tilde V}\gamma' - L_{\tilde V'}\gamma \right). \qquad \square$$

<u>Corollary</u> The commutator of two generalized symmetry fields is also a symmetry field.

e) The coordinate expression of the symmetry condition

Let $\Lambda = \mathcal{L}(x, u^{(n)})\, \underbrace{d^D x}_{dx^1 \dots dx^D}$

$$\gamma = \Gamma^\mu (x, u^{(m)})\, \underbrace{d^{D-1}_\mu x}_{(-1)^{\mu+1} dx^1 \dots \widehat{dx^\mu} \dots dx^D}$$

$\delta\Lambda = \mathcal{E}_A(\mathcal{L})\, du^A\, d^D x$, where

$$\mathcal{E}_A(\mathcal{L})(x, u^{(2n)}) = (-\mathcal{D})_{\underline{\sigma}}\, \partial_{u^A_{\underline{\sigma}}} \mathcal{L}$$

Then $L_{\tilde{V}} \Lambda = d^h \gamma$

$$\Longleftrightarrow \quad V^A(x, u^{(k)}) \cdot \mathcal{E}_A(\mathcal{L})(x, u^{(2n)}) = \mathcal{D}_{x^\mu} \Gamma^\mu (x, u^{(m)})$$