A New Version of Orthonormal Polynomial Expansion Method

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Abstract. The proposed version of our Orthonormal Polynomial Expansion Method presents the combination of two new features: including the errors in independent variable with the help of new total variance and using the usual coefficients for calculating the approximating values.

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INTRODUCTION

In our previous papers we have constructed the algorithm, named Orthonormal Polynomial Expansion Method (OPEM). This algorithm proposes a new description of function $y = f(x)$, given in arbitrary points $\{x_i\}$ in the interval $[a,b]$ as values $\{y_i\}$ with errors in every point $\{sy_i\}$ for $i = 1,2...,M$. This description is carried out in ortonormal polynomials, first generated by Forsythe [1]. All calculations for receiving orthonormal description are made in a new point set in the interval $[-1 \geq x \leq 1]$, by linear procedure from [a,b]. We have successfully generalized this construction with fourth term, giving us derivatives and integrals of functions in given points [2],[3],[4] and for many-dimensional case [5]. The principal relation for one-dimensional generation of orthonormal polynomials $\Psi_k^{(0)}, k = 1,2,\dots$ and their derivatives $\Psi_k^{(m)}, m = 1,2,\dots$, in OPEM is:

$$\begin{aligned}\Psi_{k+1}^{(m)}(x) &= \gamma_{k+1}[(x-\alpha_{k+1})\Psi_k^{(m)}(x) - \\ &\quad (1-\delta_{k0})\beta_k\Psi_{k-1}^{(m)}(x) + m\Psi_k^{(m-1)}(x)]. \quad (1)\end{aligned}$$

THE IMPROVEMENT OF THE OPEM

We generalize the method in two directions:
1) Including the errors in independent variables sx_i and defining a total variance:

$$S_i^2 = sy_i^2 + (dy/dx)^2 sx_i^2, i = 1,...M. \quad (2)$$

Here the Bevington's [6] proposal to combine both variable uncertainties and assign them to the dependent variable, is used. The procedure is carried in two stages. First stage is minimizing the normalized χ^2 (N is a polynomial degree)

$$\chi^2 = \sum_{i=1}^{M} w_i[f_i^{appr} - y_i]^2/(M-N-1). \quad (3)$$

using the usual weights $w_i = 1/sy_i^2$. The second stage is the minimization of χ^2 with the new weights $W_i = 1/S_i^2$. The result of the n-th iteration for optimal degree N is selected automatically by two criteria: the graph of the fitting curve lies in the big error corridor of dependent variable $[y + Sy_i, y - Sy_i]$ and the $min\chi^2$ is reached.
2)Receiving the ordinary (usual) coefficients from orthonormal ones using the relation between the orthonormal and usual descriptions of function:

$$f^{(m)appr}(x) = \sum_{j=0}^{k} a_j\Psi_j^{(m)}(x) = \sum_{j=0}^{k} c_j x'^j. \quad (4)$$

In the future, the prime sign is omitted for the sake of brevity in the text. The last feature is very important to present the results in given point interval.

Here we present in details the relation between orthogonal and ordinary coefficients, first giving in JINR Communication [2](in Russian). Let us write the polynomials Ψ_j in the ordinary basis:

$$\Psi_j = \sum_{i=0}^{j} c_i^{(j)} x^i, j = 0,...k. \quad (5)$$

Here the superscript indicates the polynomial order and evidently

$$c_j^{(j)} \neq 0, j = 0,1,2...k.$$

The substitution of above Eq.(5) in first relation gives for new coefficients $c_j^{(i)}, i = 1,2,...k-1; j = 0,1,...i$

$$c_j^{(i+1)} \quad = \quad 1/\beta_{i+1}[(1-\delta_{0,j})\alpha_{j-1}^{(i)} -$$

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$$(1-\delta_{i+1,j})\alpha_{i+1}\alpha_j^{(i)} -$$
$$(1-\delta_{i,j})(1-\delta_{i+1,j})\beta_i\alpha_j^{(i-1)}], \qquad (6)$$

and the first term is $c_0^{(0)} = 1/\beta_0$. After $c_j^{(i)}$ being computed, Eq.(5) can be substituted in Eq.(4)

$$\sum_{j=0}^{k} a_j \sum_{i=0}^{j} c_i^{(j)} x^i = \sum_{j=0}^{k} c_j x^j. \qquad (7)$$

Then we obtain the relation between orthonormal a_i and ordinary coefficients c_j

$$c_j = \sum_{i=j}^{k} a_i c_j^{(i)}, j = 0, 1...k. \qquad (8)$$

Finally, from Eq.(8) the accuracies δc_j can be expressed from δa_i

$$\delta c_j = [\sum_{i=j}^{k} (c_j^{(i)})^2]^{1/2} \delta a_i, j = 0, 1...k \qquad (9)$$

and the errors in orthonormal coefficients are equal, $\delta a_i = \theta$, (see [2]). The inherited-error equation is presented.

NUMERICAL RESULTS

Some interesting numerical results about receiving of usual coefficients without including errors in independent variable sx_i for cryogenic thermometry data are given in paper [4].

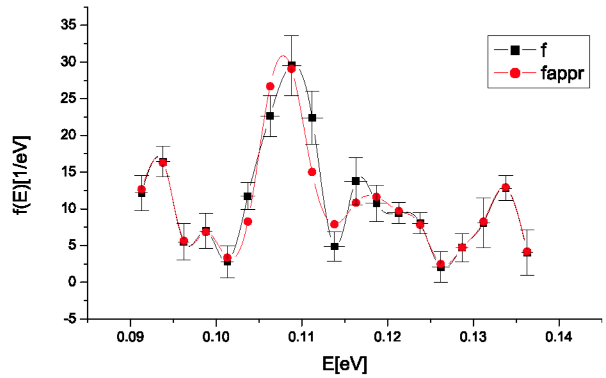
We present some new results for test example of quadratic function $y = c_2x^2 + c1x + c_0$ (M=5), from our previous paper by Bogdanova and Todorov [3]. Here the improvements in OPEM coefficients are given for 5-*th* iteration in Table 1. OPEM total variance is close to popular method but it is very simple and flexible.

The second example is from our previous paper [7] for experimental data (M=19,squares) of energy spectrum of the river water from Rila mountain. The approximation of the spectrum with $N = 17$ polynomial degree is presented there. Here the respective uncertainties sx_i, syi are denoted by horizontal and vertical bars in Fig. 1. Now we have received the new approximating curve(circles), with the help of selected usual coefficients for $N = 15$ order of polynomials and square root of normalized $\chi^2 = 1.$ with second iteration. The calculating values are in the big error corridor. We have succeeded to combine the two new features of the new algorithm - OPEM total variance: including the errors in both variables and calculating the function by usual coefficients.

TABLE 1. Effective Variance Method, full Minuit and Total Variance OPEM approximation-quadratic function test

	Effect. var.		**Minuit**		**OPEM**	
	$\chi^2 = 4.34$		$\chi^2 = 4.17$		$\chi^2 = 4.34$	
	Value	$\delta c/c$ [%]	Value	$\delta c/c$ [%]	Value	$\delta c/c$ [%]
c_2	0.99	61.4	1.16	57.1	0.93	61.7
c_1	2.21	26.0	2.06	30.4	2.26	42.2
c_0	-0.03	240.3	-0.02	324.2	-0.03	105.8

Source: Bogdanova,Todorov(2001)



Source: Bogdanova,Todorov,2000-Rila mountain

FIGURE 1. Total OPEM Approximation of River Water Energy Spectrum

REFERENCES

1. G.Forsythe, *J. Soc. Indust. Applied Mathhematics* **5**, 75–88 (1957).
2. V.Gadjokov, and N. Bogdanova, Data Fitting by Means of Orthonormal Polynomials (in Russian), Tech. rep., Dubna-JINR-R11-80-781 (1980).
3. N.Bogdanova, and S. Todorov, *Int.J.Modern Physics C* **12**, 119–126 (2001).
4. N.Bogdanova, B. Terziyska, and H. Madge, *Review of Scientific Instruments* **76**, 111104 (2005).
5. N.Bogdanova, and T. Kupenova, *Computer Physics Communications* **73**, 170–178 (1992).
6. P. Bevington, *Data Reduction and Error Analysis for the Physical Sciences*, McGrow Hill, New York, 1969.
7. N.Bogdanova, and S. Todorov, *Bulg. J. of Meteorology and Hydrology* **11**, 95–99 (2000).