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FIELD THEORY MODELS WITH INFINITE-DIMENSIONAL SYMMETRIES, INTEGRABILITY AND SUPERSYMMETRY

T H E S I S

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The present thesis, submitted for obtaining the scientific degree *"Doctor of Sciences"*, consists of introductory, main and reference parts. The short introductory part outlines the place, the objectives and the significance of the research topics and the main results of the thesis within the framework of contemporary theoretical physics. The main (proper) part provides a systematic and detailed description of the contents and principal results in the included full-text copies of author's publications, which are an inseparable integral part of the thesis. The latter represent *33 selected papers* (out of the 99 works in the full list of author's scientific papers), with *331 independent citations* (out of the 958 independent citations of all author's works) and with *impact-factor 68.621* (out of the total impact-factor 175.350 of all author's works).

The unifying theme of all included scientific papers are various related aspects of the modern theory of integrable systems under the following topical subdivision:

(a) Higher quantum conservation laws in integrable two-dimensional models of quantum field theory – these include 7 works [A1–A7] with 118 independent citations and impact-factor 7.982.

(b) Geometric field theory models on coadjoint orbits of infinite-dimensional groups with central extensions – these include 9 works [B1–B9] with 103 independent citations and impact-factor 32.358.

(c) Integrable hierarchies of Kadomtsev-Petviashvili type – reductions, method of squared eigenfunction potentials, additional non-isospectral symmetries and supersymmetric generalizations. Here 17 papers [C1–C17] are included with 110 independent citations and impact-factor 28.281.

The concluding reference part contains the list of the principal scientific contributions in the thesis, the list of the selected included author's papers, citation and impact-factor indices.

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Contents

1 Objectives and Significance of the Main Results

The theory of integrable systems has a long history in the development of theoretical physics during the past 20-th century. Today it ranks as one of the most important mathematical tools in theoretical studies in all principal areas of contemporary physics – from nonlinear hydrodynamics and nonlinear optics, through condensed matter physics and theoretical biophysics, up to modern string theory of elementary particles and fundamental forces of Nature at ultra-high energies. In fact, according to one of the main modern concepts in elementary particle theory all significant dynamical properties of particle interactions can be described adequately in terms of quantum field theory models in lower space-time dimensions. Concurrently these lower-dimensional field theory models possess infinite-dimensional symmetries and, accordingly, in many cases they are integrable (exactly solvable).

The quintessence of the basic ideas and concepts in the theory of integrable systems is the notion of *"soliton"*. Historically solitons appeared for the first time as solutions of certain special nonlinear evolution equations describing solitary localized stable (non-decaying) waves where the nonlinearity of the media completely compensates for the dispersion, so that the solitary wave propagates with constant group velocity and constant wave profile. The most remarkable property of soliton dynamics is the unique character of their scattering: any scattering process of arbitrary number of incoming soliton waves reduces to exchange of their *conserving* group velocities (*i.e.*, only a time delay/phase shift) and, furthermore, soliton scattering interaction is always *factorized*, *i.e.*, it reduces to a series of subsequent two-body scattering subprocesses.

From mathematical point of view the soliton nonlinear evolution equations can be considered as equations of motion of infinite-dimensional (field-theoretic) dynamical systems possessing the outstanding additional property of being completely integrable Hamiltonian systems. The latter are characterized by the presence of infinite complete sets of independent mutually (Poisson-)commuting integrals of motion (conservation laws). From physical point of view it is precisely the complete integrability of soliton dynamics which explains the exceptional soliton stability and the striking properties of their interactions.

In the case of ordinary finite-dimensional (classical mechanics') dynamical systems the property of complete integrability allows, via the classic Liouville theorem, for exact solutions of the corresponding equations of motion in terms of a canonical transformation to a special new set of canonical variables called *"action-angle variables"*.

Generalizing the above standard procedure for constructing exact solutions of integrable systems to the case of infinite-dimensional field theory integrable models turns out to be a highly non-trivial task. In this context the Lax operator formulation of integrable field-theoretic soliton equations and the resulting profound mathematical schemes for their exact solution rank among the main achievements of 20-th century's mathematical physics.

Regarding physically relevant examples of integrability and solitons, we should stress that their number is overwhelming. Let us very briefly mention in this context their crucial role in modern string theory of fundamental interactions of elementary particles at ultra-high energies. Here the very notion of "soliton" surpasses by far its original meaning as solitary localized nonlinear wave. The fundamental role of integrability reveals itself in several different manifestations, the most important being as follows:

(a) The so called *"p-branes"* (p-dimensional membranes) can be viewed as "topological defects" or "domain walls" and they arise as soliton solutions of the supersymmetric string equations of motion in the low-energy limit;

(b) The partition functions of the matrix nonperturbative string models are identified as special soliton solutions (tau-functions) of integrable systems of Kadomtsev-Petviashvili type;

(c) Integrable structures arising in the context of anti-de-Sitter/conformal-field-theory dualities.

The above very sketchy review convincingly demonstrates that the theme and objectives of the present thesis fit entirely within the framework of some of the most actively developing research areas of modern theoretical and mathematical physics.

One of the central features characterizing the classical (non-quantized) field-theoretic integrable systems is the presence of infinite systems of independent integrals of motion in involution (conservation laws). Upon the first straightforward naive application of the standard quantization procedure to study the quantum versions of the classical integrable models by other authors, it has been observed that this quantization procedure yields quantum anomalies (as a result of the inherent regularization and renormalization of the ultraviolet divergences – a phenomenon common to any quantum field theory model), which by themselves break the conservation of the pertinent quantized integrals of motion, and which lead (at first) the latter authors to the (incorrect) conclusion about breakdown of the complete integrability on quantum level causing certain discomfort among the theoretical physics community at that time. For the first time in the literature a solution to the above serious problem has been found in a series of our works [A1–A7]. We have proposed there a general scheme for a systematic renormalization of the quantized integrals of motion in classically completely integrable field theory models which guarantees their conservation in involution also on quantum level. The new approach proposed in [A1–A7] is based on correct generalization and application of the renormalization procedure for products of fundamental quantum fields (renormalized products of quantum fields is *not* equal to the product of the individually renormalized original fields). In this way our works [A1–A7] provided the first proof of complete integrability on quantum level and they laid the ground for the subsequent derivation of the exact quantum S-matrices of the pertinent integrable field theory models.

An important class of integrable systems, relevant both from physical and mathematical points of view, is the class of geometric dynamical systems with phase spaces on coadjoint orbits of infinite-dimensional Lie groups with central extensions. Among the most noteworthy members of this class let us mention the famous conformal models of Wess-Zumino-Novikov-Witten, which describe various non-perturbative ground states in superstring theory. In our series of papers [B1-B9] we have proposed for the first time in the literature a systematic general scheme for explicit construction of geometric dynamical systems on coadjoint orbits of *arbitrary* infinite-dimensional groups in terms of only two fundamental geometric objects – Maurer-Cartan one-form and one-cocycle on the group with values in the dual space of the corresponding infinite-dimensional Lie algebra. This general scheme is worked out in detail for a number of specific physically relevant geometric models, also proposed for the first time in our papers [B1-B9], which describe induced two-dimensional supergravity, anomalous effects in quantized membrane theory, *etc.* As one of its most significant results our scheme allows for exact quantum solvability of *any* geometric model.

In a series of papers [C1–C17] we have systematically studied a number of important aspects of the classical (non-quantized) integrable systems of Kadomtsev-Petviashvili (KP) type, and more specifically – classes of constrained (reduced) KP hierarchies, including their supersymmetric extensions, in the context of their fundamental role in modern string theory. We have obtained complete description of all representations of constrained KP hierarchies in terms of finite even numbers of free bosonic fields, whereby proving that the constrained KP hierarchies arise as well-defined Poisson reductions of the original full unconstrained KP hierarchy within the framework of the R-matrix scheme of Adler-Kostant-Symes-Reyman-(Semenov-Tyan-Shansky). As a byproduct we have found series of representations in terms of finite even numbers of free bosonic fields of both the classical infinite-dimensional Lie algebra $\mathbf{W_{1+\infty}}$, as well as of the classical infinite–dimensional nonlinear Zamolodchikov's algebra $\widetilde{\mathbf{W}}_\infty$.

We have developed a new approach for description of KP-type integrable hierarchies called *"method of squared eigenfunction potentials"* which is shown to be equivalent to the standard approaches (Lax pseudo-differential operator formalism, bilinear Hirota identities, *etc*). The value of this new method consists in its significant role in the description of KP hierarchies in terms of infinite-dimensional Sato Grassmannians, the study of additional non-isospectral symmetries and the formulation of a new type of *binary* Darboux-Bäcklund transformations whose orbits correspond to a new type of *square* Toda lattices.

For the first time in the literature we provide a complete systematic description of all additional non-Hamiltonian (non-isospectral) symmetries of integrable hierarchies of KP type, including their supersymmetric generalizations. In particular, we have discovered a new kind of additional symmetries spanning non-trivial infinite-dimensional loop algebras. Also we have shown that the above mentioned string-theory-related conformal models of Wess-Zumino-Novikov-Witten, as well as a number of other nonlinear evolution equations with important applications in nonlinear hydrodynamics, nonlinear optics and plasma physics, are contained as special particular subsystems within the KP-type integrable hierarchies. More specifically, they are obtained as evolution equations w.r.t. certain special subsets of the pertinent additional non-isospectral symmetry flows. Another physically relevant result is the interesting interpretation of a new class of tau-functions (soliton-like solutions) of KP-type integrable hierarchies in the language of condensed matter theory as joint distribution functions of a new so far unknown type of random matrix models, where an attractive two-body potential appears among the corresponding quasi-particles.

For a complete list of the all scientific contributions in the thesis, please, consult the last Section 7.

Remark about the cited literature. Due to the obvious unfeasibility to supply a complete exhaustive list of all relevant references pertaining to the material in the present thesis, we have restricted ourselves in providing only citations of ground-laying, key or review publications. References to the selected author's papers included as part of the thesis are given according to the numbering in the attached list in Section 6.

2 Higher Quantum Conservation Laws in D = 2 Integrable Field Theories

In the last few decades two-dimensional completely integrable field-theory models gained substantial importance in elementary particle physics primarily due to:

(i) their non-perturbative particle spectra – exhibiting "extended particle solutions" (solitons) [1, 2, 3];

(ii) their exact solvability – exact S-matrices [4];

(iii) the deep analogies between some of the most interesting among them (generalized non-linear sigma-models [5]) and the realistic four-dimensional gauge theories (dynamical symmetry breakdown, dimensional transmutation, asymptotic freedom, instantons);

(iv) their role in non-perturbative supersymmetric gauge theories (Seiberg-Witten approach) [6];

(v) their role in modern string theory – in the context of random matrix model description [7] and anti-de-Sitter/conformal-field-theory dualities [8].

The main ingredient of the exact solvability of $D = 2$ completely integrable models is the existence of infinite sets of higher local conserved currents giving rise to an infinite number of conserved charges in involution. The latter impose severe restrictions on the dynamics:

$$\sum_{j=1}^{N_{in}} \left(p_{j,\pm}^{(in)}\right)^{2n+1} = \sum_{j=1}^{N_{out}} \left(p_{j,\pm}^{(out)}\right)^{2n+1} \quad , \quad n = 0, 1, 2, \ldots \ , \tag{1}$$

for any scattering process, where $p_{j,\pm}^{(in)}$ ($p_{j,\pm}^{(out)}$) are the light-cone combinations of energies and momenta of the asymptotic in-coming (out-going) particles. Relations (1) imply:

(a) absence of multi-particle production, *i.e.*, $N_{in} = N_{out}$;

(b) Factorization of multi-particle scattering, *i.e.*, factorization of any N-particle ($N \geq 3$) scattering process into a sequence of purely elastic two-particle scattering subprocesses[1].

Properties (a)-(b) are the main axiomatic input in Zamolodchikov's construction [4] of exact quantum S-matrices for $D = 2$ completely integrable models. Therefore, it is of primary importance for the self-consistency of the exact S-matrices constructions to show that the pertinent higher local conserved currents "survive" quantization. Furthermore, this will constitute the proof of quantum integrability of the underlying (classically) integrable field theory model.

The latter task is by no means a trivial one since, as it is well known, for any quantum field theory model the local algebra of functions of the fundamental field operators acquires a new structure in comparison with the corresponding classical (non-quantized) one as a result of the *non-multiplicative renormalization* of composite quantum fields due to the ultraviolet (UV) divergencies. Naive quantization by straightforward conversion of the classical functional expressions for the higher local conserved currents into quantum composite operators leads to "anomalies", *i.e.*, non-conservation of the corresponding currents on quantum level. Thus, one cannot apply in the quantum case the standard formalism for construction of

[1] In fact, one can show that the first non-trivial higher conservation law ($n = 1$ in Eq.(1)) is already sufficient to prove absence of multi-particle production and factorization of multi-particle scattering [9].

higher quantum local conserved currents based on the classical inverse scattering method [10, 11, 12][2].

The most appropriate setting turns out to be Zimmermann's normal product formalism [14] for renormalization of composite operators based on the standard *BPHZ* (Bogolyubov-Parasiuk-Hepp-Zimmermann) renormalization scheme [15] and its mass-independent ("soft-mass") extension [16]. In Refs.[A1]–[A7] we have developed a systematic approach for explicit construction of higher local quantum conserved currents in integrable field theory models using Zimmermann's normal product formalism which is essentially algebraic and relies on the following main ingredients:

(i) renormalized quantum equations of motion and, in the case of nonlinear sigma models, the renormalized quantum nonlinear constraints on the fields;

(ii) Zimmermann identities which relate normal products of quantum fields involving non-canonical UV subtractions ("over-subtractions", the latter do appear in the quantum equations of motion) with the canonical normal products.

2.1 Sine-Gordon Model

First, we will consider the most famous $D = 2$ relativistic integrable model – the Sine-Gordon model. Let us note, that apart from its importance in the QFT context the Sine-Gordon model has a wide range of applications in other areas such as solid state physics and non-linear optics (propagation of dislocations, magnetic flux in Josephson junctions, ultrashort optical pulses, Block-wall motion in magnetic crystals) [2].

The Sine-Gordon Lagrangian reads:

$$\mathcal{L}_{SG} = -\frac{1}{2}\left(\partial_\mu\varphi\right)^2 + \frac{m^2}{\beta^2}\left(\cos\beta\varphi - 1\right) \ . \tag{2}$$

In what follows we will use the standard $D = 2$ light-cone notations:

$$\mathcal{B}_\pm = \frac{1}{2}\left(\mathcal{B}_0 \pm \mathcal{B}_1\right) \quad , \quad \partial_\pm\varphi \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)\varphi \quad \text{etc} \ , \tag{3}$$

where $\mathcal{B}_\mu$ is arbitrary $D = 2$ vector.

At the classical level (2) is exactly solvable via the inverse scattering method, it is an infinite-dimensional completely integrable Hamiltonian system possessing an infinite number of conservation laws in involution [1]. Here we will write explicitly the first non-trivial higher local conserved current of (2) :

$$J^{(3)}_+ = \partial_+\varphi\partial^3_+\varphi + \frac{\beta^2}{4}(\partial_+\varphi)^4 \quad , \quad J^{(3)}_- = \frac{m^2}{4\beta}\partial^2_+\varphi\sin\beta\varphi \quad , \quad \partial_- J^{(3)}_+ + \partial_+ J^{(3)}_- = 0 \ . \tag{4}$$

In order to construct the quantum analogues of the higher conserved current (4) we need the renormalized quantum equations of motion for the Sine-Gordon model written in terms

[2] In fact, there exists the powerful *quantum inverse scattering method* [13] allowing exact quantum computation of on-mass-shell quantities, while it is still inapplicable for off-shell quantum correlation functions such as those containing composite operators of the higher conserved currents.

of time-ordered correlation functions:

$$2\left\langle \mathcal{N}\left[\mathcal{B}\partial_x^n\left(\partial_-\partial_+\varphi\right)\right](x)\,\mathcal{X}\right\rangle = -m^2\left\langle \mathcal{N}\left[\mathcal{B}\left\{\partial_x^n\varphi\right\}\right](x)\,\mathcal{X}\right\rangle$$
$$+\left\langle \mathcal{N}\left[\mathcal{B}\partial_x^n\left(m^2\varphi - \frac{\widetilde{m}^2(\beta)}{\beta}\sin\beta\varphi\right)\right](x)\,\mathcal{X}\right\rangle - i\sum_{j=1}^{k}\left(\partial_x^n\delta(x-y_j)\right)\left\langle \mathcal{N}\left[\mathcal{B}\right](x)\,\widehat{\mathcal{X}}^j\right\rangle\ , \tag{5}$$
$$\mathcal{X}\equiv\prod_{i=1}^{k}\varphi(y_i)\quad,\quad \widehat{\mathcal{X}}^j\equiv\prod_{i=1,\,i\neq j}^{k}\varphi(y_i)$$

Here the following notations are used: $\mathcal{B}$ denotes arbitrary monomial in φ and its derivatives; the symbol $\mathcal{N}[\ldots]$ (without curly brackets inside) denotes canonical Zimermann's normal product; the curly brackets inside the normal product symbol $\mathcal{N}[\ldots\{Q\}]$ indicate non-canonical Zimermann's normal product with 2 oversubtractions in every UV divergent Feynman diagram containing internal lines corresponding to the composite operator Q; $\widetilde{m}^2(\beta)\equiv m^2+\delta m^2(\beta)$ where $\delta m^2(\beta)$ takes into account the renormalization ambiguity (for details, see [A3]). The explicit form of the last delta-function ("contact") term in (5) is irrelevant for the issue of construction of higher local quantum conserved currents and henceforth will be omitted.

The quantum analogue of the divergence of the classically conserved higher local current (4) upon using the renormalized quantum equations of motion (5) takes the form:

$$\partial_-\left\langle \mathcal{N}\left[J_+^{(3)}\right](x)\,\mathcal{X}\right\rangle + \partial_+\left\langle \mathcal{N}\left[J_-^{(3)}\right](x)\,\mathcal{X}\right\rangle =$$
$$+\frac{m^2}{2}\left\langle \mathcal{N}\left[\partial_+^3\varphi\left(\varphi-\{\varphi\}\right)+\partial_+\varphi\left(\partial_+^2\varphi-\{\partial_+^2\varphi\}\right)\right](x)\,\mathcal{X}\right\rangle$$
$$+\frac{\beta^2m^2}{2}\left\langle \mathcal{N}\left[\left(\partial_+\varphi\right)^3\left(\varphi-\{\varphi\}\right)\right](x)\,\mathcal{X}\right\rangle + \text{contact terms}\ . \tag{6}$$

The first two terms on the r.h.s. of (6) spoil the quantum conservation of (4). They are an example of quantum "anomalies". However, upon using Zimmermann identities for the non-canonical (oversubtracted) normal products one can show that both "anomalous" terms are in fact total derivatives, so that they can be absorbed in an appropriate *redefinition* of the quantum version of the current (4):

$$\widetilde{J}_+^{(3)} = \mathcal{N}\left[\partial_+\varphi\partial_+^3\varphi + \frac{\beta^2}{4}\left(\partial_+\varphi\right)^4\left(1-3\beta^2(c_3-\frac{1}{2}\beta^2c_2)\left(1-3\beta^2(c_3-\frac{1}{2}\beta^2c_2)\right)^{-1}\right)\right]\ , \tag{7}$$
$$\widetilde{J}_-^{(3)} = \frac{\widetilde{m}^2(\beta)}{4\beta}\mathcal{N}\left[\partial_+^2\varphi\sin\beta\varphi\,\left(1+c_3\beta^2+c_1\beta^4\right)+\beta\left(c_3-\frac{3}{2}\beta^2+c_1\beta^4\right)\left(\partial_+\varphi\right)^2\cos\beta\varphi\right]\ , \tag{8}$$

which is now conserved on quantum level:

$$\partial_-\left\langle \mathcal{N}\left[\widetilde{J}_+^{(3)}\right](x)\,\mathcal{X}\right\rangle + \partial_+\left\langle \mathcal{N}\left[\widetilde{J}_-^{(3)}\right](x)\,\mathcal{X}\right\rangle = \text{contact terms}\ .$$

In Eqs.(7)–(8) the coefficients $c_{1,2,3}$ denote numerical constants expressed in terms of derivatives of one-particle-irreducible Feynman diagrams w.r.t. the external particle momenta at zero values of the momenta, which enter the above mentioned Zimmermann identities for non-canonical normal products for composite operators. For further details, see Refs.[A1,A3].

2.2 Massive Thirring Model

The next $D = 2$ integrable field theory model we consider is the massive Thirring model (MTM):

$$\mathcal{L}_{MTM} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - \frac{g}{4}\left(\bar{\psi}\gamma^{\mu}\psi\right)^2 \ . \tag{9}$$

Here $\psi = (\psi_1, \psi_2)$ is a two-component $D = 2$ Dirac fermionic spinor, γ^{μ} are the $D = 2$ Dirac matrices and $\bar{\psi} = \psi^* \gamma^0$. The MTM is physically relevant in many different aspects. Its renormalization group beta-function is identically zero (no running coupling constant g) meaning that its massless limit is conformally invariant for any value of g. Further, (9) is equivalent (on the quantum level) via the celebrated boson-fermion correspondence to the previously discussed Sine-Gordon model (2). In particular, the fundamental MTM fermionic field ψ is the quantum interpolating operator for the Sine-Gordon soliton. The main interest towards MTM from the point of view of particle phenomenology stems from the fact that it serves as a $D = 2$ analog of low-energy QCD. Apart from its importance in the QFT and particle physics framework, the MTM plays significant role in the description of statistical mechanics of the classical two-dimensional Coulomb gas, the $(1+1)$-dimensional fermion gas at finite temperature, *etc.*

In our papers [A2,A3] we have explicitly constructed an infinite series of higher local (classical) conserved currents which explicitly exhibit the complete integrability of (9). They are given through the following recurrence relations:

$$j_-^{(n)} = \psi_1^* b_n - h.c. \quad , \quad j_+^{(n)} = -im\left(\psi_2^* b_{n-1} - h.c.\right) \quad , \quad \partial_+ j_-^{(n)} + \partial_- j_+^{(n)} = 0 \ , \tag{10}$$

$$b_{n+1} = \partial_- b_n - ig\psi_1^*\psi_1 b_n - 2ig \sum_{k+l=n,\, k\neq 0} b_k^* \psi_1 b_l + \psi_1 \delta_{n+1,0} \ ; \ b_n = 0 \text{ for } n < 0 \ . \tag{11}$$

There is a second infinite set of higher local (classical) conserved currents obtained from (10)–(11) by means of the interchanges $1 \longleftrightarrow 2$, $(+) \longleftrightarrow (-)$ (all currents for even n are trivial total derivatives). Proceeding along similar lines as in the case of the quantum Sine-Gordon model, we have shown in [A3] that all quantum "anomalies" spoiling the conservations of the naively quantized MTM higher local conserved currents (10) can be absorbed into appropriate *redefinitions* of $j_{\pm}^{(n)}$ where the quantum corrections are given in terms of derivatives of one-particle-irreducible Feynman diagrams w.r.t. the external particle momenta at zero values of the momenta, which enter the pertinent Zimmermann identities for non-canonical normal products for composite operators – functions of ψ, $\bar{\psi}$ and their derivatives. For further details, see Ref.[A3]. Also, in Ref.[A5] we have derived Bäcklund transformations and an infinite set of higher conserved currents in the bosonic version of the MTM using the Wahlquist-Estabrook pseudo-potential method [17].

2.3 Non-Linear Sigma-Model

The third and physically most interesting example of $D = 2$ completely integrable field theory models (on classical level) is the $O(N)$ non-linear sigma model (NLSM) whose Lagrangian is given by:

$$\mathcal{L}_{NLSM} = -\frac{1}{2}\left(\partial_\mu \varphi^a\right)^2 \equiv 2\partial_+ \varphi^a \partial_- \varphi_a \quad , \quad \vec{\varphi}^{\,2} = N/g \; , \tag{12}$$
$$\partial_\pm = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \quad , \quad \vec{\varphi} = \left(\varphi^1, \ldots \varphi^N\right) \; .$$

Apart from its deep analogies with the realistic $D = 4$ gauge theories, NLSM (12) describes quantum spin chains, effective low energy degrees of freedom of the high-T_c superconductors *etc.* NLSM (12) and its generalizations (with more complicated target spaces for the fundamental fields $\varphi^a(x)$, such as complex projective and Grassmannian manifolds [5]) play fundamental role in the general field theoretic description of phase transitions and critical phenomena.

The quantum $O(N)$ NLSM can be systematically treated within the *semi-nonperturbative* large-N expansion which is obtained from the generating functional of the time-ordered correlation functions:

$$Z[J] = \int \mathcal{D}\vec{\varphi} \prod_x \delta\left(\vec{\varphi}^{\,2} - N/g\right) \exp\left\{ i \int d^2x \left[\partial_+ \vec{\varphi}\, \partial_- \vec{\varphi} + \left(\vec{J}, \vec{\varphi}\right)\right]\right\} \tag{13}$$
$$= \int \mathcal{D}\vec{\varphi}\, \mathcal{D}\alpha \exp\left\{ i \int d^2x \left[\partial_+ \vec{\varphi}\, \partial_- \vec{\varphi} - \frac{1}{2}\alpha\left(\vec{\varphi}^{\,2} - N/g\right) + \left(\vec{J}, \vec{\varphi}\right)\right]\right\}$$
$$= \int \mathcal{D}\alpha \exp\left\{ -\frac{N}{2} S_1[\alpha] + \frac{i}{2}\int d^2x\, d^2y \left(\vec{J}(x), (-\partial^2 + \alpha)^{-1}\vec{J}(y)\right)\right\} \; ,$$

$$S_1[\alpha] \equiv \mathrm{T}r \ln(-\partial^2 + \alpha) - \frac{i}{g}\int d^2x\, \alpha \quad , \quad \partial^2 = -\left(\frac{\partial}{\partial x^0}\right)^2 + \left(\frac{\partial}{\partial x^1}\right)^2 = -2\partial_+\partial_- \; , \tag{14}$$

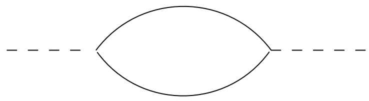
by expanding the effective α-field action $S_1[\alpha]$ (14) around its constant saddle point $\widehat{\alpha} \equiv m^2$, *i.e.*, $\alpha(x) = m^2 + \frac{1}{\sqrt{N}}\tilde{\alpha}(x)$. From the stationary equation $\delta S_1[\alpha]/\delta\alpha \;\big|_{\alpha=m^2} = 0$ one obtains $m^2 = \hat{\mu}^2 e^{-4\pi/g}$, where $\hat{\mu}$ is a renormalization mass scale appearing due to renormalization of the UV divergence coming from the first term in (14):

$$i\frac{\delta}{\delta\alpha}\,\mathrm{T}r \ln\left(-\partial^2 + \alpha\right)\Big|_{\widehat{\alpha}=m^2} = \left\{ i \int d^Dp/(2\pi)^D \left[m^2 + p^2\right]^{-1}\right\}^{\mathrm{ren}}$$
$$= i\int d^Dp/(2\pi)^D \left[\left(m^2 + p^2\right)^{-1} - \left(\hat{\mu}^2 + p^2\right)^{-1}\right] = \frac{1}{4\pi}\ln\left(m^2/\hat{\mu}^2\right) \; . \tag{15}$$

Thus, the "Goldstone" fields $\vec{\varphi}$ acquire *dynamically generated* mass (squared) $\widehat{\alpha} \equiv m^2$, classical conformal invariance of (13) is broken due to the *dimensional transmutation* (the dimensionless coupling g is replaced by m^2), and the classically nonlinearly realized $O(N)$ "flavor" symmetry becomes linearly realized on the quantum level. From (14) one arrives at the large-N diagram technique with (free) propagators in momentum space:

$$\left\langle \varphi^a\, \varphi^b \right\rangle_{(0)} = -i\left(m^2 + p^2\right)^{-1}\delta^{ab} \quad , \quad \langle \tilde{\alpha}\, \tilde{\alpha}\rangle_{(0)} = 2\Big(\Sigma\left(p^2\right)\Big)^{-1} \; , \tag{16}$$
$$\Sigma\left(p^2\right) = -\int \frac{d^2k}{(2\pi)^2}\left[\left(m^2 + k^2\right)\left(m^2 + (p-k)^2\right)\right]^{-1} \; ,$$

and tri-linear $\tilde{\alpha}\varphi\varphi$-vertices, where one-loop φ-tadpoles and subdiagrams of the form in the picture below are forbidden (solid lines depict φ propagators, dashed lines depict $\tilde{\alpha}$ propagators).



The diagrams of the large-N expansion still contain UV divergences which can be systematically renormalized [18] both in $D = 2$ and $D \geq 3$ by a version of the mass-independent ("soft-mass") momentum-space subtraction procedure of Zimmermann-Lowenstein [16].

A remarkable property of the large-N expansion in nonlinear sigma models is that the nonlinearity of the "Goldstone" field $\vec{\varphi}(x)$ is preserved on the quantum level as an identity on the correlation functions, in spite of the manifest *linear* $O(N)$ symmetry of the large-N diagrams:

$$\left\langle \mathcal{N}\Big[\vec{\varphi}^2\mathcal{B}(\vec{\varphi},\partial\vec{\varphi})\Big](x)\mathcal{X}\right\rangle = \mathrm{const}\left\langle \mathcal{N}\Big[\mathcal{B}(\vec{\varphi},\partial\vec{\varphi})\Big](x)\mathcal{X}\right\rangle \quad , \quad \mathcal{X} \equiv \prod_{i=1}^{k} \varphi^{a_i}(x_i)\ , \tag{17}$$

where $\mathcal{B}(\vec{\varphi},\partial\vec{\varphi})$ is arbitrary local polynomial of the fundamental fields and their derivatives, and $\mathcal{N}[\ldots]$ indicates *BPHZL-renormalized* normal product of the corresponding composite fields within the $1/N$ expansion.

Using the BPHZL-renormalized large-N expansion one can explicitly construct the higher quantum local conserved currents $\mathcal{J}^{(s)}_{\pm}$ for the model (12):

$$\partial_+\left\langle \mathcal{J}^{(s)}_{-}(x)\mathcal{X}\right\rangle + \partial_-\left\langle \mathcal{J}^{(s)}_{+}(x)\mathcal{X}\right\rangle = \text{contact terms} \quad , \quad s = 3,5,\ldots\ ,$$

where s indicates the $D = 2$ Lorentz spin of the corresponding higher conserved charge. Their existence is of profound importance as they imply *quantum integrability* of the $O(N)$ nonlinear sigma-model (12). The first non-trivial higher quantum local conserved current is of the form:

$$\mathcal{J}^{(3)}_{-} = \mathcal{N}\left[\left(\partial_-^2\vec{\varphi}\right)^2\right] + a_1\mathcal{N}\left[\left((\partial_-\vec{\varphi})^2\right)^2\right] \quad , \quad \mathcal{J}^{(3)}_{+} = \left(\frac{1}{2} + a_2\right)\mathcal{N}\left[(\partial_-\vec{\varphi})^2\,\alpha\right] + a_3\partial_-^2\alpha\ , \tag{18}$$

where all coefficients $a_{1,2,3} = O(1/N)$ are expressed in terms of one-particle irreducible correlation functions and their derivatives in momentum space at zero external momenta. Their explicit form can be found order by order in $1/N$ from the renormalized large-N diagram technique described above. It is very important to stress, that (18) as well as its higher counterparts $\mathcal{J}^{(s)}_{\pm}$ for $s = 5,7,\ldots$ *do not have* analogues in the classical conformally invariant $O(N)$ nonlinear sigma-model (12). Their existence is entirely due to non-perturbative quantum effects. For further details, see [A4].

2.4 Supersymmetric Sine-Gordon Model

We can use the formalism in Refs.[A1]–[A4] to prove the quantum complete integrability of the *supersymmetric* generalizations of $D = 2$ integrable field theory models. Here we will consider the supersymmetric version of the Sine-Gordon model (2) – the super-Sine-Gordon model.

We will use the standard notions of the superspace approach (see *e.g.* [19]) appropriately accommodated to the case of $D = 2$ space-time (cf. (3)) :

$$\phi(x,\theta) = \varphi_(x) + i\bar{\theta}^\alpha\psi_\alpha(x) + \frac{1}{2}i\bar{\theta}^\alpha\theta_\alpha\, F(x) \quad , \quad \bar{\theta}^\alpha = C^{-1\,\alpha\beta}\theta_\beta \quad (\alpha,\beta = 1,2)\ , \tag{19}$$

$$\mathcal{D}_+ = -\frac{\partial}{\partial\theta_1} - 2i\theta_1\partial_+ \quad , \quad \mathcal{D}_- = \frac{\partial}{\partial\theta_2} - 2i\theta_2\partial_-\ . \tag{20}$$

Here $\varphi(x)$, $F(x)$ are ordinary (pseudo)-scalar bosonic fields; $\psi_\alpha(x)$ and θ_a are two-component fermionic Majorana spinors with $C_{\alpha\beta}$ – the charge conjugation matrix; $\mathcal{D}_\pm$ is the fermionic super-derivative ("square-root" of the ordinary space-time derivatives $(\mathcal{D}_\pm)^2 = \pm 2i\partial_\pm$).

In the notations (19)–(20) the superspace Lagrangian and the classical superspace equations of motion of the super-Sine-Gordon model take the following form:

$$\mathcal{L}_{SSG}(x,\theta) = \frac{1}{2}\mathcal{D}_+\phi\,\mathcal{D}_-\phi - \frac{m}{\beta^2}\cos\beta\phi \quad , \quad \mathcal{D}_+\mathcal{D}_-\phi - \frac{m}{\beta}\sin\beta\phi = 0 \tag{21}$$

or, equivalently, in components:

$$\mathcal{L}_{SSG}(x) = 2\partial_+\varphi\,\partial_-\varphi + i\psi_1\partial_+\psi_1 - i\psi_2\partial_-\psi_2 - \frac{1}{2}F^2 - m\,\psi_1\psi_2\,\cos\beta\varphi + \frac{im}{\beta}F\,\sin\beta\varphi\ , \tag{22}$$

$$4\partial_+\partial_-\varphi = m\beta\,\psi_1\psi_2\,\sin\beta\varphi + im\,F\,\cos\beta\varphi \quad , \quad F = \frac{im}{\beta}\sin\beta\varphi \tag{23}$$

$$2i\partial_+\psi_1 = m\psi_2\cos\beta\varphi \quad , \quad 2i\partial_-\psi_2 = m\psi_1\cos\beta\varphi\ . \tag{24}$$

From (21) one can readily deduce the elements of the superspace Feynman graphs' expansion (super-graphs). In particular, the superfield propagator corresponding to internal ϕ-lines read in momentum space representation:

$$D\,(p;\theta,\theta') = \exp\left\{-\bar{\theta}\gamma^\mu p_\mu\theta\right\}\,(1 - m\delta(\theta-\theta'))\,[m^2 + p^2 - i0]^{-1}\ . \tag{25}$$

To renormalize UV divergent super-graphs we employ a partially "soft" mass version of the BPHZ renormalization scheme. Namely:

(i) the mass m in the numerator of (25) is replaced by sm where s ($0 \le s \le 1$) is an auxiliary parameter;

(ii) UV subtractions are performed as zero external graph momenta *and* at $s = 0$ (partially zero mass), and only at the end one sets $s = 1$.

As in the ordinary bosonic case, the main ingredients in the construction of the higher local quantum conserved currents are the quantum equations of motion and the Zimmermann identities for normal products of composite operators with UV oversubtractions. We will only write down the explicit expression for the first non-trivial higher quantum conserved current of the super-Sine-Gordon model $\mathcal{J}^{1,2}_{7/2}$ (it is fermionic and the subscript 7/2 indicates its canonical dimension):

$$\mathcal{D}_+\Big\langle\mathcal{J}^1_{7/2}(x,\theta)\prod_{i=1}^{L}\phi(x_i,\theta_i)\Big\rangle - \mathcal{D}_-\Big\langle\mathcal{J}^2_{7/2}(x,\theta)\prod_{i=1}^{L}\phi(x_i,\theta_i)\Big\rangle = \text{contact terms}\ , \tag{26}$$

where:

$$\mathcal{J}^1_{7/2}(x,\theta) = \mathcal{N}\left[\mathcal{D}^3_-\phi\,\mathcal{D}^4_-\phi\right] + A_0(\beta^2)\,\mathcal{N}\left[\mathcal{D}_-\phi\left(\mathcal{D}^2_-\phi\right)^3\right] \ , \quad (27)$$

$$\mathcal{J}^2_{7/2}(x,\theta) = A_1(\beta^2)\frac{\widetilde{m}(\beta^2)}{\beta}\mathcal{N}\left[\mathcal{D}^5_-\phi\,\sin\beta\phi\right] + A_2(\beta^2)\frac{\widetilde{m}(\beta^2)}{\beta}\mathcal{N}\left[\mathcal{D}_-\phi\left(\mathcal{D}^2_-\phi\right)^2\sin\beta\phi\right]$$
$$+A_3(\beta^2)\,\widetilde{m}(\beta^2)\,\mathcal{N}\left[\mathcal{D}_-\phi\,\mathcal{D}^4_-\phi\,\cos\beta\phi\right] + A_4(\beta^2)\,\widetilde{m}(\beta^2)\,\mathcal{N}\left[\mathcal{D}^2_-\phi\,\mathcal{D}^3_-\phi\cos\beta\phi\right] \ . \quad (28)$$

Here $\widetilde{m}(\beta^2) = m + \delta m(\beta^2)$, where $\delta m(\beta^2$ like in the ordinary bosonic Sine-Gordon case takes into account the renormalization ambiguity. $A_{0,\ldots,4}(\beta^2)$ are rational functions of β^2 with coefficients which, similarly to the ordinary Sine-Gordon case (7)–(8), are numerical constants expressed in terms of derivatives of one-particle-irreducible Feynman super-graphs w.r.t. the external particle momenta at zero values of the momenta (the latter enter the Zimmermann identities for non-canonical (oversubtracted) normal products for composite superfield operators). For details, see Refs.[A6,A7].

An interesting phenomena occurs with the quantum version of the lowest member $\mathcal{J}^{1,2}_{3/2}(x,\theta)$ of the infinite set of higher local quantum conserved currents $\mathcal{J}^{1,2}_{n+\frac{1}{2}}(x,\theta)$ ($n = 1,3,5,\ldots$) which embody the quantum complete integrability of the super-Sine-Gordon model. This is the conserved spin-vector supercurrent which contains the energy-momentum tensor $T_{\mu\nu}(x)$ among its components. The classical form of the spin-vector current reads:

$$J^1_{3/2\,cl}(x,\theta) = \mathcal{D}_-\phi\mathcal{D}^2_-\phi \quad , \quad J^2_{3/2\,cl}(x,\theta) = -\frac{m}{\beta^2}\mathcal{D}_-\left(\cos\beta\phi\right) \ , \quad (29)$$

whereas for its quantum renormalized form we obtain:

$$\mathcal{J}^1_{3/2}(x,\theta) = \mathcal{N}\left[\mathcal{D}_-\phi\mathcal{D}^2_-\phi\right] \quad , \quad \mathcal{J}^2_{3/2}(x,\theta) = -\left(1-\frac{\beta^2}{4\pi}\right)\frac{\widetilde{m}(\beta^2)}{\beta^2}\mathcal{N}\left[\mathcal{D}_-\left(\cos\beta\phi\right)\right] \ . \quad (30)$$

In particular, Eq.(30) implies for the trace of the quantum energy momentum-tensor of the super-Sine-Gordon model (in terms of the component fields (22)):

$$\widetilde{T}^\mu_\mu = \left(1-\frac{\beta^2}{4\pi}\right)\widetilde{m}(\beta^2)\,\mathcal{N}\left[\frac{iF}{\beta}\sin\beta\phi - \psi_1\psi_2\cos\beta\phi\right] \ . \quad (31)$$

From (30) (or (31)) we conclude that $\beta^2 = 4\pi$ is a critical point (point of scale invariance) for the super-Sine-Gordon model – a new phenomena which parallels the well-known property in the ordinary bosonic Sine-Gordon model where the critical point occurs at $\beta^2 = 8\pi$. For further details, see Refs.[A7].

3 Geometric Field Theory Models on Coadjoint Orbits of Infinite-Dimensional Groups With Central Extensions

In the previous section we have discussed *massive* completely integrable $D = 2$ field theory models. On the other hand there exists another huge class of integrable models – the $D = 2$ conformal field theories [20]. Typical examples of the latter are the *rational* conformal field theory models of which the most extensively studied are the Wess-Zumino-Novikov-Witten (WZNW) models [21]. WZNW models originally appeared in the literature (in more that 2 space-time dimensions) as effective actions incorporating the anomalies in gauge theories with quantized chiral fermionic matter fields. Subsequently, with the advent of string theory WZNW models and their gauged versions [22] have been recognized as fundamental building blocks of the latter describing various ground states of string dynamics.

Standard $D = 2$ WZNW models and their gauged versions are themselves special cases (when the dynamical fields take values in Kac-Moody groups) of a much broader class of *geometric dynamical (Hamiltonian) models on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions*. These coadjoint-orbit models can also be viewed as anomalous quantum effective actions of matter fields with gauged (infinite-dimensional) Noether symmetry groups. Moreover, all basic properties of standard (gauged) WZNW models, in particular, the fundamental Polyakov-Wiegmann group composition law [23], have their natural extensions [B2,B4,B5] in the general setting of geometric coadjoint-orbit models. Nontrivial examples of such more general geometric models include the effective action of induced $D = 2$ gravity [24] and its (extended) supersymmetric generalizations [B1,B2,B7],[25, 26]; the effective WZNW-type action of the toroidal membrane [B3,B6]; the effective WZNW-type action of induced $\mathbf{W_{1+\infty}}$-gravity [B8,B9], WZNW models based on two-loop Kac-Moody algebras [27], *etc.* .

In the framework of the general formalism for geometric actions on coadjoint orbits of (infinite-dimensional) groups with central extensions proposed in Refs.[B2–B6] each physical model is fully characterized by few fundamental ingredients: (i) pairing $\langle\cdot|\cdot\rangle$ between Lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$; (ii) (non-)trivial Lie algebra two-cocycle $\omega(\cdot,\cdot)$ yielding the central extension and a $\mathcal{G}^*$*-valued group one-cocycle* associated with the latter; (iii) the fundamental $\mathcal{G}$-valued *Maurer-Cartan form* on G. These ingredients carry the whole information about the symmetry structure of the models in question and enter into the general recipe for constructing the pertinent coadjoint-orbit geometric action. Classical r-matrices and Yang-Baxter equations appear naturally in this geometric setting [B7],[25].

3.1 Coadjoint Orbits: Basic Ingredients

Consider arbitrary (infinite-dimensional) group G with a Lie algebra $\mathcal{G}$ and its dual space $\mathcal{G}^*$. The adjoint and coadjoint actions of G and $\mathcal{G}$ on $\mathcal{G}$ and $\mathcal{G}^*$ are given by:

$$Ad(g)(X) = gXg^{-1} \quad , \quad ad(X_1)X_2 = \Big[X_1 \, , \, X_2 \Big] \, , \tag{32}$$

$$\langle Ad^*(g)U|X\rangle = \Big\langle U|Ad(g^{-1})X\Big\rangle \quad , \quad \langle ad^*(X_1)U|X_2\rangle = -\,\langle U|ad(X_1)X_2\rangle \ . \tag{33}$$

Here $g \in G$ and $X,\, X_{1,2} \in \mathcal{G}$, $U \in \mathcal{G}^*$ are arbitrary elements, whereas $\langle\cdot|\cdot\rangle$ indicates the natural bilinear form "pairing" $\mathcal{G}$ and $\mathcal{G}^*$.

Our primary interest is in infinite-dimensional Lie algebras with a central extension $\widetilde{\mathcal{G}} = \mathcal{G} \oplus \mathbb{R}$ of $\mathcal{G}$ and, correspondingly, an extension $\widetilde{\mathcal{G}}^* = \mathcal{G}^* \oplus \mathbb{R}$ of the dual space $\mathcal{G}^*$. The central extension is given by a linear operator $\hat{s} : \mathcal{G} \longrightarrow \mathcal{G}^*$ satisfying:

$$\hat{s}([X_1, X_2]) = ad^*(X_1)\hat{s}(X_2) - ad^*(X_2)\hat{s}(X_1) \;, \tag{34}$$

which defines a nontrivial two-cocycle on the Lie algebra $\mathcal{G}$:

$$\omega(X_1, X_2) \equiv -\lambda \left\langle \hat{s}(X_1) \,\middle|\, X_2 \right\rangle \qquad \text{for } \forall\, X_{1,2} \in \mathcal{G} \;, \tag{35}$$

where λ is a numerical normalization constant. The Jacobi identity (34) can be integrated ($X_2 \longrightarrow g = \exp X_2$) to get a unique nontrivial $\mathcal{G}^*$-valued group one-cocycle $S(g)$ in terms of the Lie-algebra cocycle operator $\hat{s}$ (provided $H^1(G) = \emptyset$, $\dim H^2(G) = 1$; see [28]) :

$$ad^*(X)S(g) = Ad^*(g)\, \hat{s}\left(Ad(g^{-1})X\right) - \hat{s}(X) \qquad \text{for } \forall\, X \in \mathcal{G} \;, \tag{36}$$

satisfying the relations :

$$\hat{s}(X) = \frac{d}{dt} S(e^{tX}) \Bigg|_{t=0} \quad , \quad S(g_1 g_2) = S(g_1) + Ad^*(g_1) S(g_2) \;. \tag{37}$$

One can easily generalize the adjoint and coadjoint actions of G and $\mathcal{G}$ to the case with a central extension (acting on elements $(X, n), (X_{1,2}, n_{1,2}) \in \widetilde{\mathcal{G}}$ and $(U, c) \in \widetilde{\mathcal{G}}^*$; see *e.g.* [B2]):

$$\widetilde{Ad}(g)\,(X, n) = \left(Ad(g)X \;,\; n + \lambda \left\langle S(g^{-1}) \,|\, X \right\rangle\right) \;, \tag{38}$$

$$\widetilde{ad}(X_1, n_1)\;(X_2, n_2) \equiv [(X_1, n_1)\,,\,(X_2, n_2)] = \Big(ad(X_1)\; X_2 \;,\; -\lambda \left\langle \hat{s}(X_1) | X_2 \right\rangle\Big) \;, \tag{39}$$

$$\begin{aligned} \widetilde{Ad}^*(g)\,(U, c) &= (Ad^*(g)U + c\lambda\, S(g) \;,\; c) \;, \\ \widetilde{ad}^*(X, n)(U, c) &= (ad^*(X)\, U + \, c\, \lambda\, \hat{s}(X) \;,\; 0\;) \;. \end{aligned} \tag{40}$$

Also, the bilinear form $\langle\cdot|\cdot\rangle$ on $\mathcal{G}^* \otimes \mathcal{G}$ can be extended to a bilinear form on $\widetilde{\mathcal{G}}^* \otimes \widetilde{\mathcal{G}}$ as :

$$\left\langle (U, c) \,\middle|\, (\xi, n) \right\rangle = \left\langle U \,\middle|\, \xi \right\rangle + c\, n \;. \tag{41}$$

From physical point of view the interpretation of the $\mathcal{G}$-cocycle $\hat{s}$ is that of "anomaly" of the Lie algebra (i.e., existence of a c-number term in the commutator (39)), whereas the group cocycle $S(g)$ is the integrated "anomaly', *i.e.*, the "anomaly" for finite group transformations (see Eqs.(38) and (37)).

Another basic geometric object is the fundamental $\mathcal{G}$-valued Maurer-Cartan one-form $Y(g)$ on G with values in $\mathcal{G}$ satisfying:

$$d\,Y(g) = \frac{1}{2}\,[\,Y(g)\,,\,Y(g)\,] \;. \tag{42}$$

It is related to the group one-cocycle $S(g)$ through the equation :

$$d\,S(g) = ad^*(Y(g))\,S(g) + \hat{s}(Y(g)) \;, \tag{43}$$

and possesses group one-cocycle property similar to that of $S(g)$ (37) :

$$Y(g_1 g_2) = Y(g_1) + Ad(g_1)Y(g_2) \;. \tag{44}$$

Using (36) one can rewrite relation (43) in another useful form:

$$d\,S(g) = -Ad^*(g)\hat{s}\big(Y(g^{-1})\big) \;. \tag{45}$$

The group- and algebra-cocycles $S(g)$ and $\hat{s}(X)$ can be generalized to include trivial (co-boundary) parts ((U_0, c) being an arbitrary point in the extended dual space $\widetilde{\mathcal{G}}^*$) :

$$\Sigma(g) \;\equiv\; \Sigma\Big(g; (U_0, c)\Big) = c\lambda S(g) + Ad^*(g)U_0 - U_0 \;, \tag{46}$$

$$\hat{\sigma}(X) \;\equiv\; \hat{\sigma}(X; (U_0, c)) = ad^*(X)U_0 + c\lambda\,\hat{s}(X) = \frac{d}{dt}\Sigma(e^{tX})\bigg|_{t=0} \;. \tag{47}$$

The generalized cocycles (46) and (47) satisfy the same relations as (37), (43) and (36).

The coadjoint orbit of G, passing through the point (U_0, c) of the dual space $\widetilde{\mathcal{G}}^*$, is defined as (cf. (40)) :

$$\mathcal{O}_{(U_0,c)} \equiv \left\{ \Big(U(g), c\Big) \in \widetilde{\mathcal{G}}^* \;;\; U(g) = U_0 + \Sigma(g) = Ad^*(g)U_0 + c\lambda S(g) \right\} . \tag{48}$$

The orbit (48) is a right coset $\mathcal{O}_{(U_0,c)} \simeq G/G_{stat}$ where G_{stat} is the stationary subgroup of the point (U_0, c) w.r.t. the coadjoint action (40) :

$$G_{stat} = \left\{ k \in G \;;\; \Sigma(k) \equiv c\lambda S(k) + Ad^*(k)U_0 - U_0 = 0 \right\} . \tag{49}$$

The Lie algebra corresponding to G_{stat} is :

$$\mathcal{G}_{stat} \equiv \left\{ X_0 \in \mathcal{G} \;;\; \hat{\sigma}(X_0) \equiv ad^*(X_0)U_0 + c\lambda\,\hat{s}(X_0) = 0 \right\} . \tag{50}$$

Now we can express the Kirillov-Konstant symplectic form Ω_{KK} [29] on $\mathcal{O}_{(U_0,c)}$ for any infinite-dimensional (centrally extended) group G in a simple compact form [B2]–[B6]. Namely, introducing the centrally extended objects :

$$\widetilde{\Sigma}(g) \equiv (\Sigma(g), c) \in \widetilde{\mathcal{G}}^* \quad , \quad \widetilde{Y}(g) \equiv \Big(Y(g), m_Y(g)\Big) \in \widetilde{\mathcal{G}} \;, \tag{51}$$

$$d\widetilde{\Sigma}(g) = \widetilde{a}d^*\Big(\widetilde{Y}(g)\Big)\widetilde{\Sigma}(g) \quad , \quad d\widetilde{Y}(g) = \frac{1}{2}\left[\widetilde{Y}(g)\,,\,\widetilde{Y}(g)\right] \;, \tag{52}$$

we obtain (using (41) and (43)) :

$$\Omega_{KK} = -d\bigg(\Big\langle \widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \Big\rangle\bigg) = -\frac{1}{2}\left\langle d\widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \right\rangle \;. \tag{53}$$

3.2 Geometric Actions and Symmetries

The geometric action on a coadjoint orbit $\mathcal{O}_{(U_0,c)}$ of arbitrary infinite-dimensional (centrally extended) group G can now be written down compactly as [B2,B4,B5] :

$$W[g] = \int d^{-1}\Omega_{KK} = -\int \Big\langle \widetilde{\Sigma}(g) \,\Big|\, \widetilde{Y}(g) \Big\rangle = -\int \Big[\Big\langle \Sigma(g) \,|\, Y(g) \Big\rangle - \frac{1}{2} d^{-1}\Big(\langle \hat{\sigma}(Y(g)) \,|\, Y(g) \rangle \Big) \Big] \,, \tag{54}$$

or, in more detail, introducing the explicit expressions (51), (52), (46) and (47) :

$$W[g] = \int \langle U_0 \mid Y(g^{-1}) \rangle - c\lambda \int \Big[\Big\langle S(g) \mid Y(g) \Big\rangle - \frac{1}{2} d^{-1}\Big(\Big\langle \hat{s}(Y(g)) \mid Y(g) \Big\rangle \Big) \Big] \,. \tag{55}$$

The integral in (54), (55) is over one-dimensional curve on the phase space $\mathcal{O}_{(U_0,c)}$ with a "time-evolution" parameter t . Along the curve the exterior derivative becomes $d = dt\partial_t$ and the projection of the Maurer-Cartan one-form $Y(g)$ is : $Y(g) = dtY_t(g)$.

The fundamental Poisson brackets resulting from the geometric action (55) read:

$$\Big\{ \Big\langle \widetilde{\Sigma}(g) \mid \widetilde{X}_1 \Big\rangle \,,\, \Big\langle \widetilde{\Sigma}(g) \mid \widetilde{X}_2 \Big\rangle \Big\}_{PB} = -\Big\langle \widetilde{\Sigma}(g) \mid [\widetilde{X}_1 \,,\, \widetilde{X}_2] \Big\rangle \quad , \quad \widetilde{X}_{1,2} \equiv (X_{1,2} \,,\, n_{1,2}) \,, \tag{56}$$

or in the case of orbits $\mathcal{O}_{(U_0=0,c)}$ (cf. (48)) :

$$\{ \langle S(g) \mid X_1 \rangle \,,\, \langle S(g) \mid X_2 \rangle \}_{PB} = -\langle S(g) \mid [X_1 \,,\, X_2] \rangle + c\lambda \, \langle \hat{s}(X_1) \mid X_2 \rangle \,. \tag{57}$$

In particular, (56) shows that $\widetilde{\Sigma}(g)$ is an equivariant moment map.

Using the group cocycle properties of $S(g)$ and $Y(g)$ (Eqs.(37) and (44)) we derive the following fundamental group composition law [B4,B5] (with $\Sigma(g)$ as in (46)) :

$$W[g_1 g_2] = W[g_1] + W[g_2] + \int \langle \Sigma(g_2) \mid Y(g_1^{-1}) \rangle \,. \tag{58}$$

Eq.(58) is a generalization of the famous Polyakov-Wiegmann composition law [23] in WZNW models to geometric actions on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions.

Eq.(58) contains the whole information about the symmetries of the geometric action (55). Under arbitrary left and right infinitesimal group translations:

$$g \to (\mathbb{1} + \varepsilon_L) g \quad , \quad g \to g(\mathbb{1} + \varepsilon_R) \quad , \quad \varepsilon_{L,R} \in \mathcal{G} \,, \tag{59}$$

we obtain using (58) :

$$\delta_L W[g] = -\int \langle \Sigma(g) \mid d\varepsilon_L \rangle \,, \tag{60}$$

i.e., $\Sigma(g)$ (46) is a Noether conserved current $\partial_t \Sigma(g) = 0$, and:

$$\delta_R W[g] = \int \langle \hat{\sigma}(\varepsilon_R) \mid Y(g^{-1}) \rangle = -\int \langle \hat{\sigma}(Y(g^{-1})) \mid \varepsilon_R \rangle \,. \tag{61}$$

Recalling (50) we find "gauge" invariance of $W[g]$ under right group translations from the stationary subgroup G_{stat} (49) of the orbit $\mathcal{O}_{(U_0,c)}$ (48) : $\delta_R W[g] = 0$ for $\forall \varepsilon_R \in \mathcal{G}_{stat}$ (50). This reveals the geometric meaning of "hidden" local symmetries [24] in models with arbitrary infinite-dimensional Noether symmetry groups (for further details, see Ref.[B5]).

The fundamental importance of the generalized group composition law (58) is further underlined by the fact that as a result of (58) we obtain Ward identities allowing for the exact quantum solvability of *any* geometric model (54). Indeed, let us consider the quantum generating functional for the correlation functions $\ll y_{t_1}(g^{-1})...y_{t_n}(g^{-1}) \gg$ of the group-covariant "composite" fields $y_t(g^{-1})$ (with the notation for the one-form $y(g^{-1}) = dt\, y_t(g^{-1})$ along a curve on the orbit $\mathcal{O}_{(0,c)}$ with parameter t) :

$$Z[j] \equiv \exp\{i\mathcal{W}[j]\} = \int \mathcal{D}g \, \exp i \left\{ W[g] + \int \langle j | y(g^{-1}) \rangle \right\} \, . \tag{62}$$

Performing change of variables $g \to g(I + \varepsilon_R)$ in (62), using the invariance of the measure and accounting for (61) we obtain the following Ward identity :

$$\partial_t j - ad^* \left(\frac{\delta \mathcal{W}}{\delta j} \right) j - c\lambda \, \hat{s} \left(\frac{\delta \mathcal{W}}{\delta j} \right) = 0 \, . \tag{63}$$

We note that (63) represents a closed system of equations allowing recursive determination of higher order correlation function in terms of the lower ones. For the lowest non-trivial correlation function one gets :

$$ic\lambda \hat{s}^{(1)} \ll y_t(g^{-1}) \otimes y_{t'}(g^{-1}) \gg - \partial_t \delta(t - t') I \otimes I = \, 0 \, . \tag{64}$$

Here $\hat{s}^{(1)}$ indicates action of the infinitesimal cocycle $\hat{s}$ on the first member in the tensor product $\mathcal{G} \otimes \mathcal{G}$.

Using the infinitesimal versions of the generalized group composition law Eqs.(59)–(61) we find the important relations:

$$\frac{\delta W[g]}{\delta S(g)} = -c\lambda \, y_t(g) \quad , \quad \frac{\delta W[g]}{\delta y_t(g^{-1})} = -c\lambda \, S(g^{-1}) \, . \tag{65}$$

Now, parametrizing the source $j \in \mathcal{G}^*$ in (62)–(63) as $j \equiv c\lambda \, S(\widetilde{g})$ for some $\widetilde{g} \in G$, and using the first relation in (65), we obtain the *explicit* solution of the Ward identities (63) in the form:

$$\mathcal{W}\big[j \equiv c\lambda \, S(\widetilde{g})\big] = -W[\widetilde{g}] \, , \tag{66}$$

with $W[\cdot]$ as in (55) (with $U_0 = 0$). Similarly, for the quantum effective action (Legendre transform of the "free energy" functional $\mathcal{W}[j]$) $\Gamma[y] = \mathcal{W}[j] - \int \langle j \mid \delta\mathcal{W}/\delta j \rangle$ with $y \equiv \delta\mathcal{W}/\delta j$ we get:

$$\Gamma\big[y \equiv y_t(\widetilde{g})\big] = W[\widetilde{g}] \, . \tag{67}$$

For further details on Eqs.(62)–(67) and application to $D = 2$ induced gravity, see Ref.[B4].

3.3 Examples of Geometric Actions on Coadjoint Orbits

3.3.1 Kac-Moody Groups

The Kac-Moody group elements $g \simeq g(x)$ are smooth mappings $S^1 \longrightarrow G_0$, where G_0 is a finite-dimensional Lie group with generators $\{T^A\}$. The explicit form of (38)-(40) reads in this case:

$$Ad(g)X = g(x)X(x)g^{-1}(x) \quad , \quad ad(X_1)X_2 = [X_1(x)\, , \, X_2(x)] \quad , \quad X_{1,2}(x) = X^A_{1,2}(x)T_A$$
$$Ad^*(g)U = g(x)U(x)g^{-1}(x) \quad , \quad ad^*(X)U = [X(x)\, , \, U(x)] \quad , \quad U(x) = U_A(x)T^A$$

$$\hat{s}(X) = \partial_x X(x) \quad , \quad S(g) = \partial_x g(x)\, g^{-1}(x) \quad , \quad Y(g) = dg(x)\, g^{-1}(x)\ . \tag{68}$$

Plugging (68) into (55) one obtains the well-known WZNW action [21] for G_0-valued chiral fields coupled to an external "potential" $U_0(x)$, whereas Eq.(58) reduces to the Polyakov-Wiegmann group composition law for WZNW actions [23].

3.3.2 Virasoro Group

The Virasoro group elements $g \simeq F(x)$ are smooth diffeomorphisms of the circle S^1. Group multiplication is given by composition of diffeomorphisms in inverse order : $g_1 \cdot g_2 = F_2 \circ F_1\,(x) = F_2\Big(F_1(x)\Big)$. Eqs.(38)-(40) have now the following explicit form :

$$Ad(F)X = \Big(\partial_x F\Big)^{-1} X\left(F(x)\right) \quad , Ad^*(F)U = \Big(\partial_x F\Big)^2 U\left(F(x)\right)\ ;$$
$$ad(X_1)X_2 \equiv [X_1, X_2] = X_1\partial_x X_2 - (\partial_x X_1)X_2 \quad , \quad ad^*(X)U = X\partial_x U + 2(\partial_x X)U\ ;$$

$$\hat{s}(\xi) = \partial_x^3\xi \quad , \quad S(F) = \frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2}\left(\frac{\partial_x^2 F}{\partial_x F}\right)^2 \quad , \quad Y(F) = \frac{dF}{\partial_x F}\ . \tag{69}$$

Here $S(F)$ is the well-known Schwarzian. Plugging (69) into the general expressions (55) and (58) one reproduces the well-known Polyakov $D = 2$ gravity action (coupled to an external stress-tensor $U_0(x)$) :

$$W[F] = \int dtdx \left[-U_0(F(t,x))\, \partial_x F\, \partial_t F + \frac{c}{48\pi}\frac{\partial_t F}{\partial_x F}\left(\frac{\partial_x^3 F}{\partial_x F} - 2\frac{(\partial_x^2 F)^2}{(\partial_x F)^2}\right)\right] \tag{70}$$

and its group composition law [24].

3.3.3 $(\mathbf{N}, \mathbf{0})$ Super-Virasoro Group $(\mathbf{N} \leq \mathbf{4})$

Here we shall use the manifestly $(N, 0)$ supersymmetric formalism. The points of the $(N, 0)$ superspace are labeled as (t, z), $z \equiv (x, \theta^i)$, $i = 1, .., N$. The group elements are given by superconformal diffeomorphisms :

$$z \equiv (x, \theta^j) \ \longrightarrow\ \widetilde{Z} \equiv \Big(F(x, \theta^j), \widetilde{\Theta}^i(x, \theta^j)\Big) \tag{71}$$

obeying the superconformal constraints:

$$D^j F - i\widetilde{\Theta}^k D^j \widetilde{\Theta}_k = 0 \quad , \quad D^j \widetilde{\Theta}^l D^k \widetilde{\Theta}_l - \delta^{jk}\Big[D\widetilde{\Theta}\Big]^2_N = 0 \; , \tag{72}$$

with the following superspace notations:

$$D^i = \frac{\partial}{\partial \theta_i} + i\theta^i \partial_x \quad , \quad D^N \equiv \frac{1}{N!}\epsilon_{i_1 \cdots i_N} D^{i_1} \cdots D^{i_N} \quad , \quad \Big[D\widetilde{\Theta}\Big]^2_N \equiv \frac{1}{N} D^m \widetilde{\Theta}^n D_m \widetilde{\Theta}_n \; . \tag{73}$$

The $(N,0)$ supersymmetric analogues of (69) read:

$$Ad(\widetilde{Z})X = \Big(\Big[D\widetilde{\Theta}\Big]^2_N\Big)^{-1} X(\widetilde{Z}(z)) \quad , \quad Ad^*(\widetilde{Z})U = \Big(\Big[D\widetilde{\Theta}\Big]^2_N\Big)^{2-\frac{N}{2}} U(\widetilde{Z}(z)) \; ,$$

$$ad(X_1)X_2 \equiv [X_1, X_2] = X_1 \partial_x X_2 - (\partial_x X_1) X_2 - \frac{i}{2} D_k X_1 D^k X_2 \; ,$$

$$ad^*(X)U = X\partial_x U + (2 - \frac{N}{2})(\partial_x X)U - \frac{i}{2} D_k \xi D^k U \; ,$$

$$\hat{s}_N(X) = i^{N(N-2)} D^N \partial_x^{3-N} X \quad , \quad Y_N(\widetilde{Z}) = \Big(dF + i\widetilde{\Theta}^j d\widetilde{\Theta}_j\Big)\Big(\Big[D\widetilde{\Theta}\Big]^2_N\Big)^{-1} \; . \tag{74}$$

The associated $\mathcal{G}^*$-valued group one-cocycles $S_N(\widetilde{Z})$ coincide with the well-known [30] $(N,0)$ super-Schwarzians. Inserting the latter and (74) into (55) one obtains the $(N,0)$ supersymmetric generalization of the Polyakov $D=2$ gravity action for any $N \leq 4$ [B2,B7] (see also [25, 26]) :

$$W_N[\widetilde{Z}] = \int dt\; (dz)\Big[\partial_t\Big(\ln\Big[D\widetilde{\Theta}\Big]^2_N\Big) D^N \partial_x^{1-N}\Big(\Big[D\widetilde{\Theta}\Big]^2_N\Big) - U_0(\widetilde{Z})\Big(\Big[D\widetilde{\Theta}\Big]^2_N\Big)^{2-\frac{N}{2}} Y_N(\widetilde{Z})\Big] \; . \tag{75}$$

3.3.4 Group of Area-Preserving Diffeomorphisms on Torus with Central Extension $\widetilde{\mathbf{SDiff}}(\mathbf{T^2})$

The elements of $\widetilde{SDiff}\,(T^2)$ are described by smooth diffeomorphisms $T^2 \ni \vec{x} \equiv (x^1, x^2) \longrightarrow F^i(\vec{x}) \in T^2$ $(i = 1,2)$, such that $\det \| \frac{\partial F^i}{\partial x^j} \| = 1$. The Lie algebra of $\widetilde{SDiff}\,(T^2)$ reads : $\Big[\hat{\mathcal{L}}(\vec{x}), \hat{\mathcal{L}}(\vec{y})\Big] = -\epsilon^{ij}\partial_i \hat{\mathcal{L}}(\vec{x})\partial_j \delta^{(2)}(\vec{x}-\vec{y}) - a^i \partial_i \delta^{(2)}(\vec{x}-\vec{y})$, where $\vec{a} \equiv (a^1, a^2)$ are the “central charges” [31] . The general Eqs.(38)-(40) now specialize to (see [B3]) :

$$Ad(\vec{F})X = X(\vec{F}(\vec{x})) \quad , \quad ad(X_1)X_2 \equiv [X_1, X_2](\vec{x}) = \epsilon^{ij}\partial_i X_1(\vec{x})\partial_j X_2(\vec{x}) \; ,$$

$$Ad^*(\vec{F})U = U(\vec{F}(\vec{x})) \quad , \quad ad^*(X)U = \epsilon^{ij}\partial_i X(\vec{x})\partial_j U(\vec{x}) \; ,$$

$$\hat{s}(X) = a^i \partial_i X(\vec{x}) \quad , \quad S(\vec{F}) = a^i \epsilon_{ij}\Big(F^j(\vec{x}) - x^j\Big) \quad , \quad Y(\vec{F}) = \frac{1}{2}\epsilon_{ij} F^i dF^j + d\rho(\vec{F}) \; , \tag{76}$$

where $\partial_i \rho(\vec{F}) = -\frac{1}{2}\left(\epsilon_{kl} F^k \partial_i F^l + \epsilon_{ij} x^j\right)$.

Plugging (76) into (55) we get the $\widetilde{SDiff}\,(T^2)$ co-orbit geometric action [B3]:

$$W_{\widetilde{SDiff}\,(T^2)}[\vec{F}] = -\frac{1}{3}\int dt\, dx^2 \left(a^k \epsilon_{kl} F^l\right)\, \epsilon_{ij} F^i \partial_t F^j \; . \tag{77}$$

In Ref.[B6] it was shown that (77) is the Wess-Zumino anomalous effective action for the toroidal membrane in the light-cone gauge.

3.3.5 $\mathbf{W_{1+\infty}}$-Gravity Effective Action

The $\mathbf{W_{1+\infty}}$-algebra is isomorphic to the algebra of all differential operators on the circle $\mathcal{DOP}(S^1) = \left\{X \equiv X(x,\partial) = \sum_{i\geq 0} X_i \partial^i\right\}$. Accordingly, the dual space is the space of all purely pseudo-differential operators:

$$\mathcal{DOP}^*(S^1) = \left\{ U \equiv U(x,\partial) = \sum_{j\geq 1} u_j \partial^{-j} \right\}, \tag{78}$$

where the bilinear pairing is defined by:

$$\left\langle U \mid X \right\rangle = \int dx \mathrm{Res}(U\, X) \quad ; \quad \mathrm{Res}\mathcal{A} \equiv a_{-1} \quad \text{for any} \;\; \mathcal{A} = \sum_k a_k \partial^k \; . \tag{79}$$

The corresponding Lie group $DOP(S^1)$ is defined as formal exponentiation of the Lie algebra $\mathcal{DOP}(S^1)$, where the group elements $g(x,\partial) = \exp X(x,\partial)$ are understood again in the sense of pseudo-differential operator calculus. The relevant objects from the coadjoint orbit formalism are given in this case as follows:

$$\begin{gathered}
Ad(g)X = g(x,\partial)X(x,\partial)g^{-1}(x,\partial) \quad , \quad ad(X_1)X_2 = [X_1(x,\partial)\, ,\, X_2(x,\partial)] \, , \\
Ad^*(g)U = \left(g(x,\partial)U(x,\partial)g^{-1}(x,\partial)\right)_- \quad , \quad ad^*(X)U = \Big[X(x,\partial)\, ,\, U(x,\partial) \Big]_- \, , \\
\hat{s}(X) = -\Big[\ln\partial \, ,\, X(x,\partial) \Big]_- \;\; , \;\; S(g) = -\left(\Big[\ln\partial \, ,\, X(x,\partial) \Big] g^{-1}(x,\partial) \right)_- \; , \\
Y(g) = dg(x,\partial)\, g^{-1}(x,\partial) \; .
\end{gathered} \tag{80}$$

Everywhere in (80) products are understood in the sense of pseudo-differential operator calculus, and the subscript $(-)$ indicates taking the purely pseudo-differential part.

Now, the geometric action on a coadjoint orbit of $DOP(S^1) \simeq \mathbf{W_{1+\infty}}$, which is the WZNW effective action of induced $\mathbf{W_{1+\infty}}$-gravity, is given as [B8,B9] (for brevity we suppress the indication of arguments in $g = g(x,\partial)$ etc.) :

$$\begin{gathered}
W[g] \equiv W_{DOP(S^1)}[g] = -\int dt dx \mathrm{Res} \left(U_0\, g^{-1}\, \partial_t g\right) + \\
\frac{c}{4\pi} \int\!\!\int dx \mathrm{Res} \left([\ln\partial \, , \, g]\, g^{-1}\, \partial_t g\, g^{-1} - \frac{1}{2} d^{-1} \Big\{ [\ln\partial \, , \, dg\, g^{-1}] \wedge \left(dg\, g^{-1}\right) \Big\} \right) \, ,
\end{gathered} \tag{81}$$

whereas the group composition law reads [B8,B9]:

$$W[g\, h] = W[g] + W[h] + \int dt dx \Big\{ \Big(h\, U_0\, h^{-1} - \frac{c}{4\pi} [\ln\partial \, , \, h]\, h^{-1} \Big)\, g^{-1}\, \partial_t g \Big\} \, . \tag{82}$$

It has been shown in the [B8] that the stationary subgroup (49) of the pertinent $DOP(S^1) \simeq \mathbf{W_{1+\infty}}$ coadjoint orbit is $SL(\infty)$, which thereby appears as “hidden” symmetry of the $\mathbf{W_{1+\infty}}$ geometric action (81), and that the energy-momentum component T_{++} possesses $SL(\infty)$ Sugawara form.

3.4 Gauging of Geometric Actions

Let us now return to the general case of arbitrary (infinite-dimensional) groups G with central extensions. Henceforth, for simplicity we will consider geometric actions (55) on coadjoint orbits (48) with $U_0 = 0$[3], and also we will set the normalization constant in (55) $-c\lambda = 1$:

$$W[g] = \int \Big[\Big\langle S(g) \bigm| Y(g) \Big\rangle - \frac{1}{2} d^{-1} \Big(\Big\langle \hat{s}(Y(g)) \bigm| Y(g) \Big\rangle \Big) \Big] \; , \tag{83}$$

whereupon the generalized group composition law (58) simplifies to:

$$W[g_1 g_2] = W[g_1] + W[g_2] - \int \left\langle S(g_2) \bigm| Y(g_1^{-1}) \right\rangle \; . \tag{84}$$

Now, suppose that there exist two fixed elements $E_- \in \mathcal{G}$ and $\mathcal{E}_+ \in \mathcal{G}^*$ such that they define splitting (as vector spaces) of the Lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$ with the following properties:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \quad , \quad \mathcal{G}^* = \mathcal{H}^* \oplus \mathcal{M}^* \quad , \quad \mathcal{M}^* \equiv \Big\{ U \Big| \langle U \mid X_H \rangle = 0 \;\; \text{for} \; \forall X_H \in \mathcal{H} \Big\} \; , \tag{85}$$

where:

$$\mathcal{H} = \mathrm{Ker}\Big(\mu\left(\mathcal{E}_+\right) \Big) \quad , \quad E_- \in \mathcal{H} \quad , \quad \hat{s}(E_-) = 0 \;\; (\; i.e. \; E_- \in \mathcal{G}_{stat} \; , \; \text{cf. } (50) \;) \; , \tag{86}$$

and $\mu(U)(\cdot)$ is the mapping:

$$\mu(U) \; : \; \mathcal{G} \to \mathcal{G}^* \quad , \quad \mu(U)X = ad^*(X)U \quad \text{for} \;\; X \in \mathcal{G} \;\; , \;\; U \in \mathcal{G}^* \; . \tag{87}$$

Properties (86) imply that $\mathcal{H}$ is a subalgebra of $\mathcal{G}$ and that:

$$\mu\left(\mathcal{E}_+\right) \mathcal{G} \equiv ad^*\left(\mathcal{G}\right) \mathcal{E}_+ \subset \mathcal{M}^* \quad , \quad ad^*\left(\mathcal{H}\right) \mathcal{M}^* \subset \mathcal{M}^* \; . \tag{88}$$

In particular, the first two relations (86) show that the fixed elements E_- and $\mathcal{E}_+$ mutually "commute":

$$ad^*(E_-)\mathcal{E}_+ = 0 \; . \tag{89}$$

The properties (86) arise as sufficient conditions for consistency and validity of the *generalized "zero-curvature" representation* on the group coadjoint orbit of the equations of motion of the gauged geometric actions (see Eqs.(96) below).

In the special case of G being Kac-Moody group (cf. subsection 3.3.1 above) where Lie algebras and dual spaces are identified, relations (85)–(86) acquire the following meaning:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \quad , \quad \mathcal{H} = \mathrm{Ker}\left(ad(E_+)\right) \; , \tag{90}$$

[3]Note that the first term in (55) containing U_0 can be interpreted as coupling to an external background field.

where $\mathcal{E}_+ \equiv E_+$ and E_- are two mutually commuting fixed Kac-Moody algebra elements belonging to $\mathcal{H}$. In order to make contact with integrable models, one requires in addition to (90) that E_+ is semisimple element, *i.e.* :

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \quad , \quad \mathcal{H} = \mathrm{Ker}\big(ad(E_+)\big) \quad , \quad \mathcal{M} = \mathrm{Im}\big(ad(E_+)\big) \quad \longrightarrow \quad \Big[\mathcal{H}\,,\,\mathcal{M}\Big] \subset \mathcal{M} \,. \tag{91}$$

Going back to the general case of coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions, let us consider "gauge" fields – the one-form $\mathcal{A}_+ \in \mathcal{H}$ and $A_- \in \mathcal{H}^*$ which are parametrized in terms of the group elements h_L, $h_R \in H$ as:

$$\mathcal{A}_+ = Y(h_L) \quad , \quad A_- = S(h_R^{-1}) \,, \tag{92}$$

where $Y(\cdot)$ and $S(\cdot)$ are the fundamental Maurer-Cartan form (43)–(44) and the nontrivial group cocycle (36) restricted on the subgroup H.

Using the above machinery we find the following new geometric action which is a *"massive" gauge-invariant generalization* of (83) :

$$W[g, A_+, A_-] \equiv W[h_L^{-1} g h_R^{-1}] - W[h_L^{-1} h_R^{-1}] + \int dt \Big\langle Ad^*(h_R g^{-1} h_L)\mathcal{E}_+ \mid E_- \Big\rangle \tag{93}$$

$$= \int \Big[\Big\langle S(g) \mid Y(g) \Big\rangle - \frac{1}{2} d^{-1}\Big(\Big\langle \hat{s}(Y(g)) \mid Y(g) \Big\rangle \Big) \Big] -$$
$$- \int \Big[\Big\langle S(g) \mid \mathcal{A}_+ \Big\rangle + \Big\langle A_- \mid Y(g^{-1}) \Big\rangle \Big] -$$
$$- \int \Big[\Big\langle Ad^*(g) A_- \mid \mathcal{A}_+ \Big\rangle - \Big\langle A_- \mid \mathcal{A}_+ \Big\rangle \Big] - \int dt \Big\langle Ad^*(h_R g^{-1} h_L)\mathcal{E}_+ \mid E_- \Big\rangle \,, \tag{94}$$

where $\mathcal{A}_+$, A_- are as in (92). Along the curve of "time" integration in (94) the Maurer-Cartan $\mathcal{H}$-valued one-form $\mathcal{A}_+ = Y(h_L)$ becomes $\mathcal{A}_+ = A_+ dt$ similar to the fundamental $\mathcal{G}$-valued Maurer-Cartan form $Y(g) = Y_t(g)dt$ as pointed out above. Let us stress that, although the gauged geometric action (94) formally resembles ordinary G/H gauged WZNW models where G, H are Kac-Moody groups, the action (94) is valid in the much more general setting of arbitrary (infinite-dimensional) groups with central extensions (cf. the examples in subsection 3.3).

The action (94) exhibits manifest "vector-like" gauge invariance under:

$$g \to h^{-1} g h \quad , \quad h_L \to h^{-1} h_L \quad , \quad h_R \to h_R h \quad , \quad h \in H \,. \tag{95}$$

In particular, we note that the last "mass" term on the r.h.s. of (94) is gauge-invariant by itself. Also, let us emphasize that the second explicit form (94) is obtained from the first defining form (93) by using the general group composition law identities (84). The gauged geometric action (94) is a generalization of the well-known gauged WZNW actions [22] to the case of arbitrary (infinite-dimensional) groups with central extensions.

The gauge-invariant equations of motion resulting from the action (94) acquire the form:

$$\partial_t A_- - \hat{s}(A_+) - ad^*(A_+) A_- = 0 \,. \tag{96}$$

Eq.(96) has a very important meaning, namely it can be viewed as a *generalized "zero-curvature" equation* where we remind that $A_+ \in \mathcal{H}$ whereas $A_- \in \mathcal{H}^*$ (cf. (92)). Indeed, in the special case of Kac-Moody groups (68) it reduces to the ordinary zero-curvature equation on the subalgebra $\mathcal{H}$:

$$\partial_t A_- - \partial_x A_+ - \Big[A_+ \, , \, A_- \Big] = 0 \quad , \quad A_- = \partial_x h_R^{-1} \, h_R \;\; , \;\; A_+ = \partial_t h_L \, h_L^{-1} \; . \tag{97}$$

For further details, see Refs.[B5],[26].

3.5 Classical r-matrices and Poisson Bracket Structures on Infinite-Dimensional Groups

In Ref.[B7], starting with a canonical symplectic structure defined on the cotangent bundle T^*G we derive, via Dirac Hamiltonian reduction, Poisson brackets (PB's) on arbitrary infinite-dimensional group G (admitting central extension). The PB structures are given in terms of an *r-operator kernel* related to the two-cocycle of the underlying Lie algebra and satisfying a *differential classical Yang-Baxter equation.*

It is a basic result of classical differential geometry [32] that the cotangent bundle $T^*\mathcal{M}$ of any Riemannian manifold $\mathcal{M}$ possesses a symplectic structure on itself, *i.e.*, any $T^*\mathcal{M}$ can be interpreted as a phase space of a classical hamiltonian mechanics with $\mathcal{M}$ being the configuration space. Taking $\mathcal{M} = G$, where G is a Lie group, and denoting a generic point of T^*G as (U, g), one has the following canonical symplectic two-form :

$$\Omega(U, g) = -d\left(\Big\langle U \mid Y(g) \Big\rangle\right) + \frac{1}{2} c\lambda \Big\langle \hat{s}(Y(g)) \mid Y(g) \Big\rangle \tag{98}$$

Again here and below the notions and notations from subsections IIB.1 and IIB.2 are used. The symplectic form (98) is a generalization to centrally extended (infinite-dimensional) groups [B7] of the symplectic form used in [34].

It is straightforward to deduce from (98) the explicit form of the Poisson brackets (PB's) among the canonical momenta and coordinates (U, g):

$$\begin{aligned}
\left\{ \left\langle\, U \mid X_1 \,\right\rangle \, , \, \left\langle\, U \mid X_2 \,\right\rangle \right\}_{PB} &= \left\langle\, ad^*(X_1) U + c\lambda \hat{s}(X_1) \mid X_2 \,\right\rangle && (99)\\
\left\{ \left\langle\, U \mid X \,\right\rangle \, , \, \Phi(g) \right\}_{PB} &= L_X \Phi(g) \equiv \frac{d}{dt} \Phi(e^{tX} g) \Big|_{t=0} && (100)\\
\left\{ \Phi_1(g) \, , \, \Phi_2(g) \right\}_{PB} &= 0 && (101)
\end{aligned}$$

Here $\Phi(g)$, $\Phi_{1,2}(g)$ are arbitrary smooth functions on G. In Eq.(100) L_X denotes the left Lie derivative along the vector field corresponding to $X \in \mathcal{G}$.

Let us now consider a reduction of the original phase space T^*G by the set of the following Dirac constraints:

$$\Psi_\xi \,(U, g) \equiv \left\langle \; Ad^*(g^{-1}) \Big(U - \Sigma(g) - U_0 \Big) \Big| \xi \; \right\rangle = 0 \tag{102}$$

$$\{ \, \Psi_{X_1}(U, g) \, , \, \Psi_{X_2}(U, g) \, \}_{PB} = \Psi_{[X_1 \, , \, X_2]}(U, g) - \left\langle \hat{s}(X_1) \mid X_2 \right\rangle \tag{103}$$

where in obtaining (103) we used (99)–(101) and the cocycle property of $\Sigma(g)$ ($\Sigma(g_1 g_2) = \Sigma(g_1) + Ad^*(g_1)\Sigma(g_2)$). The reduced phase space $\mathcal{O}_{(U_0,c)}$ defined by the set of Dirac constraints (102) has the same form as (48)–(49), *i.e.*, it is immediately recognized as coadjoint orbit (cf. [32]) of (the central extension of) G passing through the point (U_0, c) on the dual space $\widetilde{\mathcal{G}}^*$. From (49) we deduce that the Dirac constraints $\Psi_X(U,g)$ (102) are combinations of second-class constraints $\Psi_\perp \equiv \Psi_{X_\perp}(U,g)$ and first-class constraints $\Psi_0 \equiv \Psi_{X_0}(U,g)$, where $X = X_0 + X_\perp$ with $X_0 \in \mathcal{G}_{stat}$ and $X_\perp \in \mathcal{G} \setminus \mathcal{G}_{stat}$.

According to the standard Dirac theory of constrained Hamiltonian systems [33], the canonical Poisson structure on $\mathcal{O}_{(U_0,c)}$ is given by the pertinent Dirac brackets:

$$\begin{aligned} \{\Phi_1(g), \Phi_2(g)\}_{DB} &= \{\Phi_1(g), \Psi^\alpha(U,g)\}_{PB} \, (\hat{s}_\perp^{-1})_{\alpha\beta} \left\{\Psi^\beta(U,g), \Phi_2(g)\right\}_{PB} \\ &= -r_{\alpha\beta} R^\alpha \Phi_1(g) R^\beta \Phi_2(g) \end{aligned} \tag{104}$$

where, for simplicity, we have used component notations w.r.t. specific basis $\{T^\alpha\}$ of the Lie algebra $\mathcal{G}$ and its dual basis $\{T^*_\alpha\}$ in $\mathcal{G}^*$, *i.e.*:

$$X = T^\alpha X_\alpha \quad , \quad \Psi_X = X_a \Psi^\alpha \quad , \quad \left\langle \hat{s}(X_1) \,\middle|\, X_2 \right\rangle = X_{1\,\alpha} \hat{s}^{\alpha\beta} X_{2\,\alpha} \tag{105}$$

Further:

$$R_X \Phi(g) = X_\alpha R^\alpha \Phi(g) = L_{Ad(g)X} \Phi(g) = \frac{d}{dt} \Phi\left(g e^{tX}\right)\Big|_{t=0} \tag{106}$$

denotes the right Lie derivative along X, and:

$$r_{\alpha\beta} = (\hat{s}_\perp^{-1})_{\alpha\beta} \tag{107}$$

is the operator kernel of the inverse operator of the cocycle operator $\hat{s} : \mathcal{G} \longrightarrow \mathcal{G}^*$ (47) restricted on the non-zero-mode subspace $\mathcal{G} \setminus \mathcal{G}_{stat}$.

Let us emphasize that the Dirac brackets (104) are precisely the fundamental Poisson brackets corresponding to the geometric action (54).

It is a simple exercise to show that the Jacobi identities for the Dirac brackets (104) imply the following equation obeyed by the kernel $r_{\alpha\beta}$ (107):

$$[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] + [r^{(13)}, r^{(23)}] = 0 \quad \text{with} \quad r^{(12)} = -r^{(21)} \tag{108}$$

where $r^{(12)} \equiv r_{\alpha\beta} T^\alpha \otimes T^\beta \otimes 1\!\!1 \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$ and similarly for $r^{(13)}$ and $r^{(23)}$, with $\mathcal{U}(\mathcal{G})$ denoting the universal envelopping algebra of $\mathcal{G}$. Eq.(108) completely resembles the standard classical Yang-Baxter equation except that in the case of infinite-dimensional (centrally-extended) groups G it is a *diffential* equation for a kernel $r(x,y)$.

As a non-trivial example let us consider G being the Virasoro group (cf. subsubsection 3.3.2). In this case we have:

$$\begin{aligned} & c\lambda \partial_x^3 r(x,y) = \delta(x-y) \qquad ; \qquad r(x,y) = -r(y,x) \\ & \sum_{cyclic\ (1,2,3)} \left\{ r(x_1,x_2)\partial_{x_2} r(x_2,x_3) - \partial_{x_2} r(x_1,x_2) r(x_2,x_3) \right\} = 0 \end{aligned} \tag{109}$$

the latter one being the classical Yang-Baxter equation for the Virasoro group. The normalization constant $\lambda = -\frac{1}{24\pi}$ (this is true for all $(N,0)$ (super-)Virasoro groups). From (109) one finds:

$$r(x,y) = \frac{1}{c\lambda}\left[\frac{1}{4}(x-y)^2\varepsilon(x-y) + b_0\,(x^2-y^2) + b_1\,(x-y) + b_2\,xy(x-y)\right] \;, \tag{110}$$

where b_0, b_1, b_2 are arbitrary constants subject to the constraint $b_0^2 - b_1 b_2 = 1/16$. It is easy to check that $r(x,y)$ preserves its form (110) under $SL(2;\mathbb{R})$ fractional-linear transformation on x and y.

The above results can be generalized to the case of $(N,0)$ extended super-Virasoro groups for any $N \leq 4$ (see subsection 3.3.3). The $(N,0)$ supersymmetric r-operator kernel satisfies:

$$i^{N(N-2)} c\lambda D_1^N \partial_{x_1}^{3-N} r_N(z_1,z_2) = \delta^{(N)}(z_1-z_2) \equiv \delta(x_1-x_2)\delta^{(N)}(\theta_1-\theta_2) \tag{111}$$

and obeys the $(N,0)$-*supersymmetric classical Yang-Baxter equation* :

$$\sum_{cyclic\ (1,2,3)} \Big\{ r_N(z_1,z_2)\partial_{x_2} r_N(z_2,z_3) - \partial_{x_2} r_N(z_1,z_2)\, r_N(z_2,z_3) - \frac{i}{2} D_2^j r_N(z_1,z_2) D_{2j} r_N(z_2,z_3)\Big\} = 0 \tag{112}$$

The general solution of (111), accounting for (112), reads:

$$r_N(z_1,z_2) = \frac{1}{4c\lambda}\Big[\varepsilon(x_1 - x_2 - i\theta_1^k\theta_{2k}) + x_1^2 - x_2^2 - 2i\theta_1^k\theta_{2k}(x_1+x_2)\Big] \tag{113}$$

Similarly, exact expressions for the classical r-operator kernel and the associated differential classical Yang-Baxter equation can be obtained for the Wess-Zumino geometric action of the toroidal membrane (cf. subsection IIB.3.d).

4 Integrable Hierarchies of Kadomtsev-Petviashvili Type – Method of Squared Eigenfunction Potentials, Additional Nonisospectral Symmetries and Supersymmetry

Kadomtsev-Petviashvili (KP) hierarchy of integrable soliton evolution equations, together with its reductions and multi-component (matrix) generalizations, describe a variety of physically important nonlinear phenomena (for a review, see *e.g.* [11, 12, 35, 36]). Constrained (reduced) KP models are intimately connected with the matrix models in non-perturbative string theory of elementary particles at ultra-high energies ([37] and references therein). They provide an unified description of a number of basic soliton equations such as (modified) Korteveg-de-Vries, nonlinear Schrödinger (AKNS hierarchy in general), Yajima-Oikawa, coupled Boussinesq-type equations *etc.* Dispersionless limits of KP models were recently found [38] to play a fundamental role in the description of interface dynamics (the so called Laplacian growth problem). Furthermore, multi-component (matrix) KP hierarchies are known to contain such physically interesting systems as 2-dimensional Toda lattice, Davey-Stewartson, N-wave resonant system *etc.* It has been shown recently in Ref.[39] that multi-component KP tau-functions provide solutions to the basic Witten-Dijkgraaf-Verlinde-Verlinde equations in topological field theory.

Multi-component (matrix) KP hierarchies can be identified as ordinary (scalar) one-component KP hierarchies supplemented with a special set of commuting additional symmetry flows, namely, the Cartan subalgebra of the underlying loop algebra of additional symmetries. This construction was first proposed in Refs.[C12,C15]. In particular, Davey-Stewartson [40, 11] and N-wave resonant systems are shown to arise as symmetry flows of one-component reduced $cKP_{R,M}$ hierarchies (see below). The above identification leads to new systematic methods of constructing soliton-like solutions of multi-component KP hierarchies by employing the well-established techniques of Darboux-Bäcklund transformations in ordinary one-component KP hierarchies. In Refs.[C15,C17] we presented the explicit construction of Darboux-Bäcklund orbits of solutions for the tau-function (the *soliton-like solutions*) of $cKP_{R,M}$ hierarchies which simultaneously preserve the additional (non-isospectral) symmetries of the latter. The pertinent tau-functions are given in terms of generalized Wronskian-like determinants which contain among themselves solutions of the $SL(M+1)/U(1) \times SL(M)$ gauged WZNW field equations (see, also our paper [26]). A subclass of the generalized Wronskian-like determinant solutions appears in the form of *multiple Wronskians* [C17],[26] which contain as special cases the well-known (multi-)dromion solutions [41] of Davey-Stewartson equations.

4.1 AKS Approach to KP Hierarchy

Let us first recall how the Adler-Kostant-Symes-Reyman-(Semenov-Tyan-Shansky) [42, 43] formalism (based on the group coadjoint orbit method; cf. Section 3) associates three KP-type integrable systems labeled by the index $\ell = 0, 1, 2$ to three possible decompositions of the Lie algebra $\mathcal{G}$ of pseudo-differential operators on the circle into a linear sum of two subalgebras. Writing an arbitrary pseudo-differential operator $X \in \mathcal{G}$ as $X = \sum_{k \geq -\infty} D^k X_k(x)$ [4] we can decompose $\mathcal{G}$ as $\mathcal{G} = \mathcal{G}_+^\ell \oplus \mathcal{G}_-^\ell$ with:

$$\mathcal{G}_+^\ell = \{ X_{\geq \ell} = \sum_{i=\ell}^{\infty} D^i X_i(x) \} \quad ; \quad \mathcal{G}_-^\ell = \{ X_{<\ell} = \sum_{i=-\ell+1}^{\infty} D^{-i} X_{-i}(x) \} \tag{114}$$

for $\ell = 0, 1, 2$. The corresponding dual spaces with respect to the Adler bilinear pairing $\langle L \mid X \rangle = \mathrm{T}r\,(L\,X) = \int dx\, \mathrm{Res}\,(L\,X)$ are given by:

$$\mathcal{G}_+^{\ell\,*} = \{ L_{<-\ell} = \sum_{i=\ell+1}^{\infty} u_{-i}(x) D^{-i} \} \;\; ; \;\; \mathcal{G}_-^{\ell\,*} = \{ L_{\geq -\ell} = \sum_{i=-\ell}^{\infty} u_i(x) D^i \} \tag{115}$$

Note the opposite ordering of D's and coefficient functions in (114) and (115). Denoting the projections on the subalgebras in (114) by $\mathcal{P}_\pm^\ell$ we can define the R-matrix operator [43] on $\mathcal{G}$ as $R_\ell \equiv \mathcal{P}_+^\ell - \mathcal{P}_-^\ell$. There exists a new Lie commutator on $\mathcal{G}$ associated to each R_ℓ-matrix and defined by:

$$[X, Y]_{R_\ell} \equiv \frac{1}{2} [R_\ell X, Y] + \frac{1}{2} [X, R_\ell Y] = [X_{\geq \ell}, Y_{\geq \ell}] - [X_{<\ell}, Y_{<\ell}] \; . \tag{116}$$

The Poisson structure on $\mathcal{G}^*$ follows naturally by generalizing the Kirillov-Kostant formula (99) to the R_ℓ-commutator (116) as follows:

$$\{ F_1(L) \, , \, F_2(L) \}_{R_\ell}(L) = - \langle L \mid [\, \nabla F_1(L) \, , \, \nabla F_2(L) \,]_{R_\ell} \rangle \tag{117}$$

In particular, for the simplest case of linear functions $F_{1,2} = \langle L \mid X_{1,2} \rangle$:

$$\Big\{ \langle L \mid X_1 \rangle \, , \, \langle L \mid X_2 \rangle \Big\}_{R_\ell} = - \langle L \mid [X_1, \, X_2]_{R_\ell} \rangle \tag{118}$$

Now, one can construct three different integrable systems called KP_ℓ hierarchies in the following way (see [C1] for details). Consider the Casimir functions on $\mathcal{G}^*$ defined as functions, which are invariant under coadjoint action of the corresponding Lie group G. The Casimir functions constitute a set of functions in involution w.r.t. Poisson structure (117). A convenient choice of Casimirs is:

$$H_n = \frac{1}{n+1} \mathrm{T}r\, L^{n+1} \quad , \quad \nabla H_n = (L^n)_{\geq \ell} \; . \tag{119}$$

The Hamiltonian equations of motion on $(\mathcal{G}^*, \{\cdot, \cdot\}_{R_\ell})$ associated to these Casimir functions:

$$\frac{\partial L}{\partial t_n} = \{ \, H_n \, , \, L \, \}_{R_\ell} \tag{120}$$

take, according to (117), the form of *Lax evolution equations* on $\mathcal{G}^*$ for all three integrable KP_ℓ systems:

$$\frac{\partial L}{\partial t_n} = [\, (L^n)_{\geq \ell} \, , \, L \,] \qquad \ell \, , = 0, 1, 2 \tag{121}$$

[4]In what follows D denotes the differential operator $D = \frac{\partial}{\partial x}$, whereas derivative acting on a function will be denoted by $\partial_x f$.

where the elements L of $\mathcal{G}^*$ assume the role of pseudo-differential *Lax operators.* Further, the R_ℓ-Poisson bracket (117) is the first Hamiltonian structure for the corresponding KP_ℓ hierarchy. Let us also note that Eqs.(120)–(121) can equivalently obtained from the following coadjoint orbit geometric action [C2]:

$$\mathcal{W}[L] = -\int \langle L \mid \mathcal{Y}_{R_\ell}(L)\rangle - \int dt_n\, H_n[L] \tag{122}$$

$$dL = ad^*_{R_\ell}\left(\mathcal{Y}_{R_\ell}(L)\right) U \quad \longrightarrow \quad d\mathcal{Y}_{R_\ell} = \frac{1}{2}\left[\mathcal{Y}_{R_\ell}\, ,\, \mathcal{Y}_{R_\ell}\right]_{R_\ell} \tag{123}$$

where the integrals in (122) are along arbitrary smooth curve on the phase space $\mathcal{G}^*$ with evolution parameter t_n; $\mathcal{Y}_{R_\ell}(L)$ is Maurer-Cartan one-form on $\mathcal{G}_R$ (the Lie algebra with commutator (116)) and a function of $L \in \mathcal{G}^*$ determined by the first eq.(123), with the R-coadjoint action :

$$ad^*_{R_\ell}(X)U = \frac{1}{2}\Big[\, R_\ell X\, ,\, L\,\Big] - \frac{1}{2}R_\ell\Big(\Big[\, X\, ,\, L\,\Big]\Big) \tag{124}$$

In ref.[C1] we have shown that there is a way of relating Lax operators $L \equiv L^{(\ell)}$ of different KP_ℓ hierarchies by a map, which plays a role of *symplectic* gauge transformation meaning that it maps the R_ℓ-Poisson bracket structure for KP_ℓ to the $R_{\ell'}$-bracket structure for $\mathrm{KP}_{\ell'}$. The explicit form of $L \equiv L^{(\ell)}$ and the pertinent gauge transformations are given by:

$$L \equiv L^{(0)} = D + \sum_{i=1}^{\infty} u_i D^{-i} \quad , \quad L^{(\ell=1)} = D + v_0 + v_1 D^{-1} + \sum_{i\geq 2} v_i\, D^{-i}\ , \tag{125}$$

$$L^{(\ell=2)} = w_{-1}D + w_0 + w_1 D^{-1} + w_2 D^{-2} + \sum_{i\geq 3} w_i D^{-i}\ , \tag{126}$$

$$L^{(0)} = G^{-1}L^{(\ell=1)}G \quad , \quad G \equiv \exp\Big\{-\int^x v_0 dx'\Big\}\ ,$$

$$\exp\left(\phi(x)D\right) L^{(\ell=2)} \exp\left(-\phi(x)D\right) = L^{(\ell=1)}\ , \tag{127}$$

where $\phi(x)$ is chosen in such a way that $w_{-1}\left(F_\phi(x)\right) = \partial_x F_\phi(x)$ with $F_\phi(x) = \exp(\phi(x)\partial_x)x$ representing a finite conformal transformation.

In (125) $L \equiv L^{(0)}$ describes the standard KP hierarchy. The above result established full gauge equivalence between all three KP_ℓ integrable systems defined by (121).

4.2 Sato Pseudo-Differential Operator Formulation. Squared Eigenfunction Potential

The standard scalar (one-component) KP hierarchy is defined through the pseudo-differential operator L from (125) obeying Sato evolution equations (also known as isospectral flow equations; for a systematic exposition, see [36]) :

$$L = D + \sum_{k=1}^{\infty} u_k D^{-k} \quad , \quad \frac{\partial}{\partial t_n} L = \Big[\, (L^n)_+\, ,\, L\,\Big] \ \ , \ \ n = 1, 2, \ldots\ , \tag{128}$$

with Sato dressing operator W :

$$L = WDW^{-1} \quad , \quad \frac{\partial}{\partial t_n} W = -\left(WD^n W^{-1}\right)_- W \quad , \quad W = \sum_{k=0}^{\infty} \frac{p_k(-[\partial])\tau(t)}{\tau(t)} D^{-k} \; , \tag{129}$$

(adjoint) Baker-Akhiezer (BA) wave functions $\psi^{(*)}_{BA}(t,\lambda)$ and tau-function $\tau(t)$:

$$L^{(*)}\psi^{(*)}_{BA} = \lambda\psi^{(*)}_{BA} \quad , \quad \frac{\partial}{\partial t_n}\psi^{(*)}_{BA} = \pm\left(L^{(*)^n}\right)_+ (\psi^{(*)}_{BA}) \; , \tag{130}$$

$$\psi^{(*)}_{BA}(t,\lambda) = W^{(*\,-1)}\left(e^{\pm\xi(t,\lambda)}\right) = \frac{\tau(t\mp[\lambda^{-1}])}{\tau(t)} e^{\pm\xi(t,\lambda)} \quad , \quad \xi(t,\lambda) \equiv \sum_{\ell=1}^{\infty} t_\ell \lambda^\ell \; . \tag{131}$$

Here and below we employ the following short-hand notations:

$$(t) \equiv (t_1 \equiv x, t_2, \ldots) \tag{132}$$

for the set of isospectral time-evolution parameters;

$$[\partial] \equiv \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\right) \tag{133}$$

$$[\lambda^{-1}] \equiv \left(\lambda^{-1}, \frac{1}{2}\lambda^{-2}, \frac{1}{3}\lambda^{-3}, \ldots\right) \; ; \tag{134}$$

and $p_k(.)$ indicate the well-known Schur polynomials:

$$\exp\Bigl\{\sum_{l\geq 1}\lambda^l t_l\Bigr\} = \sum_{k=0}^{\infty}\lambda^k\, p_k(t_1,t_2,\ldots) \tag{135}$$

The tau-function $\tau(t)$ is related to the coefficients of the Lax operator (128) through the relation:

$$\partial_x \frac{\partial}{\partial t_n} \ln\tau = \mathrm{R}esL^n \; , \tag{136}$$

where the terms on the r.h.s. of (136) are the densities of the conserved quantities.

There exist few other objects in Sato formalism for integrable hierarchies which play fundamental role in our construction. (Adjoint) eigenfunctions $\Phi(t)$ ($\Psi(t)$, respectively) are those functions of KP "times" (t) satisfying:

$$\frac{\partial}{\partial t_n}\Phi = (L^n)_+(\Phi) \quad , \quad \frac{\partial}{\partial t_n}\Psi = -(L^n)^*_+(\Psi) \; . \tag{137}$$

According to second Eq.(130), (adjoint) BA functions are special cases of (adjoint) eigenfunctions, which in addition satisfy spectral equations (first Eq.(130)).

It has been shown in Ref.[C11] that any (adjoint) eigenfunction possesses a spectral representation of the form[5] :

$$\Phi(t) = \int d\lambda\,\varphi(\lambda)\,\psi_{BA}(t,\lambda) \quad , \quad \Psi(t) = \int d\lambda\,\psi(\lambda)\,\psi^*_{BA}(t,\lambda) \; , \tag{138}$$

[5]Integrals over spectral parameters are understood as: $\int d\lambda \equiv \oint_0 \frac{d\lambda}{2i\pi} = \mathrm{Res}_{\lambda=0}$.

with appropriate spectral densities $\varphi(\lambda)$ and $\psi(\lambda)$ which are formal Laurent series in λ. Clearly, any KP hierarchy possesses an infinite set of independent (adjoint) eigenfunctions in one-to-one correspondence with the space of all independent formal Laurent series in λ.

The next important object is the so called squared eigenfunction potential (SEP) – a function $S\left(\Phi(t), \Psi(t)\right)$ associated with an arbitrary pair of (adjoint) eigenfunctions $\Phi(t), \Psi(t)$ which possesses the following characteristics:

$$\frac{\partial}{\partial t_n} S\left(\Phi(t), \Psi(t)\right) = \mathrm{Res}\left(D^{-1}\Psi(L^n)_+ \Phi D^{-1}\right) \ . \tag{139}$$

In particular, for $n = 1$ Eq.(139) implies $\partial_x S\left(\Phi(t), \Psi(t)\right) = \Phi(t)\,\Psi(t)$ (recall $\partial_x \equiv \partial/\partial t_1$). Eq.(139) determines $S\left(\Phi(t), \Psi(t)\right) \equiv \partial^{-1}(\Phi(t)\,\Psi(t))$ up to a shift by a trivial constant which is uniquely fixed by the fact that any SEP obeys the following double-spectral representation [C11] :

$$\begin{aligned}\partial^{-1}(\Phi(t)\,\Psi(t)) &= -\int\!\!\int d\lambda d\mu\, \psi(\lambda)\,\varphi(\mu)\,\frac{1}{\lambda}\psi^*_{BA}(t,\lambda)\psi_{BA}(t+[\lambda^{-1}],\mu) \\ &= -\int\!\!\int d\lambda d\mu\, \frac{\psi(\lambda)\,\varphi(\mu)}{\lambda-\mu} e^{\xi(t,\mu)-\xi(t,\lambda)}\frac{\tau(t+[\lambda^{-1}]-[\mu^{-1}])}{\tau(t)} \ ,\end{aligned} \tag{140}$$

with $\varphi(\lambda)$, $\psi(\lambda)$ being the respective spectral densities in (138). It is in this well-defined sense that inverse space derivatives ∂^{-1} will appear throughout our construction below.

As shown in Ref.[C11], the concept of SEP plays fundamental role in all subsequent constructions in the present Section: definition of constrained (reduced) KP hierarchies, description of the algebra of additional non-isospectral (non-Hamiltonian) symmetries, Darboux-Bäcklund transformations *etc.*

Let us also recall that the space of KP Lax operators (128) is endowed with bi-Hamiltonian Poisson bracket structures (another expression of its integrability) which result from the two compatible Hamiltonian structures on the algebra of pseudo-differential operators [43]. The first one is given by (118):

$$\left\{\, \langle L|\, X\rangle\, ,\, \langle L|\, Y\rangle\, \right\}_1 = -\left\langle L \,\middle|\, [X,\, Y]\right\rangle \tag{141}$$

whereas the second one reads:

$$\left\{\, \langle L|\, X\rangle\, ,\, \langle L|\, Y\rangle\, \right\}_2 = \mathrm{Tr}\left((LX)_+\, LY - (XL)_+\, YL\right) + \int dx\, \mathrm{Res}\big([\, L\, ,\, X\,]\big)\partial^{-1}\mathrm{Res}\big([\, L\, ,\, Y\,]\big) \tag{142}$$

In terms of the Lax coefficient functions $u_1, u_2, \ldots$, the first Poisson bracket structure (141) takes the form of the infinite-dimensional Lie algebra $\mathbf{W}_{1+\infty}$ [44]. Its Cartan subalgebra contains the infinite set of (Poisson-)commuting KP integrals of motion $H_{n-1} = \frac{1}{n}\mathrm{Tr}L^n$ whose densities are expressed in terms of the τ-function through (136).

In turn, the second Poisson bracket structure (142) spans a nonlinear (quadratic) algebra called $\widehat{\mathbf{W}}_\infty$ [45], which is an infinite-dimensional generalization of Zamolodchikov's W_N conformal algebras [46].

4.3 Constrained (Reduced) KP Hierarchies

In the previous subsection we have considered the general case of unconstrained (scalar) KP hierarchy (128). In Refs.[C9,C10,C11] we have introduced a broad class of constrained (reduced) KP hierarchies called $\mathsf{cKP}_{R,M}$ (an acronym for *"constrained KP hierarchies of (R,M) type"*), where the analogues of the Lax Eqs.(128),(130),(137) take the form:

$$\frac{\partial}{\partial t_n}\mathcal{L} = \left[(\mathcal{L})_+^{\frac{n}{R}} \, , \, \mathcal{L}\right] \quad , \quad \mathcal{L} \equiv \mathcal{L}_{R,M} \;\; , \;\; n = 1,2,\ldots \; , \tag{143}$$

$$\mathcal{L}^{(*)}\psi_{BA}^{(*)} = \lambda^R \psi_{BA}^{(*)} \quad , \quad \frac{\partial}{\partial t_n}\psi_{BA}^{(*)} = \pm\left(\mathcal{L}^{(*)\frac{n}{R}}\right)_+ (\psi_{BA}^{(*)}) \; , \tag{144}$$

$$\frac{\partial}{\partial t_n}\Phi = (\mathcal{L}^{\frac{n}{R}})_+(\Phi) \quad , \quad \frac{\partial}{\partial t_n}\Psi = -(\mathcal{L}^{\frac{n}{R}})_+^*(\Psi) \; . \tag{145}$$

The pertinent Lax operators can be written in several equivalent forms[6] :

$$\mathcal{L} \equiv \mathcal{L}_{R,M} = D^R + \sum_{i=0}^{R-2} u_i D^i + \sum_{j=1}^{M} \Phi_j D^{-1} \Psi_j \; , \tag{146}$$

$$\mathcal{L} \equiv \mathcal{L}_{R,M} = D^R + \sum_{i=0}^{R-2} u_i D^i + \sum_{j=1}^{M} a_j \left(D - b_j\right)^{-1} \quad , \quad \Phi_j = a_j e^{\int b_j} \;\; , \;\; \Psi_j = e^{-\int b_j} \; , \tag{147}$$

$$\mathcal{L} \equiv \mathcal{L}_{R,M} = D^R + \sum_{i=0}^{R-2} u_i D^i + \sum_{j=1}^{M} A_j \left(D + \tilde{v}_j\right)^{-1} \ldots \left(D + \tilde{v}_M\right)^{-1} \; , \tag{148}$$

$$\mathcal{L} \equiv \mathcal{L}_{R,M} = L_{M+R} L_M^{-1} \; , \tag{149}$$

where L_{M+R} and L_M are two purely differential operators:

$$\begin{aligned} L_{M+R} &= (D + v_{M+R})\,(D + v_{M+R-1}) \cdots (D + v_1) \\ L_M &= (D + \tilde{v}_M)\,(D + \tilde{v}_{M-1}) \cdots (D + \tilde{v}_1) \end{aligned} \tag{150}$$

with coefficients $v_i, \tilde{v}_i$ subject to a constraint:

$$\sum_{j=1}^{M+R} v_j - \sum_{l=1}^{M} \tilde{v}_l = 0 \tag{151}$$

Before proceeding let us note that the class of $\mathsf{cKP}_{R,M}$ hierarchies (143) with (146)–(150) contains various well-known integrable hierarchies as special cases: mKdV hierarchies (for $M = 0$); AKNS hierarchy (for $R = 1$, $M = 1$); Fordy-Kulish hierarchies [47] (for $R = 1$, M arbitrary); Yajima-Oikawa equations (for $R = 2$, $M = 1$), *etc.* On the other hand $\mathsf{cKP}_{R,M}$ hierarchies appeared in different disguises from various parallel developments such as:

[6] Henceforth we shall employ the short-hand notation $\mathcal{L}$ for $\mathcal{L}_{R,M}$ (146) whenever this will not lead to a confusion.

(i) free-field realizations, in terms of finite number of fields, of both compatible first and second KP Hamiltonian structures (141)–(142), see Refs.[C3,C4,C6] and below;

(ii) a method of extracting continuum integrable hierarchies from generalized Toda-like lattice hierarchies underlying (multi-)matrix models in string theory [48],[C5].

The relation between the coefficients of (146) and (148) are as follows [C10]:

$$A_k = (-1)^{M-k} \sum_{s=1}^{k} \Phi_s \frac{W_{M-k+1}\left[\Psi_M, \ldots, \Psi_{k+1}, \Psi_s\right]}{W_{M-k}\left[\Psi_M, \ldots, \Psi_{k+1}\right]} \tag{152}$$

$$\tilde{v}_k = \partial_x \ln \frac{W_{M-k+1}\left[\Psi_M, \ldots, \Psi_{k+1}, \Psi_k\right]}{W_{M-k}\left[\Psi_M, \ldots, \Psi_{k+1}\right]} \tag{153}$$

where $W_k\left[f_1, \ldots, f_k\right]$ denotes Wronskian determinant of the functions $\{f_1, \ldots, f_k\}$.

Similarly we can express the Lax coefficients u_i, A_j entering (148) in terms of the Lax coefficients $v_j, \tilde{v}_l$ of $\mathcal{L}_{R,M}$ in the form (149)–(150) [C6,C10] :

$$\begin{aligned}
&\sum_{n_s=1}^{M+R-s} \left(\partial + v_{n_s+s} - \tilde{v}_{n_s}\right) \sum_{n_{s-1}=1}^{n_s} \left(\partial + v_{n_{s-1}+s-1} - \tilde{v}_{n_{s-1}}\right) \times \cdots \times \\
&\times \sum_{n_2=1}^{n_3} \left(\partial + v_{n_2+2} - \tilde{v}_{n_2}\right) \sum_{n_1=1}^{n_2} \left(\partial + v_{n_1+1} - \tilde{v}_{n_1}\right) \Big(\sum_{l=n_1+1}^{M+R} \left(\tilde{v}_l - v_l\right) \Big) \\
&= \begin{cases} u_{r-1-s} \;, & \text{for } \; s = 1, \ldots, R-1 \\ A_{M+R-s} \;, & \text{for } \; s = R, \ldots, m+R-1 \end{cases}
\end{aligned} \tag{154}$$

(here it is understood that $\tilde{v}_j \equiv 0$ for $j \geq m+1$).

Let us list the following important properties of $\mathsf{cKP}_{R,M}$ hierarchies:

- As shown in Refs.[C3,C4,C6] one can view $\mathsf{cKP}_{R,M}$ in either of their equivalent forms (146)–(149), as consistent *Hamiltonian reductions* of the original full (unconstrained) KP hierarchy (128) w.r.t. the bi-Hamiltonian Poisson structures (141)–(142). Moreover, $\mathcal{L} \equiv \mathcal{L}_{R,M}$ obeys the same fundamental Poisson bracket algebras (141)–(142) as the original unconstrained L (128).

- The first Hamiltonian structure (141) for $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (146)–(147) is equivalent to the following *free-field* fundamental Poisson brackets:

$$\left\{ \Phi_i(x) \,,\, \Psi_j(x') \right\}_1 = -\delta_{ij}\delta(x-x') \quad , \quad \text{rest} = 0 \tag{155}$$

  or, equivalently, in terms of the fields a_i, b_i (147) :

$$\left\{ a_i(x) \,,\, b_j(x') \right\}_1 = -\delta_{ij}\partial_x\delta(x-x') \quad , \quad \text{rest} = 0 \tag{156}$$

  Expanding (146)–(147) in a standard pseudo-differential operator series:

$$\mathcal{L}_{R,M} = L_+ + \sum_{k=1}^{\infty} U_k(\Phi,\Psi)\, D^{-k} = L_+ + \sum_{k=1}^{\infty} U_k(a,b)\, D^{-k} \tag{157}$$

$$U_k(\Phi,\Psi) \equiv \sum_{i=1}^{m} \Phi_i(-\partial_x)^{k-1}\Psi \;,\; U_k(a,b) \equiv \sum_{i=1}^{m} a_i(-1)^{k-1}P_{k-1}(-b_i) \tag{158}$$

where $P_k(\cdot)$ are the familiar Faá di Bruno polynomials, we obtain a series of free-field realizations of the $\mathbf{W}_{1+\infty}$ algebra of the Lax coefficient fields (158) in terms of the free bosonic fields Φ_i, Ψ_i (155) or a_i, b_i (156), respectively.

- The second Hamiltonian structure (142) for $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (149)–(151) is equivalent to the following free-field fundamental Poisson brackets:

$$\{ v_i(x) , v_j(x') \}_2 = \left(\delta_{ij} - \frac{1}{R}\right) \partial_x \delta(x-x') , \qquad i,j = 1,\ldots,m+r$$
$$\{ \tilde{v}_k(x) , \tilde{v}_l(x') \}_2 = -\left(\delta_{kl} + \frac{1}{R}\right) \partial_x \delta(x-x') , \qquad k,l = 1,\ldots,m$$
$$\{ v_i(x) , \tilde{v}_l(x') \}_2 = \frac{1}{R} \partial_x \delta(x-x') \tag{159}$$

This algebra was demonstrated in [C6] to be precisely the Cartan subalgebra of the graded $SL(M+R|M)$ Kac-Moody algebra. Expanding (149)–(150) in a standard pseudo-differential operator series:

$$L_{R,M} = D^R + \sum_{l=0}^{R-2} u_l(v,\tilde{v}) D^l + \sum_{j=1}^{\infty} U_j(v,\tilde{v}) D^{-j} \tag{160}$$
$$U_j(v,\tilde{v}) \equiv \sum_{s=0}^{\min(M-1,j-1)} (-1)^{j-1-s} A_{m-s}(v,\tilde{v}) P^{(s+1)}_{j-1-s}(\tilde{v}_{m-s},\ldots,\tilde{v}_m) \tag{161}$$

where $u_l(v,\tilde{v})$, $A_s(v,\tilde{v})$ are given by (154) and

$$P^{(s)}_n(f_1,\ldots,f_s) = \sum_{l_1+\cdots+l_s=n} (\partial + f_1)^{l_1} \ldots (\partial + f_s)^{l_s}\, 1 \tag{162}$$

are the multiple Faá di Bruno polynomials, we obtain a series of free-field realizations of the nonlinear $\hat{W}^{(R)}_{\infty}$ algebra [46] spanned by the Lax coefficient fields $u_l(v,\tilde{v}), U_j(v,\tilde{v})$ (154),(161) in terms of the free bosonic fields $v_i, \tilde{v}_l$ (159).

Now, is straightforward to show that the functions $\{\Phi_j,\, \Psi_j\}_{j=1}^M$ entering the pseudo-differential part of $\mathcal{L}$ (146) form a set of (adjoint) eigenfunctions of the latter. In what follows we will also need the explicit form of inverse powers of $\mathcal{L}$. To this end we will use the inverses of the underlying purely differential operators which are given by:

$$L_M^{-1} = \sum_{i=1}^{M} \varphi_i D^{-1} \psi_i \quad , \quad L_{M+R}^{-1} = \sum_{a=1}^{M+R} \bar{\varphi}_a D^{-1} \bar{\psi}_a \,, \tag{163}$$

Here the functions $\{\varphi_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$ span $Ker(L_M)$ and $Ker(L_M^*)$, respectively, whereas $\{\bar{\varphi}_a\}_{a=1}^{M+R}$ and $\{\bar{\psi}_a\}_{a=1}^{M+R}$ span $Ker(L_{M+R})$ and $Ker(L_{M+R}^*)$, respectively. Therefore we have:

$$\mathcal{L}^{-1} = \sum_{a=1}^{M+R} L_M(\bar{\varphi}_a) D^{-1} \bar{\psi}_a \,, \tag{164}$$

and more generally ($N \geq 1$):

$$\mathcal{L}^{-N} = \sum_{a=1}^{M+R} \sum_{s=0}^{N-1} \mathcal{L}^{-(N-1)+s} \big(L_M(\bar{\varphi}_a)\big) D^{-1} \big(\mathcal{L}^{-s}\big)^* (\bar{\psi}_a) \; . \tag{165}$$

Let us also note that the following simple consequences from the definitions of the corresponding objects will play essential role for the consistency of the constructions involving inverse powers of $\mathcal{L}$:

$$\Phi_j = L_{M+R}(\varphi_j) \quad , \quad \Psi_j = \psi_j \; , \tag{166}$$

$$\mathcal{L}\big(L_M(\bar{\varphi}_a)\big) = 0 \quad , \quad \mathcal{L}^*(\bar{\psi}_a) = 0 \quad , \quad \mathcal{L}^{-1}(\Phi_j) = 0 \quad , \quad \big(\mathcal{L}^{-1}\big)^* (\Psi_j) = 0 \; . \tag{167}$$

Applying the isospectral flow Eqs.(143) to $\mathcal{L}^{-1}$, *i.e.*, $\partial/\partial t_n \mathcal{L}^{-1} = \Big[\big(\mathcal{L}^{\frac{n}{R}}\big)_+ , \mathcal{L}^{-1} \Big]$ and taking into account the explicit form of $\mathcal{L}^{-1}$ (164) we also find that $L_M(\bar{\varphi}_a)$ and $\bar{\psi}_a$ are (adjoint) eigenfunctions of $\mathcal{L}$ (cf. (145)) :

$$\frac{\partial}{\partial t_n} L_M(\bar{\varphi}_a) = (\mathcal{L}^{\frac{n}{R}})_+ \big(L_M(\bar{\varphi}_a)\big) \quad , \quad \frac{\partial}{\partial t_n} \bar{\psi}_a = -(\mathcal{L}^{\frac{n}{R}})^*_+ (\bar{\psi}_a) \; . \tag{168}$$

4.4 Virasoro and Loop-Algebra Symmetries of Constrained KP Hierarchies

The original unconstrained KP hierarchy (128) possesses an infinite-dimensional $\mathbf{W_{1+\infty}}$ algebra of additional non-isospectral (non-Hamiltonian) symmetries. We will specifically consider the Virasoro subalgebra of additional symmetry flows defined as [49]:

$$\delta^V_n L = \Big[-(ML^n)_- \, , L \Big] \quad , \quad \Big[\delta^V_n \, , \delta^V_m \Big] = -(n-m)\delta^V_{n+m-1} \tag{169}$$

where $\delta^V_n \simeq -L_{n-1}$ (in terms of standard Virasoro notations). Here:

$$\Big[L \, , M \Big] = 1\!\!1 \quad , \quad M = \sum_{k \geq 1} k t_k L^{k-1} + \sum_{j \geq 1} \left(-j p_j(-[\partial]) \ln \tau\right) L^{-j-1} \tag{170}$$

where the standard notations introduced in subsection 4.2 are used. On the other hand, we have shown in ref.[C9,C10] that the original Orlov-Schulman additional symmetry flows (169)–(170) applied to the class $\mathsf{cKP}_{R,M}$ of constrained (reduced) KP hierarchies (143),(146) *do not* yield symmetries of the latter, namely, δ^V_n maps the class $\mathsf{cKP}_{R,M}$ into a *different* class $\mathsf{cKP}_{R,M(n-1)}$.

Solution to the problem of constructing additional non-isospectral Virasoro symmetry flows for the constrained $\mathsf{cKP}_{R,M}$ hierarchies has been given for the first time in the literature in refs.[C9,C10,C15]. It is achieved via the following non-trivial modification of the original Orlov-Schulman flows:

$$\bar{\delta}^V_n \mathcal{L} = \Big[-(\mathcal{M}\mathcal{L}^n)_- + \mathcal{X}_n \, , \mathcal{L} \Big] \quad , \quad \mathcal{L} \equiv \mathcal{L}_{R,M} \text{ (Eq.(146))} \; , \tag{171}$$

$$\Big[\mathcal{L} \, , \mathcal{M} \Big] = 1\!\!1 \quad , \quad \mathcal{M} = \sum_{k \geq 1} k t_k \mathcal{L}^{k-1} + \sum_{j \geq 1} \left(-j p_j(-[\partial]) \ln \tau\right) \mathcal{L}^{-j-1} \; , \tag{172}$$

where the crucial $\mathcal{X}_n$ operator term in the Virasoro flow definition (171) is given by:

$$\mathcal{X}_n = \sum_{i=1}^{M}\sum_{j=0}^{n-2}\bigl(j - \frac{1}{2}(n-2)\bigr)\mathcal{L}^{n-2-j}(\Phi_i)D^{-1}\left(\mathcal{L}^j\right)^*(\Psi_i) \quad \text{for } n \geq 0 \tag{173}$$

$$\mathcal{X}_{(-n)} = \sum_{a=1}^{M+R}\sum_{j=0}^{n}\bigl(\frac{n}{2} - j\bigr)\mathcal{L}^{-(n-j)}\bigl(L_M(\bar{\varphi}_a)\bigr)D^{-1}\left(\mathcal{L}^{-j}\right)^*(\bar{\psi}_a) \quad \text{(for the "negative" flows)} \tag{174}$$

with notations introduced in the previous subsection. To obtain (171)–(174) an extensive use is made of the relations (164)–(168). As proved in refs.[C9,C10,C15], the new non-isospectral flows $\bar{\delta}^V_n$ preserve the constrained form of each $\mathsf{cKP}_{R,M}$ hierarchy and span Virasoro algebra.

Let us now consider the following infinite sets of (adjoint) eigenfunctions of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (146) :

$$\Phi_i^{(n)} \equiv \mathcal{L}^{n-1}(\Phi_i) \quad , \quad \Psi_i^{(n)} \equiv \left(\mathcal{L}^*\right)^{n-1}(\Psi_i) \quad , \quad n = 1,2,\ldots\,; \;\; i = 1,\ldots,M \;, \tag{175}$$

$$\Phi_a^{(-m)} \equiv \mathcal{L}^{-(m-1)}\bigl(L_M(\bar{\varphi}_a)\bigr) \quad , \quad \Psi_a^{(-m)} \equiv \left(\mathcal{L}^{-(m-1)}\right)^*(\bar{\psi}_a) \;, \qquad m = 1,2,\ldots \;, \;\; a = 1,\ldots,M+R \;. \tag{176}$$

Using (175) we can build the following infinite set of additional symmetry flows:

$$\delta_A^{(n)}\mathcal{L} = \left[\mathcal{M}_A^{(n)}\,,\,\mathcal{L}\right] \quad , \quad \mathcal{M}_A^{(n)} \equiv \sum_{i,j=1}^{M} A_{ij}^{(n)}\sum_{s=1}^{n}\Phi_j^{(n+1-s)}D^{-1}\Psi_i^{(s)} \;, \tag{177}$$

where $A_{ij}^{(n)}$ is an arbitrary constant $M \times M$ matrix, *i.e.*, $A^{(n)} \in Mat(M)$. The flows (177) define symmetries since they commute with the isospectral flows $\frac{\partial}{\partial t_l}$:

$$\left[\delta_\alpha\,,\,\frac{\partial}{\partial t_l}\right] = 0 \quad \longleftrightarrow \quad \frac{\partial}{\partial t_l}\mathcal{M}_A^{(n)} = \left[(\mathcal{L}^l)_+\,,\,\mathcal{M}_A^{(n)}\right]_- \;. \tag{178}$$

In particular, from (146) and (177) we find that the flows $\delta^{(n)}_{A=1\!\!1}$ coincide up to a sign with the ordinary isospectral flows of $\mathsf{cKP}_{R,M}$ modulo R: $\delta^{(n)}_{A=1\!\!1} = -\frac{\partial}{\partial t_{nR}}$. Thereby the flows $\delta_A^{(n)}$ (177) will be called "positive" for brevity.

Similarly, using (176) we obtain another infinite set of "negative" symmetry flows which parallels completely the set of "positive" flows (177) :

$$\delta_{\mathcal{A}}^{(-n)}\mathcal{L} = \left[\mathcal{M}_{\mathcal{A}}^{(-n)}\,,\,\mathcal{L}\right] \quad , \quad \mathcal{M}_{\mathcal{A}}^{(-n)} \equiv \sum_{a,b=1}^{M+R}\mathcal{A}_{ab}^{(-n)}\sum_{s=1}^{n}\Phi_b^{(-n-1+s)}D^{-1}\Psi_a^{(-s)} \;, \tag{179}$$

where $\mathcal{A}_{ab}^{(-n)}$ is an arbitrary constant $(M+R)\times(M+R)$ matrix, *i.e.*, $\mathcal{A}^{(-n)} \in Mat(M+R)$. In fact, since according to (165) we have $\mathcal{M}^{(-n)}_{\mathcal{A}=1\!\!1} = \mathcal{L}^{-n}$, the flows $\delta^{(-n)}_{\mathcal{A}=1\!\!1}$ vanish identically, *i.e.*, $\delta^{(-n)}_{\mathcal{A}=1\!\!1} = 0$, therefore, we restrict $\mathcal{A}^{(-n)} \in SL(M+R)$.

The new infinite set of additional symmetries for KP-type hierarchies (177)–(179), proposed for the first time in the literature in refs.[C12,C15], has been shown there to be compatible with the reductions of the KP hierarchies, *i.e.*, they preserve explicitly the constrained form of $\mathsf{cKP}_{R,M}$ hierarchies and, furthermore, they span the following infinite-dimensional loop algebra:

$$\left(\widehat{U}(1) \oplus \widehat{SL}(M)\right)_+ \oplus \left(\widehat{SL}(M+R)\right)_- , \tag{180}$$

where also the ordinary isospectral-flow symmetries are included. The subscripts $(\pm)$ in (180) indicate taking positive/negative-grade subalgebras of the corresponding loop-algebras in the direct sum.

Applying (177)–(179) to relation (136) we deduce that the action of additional symmetry flows on the tau-function is given by the action of the "smeared" Kyoto school bilocal vertex operator $\widehat{\mathcal{X}}(\lambda,\mu)$ [50] :

$$\delta_A^{(n)}\tau(t) = \int\!\!\int d\lambda d\mu\, \rho_A^{(n)}(\lambda,\mu)\widehat{\mathcal{X}}(\lambda,\mu)\,\tau(t) , \tag{181}$$

$$\delta_{\mathcal{A}}^{(-n)}\tau(t) = \int\!\!\int d\lambda d\mu\, \rho_{\mathcal{A}}^{(-n)}(\lambda,\mu)\widehat{\mathcal{X}}(\lambda,\mu)\,\tau(t) . \tag{182}$$

Here:

$$\rho_A^{(n)}(\lambda,\mu) \equiv \frac{\lambda^n-\mu^n}{\lambda-\mu}\sum_{i,j=1}^{M} A_{ij}^{(n)}\psi_i(\lambda)\varphi_j(\mu) , \tag{183}$$

$$\rho_{\mathcal{A}}^{(-n)}(\lambda,\mu) \equiv \frac{\lambda^{-n}-\mu^{-n}}{\lambda^{-1}-\mu^{-1}}\sum_{a,b=1}^{M+R} \mathcal{A}_{ab}^{(-n)}\psi_a^{(-1)}(\lambda)\varphi_b^{(-1)}(\mu) , \tag{184}$$

with $\varphi_i(\lambda)$, $\psi_i(\lambda)$ and $\varphi_a^{(-1)}(\lambda)$, $\psi_a^{(-1)}(\lambda)$ indicating the spectral densities (cf. (138)) of the (adjoint) eigenfunctions Φ_i, Ψ_i and $\Phi_a^{(-1)} \equiv L_M(\bar{\varphi}_a)$, $\Psi_a^{(-1)} \equiv \bar{\psi}_a$, respectively. Further, $\widehat{\mathcal{X}}(\lambda,\mu)$ denotes the Kyoto school bilocal vertex operator [50] :

$$\widehat{\mathcal{X}}(\lambda,\mu) = \frac{1}{\lambda-\mu} : e^{\hat{\theta}(\lambda)-\hat{\theta}(\mu)} := \frac{1}{\lambda-\mu}\, e^{\xi(t,\mu)-\xi(t,\lambda)}\, e^{\sum_1^\infty \frac{1}{l}\left(\lambda^{-l}-\mu^{-l}\right)\frac{\partial}{\partial t_l}} \tag{185}$$

for $|\mu| \leq |\lambda|$, where:

$$\hat{\theta}(\lambda) \equiv -\sum_{l=1}^{\infty}\lambda^l t_l + \sum_{l=1}^{\infty}\frac{1}{l}\lambda^{-l}\frac{\partial}{\partial t_l} , \tag{186}$$

and the columns $:\ldots:$ in (185) indicate Wick normal ordering w.r.t. the creation/annihilation "modes" t_l and $\frac{\partial}{\partial t_l}$, respectively.

Since the tau-function contains all solutions of the underlying integrable hierarchy, Eqs.(181)–(182) describe the action of loop-algebra (180) additional symmetries on the space of (soliton-like) solutions of $\mathsf{cKP}_{R,M}$ hierarchies (146).

Remark. The general unconstrained KP hierarchy defined by (128) possesses a huge loop-algebra of additional non-isospectral symmetries of the form (180) for any values of M, R.

Indeed, the construction above can be straightforwardly extended to the case of the original unconstrained KP hierarchy where all relations (177)–(179) remain intact with:

$$\left\{\Phi_i^{(n)},\,\Psi_i^{(n)}\right\}_{i=1,\ldots,M}^{n=1,2,\ldots} \quad , \quad \left\{\Phi_a^{(-n)},\,\Psi_a^{(-n)}\right\}_{a=1,\ldots,M+R}^{n=1,2,\ldots} \tag{187}$$

forming an infinite system of independent (adjoint) eigenfunctions of the general Lax operator (128) (M, R – arbitrary positive integers).

4.5 Multicomponent (Matrix) KP Hierarchies From One-Component Ones

Let us now consider the following subset of "positive" flows $\delta_{E_k}^{(n)}$ (177) for the general KP hierarchy (128) corresponding to:

$$E_k = diag(0,\ldots 0,1,0,\ldots,0) \quad , \quad i.e. \;\; \mathcal{M}_{E_k}^{(n)} = \sum_{s=1}^{n} \Phi_k^{(n+1-s)} D^{-1} \Psi_k^{(s)} \; . \tag{188}$$

The flows $\delta_{E_k}^{(n)}$ span an infinite-dimensional Abelian algebra and, by construction, they commute with the original isospectral flows $\frac{\partial}{\partial t_n}$ as well. Using the extended set of mutually commuting flows:

$$\frac{\partial}{\partial t_n} \equiv \partial/\partial \overset{(1)}{t_n} \quad , \quad \delta_{E_k}^{(n)} \equiv \partial/\partial \overset{(k+1)}{t_n} \quad , \quad k=1,\ldots,M \; , \tag{189}$$

we can construct the following extended KP integrable hierarchy starting from (128):

$$\partial/\partial t_n L = \Big[(L^n)_+ \, , \, L \Big] \quad , \quad \partial/\partial \overset{(k)}{t_n} L = \Big[\mathcal{M}_{E_k}^{(n)} \, , \, L \Big] \tag{190}$$

with $\mathcal{M}_{E_k}^{(n)}$ as in (188), where the additional sets of "isospectral" flows act on the constituent (adjoint) eigenfunctions as:

$$\partial/\partial \overset{(k)}{t_n} \Phi_i^{(m)} = \mathcal{M}_{E_k}^{(n)}(\Phi_i^{(m)}) \quad , \quad \partial/\partial \overset{(k)}{t_n} \Psi_i^{(m)} = -\left(\mathcal{M}_{E_k}^{(n)}\right)^* (\Psi_i^{(m)}) \quad \text{for } i \neq k \; , \tag{191}$$

$$\begin{aligned} \partial/\partial \overset{(k)}{t_n} \Phi_k^{(m)} &= \mathcal{M}_{E_k}^{(n)}(\Phi_k^{(m)}) - \Phi_k^{(n+m)} \; , \\ \partial/\partial \overset{(k)}{t_n} \Psi_k^{(m)} &= -\left(\mathcal{M}_{E_k}^{(n)}\right)^* (\Psi_k^{(m)}) + \Psi_k^{(n+m)} \; , \end{aligned} \tag{192}$$

with $i,k = 1,\ldots,M$ and using short-hand notations (175).

Such extended KP hierarchies have been proposed for the first time in Refs.[C12,C15],[51]. As shown there, we can identify the extended set of "isospectral" flows (189) with the set of isospectral flows $\left\{\overset{(\ell)}{t_n}\right\}_{n=1,2,\ldots}^{\ell=1,\ldots,M+1}$ of the (unconstrained) $M{+}1$-*component matrix* KP hierarchy.

The latter is defined in terms of the $M+1 \times M+1$ matrix Hirota bilinear identities (see [C12,C15]) :

$$\sum_{k=1}^{M+1} \varepsilon_{ik}\,\varepsilon_{jk} \int d\lambda\, \lambda^{\delta_{ik}+\delta_{jk}-2}\, e^{\xi(\overset{(k)}{t} - \overset{(k)}{t'},\lambda)} \times$$

$$\times\, \tau_{ik}\big(\ldots, \overset{(k)}{t} - [\lambda^{-1}], \ldots\big)\, \tau_{kj}\big(\ldots, \overset{(k)}{t'} + [\lambda^{-1}], \ldots\big) = 0 \; , \tag{193}$$

which are obeyed by a set of $M(M+1)+1$ tau-functions $\{\tau_{ij}\}$ expressed in terms of the single tau-function τ and the "positive" symmetry flow generating (adjoint) eigenfunctions (187) in the original one-component (scalar) KP hierarchy (128)–(136) as follows:

$$\tau_{11} = \tau_{ii} = \tau \quad , \quad \tau_{1i} = \tau\, \Phi^{(1)}_{i-1} \quad , \quad \tau_{i1} = -\tau\, \Psi^{(1)}_{i-1} \; ,$$

$$\tau_{ij} = \varepsilon_{ij} \tau\, \partial^{-1}\Big(\Phi^{(1)}_{j-1} \Psi^{(1)}_{i-1}\Big) \quad , \quad i \neq j \;\; , \;\; i,j = 2, \ldots M+1 \; . \tag{194}$$

Here $\varepsilon_{ij} = 1$ for $i \leq j$ and $\varepsilon_{ij} = -1$ for $i > j$, and δ_{ij} are the usual Kronecker symbols.

The above construction of multi-component (matrix) KP hierarchies out of ordinary one-component ones can be straightforwardly carried over to the case of constrained KP models (146) :

$$\partial/\partial t_n \mathcal{L} = \Big[\big(\mathcal{L}^{\frac{n}{R}}\big)_+ , \mathcal{L} \Big] \quad , \quad \partial/\partial \overset{(k)}{t_n} \mathcal{L} = \Big[\mathcal{M}^{(n)}_{E_k} , \mathcal{L} \Big] \quad , \quad k = 2, \ldots, M+1 \; , \tag{195}$$

using the identification (175) for the symmetry-generating (adjoint) eigenfunctions. In this case, however, there is a linear dependence among the flows (189) :

$$\sum_{k=2}^{M+1} \partial/\partial \overset{(k)}{t_n} = \delta^{(n)}_{A=1\!\!1} = -\frac{\partial}{\partial t_{nR}} \; , \tag{196}$$

therefore, the associated $cKP_{R,M}$-based extended hierarchy (195) is equivalent to $M \times M$ matrix constrained KP hierarchy.

Similarly, we can start with the subset of "negative" symmetry flows $\delta^{(-n)}_{E_k}$ (179) for $cKP_{R,M}$ hierarchy:

$$\delta^{(-n)}_{E_k} \equiv \partial/\partial \overset{(k)}{t}_{-n} \quad , \quad \mathcal{M}^{(-n)}_{E_k} = \sum_{s=1}^{n} \Phi^{(-n-1+s)}_k D^{-1} \Psi^{(-s)}_k \; , \tag{197}$$

$$k = 2, \ldots, M+R \; , \;\; n = 1, 2, \ldots \; .$$

The flow $\delta^{(-n)}_{E_k}$ for $k=1$ is excluded since $\sum_{k=1}^{M+R} \delta^{(-n)}_{E_k} = \delta^{(-n)}_{\mathcal{A}=1\!\!1}$ which vanishes identically as explained in the previous subsection.

The flows (197) also span an infinite-dimensional Abelian algebra commuting with the isospectral flows. Using (197) we now construct another extended KP-type hierarchy analogous to (190)–(192) and based on $cKP_{R,M}$ (146) :

$$\partial/\partial t_n \mathcal{L} = \Big[\big(\mathcal{L}^{\frac{n}{R}}\big)_+ , \mathcal{L} \Big] \quad , \quad \partial/\partial \overset{(k)}{t}_{-n} \mathcal{L} = \Big[\mathcal{M}^{(-n)}_{E_k} , \mathcal{L} \Big] \; , \tag{198}$$

$$\partial/\partial \overset{(k)}{t}_{-n}\Phi_a^{(-m)} = \mathcal{M}_{E_k}^{(-n)}(\Phi_a^{(-m)}) \quad , \quad \partial/\partial \overset{(k)}{t}_{-n}\Psi_a^{(-m)} = -\left(\mathcal{M}_{E_k}^{(-n)}\right)^*(\Psi_a^{(-m)}) \tag{199}$$

for $a \neq k$,

$$\partial/\partial \overset{(k)}{t}_{-n}\Phi_k^{(-m)} = \mathcal{M}_{E_k}^{(-n)}(\Phi_k^{(-m)}) - \Phi_k^{(-n-m)}$$
$$\partial/\partial \overset{(k)}{t}_{-n}\Psi_k^{(-m)} = -\left(\mathcal{M}_{E_k}^{(-n)}\right)^*(\Psi_k^{(-m)}) + \Psi_k^{(-n-m)} \ , \tag{200}$$

where now $a = 1,\ldots,M+R$, $k = 2,\ldots,M+R$ and we have used short-hand notations (176).

Then, following the steps of our construction in Refs.[C12,C15],[51] we arrive at $(M+R)$-*component* constrained KP hierarchy given in terms of $(M+R)(M+R-1)+1$ tau-functions $\{\widetilde{\tau}_{ab}\}$ obeying the corresponding $(M+R)\times(M+R)$ matrix Hirota bilinear identities (cf. (193)). The latter tau-functions are expressed in terms of the original single tau-function τ and the "negative" symmetry flow generating (adjoint) eigenfunctions (176) in the original ordinary $cKP_{R,M}$ hierarchy (146) as follows:

$$\widetilde{\tau}_{11} = \widetilde{\tau}_{aa} = \tau \quad , \quad \widetilde{\tau}_{1a} = \tau\, L_M(\bar{\varphi}_a) \quad , \quad \widetilde{\tau}_{a1} = -\tau\,\bar{\psi}_a \ ,$$
$$\widetilde{\tau}_{ab} = \varepsilon_{ab}\tau\,\partial^{-1}\big(L_M(\bar{\varphi}_b)\bar{\psi}_a\big) \quad , \quad a \neq b \ , \ a,b = 2,\ldots M+R \ . \tag{201}$$

Let us note that there exist alternative representation of (constrained) multi-component KP hierarchies based on matrix generalization of Sato pseudo-differential operator formalism [52]. The present construction [C12,C15],[26] of $cKP_{R,M}$-based extended (multi-component KP) hierarchies (195) and (198) has an advantage over the matrix Sato formulation since it allows us to employ the well-known Darboux-Bäcklund techniques from ordinary one-component (scalar) KP hierarchies (full or constrained) in order to obtain new soliton-like solutions of multi-component (matrix) KP hierarchies.

4.6 Higher-Dimensional Nonlinear Evolution Equations as Symmetry Flows of $\mathsf{cKP}_{R,M}$ Hierarchies

Let us recall that multi-component (matrix) KP hierarchies (193) contain various physically interesting nonlinear systems such as Davey-Stewartson and N-wave systems, which now can be written entirely in terms of objects belonging to ordinary one-component (constrained) KP hierarchy. Thereby the lowest-grade additional symmetry flow parameters acquire the meaning of coordinates for additional space dimensions.

For instance, the N-wave resonant system ($N = M(M+1)/2$) is given by:

$$\partial_k f_{ij} = f_{ik}f_{kj} \quad , \quad i \neq j \neq k \quad , \quad i,j,k = 1,\ldots,M+1 \ , \tag{202}$$

$$\partial_k \equiv \partial/\partial \overset{(k)}{t}_1 \quad , \quad f_{1i} \equiv \Phi_{i-1}^{(1)} \quad , \quad f_{i1} \equiv -\Psi_{i-1}^{(1)} \ ,$$
$$f_{ij} \equiv \varepsilon_{ij}\partial^{-1}\left(\Phi_{j-1}^{(1)}\Psi_{i-1}^{(1)}\right) \quad , \quad i \neq j \ , \ i,j = 2,\ldots M+1 \ . \tag{203}$$

As a further example, let us demonstrate in some detail that the well-known Davey-Stewartson system [40] arises as particular subset of symmetry flow equations obeyed by any pair of adjoint eigenfunctions $\left(\Phi_i, \Psi_i\right)$ (i=fixed) or $\left(L_M(\bar{\varphi}_a), \bar{\psi}_a\right)$ (a=fixed). The derivation for $\left(\Phi_i, \Psi_i\right)$ (i=fixed) has already been presented in our paper [51]. Here for simplicity we take $cKP_{1,M}$ hierarchy (the general case for $cKP_{R,M}$ hierarchy (146) is a straightforward generalization of the formulas below) and consider a pair of "negative" symmetry flow generating (adjoint) eigenfunctions$\left(\phi \equiv L_M(\bar{\varphi}_a), \psi \equiv \bar{\psi}_a\right)$ (a=fixed), which obeys the following subset of flow equations – w.r.t. $\partial/\partial t_2$, $\bar{\partial} \equiv \partial/\partial \overset{(a)}{t}_{-1}$ and $\partial/\partial \bar{t}_2 \equiv \partial/\partial \overset{(a)}{t}_{-2}$:

$$\frac{\partial}{\partial t_2}\phi = \left(\partial^2 + 2\sum_{i=1}^{M} \Phi_i \Psi_i\right)\phi \quad , \quad \frac{\partial}{\partial t_2}\psi = -\left(\partial^2 + 2\sum_{i=1}^{M} \Phi_i \Psi_i\right)\psi \, , \tag{204}$$

$$\bar{\partial}\phi = \mathcal{M}^{(-1)}(\phi) - \mathcal{L}^{-1}(\phi) \quad , \quad \bar{\partial}\psi = -\left(\mathcal{M}^{(-1)}\right)^*(\psi) + \left(\mathcal{L}^{-1}\right)^*(\psi) \, , \tag{205}$$

$$\frac{\partial}{\partial \bar{t}_2}\phi = \mathcal{M}^{(-2)}(\phi) - \mathcal{L}^{-2}(\phi) \quad , \quad \frac{\partial}{\partial \bar{t}_2}\psi = -\left(\mathcal{M}^{(-2)}\right)^*(\psi) + \left(\mathcal{L}^{-2}\right)^*(\psi) \, , \tag{206}$$

where:

$$\mathcal{M}^{(-1)} \equiv \phi D^{-1}\psi \quad , \quad \mathcal{M}^{(-2)} \equiv \mathcal{L}^{-1}(\phi) D^{-1}(\psi) + \phi D^{-1}\left(\mathcal{L}^{-1}\right)^*(\psi) \, . \tag{207}$$

Using (205) we can rewrite Eqs.(206) as purely differential equation w.r.t. $\bar{\partial}$:

$$\frac{\partial}{\partial \bar{t}_2}\phi = \left[-\bar{\partial}^2 + 2\bar{\partial}\left(\partial^{-1}(\phi\psi)\right)\right]\phi \quad , \quad \frac{\partial}{\partial \bar{t}_2}\psi = \left[\bar{\partial}^2 - 2\bar{\partial}\left(\partial^{-1}(\phi\psi)\right)\right]\psi \, . \tag{208}$$

Now, introducing new time variable $T = t_2 - \bar{t}_2$ and the short-hand notation $Q \equiv \sum_{i=1}^{M} \Phi_i \Psi_i - 2(\phi\psi) - 2\bar{\partial}\big(\partial^{-1}(\phi\psi)\big)$, and subtracting Eqs.(208) from Eqs.(204), we arrive at the following system of $(2+1)$-dimensional nonlinear evolution equations:

$$\frac{\partial}{\partial T}\phi = \Big[\frac{1}{2}(\partial^2 + \bar{\partial}^2) + Q + 2\phi\psi\Big]\phi \, , \tag{209}$$

$$\frac{\partial}{\partial T}\psi = -\Big[\frac{1}{2}(\partial^2 + \bar{\partial}^2) + Q + 2\phi\psi\Big]\psi \, , \tag{210}$$

$$\partial\bar{\partial}Q + (\partial + \bar{\partial})^2(\phi\psi) = 0 \, , \tag{211}$$

which is precisely the standard Davey-Stewartson system [40] for the "negative" (adjoint) eigenfunction pair $\left(\phi \equiv L_M(\bar{\varphi}_a),\, \psi \equiv \bar{\psi}_a\right)$ (a=fixed). For further details, see Refs.[C15],[51, 26].

4.7 Darboux-Bäcklund Transformations for $\mathsf{cKP}_{R,M}$ Hierarchies

4.7.1 Darboux-Bäcklund Transformations Preserving KP Reductions and Additional Symmetries

Let us recall that Darboux-Bäcklund (DB) transformations within Sato pseudo-differential approach are defined as "gauge" transformations of special kind on the pertinent Lax operator of the general (unconstrained) KP hierarchy:

$$L \quad \to \quad \widetilde{L} = T_\phi L T_\phi^{-1} \quad , \quad T_\phi \equiv \phi D \phi^{-1} \, , \tag{212}$$

which preserve the isospectral (Sato evolution) equations (128) :

$$\frac{\partial}{\partial t_n}\widetilde{L} = \Big[\,\frac{\partial}{\partial t_n}T_\phi\, T_\phi^{-1} + T_\phi L_+^n T_\phi^{-1}\, ,\, \widetilde{L}\,\Big] = \Big[\,\widetilde{L}_+^n\, ,\, \widetilde{L}\,\Big]\, . \tag{213}$$

For the second equality in (213) to be true, the function ϕ must be an eigenfunction (137). Similarly, one can define adjoint-DB transformation:

$$L \quad\rightarrow\quad \widehat{L} = T_\psi^{*\,-1} L T_\psi^* \quad , \quad T_\psi \equiv \psi D \psi^{-1}\, , \tag{214}$$

where the function ψ must be an adjoint eigenfunction (137). From Eq.(136) one finds the (adjoint) DB transformations of the tau-function:

$$\tau \quad\rightarrow\quad \widetilde{\tau} = \tau\,\phi \quad , \quad \tau \quad\rightarrow\quad \widehat{\tau} = -\tau\,\psi\, . \tag{215}$$

In the case of constrained $\mathsf{cKP}_{R,M}$ hierarchies (146), the (adjoint) DB transformations (212),(214) must in addition preserve also the constrained form of $\mathcal{L} \equiv \mathcal{L}_{R,M}$:

$$\widetilde{\mathcal{L}} = T_\phi \mathcal{L} T_\phi^{-1} \quad , \quad \widehat{\mathcal{L}} = T_\psi^{*\,-1} \mathcal{L} T_\psi^* \tag{216}$$

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_+ + \sum_{j=1}^{M} \widetilde{\Phi}_j D^{-1} \widetilde{\Psi}_j \quad , \quad \widehat{\mathcal{L}} = \widehat{\mathcal{L}}_+ + \sum_{j=1}^{M} \widehat{\Phi}_j D^{-1} \widehat{\Psi}_j \tag{217}$$

A thorough analysis in Refs.[C10,C11,C12,C15] (see also [26]) shows that there exist two different choices for ϕ, ψ in (216)–(217) (here and further in this subsection we use the same notations as in (146),(149),(163),(164),(175),(176)):

$$\phi = \Phi_{i_0} \quad , \quad \psi = \Psi_{j_0} \tag{218}$$

for some fixed indices i_0, j_0 between 1 and M;

$$\phi = L_M(\bar{\varphi}_{a_0}) \quad , \quad \psi = \bar{\psi}_{b_0} \tag{219}$$

for some fixed indices a_0, b_0 between 2 and $M+R$.

Furthermore, we have shown in Ref.[26] that:

(a) (Adjoint) DB transformations (216) with the choice (218) preserve (commute with) $\Big(\widehat{U}(1) \oplus \widehat{SL}(M)\Big)_+ \oplus \Big(\widehat{SL}(M+R-1)\Big)_-$ subalgebra of additional loop-algebra symmetries (180);

(b) (Adjoint) DB transformations (216) with the choice (219) preserve (commute with) $\Big(\widehat{U}(1) \oplus \widehat{SL}(M-1)\Big)_+ \oplus \Big(\widehat{SL}(M+R)\Big)_-$ subalgebra of additional loop-algebra symmetries (180).

The explicit form of the (adjoint-)DB transformed building blocks of the (adjoint-)DB transformed $\mathsf{cKP}_{R,M}$ Lax operator (217) is as follows.

- For the first choice of DB-generating (adjoint) eigenfunctions (218) we have:

$$\widetilde{\Phi}_{i_0} = T_\phi\left(\mathcal{L}(\Phi_{i_0})\right) \quad , \quad \widetilde{\Psi}_{i_0} = \frac{1}{\phi} \equiv \frac{1}{\Phi_{i_0}}\, , \tag{220}$$

$$\widetilde{\Phi}_i = T_\phi(\Phi_i) \quad , \quad \widetilde{\Psi}_i = T_\phi^{*\,-1}(\Psi_i) \quad \text{for} \;\; i \neq i_0 \; , \tag{221}$$

$$\widetilde{\Phi}_a^{(-1)} = T_\phi(\Phi_a^{(-1)}) \quad , \quad \widetilde{\Psi}_a^{(-1)} = T_\phi^{*\,-1}(\Psi_a^{(-1)}) \; , \tag{222}$$

$$\widehat{\Phi}_{j_0} = -\frac{1}{\psi} \equiv -\frac{1}{\Psi_{j_0}} \quad , \quad \widehat{\Psi}_{j_0} = -T_\psi\left(\mathcal{L}^*(\Psi_{j_0})\right) \; , \tag{223}$$

$$\widehat{\Phi}_i = -T_\psi^{*\,-1}(\Phi_i) \quad , \quad \widehat{\Psi}_i = -T_\psi(\Psi_i) \quad , \quad i \neq j_0 \; , \tag{224}$$

$$\widehat{\Phi}_a^{(-1)} = -T_\psi^{*\,-1}(\Phi_a^{(-1)}) \quad , \quad \widehat{\Psi}_a^{(-1)} = -T_\psi(\Psi_a^{(-1)}) \; . \tag{225}$$

- For the second choice of DB-generating (adjoint) eigenfunctions (219) we obtain:

$$\widetilde{\Phi}_i = T_\phi(\Phi_i) \quad , \quad \widetilde{\Psi}_i = T_\phi^{*\,-1}(\Psi_i) \quad , \quad \phi \equiv L_M(\bar{\varphi}_{a_0}) \equiv \Phi_{a_0}^{(-1)} \; , \tag{226}$$

$$\widetilde{\Phi}_{a_0}^{(-1)} = T_\phi\big(\mathcal{L}^{-1}\left(\Phi_{a_0}^{(-1)}\right)\big) \quad , \quad \widetilde{\Psi}_{a_0}^{(-1)} = \frac{1}{\phi} \equiv \frac{1}{\Phi_{a_0}^{(-1)}} \; , \tag{227}$$

$$\widetilde{\Phi}_a^{(-1)} = T_\phi\big(\Phi_a^{(-1)}\big) \quad , \quad \widetilde{\Psi}_a^{(-1)} = T_\phi^{*\,-1}\big(\Psi_a^{(-1)}\big) \quad \text{for} \;\; a \neq a_0 \; , \tag{228}$$

$$\widehat{\Phi}_i = -T_\psi^{*\,-1}(\Phi_i) \quad , \quad \widehat{\Psi}_i = -T_\psi(\Psi_i) \quad , \quad \psi \equiv \bar{\psi}_{b_0} \equiv \Psi_{b_0}^{(-1)} \; , \tag{229}$$

$$\widehat{\Phi}_{b_0}^{(-1)} = -\frac{1}{\psi} \equiv -\frac{1}{\Psi_{b_0}^{(-1)}} \quad , \quad \widehat{\Psi}_{b_0}^{(-1)} = -T_\psi\Big(\mathcal{L}\left(\Psi_{b_0}^{(-1)}\right)\Big) \; , \tag{230}$$

$$\widehat{\Phi}_a^{(-1)} = -T_\psi^{*\,-1}\big(\Phi_a^{(-1)}\big) \quad , \quad \widehat{\Psi}_a^{(-1)} = -T_\psi\big(\Psi_a^{(-1)}\big) \quad \text{for} \;\; a \neq b_0 \; . \tag{231}$$

4.7.2 Iterations of Darboux-Bäcklund Transformations. Relations to Generalized Toda Lattice Models

The general Darboux-Bäcklund orbit consists of successive applications of the allowed (adjoint) DB transformations (220)–(231). During iteration of the these DB transformations the following generalization of Wronskian determinants appears:

$$\widetilde{W}_{m;l}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_l] = \det \left\| \begin{matrix} \phi_1 & \ldots & \phi_m \\ \partial\phi_1 & \ldots & \partial\phi_m \\ \vdots & \ddots & \vdots \\ \partial^{m-l-1}\phi_1 & \ldots & \partial^{m-l-1}\phi_m \\ \partial^{-1}(\phi_1\psi_1) & \ldots & \partial^{-1}(\phi_m\psi_1) \\ \vdots & \ddots & \vdots \\ \partial^{-1}(\phi_1\psi_l) & \ldots & \partial^{-1}(\phi_m\psi_l) \end{matrix} \right\| \quad , \quad m \geq l \; , \tag{232}$$

the ordinary Wronskians being:

$$W_m[\phi_1,\ldots,\phi_m] \equiv \det \left\| \partial^{\alpha-1}\phi_\beta \right\|_{\alpha,\beta=1,\ldots,m} \; . \tag{233}$$

The simplest type of DB orbit consists of iterations of DB transformations w.r.t. any of the eigenfunctions $\phi = \Phi_j$ entering the pseudo-differential part of $\mathcal{L}$ (choice (218); see (220)–(221)). The latter has the following explicit form [C10]:

$$\widetilde{\Phi}_i^{(kM+l)} = \frac{W_{kM+l+1}\left[\Phi_1,\ldots,\Phi_M,\Phi_1^{(1)},\ldots,\Phi_M^{(1)},\ldots,\Phi_1^{(k-1)},\ldots,\Phi_M^{(k-1)},\Phi_1^{(k)},\ldots,\Phi_l^{(k)},\Phi_i^{(k\pm)}\right]}{W_{kM+l}\left[\Phi_1,\ldots,\Phi_M,\Phi_1^{(1)},\ldots,\Phi_M^{(1)},\ldots,\Phi_1^{(k-1)},\ldots,\Phi_M^{(k-1)},\Phi_1^{(k)},\ldots,\Phi_l^{(k)}\right]}$$
$$\Phi_i^{(k_+)} \equiv \Phi_i^{(k+1)} \quad \text{for} \;\; 1 \leq i \leq l \qquad ; \qquad \Phi_i^{(k_-)} \equiv \Phi_i^{(k)} \quad \text{for} \;\; l+1 \leq i \leq m \tag{234}$$

Here the following notations are used: the superscripts in the DB-transformed objects $\widetilde{\Phi}_i^{(N)}$, $\widetilde{\tau}^{(N)}$ *etc.* indicate N-th step of DB iteration; $\Phi_i^{(N)}$ are the same as in (175).

Accordingly, the DB-iteration of the τ functions reads (cf. (215)):

$$\frac{\widetilde{\tau}^{(km+l)}}{\tau} = W_{kM+l}\left[\Phi_1,\ldots,\Phi_M,\Phi_1^{(1)},\ldots,\Phi_M^{(1)},\ldots,\Phi_1^{(k-1)},\ldots,\Phi_M^{(k-1)},\Phi_1^{(k)},\ldots,\Phi_l^{(k)}\right] \tag{235}$$

As shown in [C10], any DB orbit of $\mathsf{cKP}_{R,1}$ hierarchy, *i.e.*, (146) for any $R \geq 2$ and $M = 1$), defines a structure of *two-dimensional Toda lattice* model [53] (here $\Phi \equiv \Phi_1$, $\Psi \equiv \Psi_1$):

$$\partial_x \frac{\partial}{\partial t_R} \ln \widetilde{\Phi}^{(N)} = \frac{\widetilde{\Phi}^{(N+1)}}{\widetilde{\Phi}^{(N)}} - \frac{\widetilde{\Phi}^{(N)}}{\widetilde{\Phi}^{(N-1)}} \qquad , \quad N = 0,1,2,\ldots \tag{236}$$
$$\widetilde{\Phi}^{(-1)} \equiv \Psi^{-1}$$
$$\partial_x \frac{\partial}{\partial t_R} \ln \widetilde{\tau}^{(N)} = \frac{\widetilde{\tau}^{(N+1)}\widetilde{\tau}^{(N-1)}}{\left(\widetilde{\tau}^{(N)}\right)^2} \qquad , \quad N = 1,2,\ldots \tag{237}$$

Indeed, introducing variables ψ_N, such that $\widetilde{\Phi}^{(N)} = \exp\{\psi_{N+1} - \psi_N\}$ and $\psi_N = 0$ for $N = -1,-2,\ldots$, the recurrence relation (236) for the $\mathsf{cKP}_{R,1}$ eigenfunctions turns into the familiar two-dimensional Toda lattice equations of motion, whereas the recurrence relation eq.(237) for the corresponding N-step DB-iterated $\mathsf{cKP}_{R,1}$ τ-function $\widetilde{\tau}^{(N)}(t)$ coincides with the equation for a (partial) two-dimensional Toda lattice τ-function. In the simplest $R = 1$ case (recall $t_1 \equiv x$) eqs.(236),(237) degenerate into *one-dimensional Toda lattice* structure:

$$\partial_x^2 \psi_N = e^{\psi_{N+1}-\psi_N} - e^{\psi_N - \psi_{N-1}} \quad , \quad \partial_x^2 \ln \widetilde{\tau}^{(N)} = \frac{\widetilde{\tau}^{(N+1)}\widetilde{\tau}^{(N-1)}}{\left(\widetilde{\tau}^{(N)}\right)^2} \tag{238}$$

Another more complicated example of DB-orbit is the so called “binary” DB- orbit defined by N successive DB iterations (220)–(221) plus K successive adjoint-DB iterations (223)–(224) where $\phi = \Phi_a$, $\psi = \Psi_b$ with a, b fixed. Using again notations (175) and the additional short-hand notations for the squared eigenfunction potentials (recall (140)):

$$S_{ab}^{(n,k)} \equiv \partial^{-1}\Big(\mathcal{L}^n(\Phi_a)\,\left(\mathcal{L}^*\right)^k(\Psi_b)\Big) \tag{239}$$

we obtain for the DB-orbit $\widetilde{\tau}^{N,K}$ ot the tau-function [C11]:

$$\frac{\widetilde{\tau}^{(N,K)}}{\tau} = \left((-1)^K \det_K \left\| S_{ab}^{(i-1,j-1)} \right\|\right)^{-(N-K-1)} \times$$
$$W_{N-K}\left[\det_{K+1}\left\| \begin{matrix} S_{ab}^{(i-1,j-1)} & \Phi_a^{(i-1)} \\ S_{ab}^{(K,j-1)} & \Phi_a^{(K)} \end{matrix} \right\|, \ldots, \det_{K+1}\left\| \begin{matrix} S_{ab}^{(i-1,j-1)} & \Phi_a^{(i-1)} \\ S_{ab}^{(N-1,j-1)} & \Phi_a^{(N-1)} \end{matrix} \right\|\right] \tag{240}$$
$$\text{for } N \geq K \quad , \quad i,j = 1,\ldots,K$$

$$\frac{\widetilde{\tau}^{(N,K)}}{\tau} = \left((-1)^N \det_N \left\| S_{ab}^{(i-1,j-1)} \right\|\right)^{-(K-N-1)} \times$$
$$W_{K-N}\left[\det_{N+1}\left\| \begin{matrix} S_{ab}^{(i-1,j-1)} & S_{ab}^{(i-1,N)} \\ {\Psi_b}^{(j-1)} & {\Psi_b}^{(N)} \end{matrix} \right\|, \ldots, \det_{N+1}\left\| \begin{matrix} S_{ab}^{(i-1,j-1)} & S_{ab}^{(i-1,K-1)} \\ {\Psi_b}^{(j-1)} & {\Psi_b}^{(K-1)} \end{matrix} \right\|\right] \tag{241}$$
$$\text{for } N \leq K \quad , \quad i,j = 1,\ldots,K$$

Note that the entries in the Wronskians in eqs.(240)–(241) are themselves *generalized Wronskian* determinants of the type (232).

In Ref.[C11] we have shown that the (N,K) binary Darboux-Bäcklund orbit of $\mathsf{cKP}_{R,2}$ hierarchy (Eq.(146) with R arbitrary and $M=2$, $\Phi_a \equiv \Phi_1$, $\Psi_b \equiv \Psi_2$) defines a *two-dimensional Toda square-lattice* system which describes two coupled ordinary two-dimensional Toda-lattice models corresponding to the "horizontal" $(N,0)$ and the "vertical" $(0,K)$ one-dimensional sublattices of the (N,K) binary DB square-lattice. Namely $\widetilde{\tau}^{(N,K)}$ satisfies the equation:

$$\partial_x \frac{\partial}{\partial t_R} \ln \widetilde{\tau}^{(N,K)} = \frac{\widetilde{\tau}^{(N+1,K)}\widetilde{\tau}^{(N-1,K)} - \widetilde{\tau}^{(N,K+1)}\widetilde{\tau}^{(N,K-1)}}{\left(\widetilde{\tau}^{(N,K)}\right)^2} \tag{242}$$

Furthermore, from (242) and relations (cf. the general relation (215)):

$$\widetilde{\Phi}_1^{(N,K)} = \frac{\widetilde{\tau}^{(N+1,K)}}{\widetilde{\tau}^{(N,K)}} \quad , \quad \widetilde{\Psi}_2^{(N,K)} = \frac{\widetilde{\tau}^{(N,K+1)}}{\widetilde{\tau}^{(N,K)}} \tag{243}$$

we get obtain a system of coupled equations of motion for $\widetilde{\Phi}_1^{(N,K)}$ and $\widetilde{\Psi}_2^{(N,K)}$:

$$\partial_x \frac{\partial}{\partial t_R} \ln \widetilde{\Phi}_1^{(N,K)} = \left(\frac{\widetilde{\Phi}_1^{(N+1,K)}}{\widetilde{\Phi}_1^{(N,K)}} - \frac{\widetilde{\Phi}_1^{(N,K)}}{\widetilde{\Phi}_1^{(N-1,K)}} \right) - \left(\frac{\widetilde{\Psi}_2^{(N+1,K)}}{\widetilde{\Psi}_2^{(N+1,K-1)}} - \frac{\widetilde{\Psi}_2^{(N,K)}}{\widetilde{\Psi}_2^{(N,K-1)}} \right) \tag{244}$$
$$\partial_x \frac{\partial}{\partial t_R} \ln \widetilde{\Psi}_2^{(N,K)} = -\left(\frac{\widetilde{\Psi}_2^{(N,K+1)}}{\widetilde{\Psi}_2^{(N,K)}} - \frac{\widetilde{\Psi}_2^{(N,K)}}{\widetilde{\Psi}_2^{(N,K-1)}} \right) + \left(\frac{\widetilde{\Phi}_1^{(N,K+1)}}{\widetilde{\Phi}_1^{(N-1,K+1)}} - \frac{\widetilde{\Phi}_1^{(N,K)}}{\widetilde{\Phi}_1^{(N-1,K)}} \right) \tag{245}$$

which represent a square-lattice generalization of the two-dimensional Toda model on one-dimensional lattice (237).

4.7.3 Relation to Random Matrix Models

Using the spectral representation for (adjoint) eigenfunctions (138) together with (131), as well as the following form of the *Fay identity* for τ-functions [54] :

$$\det_n \left\| \frac{\tau\big(t+[\lambda_i^{-1}]-[\mu_j^{-1}]\big)}{(\lambda_i-\mu_j)\tau(t)} \right\| = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{i>j}(\lambda_i-\lambda_j)(\mu_i-\mu_j)}{\prod_{i,j}(\lambda_i-\mu_j)} \frac{\tau\Big(t+\sum_l[\lambda_l^{-1}]-\sum_l[\mu_l^{-1}]\Big)}{\tau(t)} \tag{246}$$

we obtain an equivalent "spectral" representation for the DB-iterated $\widetilde{\tau}^{(N,N)}(t)$ (Eq.(240) with $K=N$):

$$\widetilde{\tau}^{(N,N)}(t) = \frac{(-1)^{\frac{N(N-1)}{2}}}{(N!)^2} \int \prod_{j=0}^{N-1} d\lambda_j\, d\mu_j \prod_{i>j} (\lambda_i-\lambda_j)\left(\lambda_i^R-\lambda_j^R\right) \prod_{j=0}^{N-1}\Big(\varphi_b^*(\lambda_j)e^{-\xi(t,\lambda_j)}\Big) \times$$

$$\frac{1}{\prod_{i,j}(\lambda_i-\mu_j)} \prod_{i>j}(\mu_i-\mu_j)\left(\mu_i^r-\mu_j^r\right) \prod_{j=0}^{N-1}\Big(\varphi_a(\mu_j)e^{\xi(t,\mu_j)}\Big)\,\tau\Big(t+\sum_l[\lambda_l^{-1}]-\sum_l[\mu_l^{-1}]\Big)\,. \tag{247}$$

In Eq.(247) $\xi(t,\lambda)$ is the same as in (131).

In the light of the deep relationship between continuum integrable KP-type hierarchies $\mathsf{cKP}_{R,M}$ and random (multi-)matrix models (see [C5,C6,C7,C10] and references therein) we can interpret, following Ref.[55], the τ-function (247) as a partition function of certain random multi-matrix ensemble with the following joint distribution function of eigenvalues:

$$Z_N[\{t\}] \equiv const\; \tau^{(N,N)}(t) = \int \prod_{j=0}^{N-1} d\lambda_j\, d\mu_j\, \exp\{-H(t;\{\lambda\},\{\mu\})\} \tag{248}$$

$$H(t;\{\lambda\},\{\mu\}) \equiv \sum_j \left(\bar{H}_1(\lambda_j)+H_1(\mu_j)\right) + \sum_{i>j}\left(H_2(\lambda_i,\lambda_j)+H_2(\mu_i,\mu_j)\right)$$
$$+\sum_{i,j}\widetilde{H}_2(\lambda_i,\mu_j)+H_n(\{\lambda\},\{\mu\})\;, \tag{249}$$

where the one-, two- and many-body Hamiltonians read, respectively:

$$H_1(\lambda) = -\ln\varphi(\lambda)-\xi(t,\lambda) \quad , \quad \bar{H}_1(\lambda) = -\ln\varphi^*(\lambda)+\xi(t,\lambda)\;, \tag{250}$$

$$H_2(\lambda_i,\lambda_j) = -\ln(\lambda_i-\lambda_j)^2 - \ln\Big(\sum_{s=0}^{R-1}\lambda_i^s\lambda_j^{R-1-s}\Big) \quad , \quad \widetilde{H}_2(\lambda,\mu) = \ln(\lambda-\mu)\;, \tag{251}$$

$$H_n(\{\lambda\},\{\mu\}) = -\ln\tau\Big(t+\sum_l[\lambda_l^{-1}]-\sum_l[\mu_l^{-1}]\Big)\;. \tag{252}$$

The physical implications of the above new type of joint distribution function (248)–(252) deserves further study especially regarding critical behavior of correlations. The emerging new interesting features of (248)–(252), absent in the joint distribution function derived from the conventional two-matrix model [55], are as follows:

(a) the second attractive term in the two-body potential H_2 (251) (both for λ- and μ-"particles") dominating at very long distances over the customary repulsive first term;

(b) an additional two-body attractive potential $\widetilde{H}_2$ (251) between each pair of λ- and μ-"particles";

(c) a genuine many-body potential H_n (252).

4.7.4 Multiple-Wronskian Solutions

In refs.[C11],[26] we have succeeded to derive the explicit expressions for the DB-transformed tau-function and (adjoint) eigenfunctions of $\mathsf{cKP}_{R,M}$ hierarchies (146) for the most general DB-orbit defined by arbitrary number of iterations of all allowed DB transformations (220)–(225) or (226)–(231), respectively. Elements on the general DB-orbit are labeled by two non-negative integer-valued vectors $(\vec{m}, \vec{n})$:

$$\vec{m} \equiv \Big(m_1, \ldots, m_M, \bar{m}_2, \ldots, \bar{m}_{M+R}\Big) \; , \tag{253}$$

$$\vec{n} \equiv \Big(n_1, \ldots, n_M, \bar{n}_2, \ldots, \bar{n}_{M+R}\Big) \; , \tag{254}$$

where each entry m_s in (253) indicates m_s DB steps w.r.t. $\phi = \Phi_{i_s}$, and each entry $\bar{m}_s$ indicates $\bar{m}_s$ DB steps w.r.t. $\phi = \Phi^{(-1)}_{a_s}$. Similarly, each entry n_s in (254) indicates n_s adjoint-DB steps w.r.t. $\psi = \Psi_{i_s}$, and each entry $\bar{n}_s$ indicates $\bar{n}_s$ adjoint-DB steps w.r.t. $\psi = \Psi^{(-1)}_{a_s}$.

Remark. One of the two integers in each pair (m_s, n_s) and $(\bar{m}_s, \bar{n}_s)$ in (253)–(253) must be zero due the following simple property. A pair of successive DB and adjoint-DB transformations of the first type (220)–(225) w.r.t. $\phi \equiv \Phi_{i_0}$ and $\psi \equiv \Psi_{i_0}$ (with *equal* subscripts i_0), respectively, yield an *identity* combined transformation due to (220). Similar property applies also to the pair of successive DB and adjoint-DB transformations of the second type (226)–(231) w.r.t. $\phi \equiv \Phi^{-1}_{a_0}$ and $\psi \equiv \Psi^{-1}_{a_0}$, respectively, due to (227).

We will also use further short-hand notations beyond those in (175),(176) :

$$\begin{gathered} \{\phi_1, \ldots, \phi_m\} \equiv \{\Phi\left(\vec{m}\right)\} = \\ \Big\{\Phi^{(1)}_1, \ldots, \Phi^{(m_1)}_1; \ldots; \Phi^{(1)}_M, \ldots, \Phi^{(m_M)}_M; \Phi^{(-1)}_2, \ldots, \Phi^{(-\bar{m}_2)}_2; \ldots; \Phi^{(-1)}_{\bar{M}}, \ldots, \Phi^{(-\bar{m}_{\bar{M}})}_{\bar{M}}\Big\} \; , \end{gathered} \tag{255}$$

$$\begin{gathered} \{\psi_1, \ldots, \psi_n\} \equiv \{\Psi\left(\vec{n}\right)\} = \\ \Big\{\Psi^{(1)}_1, \ldots, \Psi^{(n_1)}_1; \ldots; \Psi^{(1)}_M, \ldots, \Psi^{(n_M)}_M; \Psi^{(-1)}_2, \ldots, \Phi^{(-\bar{n}_2)}_2; \ldots; \Psi^{(-1)}_{\bar{M}}, \ldots, \Phi^{(-\bar{n}_{\bar{M}})}_{\bar{M}}\Big\} \; , \end{gathered} \tag{256}$$

with:

$$|\vec{m}| \equiv \sum_{k=1}^{M} m_k + \sum_{l=2}^{M+R} \bar{m}_l \quad , \quad |\vec{n}| \equiv \sum_{k=1}^{M} n_k + \sum_{l=2}^{M+R} \bar{n}_l \quad , \quad \bar{M} \equiv M + R \; . \tag{257}$$

Using the short-hand notations from (255)–(256) and the notation for the generalized Wronskian-like determinants (232), the general DB-iteration solution for the tau-function of $\mathsf{cKP}_{R,M}$ hierarchies (146) can be written in the following compact form:

$$\frac{\tau_{(\vec{m};\vec{n})}}{\tau} = (-1)^{|\vec{n}|(|\vec{n}|-1)/2}\, \widetilde{W}_{\vec{m};\vec{n}}\Big[\{\Phi\left(\vec{m}\right)\}; \{\Psi\left(\vec{n}\right)\}\Big] \quad , \text{ for } |\vec{m}| \geq |\vec{n}| \; , \tag{258}$$

$$\frac{\tau_{(\vec{m};\vec{n})}}{\tau} = (-1)^{|\vec{n}|(|\vec{n}|-1)/2}\, \widetilde{W}_{\vec{n};\vec{m}}\Big[\{\Psi\,(\vec{n})\}\,;\{\Phi\,(\vec{m})\}\Big] \quad , \text{ for } |\vec{n}| \geq |\vec{m}| \; , \tag{259}$$

Similar expressions (ratios of generalized Wronskians $\widetilde{W}_{\vec{m};\vec{n}}\Big[\cdot;\cdot\Big]$) are obtained for the general DB-iteration solutions for the (adjoint) eigenfunctions $\Phi_{i,(\vec{m};\vec{n})}$, $\Psi_{i,(\vec{m};\vec{n})}$ (for details, see Ref.[26]).

Let us take a closer look at the DB solution for the tau-function (258) in the case of DB orbits with $\bar{m}_2 = \ldots = \bar{m}_{M+R} = 0$ and $n_1 = \ldots = n_M = 0$ in (255)–(256). In this case the expression (258) simplifies and takes the form of a *multiple-Wronskian* determinant:

$$\mathcal{W}_{\vec{m},\vec{n}}\Big[\Phi_1^{(1)},\ldots,\Phi_M^{(m_M)},\overset{(2)}{F}_{1,1},\ldots,\overset{(2)}{F}_{M,m_M},\ldots,\overset{(\bar{M})}{F}_{1,1},\ldots,\overset{(\bar{M})}{F}_{M,m_M}\Big]$$

$$= \det \left\Vert \begin{array}{cccccc} \Phi_1^{(1)} & \cdots & \Phi_1^{(m_1)} & \cdots & \cdots & \Phi_M^{(m_M)} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \partial^{m-n-1}\Phi_1^{(1)} & \cdots & \partial^{m-n-1}\Phi_1^{(m_1)} & \cdots & \cdots & \partial^{m-n-1}\Phi_M^{(m_M)} \\ \overset{(2)}{F}_{1,1} & \cdots & \overset{(2)}{F}_{1,m_1} & \cdots & \cdots & \overset{(2)}{F}_{M,m_M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \overset{(2)}{\partial}{}^{\bar{n}_2-1}\overset{(2)}{F}_{1,1} & \cdots & \overset{(2)}{\partial}{}^{\bar{n}_2-1}\overset{(2)}{F}_{1,m_1} & \cdots & \cdots & \overset{(2)}{\partial}{}^{\bar{n}_2-1}\overset{(2)}{F}_{M,m_M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \overset{(\bar{M})}{F}_{1,1} & \cdots & \overset{(\bar{M})}{F}_{1,m_1} & \cdots & \cdots & \overset{(\bar{M})}{F}_{M,m_M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \overset{(\bar{M})}{\partial}{}^{\bar{n}_{\bar{M}}-1}\overset{(\bar{M})}{F}_{1,1} & \cdots & \overset{(\bar{M})}{\partial}{}^{\bar{n}_{\bar{M}}-1}\overset{(\bar{M})}{F}_{1,m_1} & \cdots & \cdots & \overset{(\bar{M})}{\partial}{}^{\bar{n}_{\bar{M}}-1}\overset{(\bar{M})}{F}_{M,m_M} \end{array} \right\Vert \tag{260}$$

where:

$$\overset{(k)}{F}_{i,N} \equiv \partial^{-1}\big(\Phi_i^{(N)}\Psi_k^{(-1)}\big) \qquad N \geq 1 \;\; , \;\; i = 1,\ldots,M \;\; , \;\; k = 2,\ldots,M+R \; , \tag{261}$$

$$\bar{M} \equiv M+R \quad , \quad \vec{m} = (m_1,\ldots,m_M) \quad , \quad \vec{n} = (\bar{n}_2,\ldots,\bar{n}_{\bar{M}}) \;\; ,$$
$$m = \sum_{i=1}^{M} m_i \quad , \quad n = \sum_{a=2}^{\bar{M}} \bar{n}_a \quad ; \quad \overset{(k)}{\partial} \equiv \partial/\partial \overset{(k)}{t}_{-1} \tag{262}$$

(recall $\overset{(k)}{t}_{-1}$ are additional symmetry flow parameters appearing in (197)–(198)).

Concluding this subsection, let us consider as the simplest non-trivial example the $cKP_{1,1}$-based extended KP-type hierarchy with Lax operator $\mathcal{L} = D + \Phi D^{-1}\Psi$. In this case the multiple Wronskian tau-function (260) simplifies to the following double-Wronskian form:

$$\mathcal{W}_{m,n} = \det \left\| \begin{array}{ccc} \Phi^{(1)} & \cdots & \Phi^{(m)} \\ \vdots & \ddots & \vdots \\ \partial^{m-n-1}\Phi^{(1)} & \cdots & \partial^{m-n-1}\Phi^{(m)} \\ F_1 & \cdots & F_m \\ \vdots & \ddots & \vdots \\ \bar{\partial}^{n-1}F_1 & \cdots & \bar{\partial}^{n-1}F_m \end{array} \right\| , \tag{263}$$

where:

$$\Phi^{(l)} \equiv \Phi_1^{(l)} \quad , \quad F_l \equiv \overset{(2)}{F_l} \quad , \quad m \equiv m_1 \;\; , \;\; n \equiv \bar{n}_2 \quad , \quad \bar{t}_l \equiv \overset{(2)}{t_l} \;\; , \;\; \bar{\partial} \equiv \overset{(2)}{\partial} \; , \tag{264}$$

$$\Phi^{(l)}(t) = \int d\lambda\, \lambda^l \varphi(\lambda) e^{\xi(t,\lambda)} \quad , \quad F_l(\bar{t}) = \int d\mu f_l(\mu) e^{\xi(t,\mu)} \; . \tag{265}$$

For a special delta-function choice of the spectral densities of $\Phi^{(l)}(t)$ and $F_l(\bar{t})$ (265), the double Wronskian tau-function (263) coincides with the double Wronskian tau-function of the last Ref.[41] which describes the well-known (multi-)dromion solutions of Davey-Stewartson system (209)–(211).

4.8 Algebraic (Drinfeld-Sokolov) Formulation of $\mathbf{cKP_{R,M}}$ Hierarchies and Its Relation to the Sato Formulation

Let us very briefly recall the basics of the algebraic (generalized) *Drinfeld-Sokolov* construction of integrable hierarchies [56] (see also Refs.[57]). The main ingredients are:

- Loop (or Kac-Moody) algebra with integral grading $\widehat{\mathcal{G}} = \oplus_{n\in\mathbb{Z}} \mathcal{G}^{(n)}$;
- Fixed semisimple element $E \in \mathcal{G}$ of positive grade, *e.g.* of grade 1 ($E \equiv E^{(1)}$ – the case to be considered below), meaning that $\widehat{\mathcal{G}}$ splits in a direct sum (as vector space) $\widehat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{K} \equiv \mathrm{Ker}\big(ad(E)\big)$ and $\mathcal{M} \equiv \mathrm{Im}\big(ad(E)\big)$ with $\Big[\mathcal{K}\, ,\, \mathcal{K}\Big] \subset \mathcal{K}$, $\Big[\mathcal{K}\, ,\, \mathcal{M}\Big] \subset \mathcal{M}$;
- Dynamical field $A \in \mathcal{M}$ of non-negative grade smaller than the grade of E, *e.g.* of grade 0 ($A \equiv A^{(0)}$ – the case to be considered below).

The next basic object is the transfer (monodromy) matrix T taking values in the corresponding group $\widehat{G}$ and satisfying the linear *matrix Lax* equation:

$$\left(\partial + E^{(1)} + A^{(0)}\right) T = 0 \; . \tag{266}$$

The isospectral flows are given by:

$$\left(\frac{\partial}{\partial t_n} + \left(T E^{(n)} T^{-1}\right)_+\right) T = 0 \quad , \quad \text{for } n \geq 2 \; , \tag{267}$$

where $E^{(n)}$ are elements of grade n belonging to the center of $\mathcal{K}$, and the subscripts $(\pm)$ will denote in the present subsection projection on the positive/negative-grade part of the pertinent (sub)spaces of the loop algebra $\widehat{\mathcal{G}}$.

The transfer matrix T can be represented as:

$$T = U\,S\,\exp\Bigl\{-xE^{(1)} - \sum_{\ell\geq 2} t_\ell E^{(\ell)}\Bigr\}\;, \tag{268}$$

where U and S are group-valued factors of the following form:

$$U = \exp\Bigl\{\sum_{j=1}^{\infty} u^{(-j)}\Bigr\} \quad , \quad u^{(-j)} \in \mathcal{M}^{(-j)}\;, \tag{269}$$

$$S = e^{\mathbf{s}} \quad , \quad \mathbf{s} = \sum_{j=1}^{\infty} \mathbf{s}^{(-j)} \in \mathcal{K}\;, \tag{270}$$

i.e., the exponential factors in (269)–(270) have asymptotic expansion in negative grades only. U and S have a well-defined meaning of "gauge-rotating" the matrix Lax operator in (266) to the "bare" one:

$$U\,S\left(\partial + E^{(1)}\right)S^{-1}U^{-1} = \partial + E^{(1)} + A^{(0)}\;. \tag{271}$$

As shown in Refs.[C16],[26], one can generalize the algebraic isospectral flow Eqs.(267) by considering "dressings" of more general constant algebraic elements $X^{(\pm n)}$ instead of $E^{(n)}$. In this way we explicitly construct within the algebraic Drinfeld-Sokolov formulation the whole algebra of additional non-isospectral flows of the algebraic integrable hierarchy (267). This additional symmetry algebra consists of a loop algebra isomorphic to $\mathcal{K}_+ \oplus \widehat{\mathcal{G}}_-$ and the Virasoro one.

Let us now consider the algebraic Drinfeld-Sokolov formulation of constrained $\mathsf{cKP}_{R,M}$ hierarchies (146). For simplicity here we will take the subclass of $\mathbf{cKP}_{1,M}$ hierarchies with Sato pseudo-differential Lax operator:

$$\mathcal{L}_{1,M} = D + \sum_{i=1}^{M} \Phi_i D^{-1} \Psi_i\;. \tag{272}$$

Let us introduce the column vector:

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_M \\ \psi_{M+1} \end{pmatrix} = e^{-\frac{1}{M+1}\xi(t,\lambda)} \begin{pmatrix} S_M \\ \vdots \\ S_1 \\ \psi_{BA} \end{pmatrix}\;, \tag{273}$$

where the following short-hand notations for the squared eigenfunction potentials are used (recall (140)) :

$$S_i(t,\lambda) \equiv \partial^{-1}(\psi_{BA}(t,\lambda)\Psi_i(t)) = \frac{1}{\lambda}\psi_{BA}(t,\lambda)\Psi_i(t-[\lambda^{-1}])\;,$$
$$S_i^*(t,\lambda) \equiv \partial^{-1}(\psi_{BA}^*(t,\lambda)\Phi_i(t)) = -\frac{1}{\lambda}\psi_{BA}^*(t,\lambda)\Phi_i(t+[\lambda^{-1}])\;, \tag{274}$$

with $i = 1, \ldots, M$. One can show that the linear spectral Lax equation (first Eq.(130)) for the BA wave function ψ_{BA} of (272) within the Sato formulation can be represented in the following algebraic (generalized Drinfeld-Sokolov) form:

$$\Big[D + E + A\Big] \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_M \\ \psi_{M+1} \end{pmatrix} = 0 \; , \tag{275}$$

with:

$$E \equiv E^{(1)} = \frac{\lambda}{M+1} \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -M \end{pmatrix} \equiv H^{(1)}_{\lambda_M} \; ,$$

$$A \equiv A^{(0)} = \begin{pmatrix} 0 & \cdots & 0 & -\Psi_M \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\Psi_1 \\ \Phi_M & \cdots & \Phi_1 & 0 \end{pmatrix} \; . \tag{276}$$

Here the underlying loop algebra is $\widehat{SL}(M+1)$ with standard homogeneous gradation $Q = \lambda\partial/\partial\lambda$ and the corresponding kernel $\mathcal{K} \equiv \mathrm{Ker}\big(ad(E)\big)$ and image $\mathcal{M} \equiv \mathrm{Im}\big(ad(E)\big)$ are given by:

$$\mathcal{K} = \Big\{ E^{(n)} \equiv H^{(n)}_{\lambda_M},\, H^{(n)}_1, \ldots, H^{(n)}_{M-1},\, E^{(n)}_{\pm(\alpha_{k_1}+\ldots+\alpha_{k_s})} \Big\}_{n\in\mathbb{Z}} \; , \tag{277}$$

$$\mathcal{M} = \Big\{ E^{(n)}_{\pm\alpha_M},\, E^{(n)}_{\pm(\alpha_{k_1}+\ldots+\alpha_{k_s}+\alpha_M)} \Big\}_{n\in\mathbb{Z}} \; . \tag{278}$$

λ_M is the last $SL(M+1)$ fundamental weight, $1 \leq k_1 \leq \ldots \leq k_s \leq M-1$ and $s = 1, \ldots, M-1$. The center of $\mathcal{K}$ generating the isospectral flows via (267) is $\mathcal{C}(\mathcal{K}) = \Big\{ E^{(n)} \equiv H^{(n)}_{\lambda_M} \Big\}_{n\in\mathbb{Z}}$.

In order to establish the equivalence between the algebraic Drinfeld-Sokolov and Sato formulations, we need to establish the opposite transition, *i.e.*, the transition from the basic object in the algebraic framework – the transfer matrix (266), to the objects characterizing the integrable hierarchy in Sato formalism. This transition is provided by the following formula (cf. definition (273)):

$$\begin{pmatrix} S_M \\ \vdots \\ S_1 \\ \psi_{BA} \end{pmatrix} \equiv e^{\frac{1}{M+1}\xi(t,\lambda)} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_M \\ \psi_{M+1} \end{pmatrix} = e^{\frac{1}{M+1}\xi(t,\lambda)} T \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = e^{\xi(t,\lambda)}\, U\, S \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \; , \tag{279}$$

and similarly for the adjoint objects:

$$\begin{pmatrix} S^*_M \\ \vdots \\ S^*_1 \\ \psi^*_{BA} \end{pmatrix} = e^{-\frac{1}{M+1}\xi(t,\lambda)} T^{*\,-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = e^{-\xi(t,\lambda)} (U\, S)^{*\,-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \; . \tag{280}$$

The group factors S and U in the decomposition of T (268) are of the form:

$$S = e^{\mathbf{s}(\lambda)} \quad , \quad \mathbf{s}(\lambda) = \sum_{k=1}^{M} \mathbf{s}_k(\lambda) H_k + \sum_{\beta\, ,\, \beta \neq \alpha_M} \mathbf{s}_{\pm\beta}(\lambda) E_{\pm\beta}$$
$$\longrightarrow \quad S \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = e^{-\mathbf{s}_M(\lambda)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} . \tag{281}$$

$$U = \exp \begin{pmatrix} 0 & \cdots & 0 & b_M \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & b_1 \\ a_M & \cdots & a_1 & 0 \end{pmatrix}$$
$$\longrightarrow \quad U \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \cosh(\sqrt{\vec{a}.\vec{b}}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} + \frac{\sinh(\sqrt{\vec{a}.\vec{b}})}{\sqrt{\vec{a}.\vec{b}}} \begin{pmatrix} b_M \\ \vdots \\ b_1 \\ 0 \end{pmatrix} . \tag{282}$$

Thus, for the BA wave functions (131) in the algebraic framework we obtain:

$$\psi_{BA}^{(*)}(t,\lambda) = e^{\pm\xi(t,\lambda)} \frac{\tau(t \mp [\lambda^{-1}])}{\tau(t)} = e^{\pm\xi(t,\lambda)} e^{\mp \mathbf{s}_M(\lambda)} \Big\langle (0,0,\ldots,1) \mid U^{(*-1)} \mid \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \Big\rangle \tag{283}$$

Eqs.(283), taking into account (282), imply our main result about the algebraic formalism's expression for the tau-function of $cKP_{1,M}$ hierarchy (272) :

$$\frac{\tau(t - [\lambda^{-1}])}{\tau(t + [\lambda^{-1}])} = e^{-2\mathbf{s}_M(\lambda)} , \tag{284}$$

where $\mathbf{s}_M(\lambda)$ is the coefficient in front of H_M in the expansion (281) of $\mathbf{s}(\lambda)$ in the pertinent $\mathcal{K}$ (277).

Remark. Taking the scalar product of both column vectors (279) and (280) we find the following constraint relation on the (adjoint-)BA function of $cKP_{1,M}$ hierarchy (272):

$$\psi_{BA}(t,\lambda)\psi_{BA}^{*}(t,\lambda) + \sum_{i=1}^{M} S_i(t,\lambda) S_i^{*}(t,\lambda) = 1 , \tag{285}$$

Alternatively, we can start with the general unconstrained KP hierarchy (128). Imposing the constraint relation (285) is equivalent to reduction of (128) to the $cKP_{1,M}$ hierarchy (272). Similar results hold in the more general $\mathsf{cKP}_{R,M}$ case (see Ref.[26]).

In the particular case of $cKP_{1,1}$ hierarchies, *i.e.*, with Sato Lax operator $\mathcal{L} = D + \Phi D^{-1}\Psi$ which corresponds to a (generalized) Drinfeld-Sokolov hierarchy based on $\widehat{SL}(2)$ with standard homogeneous grading and:

$$E \equiv E^{(1)} = \frac{1}{2}\lambda\sigma_3 \quad , \quad A \equiv A^{(0)} = \Phi\sigma_- - \Psi\sigma_+ \, , \tag{286}$$

we can, using (279)–(280) for $M = 1$, explicitly express all matrix elements of the transfer matrix (268) in terms of Sato-formalism objects:

$$T = \begin{pmatrix} \frac{\tau(t+[\lambda^{-1}])}{\tau(t)} & \frac{1}{\lambda}\Psi(t-[\lambda^{-1}])\frac{\tau(t-[\lambda^{-1}])}{\tau(t)} \\ \frac{1}{\lambda}\Phi(t+[\lambda^{-1}])\frac{\tau(t+[\lambda^{-1}])}{\tau(t)} & \frac{\tau(t-[\lambda^{-1}])}{\tau(t)} \end{pmatrix} e^{\left(-\frac{1}{2}\lambda x - \frac{1}{2}\sum_{j=2}^{\infty}\lambda^j t_j\right)\sigma_3} \, . \tag{287}$$

4.9 Supersymmetric Constrained KP Hierarchies. Basic Construction

4.9.1 General Supersymmetric Manin-Radul KP Hierarchy: Background and Issues

We shall use throughout the super-pseudo-differential calculus [58] with the following notations: ∂ and $\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\partial$ denote operators, whereas the symbols ∂_x and $\mathcal{D}_\theta$ will indicate application of the corresponding operators on superfield functions. As usual, (x, θ) denote superspace coordinates. For any super-pseudo-differential operator $\mathcal{A} = \sum_j a_{j/2}\mathcal{D}^j$ the subscripts ($\pm$) denote its purely differential part ($\mathcal{A}_+ = \sum_{j\geq 0} a_{j/2}\mathcal{D}^j$) or its purely pseudo-differential part ($\mathcal{A}_- = \sum_{j\geq 1} a_{-j/2}\mathcal{D}^{-j}$), respectively. For any $\mathcal{A}$ the super-residuum is defined as $\mathcal{R}es\mathcal{A} = a_{-\frac{1}{2}}$.

The general supersymmetric Manin-Radul KP hierarchy (MR-SKP) is defined through the *fermionic* Lax operator $\mathcal{L}$:

$$\mathcal{L} = \mathcal{D} + f_0 + \sum_{j=1}^{\infty} b_j \partial^{-j}\mathcal{D} + \sum_{j=1}^{\infty} f_j \partial^{-j} \tag{288}$$

expressed in terms of a *bosonic* “dressing” operator $\mathcal{W}$:

$$\mathcal{L} = \mathcal{W}\mathcal{D}\mathcal{W}^{-1} \quad , \quad \mathcal{W} = 1 + \sum_{j=1}^{\infty} \alpha_j \partial^{-j}\mathcal{D} + \sum_{j=1}^{\infty} \beta_j \partial^{-j} \tag{289}$$

where b_j, β_j are bosonic superfield functions whereas f_j, α_j are fermionic ones. The Lax evolution equations for MR-SKP w.r.t. the bosonic $\frac{\partial}{\partial t_l}$ and the *fermionic* D_n isospectral flows read:

$$\frac{\partial}{\partial t_l}\mathcal{L} = \left[\, \mathcal{L}^{2l}_+ \, , \, \mathcal{L} \,\right] \quad , \quad D_n\mathcal{L} = -\left\{\, \mathcal{L}^{2n-1}_- \, , \, \mathcal{L} \,\right\} \tag{290}$$

$$\frac{\partial}{\partial t_l}\mathcal{W} = -\left(\mathcal{W}\partial^l\mathcal{W}^{-1}\right)_- \mathcal{W} \quad , \quad D_n\mathcal{W} = -\left(\mathcal{W}\mathcal{D}^{2n-1}\mathcal{W}^{-1}\right)_- \mathcal{W} \tag{291}$$

with the short-hand notations:

$$D_n = \frac{\partial}{\partial\theta_n} - \sum_{k=1}^{\infty} \theta_k \frac{\partial}{\partial t_{n+k-1}} \quad , \quad \{ D_k \, , \, D_l \} = -2 \frac{\partial}{\partial t_{k+l-1}} \tag{292}$$

$$(t, \theta) \equiv (t_1 \equiv x, t_2, \ldots ; \theta, \theta_1, \theta_2, \ldots) \tag{293}$$

The *super-Baker-Akhiezer* (super-BA) and the *adjoint super-Baker-Akhiezer* (adjoint super-BA) wave functions are defined as:

$$\psi_{BA}(t, \theta; \lambda, \eta) = \mathcal{W}\left(\psi_{BA}^{(0)}(t, \theta; \lambda, \eta)\right) \quad , \quad \psi_{BA}^{*}(t, \theta; \lambda, \eta) = \mathcal{W}^{*-1}\left(\psi^{*(0)}_{BA}(t, \theta; \lambda, \eta)\right) \tag{294}$$

(with η being a fermionic "spectral" parameter), in terms of the "free" super-BA functions:

$$\psi_{BA}^{(0)}(t, \theta; \lambda, \eta) \equiv e^{\xi(t,\theta;\lambda,\eta)} \quad , \quad \psi^{*(0)}_{BA}(t, \theta; \lambda, \eta) \equiv e^{-\xi(t,\theta;\lambda,\eta)} \tag{295}$$

$$\xi(t, \theta; \lambda, \eta) = \sum_{l=1}^{\infty} \lambda^l t_l + \eta\theta + (\eta - \lambda\theta) \sum_{n=1}^{\infty} \lambda^{n-1} \theta_n \tag{296}$$

for which it holds:

$$\frac{\partial}{\partial t_k} \psi_{BA}^{(0)} = \partial_x^k \psi_{BA}^{(0)} \quad , \quad D_n \psi_{BA}^{(0)} = \mathcal{D}_\theta^{2n-1} \psi_{BA}^{(0)} = \partial_x^{n-1} \mathcal{D}_\theta \psi_{BA}^{(0)} \tag{297}$$

Accordingly, (adjoint) super-BA wave functions satisfy:

$$\left(\mathcal{L}^2\right)^{(*)} \psi_{BA}^{(*)} = \pm\lambda \psi_{BA}^{(*)} \quad , \quad \frac{\partial}{\partial t_l} \psi_{BA}^{(*)} = \pm \left(\mathcal{L}^{2l}\right)_+^{(*)} (\psi_{BA}^{(*)}) \quad , \quad D_n \psi_{BA}^{(*)} = \pm \left(\mathcal{L}^{2n-1}\right)_+^{(*)} (\psi_{BA}^{(*)}) \tag{298}$$

Correspondingly, the defining equations for arbitrary (adjoint-) super-eigenfunctions are :

$$\frac{\partial}{\partial t_l} \Phi = \mathcal{L}_+^{2l}(\Phi) \quad , \quad D_n \Phi = \mathcal{L}_+^{2n-1}(\Phi) \quad ; \quad \frac{\partial}{\partial t_l} \Psi = -\left(\mathcal{L}^{2l}\right)_+^{*} (\Psi) \quad , \quad D_n \Psi = -\left(\mathcal{L}^{2n-1}\right)_+^{*} (\Psi) \tag{299}$$

with supersymmetric "spectral" representations (cf. Ref.[C11]):

$$\Phi(t, \theta) = \int d\lambda \, d\eta \, \varphi(\lambda, \eta) \psi_{BA}(t, \theta; \lambda, \eta) \quad , \quad \Psi(t, \theta) = \int d\lambda \, d\eta \, \varphi^*(\lambda, \eta) \psi_{BA}^*(t, \theta; \lambda, \eta) \tag{300}$$

The *super-tau-function* $\tau(t, \theta)$ is related with the super-residues of powers of the super-Lax operator (288) as follows [C13]:

$$\mathcal{R}es \mathcal{L}^{2k} = \frac{\partial}{\partial t_k} \mathcal{D}_\theta \ln \tau \quad , \quad \mathcal{R}es \mathcal{L}^{2k-1} = D_k \mathcal{D}_\theta \ln \tau \tag{301}$$

In what follows we shall encounter objects of the form $\mathcal{D}_\theta^{-1}(\Phi\Psi) = \mathcal{D}_\theta \partial^{-1}{}_x(\Phi\Psi)$ where Φ, Ψ is a pair of super-eigenfunction and adjoint-super-eigenfunction. Similarly to the purely bosonic case (140) one can show that application of inverse derivative on such products is well-defined (upto an overall (t, θ)-independent constant). Namely, there exists [C13] a unique superfield function – *supersymmetric "squared eigenfunction potential"* (super-SEP) $S(\Phi, \Psi)$ such that: $\mathcal{D}_\theta S(\Phi, \Psi) = \Phi\Psi$. More precisely the super-SEP satisfies the relations (cf. Eqs.(139)):

$$\frac{\partial}{\partial t_k} S(\Phi, \Psi) = \mathcal{R}es\left(\mathcal{D}^{-1}\Psi\mathcal{L}^{2k}\Phi\mathcal{D}^{-1}\right) \quad , \quad D_n S(\Phi, \Psi) = \mathcal{R}es\left(\mathcal{D}^{-1}\Psi\mathcal{L}^{2n-1}\Phi\mathcal{D}^{-1}\right) \tag{302}$$

In particular, Eqs.(302) for $k = 1$ and $n = 1$ read:

$$\partial_x S(\Phi, \Psi) = \mathcal{R}es\left(\mathcal{D}^{-1}\Psi\mathcal{L}^2\Phi\mathcal{D}^{-1}\right) = \mathcal{D}_\theta(\Phi\Psi) \quad , \quad D_1 S(\Phi, \Psi) = \mathcal{R}es\left(\mathcal{D}^{-1}\Psi\mathcal{L}\Phi\mathcal{D}^{-1}\right) = \Phi\Psi \tag{303}$$

Concluding this subsubsection let us point out the following serious issue of the general MR-SKP hierarchy related to consistency of Darboux-Bäcklund transformations of the latter. Namely, consider the "gauge" transformation of $\mathcal{L}$ (288) of the form:

$$\widetilde{\mathcal{L}} = \mathcal{T}\mathcal{L}\mathcal{T}^{-1} \quad , \quad \mathcal{T} = \chi\mathcal{D}\chi^{-1} \tag{304}$$

which parallels the familiar DB-transformation in the purely bosonic case (212). Requiring the transformed Lax operator $\widetilde{\mathcal{L}}$ to obey MR-SKP evolution equations of the same form (290) as $\mathcal{L}$ implies that $\mathcal{T}$ must satisfy:

$$\frac{\partial}{\partial t_l}\mathcal{T}\,\mathcal{T}^{-1} + \left(\mathcal{T}\mathcal{L}^{2l}_{+}\mathcal{T}^{-1}\right)_{-} = 0 \quad , \quad D_n\mathcal{T}\,\mathcal{T}^{-1} - \left(\mathcal{T}\mathcal{L}^{2n-1}_{+}\mathcal{T}^{-1}\right)_{-} = -2\left(\widetilde{\mathcal{L}}^{2n-1}\right)_{-} \tag{305}$$

The first Eq.(305) is exactly analogous to the purely bosonic case (213) and implies that χ must be a super-eigenfunction (299) of $\mathcal{L}$ w.r.t. the bosonic MR-SKP flows. However, there is a problem with the second Eq.(305). Namely, for the general (unconstrained) MR-SKP hierarchy it does not have solutions for χ. In particular, if χ would be a super-eigenfunction also w.r.t. fermionic flows (cf. second Eq.(299)), then the l.h.s. of second eq.(305) would become zero whereupon we would get the contradictory relation: $\left(\widetilde{\mathcal{L}}^{2n-1}\right)_{-} = 0$.

Thus, we conclude that the DB-transformations of the general MR-SKP hierarchy preserve only the bosonic flow equations. In subsection 4.11 we will present consistent solutions of (305) in the framework of *constrained* MR-SKP systems which will be achieved thanks to a non-trivial modification of the fermionic MR-SKP flows preserving their anti-commutation algebra (292).

There is a further essential distinction of DB-transformations for MR-SKP hierarchy and its purely bosonic counterpart. Calculating the super-residues (301) of the powers of the DB-transformed super-Lax operator we obtain:

$$\mathcal{R}es\widetilde{\mathcal{L}}^s = \mathcal{D}_\theta\left(\chi^{-1}\mathcal{L}^s_{+}(\chi)\right) + (-1)^{s+1}\mathcal{R}es\mathcal{L}^s \tag{306}$$

Note the crucial sign factor in front of the second term on the r.h.s. of Eq.(306). Together with the first eq.(301) it implies for the DB-transformed super-τ function:

$$\widetilde{\tau} = \chi\,\tau^{-1} \tag{307}$$

in contrast with the bosonic case (where we have $\widetilde{\tau} = \chi\,\tau$ (215)).

4.9.2 Constrained super-KP Hierarchies $\boldsymbol{SKP}_{(R;M_B,M_F)}$ – Consistent Reductions of Manin-Radul super-KP Hierarchy

In Ref.[C13] we introduced a class of reductions of the original MR-SKP hierarchy, called $SKP_{(R;M_B,M_F)}$ constrained super-KP models, which contain the supersymmetric extensions of various basic bosonic integrable hierarchies such as (modified) Korteveg-de-Vries, nonlinear Schrödinger (AKNS hierarchy in general), Yajima-Oikawa, coupled Boussinesq-type equations *etc.* In particular, the bosonic limit of $SKP_{(R;M_B,M_F)}$ hierarchy with $R=1$ (cf. Eq.(308) below) yields new types of ordinary (non-supersymmetric) integrable models. The latter are systems of M_F-component non-linear Schrödinger hierarchies coupled to M_B-component derivative non-linear Schrödinger hierarchies in the Gerjikov-Ivanov [59] form (see Ref.[60]).

The $SKP_{(R;M_B,M_F)}$ hierarchies are defined by the following superspace Lax pseudo-differential operators:

$$\mathcal{L} \equiv \mathcal{L}_{(R;M_B,M_F)} = \mathcal{D}^R + \sum_{j=0}^{R-1} v_{\frac{j}{2}} \mathcal{D}^j + \sum_{i=1}^{M} \Phi_i \mathcal{D}^{-1} \Psi_i \quad , \quad M \equiv M_B + M_F \tag{308}$$

where $M_{B,F}$ indicate the number of bosonic/fermionic super-eigenfunctions Φ_i (299) entering the purely pseudo-differential part of $\mathcal{L}_{(R;M_B,M_F)}$. $SKP_{(R;M_B,M_F)}$ hierarchies defined by fermionic/bosonic super-Lax operators (308), for which $R \equiv 2r+1,\; M \equiv M_B + M_F \equiv 2N+1$ and $R \equiv 2r,\; M \equiv M_B + M_F \equiv 2N$, respectively, will be called in what follows "fermionic"/"bosonic" hierarchies for brevity.

For the simplest non-trivial constrained super-KP hierarchy $SKP_{1;1,0}$ the corresponding super-Lax operator $\mathcal{L} \equiv \mathcal{L}_{1;1,0}$ reads:

$$\mathcal{L} \equiv \mathcal{L}_{1;1,0} = \mathcal{D} + f_0 + \Phi \mathcal{D}^{-1} \Psi \tag{309}$$

where Φ , Ψ are bosonic (adjoint-)super-eigenfunction (299) of $\mathcal{L}$ (309).

One of the main results in [C13] was to show that the original fermionic flows D_n (290) for the general unconstrained MR-SKP hierarchy *do not* anymore define consistent flows on the space of fermionic constrained $SKP_{(R;M_B,M_F)}$ hierarchies, *i.e.*, the odd flows D_n (290) do not preserve the constrained form of fermionic super-Lax operators $\mathcal{L} \equiv \mathcal{L}_{(R;M_B,M_F)}$ (308). In Ref.[C13] we found the following consistent modification for D_n :

$$D_n \mathcal{L} = -\left\{ \mathcal{L}_{-}^{2n-1} - X_{2n-1} \,,\, \mathcal{L} \right\} = \left\{ \mathcal{L}_{+}^{2n-1} + X_{2n-1} \,,\, \mathcal{L} \right\} - 2\mathcal{L}^{2n} \,, \tag{310}$$

where:

$$X_{2n-1} \equiv 2 \sum_{i=1}^{M} (-1)^{|i|} \sum_{s=0}^{n-2} \mathcal{L}^{2(n-s)-3}(\Phi_i) \mathcal{D}^{-1} \left(\mathcal{L}^{2s+1}\right)^* (\Psi_i) \,, \tag{311}$$

$$D_n \Phi_i = \mathcal{L}_{+}^{2n-1}(\Phi_i) - 2\mathcal{L}^{2n-1}(\Phi_i) + X_{2n-1}(\Phi_i) \,, \tag{312}$$

$$D_n \Psi_i = -\left(\mathcal{L}^{2n-1}\right)_{+}^{*} (\Psi_i) + 2\left(\mathcal{L}^{2n-1}\right)^{*} (\Psi_i) - \left(X_{2n-1}\right)^{*} (\Psi_i) \,, \tag{313}$$

with the power $|i|$ in Eq.(311) and below denoting the Grassmann parity of the corresponding (adjoint) super-eigenfunction Φ_i, Ψ_i. The modified fermionic isospectral flows D_n (310)–(311) obey the anti-commutation algebra:

$$\{D_n,\, D_m\} = -2\partial/\partial t_{R(n+m-1)} \tag{314}$$

where $\frac{\partial}{\partial t_l}$ are the bosonic isospectral flows for $\mathcal{L} \equiv \mathcal{L}_{(R;M_B,M_F)}$:

$$\frac{\partial}{\partial t_l}\mathcal{L} = \Big[\Big(\mathcal{L}^{\frac{2l}{R}}\Big)_+ \, , \, \mathcal{L} \Big] \tag{315}$$

In the case of bosonic constrained $SKP_{(R;M_B,M_F)}$ (*i.e.*, $R = 2r$ and $M \equiv M_B + M_F = 2N$ in (308)), we can split the set of super-eigenfunctions entering the negative pseudo-differential part of (308) into bosonic $\big\{\Phi_a, \Psi_b\big\}_{a,b=1}^{M_{B,F}}$ and fermionic $\big\{\widetilde{\Phi}_b, \widetilde{\Psi}_a\big\}_{a,b=1}^{M_{B,F}}$ subsets, respectively, so that the expression (308) acquires the form:

$$L \equiv \mathcal{L}_{(2r;M_B,M_F)}\, \big|_{M_B+M_F=2N} = L_+ + \sum_{a=1}^{M_B} \Phi_a \mathcal{D}^{-1} \widetilde{\Psi}_a + \sum_{b=1}^{M_F} \widetilde{\Phi}_b \mathcal{D}^{-1} \Psi_b \tag{316}$$

Henceforth we will use the short-hand notation L to indicate bosonic super-Lax operators (316).

We will also need the explicit expressions for inverse powers of $SKP_{(R;M_B,M_F)}$ super-Lax operators (308). Following the same lines of construction of inverse powers of KP Lax operators as in the purely bosonic case (subsection 4.3), we can represent the super-Lax operator (308) as a ratio of two purely super-differential operators $L_{\frac{1}{2}(R+M)}$ and $L_{\frac{1}{2}M}$ of orders $\frac{1}{2}(R+M)$ and $\frac{1}{2}M$, respectively:

$$\mathcal{L}_{(R;M_B,M_F)} = L_{\frac{1}{2}(R+M)} L^{-1}_{\frac{1}{2}M} = \left\{ \begin{array}{ll} L_{N+r+1} L^{-1}_{N+\frac{1}{2}} & M \equiv M_B + M_F = 2N+1 \; , \; R = 2r+1 \\ L_{N+r} L^{-1}_{N} & \qquad M \equiv M_B + M_F = 2N \; , \; R = 2r \end{array} \right. \tag{317}$$

where the first line refers to fermionic super-Lax operator and the second line refers to bosonic super-Lax operator, respectively. According to [61] (see also Ref.[C14]), any super-differential operators L_N (bosonic, of integer order) and $L_{N+\frac{1}{2}}$ (fermionic, of half-integer order) can be parametrized through the elements of their respective kernels:

$$Ker(L_N) = \Big\{\varphi_0, \varphi_{\frac{1}{2}}, \ldots, \varphi_{N-1}, \varphi_{N-\frac{1}{2}}\Big\} \quad , \quad Ker(L_{N+\frac{1}{2}}) = \Big\{\varphi_0, \varphi_{\frac{1}{2}}, \ldots, \varphi_{N-1}, \varphi_{N-\frac{1}{2}}, \varphi_N\Big\} \; . \tag{318}$$

Furthermore, as in the purely bosonic case (163) one can show that the inverse power of L_N is given as:

$$L_N^{-1} = \sum_{\alpha=1}^{N} \Big[\varphi_\alpha \mathcal{D}^{-1} \widetilde{\psi}_\alpha + \widetilde{\varphi}_\alpha \mathcal{D}^{-1} \psi_\alpha\Big] \; , \tag{319}$$

where we have split the sets of kernels elements (318) explicitly into bosonic and fermionic (indicated by "tilde") subsets.

With the notations in (317)–(319) the inverse of bosonic super-Lax operator $L \equiv \mathcal{L}_{(R;M_B,M_F)}$ (316), where $R = 2r$ and $M \equiv M_B + M_F = 2N$, reads:

$$L^{-1} = L_N L^{-1}_{N+r} = \sum_{\beta=1}^{N+r} \Big[L_N(\varphi_\beta) \mathcal{D}^{-1} \widetilde{\psi}_\beta + L_N(\widetilde{\varphi}_\beta) \mathcal{D}^{-1} \psi_\beta\Big] \tag{320}$$

where the sets of super-functions $\{\varphi_\beta, \widetilde{\varphi}_\beta\}_{\beta=1}^{N+r}$ and $\left\{\psi_\beta, \widetilde{\psi}_\beta\right\}_{\beta=1}^{N+r}$ span the kernels $Ker(L_{N+r})$ and $Ker(L^*_{N+r})$, respectively. For later convenience it is useful to introduce the following short hand notations:

$$\Phi_a^{(m)} \equiv L^{m-1}(\Phi_a) \;\; ; \;\; \Psi_b^{(m)} \equiv \left(L^{m-1}\right)^* (\Psi_b) \;\; ; \;\; \widetilde{\Phi}_b^{(m)} \equiv L^{m-1}\big(\widetilde{\Phi}_b\big) \;\; ; \;\; \widetilde{\Psi}_a^{(m)} \equiv \left(L^{m-1}\right)^* \big(\widetilde{\Psi}_a\big) \tag{321}$$

$$\Phi_\beta^{(-m)} \equiv L^{-(m-1)}\big(L_N(\varphi_\beta)\big) \quad , \quad \Psi_\beta^{(-m)} \equiv \left(L^{-(m-1)}\right)^* (\psi_\beta) \tag{322}$$

$$\widetilde{\Phi}_\beta^{(-m)} \equiv L^{-(m-1)}\big(L_N(\widetilde{\varphi}_\beta)\big) \quad , \quad \widetilde{\Psi}_\beta^{(-m)} \equiv \left(L^{-(m-1)}\right)^* (\widetilde{\psi}_\beta) \tag{323}$$

where $m = 1, 2, \ldots$, $a = 1, \ldots, M_B$, $b = 1, \ldots, M_F$ and $\beta = 1, 2, \ldots, N + r$.

Accordingly, the explicit expression for the inverse power of fermionic super-Lax operator (308) reads:

$$\begin{aligned} \mathcal{L}^{-1} &= L_{N+\frac{1}{2}} L^{-1}_{N+r+1} = \\ &= \sum_{\alpha=1}^{N+r+1} \Big[L_{N+\frac{1}{2}}(\varphi_\alpha) \mathcal{D}^{-1} \widetilde{\psi}_\alpha + L_{N+\frac{1}{2}}(\widetilde{\varphi}_\alpha) \mathcal{D}^{-1} \psi_\alpha \Big] \end{aligned} \tag{324}$$

where:

$$\{\varphi_\alpha, \widetilde{\varphi}_\alpha\}_{\alpha=1}^{N+r+1} \equiv \{\varphi_I\}_{I=1}^{2(N+r+1)} \quad , \quad \left\{\psi_\alpha, \widetilde{\psi}_\alpha\right\}_{\alpha=1}^{N+r+1} \equiv \{\psi_I\}_{I=1}^{2(N+r+1)} \tag{325}$$

span the kernels $Ker(L_{N+r+1})$ and $Ker(L^*_{N+r+1})$, respectively, and where we have introduced further short-hand notations analogous to (321)–(323):

$$\phi_I \equiv L_{N+\frac{1}{2}}(\varphi_I) \quad , \quad \phi_I^{(-\ell/2)} \equiv \mathcal{L}^{-\ell}(\phi_I) \quad , \quad \psi_I^{(-\ell/2)} \equiv \left(\mathcal{L}^{-\ell}\right)^* (\psi_I) \tag{326}$$

$$\Phi_i^{(\ell/2)} \equiv \mathcal{L}^{\ell}(\Phi_i) \quad , \quad \Psi_i^{(\ell/2)} \equiv \left(\mathcal{L}^{\ell}\right)^* (\Psi_i) \tag{327}$$

4.10 Supersymmetric Constrained KP Hierarchies. Additional Symmetries

A systematic and detailed derivation of the additional non-isospectral symmetry algebra of $SKP_{(R;M_B,M_F)}$ hierarchies (as well as of the general unconstrained supersymmetric KP hierarchy) is given in Ref.[C17]. The logic of derivation follows the pattern for the corresponding ordinary (constrained) KP hierarchies (subsection 4.4). However, particular care has to be applied in the treatment of the fermionic additional symmetries as it was the case with the non-trivial modification of the fermionic Manin-Radul isospectral flows (subsubsection 4.9.2).

4.10.1 Superloop Superalgebra Symmetries of Constrained SKP Hierarchies

We now proceed by constructing the explicit form of additional symmetry flows in the case of fermionic constrained $SKP_{(R;M_B,M_F)}$ hierarchies (308), *i.e.*, with super-Lax operators $\mathcal{L} \equiv \mathcal{L}_{(R;M_B,M_F)}$ being fermionic (recall $R = 2r + 1$, $M \equiv M_B + M_F = 2N + 1$) :

$$\delta_{\mathcal{A}}^{(\ell/2)} \mathcal{L} = \Big[\mathcal{M}_{\mathcal{A}}^{(\ell/2)} \, , \, \mathcal{L} \Big] \quad , \quad \delta_{\mathcal{F}}^{(\ell/2)} \mathcal{L} = \Big\{ \mathcal{M}_{\mathcal{F}}^{(\ell/2)} , \, \mathcal{L} \Big\} \quad , \quad \ell = 1, 2, \ldots \tag{328}$$

where $\mathcal{M}^{(\ell/2)}_{\mathcal{A},\mathcal{F}}$ are super-pseudo-differential operators of the form:

$$\mathcal{M}^{(\ell/2)}_{\mathcal{A}} \equiv \sum_{i,j=1}^{M} \mathcal{A}^{(\ell/2)}_{ij} \sum_{s=0}^{\ell-1} (-1)^{s(|j|+\ell)} \Phi_j^{(\ell-1-s)/2} \mathcal{D}^{-1} \Psi_i^{(s/2)} \tag{329}$$

$$\mathcal{M}^{(\ell/2)}_{\mathcal{F}} \equiv \sum_{i,j=1}^{M} \mathcal{F}^{(\ell/2)}_{ij} \sum_{s=0}^{\ell-1} (-1)^{s(|j|+\ell)} \Phi_j^{(\ell-1-s)/2} \mathcal{D}^{-1} \Psi_i^{(s/2)} \tag{330}$$

Here and below the short-hand notations (321)–(323) and (326)–(327) are used. In (329)–(330) $\mathcal{A}^{(\ell/2)}$ and $\mathcal{F}^{(\ell/2)}$ are graded constant matrices specified below.

The flows (328) are well-defined, namely, they preserve the specific constrained form of the superspace Lax operator (308) provided the action of these flows on the constituent (adjoint) super-eigenfunctions is given by:

$$\delta^{(\ell/2)}_{\mathcal{A}} \Phi_i^{(m/2)} = \mathcal{M}^{(\ell/2)}_{\mathcal{A}} (\Phi_i^{(m/2)}) - \sum_{j=1}^{M} \mathcal{A}^{(\ell/2)}_{ij} \Phi_j^{((\ell+m)/2)} \tag{331}$$

$$\delta^{(\ell/2)}_{\mathcal{A}} \Psi_i^{(m/2)} = -\left(\mathcal{M}^{(\ell/2)}_{\mathcal{A}}\right)^* (\Psi_i^{(m/2)}) + \sum_{j=1}^{M} (-1)^{\ell(|j|+m-1)} \mathcal{A}^{(\ell/2)}_{ji} \Psi_j^{((\ell+m)/2)} \tag{332}$$

$$\delta^{(\ell/2)}_{\mathcal{F}} \Phi_i^{(m/2)} = \mathcal{M}^{(\ell/2)}_{\mathcal{F}} (\Phi_i^{(m/2)}) + (-1)^{m-1} \sum_{j=1}^{M} \mathcal{F}^{(\ell/2)}_{ij} \Phi_j^{((\ell+m)/2)} \tag{333}$$

$$\delta^{(\ell/2)}_{\mathcal{F}} \Psi_i^{(m/2)} = -\left(\mathcal{M}^{(\ell/2)}_{\mathcal{F}}\right)^* (\Psi_i^{(m/2)}) - \sum_{j=1}^{M} (-1)^{(\ell+1)(|j|+m-1)} \mathcal{F}^{(\ell/2)}_{ji} \Psi_j^{((\ell+m)/2)} \tag{334}$$

The graded constant matrices are of the following types:

(a) For $\ell = 2n$ the matrices $\mathcal{A}^{(n)}$ and $\mathcal{F}^{(n)}$ are purely bosonic and purely fermionic elements, respectively, belonging (as a vector space) to the superalgebra $GL(M_B, M_F)$ of graded $(M_B, M_F) \times (M_B, M_F)$ matrices:

$$\mathcal{A}^{(n)} = \begin{pmatrix} A^{(n)} & 0 \\ 0 & D^{(n)} \end{pmatrix} \quad , \quad \mathcal{F}^{(n)} = \begin{pmatrix} 0 & B^{(n)} \\ C^{(n)} & 0 \end{pmatrix} \tag{335}$$

Here the block matrices $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ and $D^{(n)}$ are of sizes $M_B \times M_B$, $M_B \times M_F$, $M_F \times M_B$ and $M_F \times M_F$, respectively.

(b) For $\ell = 2n-1$ the matrices $\mathcal{A}^{(n-\frac{1}{2})}$ and $\mathcal{F}^{(n-\frac{1}{2})}$ are purely fermionic and purely bosonic elements, respectively, belonging (as a vector space) to $\widetilde{GL}(M_B, M_F)$ – the superalgebra of $(M_B, M_F) \times (M_B, M_F)$ graded matrices in the "twisted" basis (the diagonal blocks are fermionic, whereas the off-diagonal blocks are bosonic; for a general discussion of non-standard formats of matrix superalgebras, see ref.[62]) :

$$\mathcal{F}^{(n-\frac{1}{2})} = \begin{pmatrix} B^{(n-\frac{1}{2})} & 0 \\ 0 & C^{(n-\frac{1}{2})} \end{pmatrix} \quad , \quad \mathcal{A}^{(n-\frac{1}{2})} = \begin{pmatrix} 0 & A^{(n-\frac{1}{2})} \\ D^{(n-\frac{1}{2})} & 0 \end{pmatrix} \tag{336}$$

In this case the sizes of the block matrices $A^{(n-\frac{1}{2})}$, $B^{(n-\frac{1}{2})}$, $C^{(n-\frac{1}{2})}$ and $D^{(n-\frac{1}{2})}$ are $M_B \times M_F$, $M_B \times M_B$, $M_F \times M_F$ and $M_F \times M_B$, respectively.

Thus, all graded matrices (335)-(336) are special positive-grade elements of a superloop superalgebra $\widehat{GL}(M_B, M_F)$ with half-integer grading $0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$. More generally, $\widehat{GL}(N_1, N_2)$ here will denote an infinite-dimensional algebra with half-integer grading:

$$\widehat{GL}(N_1, N_2) = \oplus_{\ell \in \mathbb{Z}} GL^{(\frac{\ell}{2})}(N_1, N_2) \tag{337}$$

whose $\frac{\ell}{2}$-grade subspaces consist of super-matrices of the following form:

$$GL^{(n)}(N_1, N_2) = \left\{ \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in GL(N_1, N_2) \right\} , \tag{338}$$

$$GL^{(n-\frac{1}{2})}(N_1, N_2) = \left\{ \begin{pmatrix} B^{(n-\frac{1}{2})} & A^{(n-\frac{1}{2})} \\ D^{(n-\frac{1}{2})} & C^{(n-\frac{1}{2})} \end{pmatrix} \in \widetilde{GL}(N_1, N_2) \right\} . \tag{339}$$

From (328)–(330) we find the following infinite-dimensional algebra of flows:

$$\left[\delta^{(\ell/2)}_{\mathcal{A}_1} , \delta^{(m/2)}_{\mathcal{A}_2} \right] = \delta^{((\ell+m)/2)}_{[\mathcal{A}_1 , \mathcal{A}_2]}$$

$$\left[\delta^{(\ell/2)}_{\mathcal{A}} , \delta^{(m/2)}_{\mathcal{F}} \right] = \delta^{((\ell+m)/2)}_{[\mathcal{A}, \mathcal{F}]} \quad \text{for } \ell = \text{even} \quad , \quad \left[\delta^{(\ell/2)}_{A} , \delta^{(m/2)}_{F} \right] = -\delta^{((\ell+m)/2)}_{\{A, F\}} \quad \text{for } \ell = \text{odd}$$

$$\begin{aligned} \left\{ \delta^{(\ell/2)}_{\mathcal{F}_1}, \delta^{(m/2)}_{\mathcal{F}_2} \right\} &= \pm \delta^{((\ell+m)/2)}_{\{\mathcal{F}_1, \mathcal{F}_2\}} \quad \text{for } (\ell, m) = (odd, odd)/(even, even) \\ \left\{ \delta^{(\ell/2)}_{\mathcal{F}_1}, \delta^{(m/2)}_{\mathcal{F}_2} \right\} &= \pm \delta^{((\ell+m)/2)}_{[\mathcal{F}_1, \mathcal{F}_2]} \quad \text{for } (\ell, m) = (odd, even)/(even, odd) \end{aligned} \tag{340}$$

Recall that $\mathcal{A}^{(\ell/2)}, \mathcal{A}^{(\ell/2)}_{1,2}$ and $\mathcal{F}^{(\ell/2)}, \mathcal{F}^{(\ell/2)}_{1,2}$ are constant graded matrices of the form (335)–(336).

In particular, from the flow definitions (329)–(330) we obtain that:

$$\delta^{(n)}_{\mathcal{A}=1\!\!1} = -\frac{\partial}{\partial t_n} \quad , \quad \delta^{(n-\frac{1}{2})}_{\mathcal{F}=1\!\!1} = -D_n \tag{341}$$

are (upto an overall minus sign) the superspace isospectral flows of the corresponding $SKP_{(R;M_B,M_F)}$ hierarchy, where the fermionic isospectral flows D_n carry the relevant modification (see Eqs.(310)–(311)) in order to preserve the specific constrained form of (308).

Relations (340) show that the algebra of symmetry flows (328) for fermionic $SKP_{(R;M_B,M_F)}$ hierarchy (308) (with $R = 2r+1$, $M \equiv M_B + M_F = 2N+1$), which contains also Manin-Radul isospectral flows according to (341), spans $\left(\widehat{GL}(M_B, M_F)\right)_+$ – the positive grade part of superloop superalgebra $\widehat{GL}(M_B, M_F)$ with half-integer grading (337)–(339).

Similarly to (328) we can construct a "negative-grade" superloop superalgebra additional symmetries for $SKP_{(R;M_B,M_F)}$ hierarchies with fermionic super-Lax operators (308) (where $R = 2r+1$, $M \equiv M_B + M_F = 2N+1$). Namely, let us define the flows:

$$\delta^{(-\ell/2)}_{\bar{\mathcal{A}}} \mathcal{L} = \left[\mathcal{M}^{(-\ell/2)}_{\bar{\mathcal{A}}} , \mathcal{L} \right] \quad , \quad \delta^{(-\ell/2)}_{\bar{\mathcal{F}}} \mathcal{L} = \left[\mathcal{M}^{(-\ell/2)}_{\bar{\mathcal{F}}} , \mathcal{L} \right] \tag{342}$$

with:

$$\mathcal{M}^{(-\ell/2)}_{\bar{\mathcal{A}}} = \sum_{I,J=1}^{2(N+r+1)} \bar{\mathcal{A}}^{(-\ell/2)}_{IJ} \sum_{s=0}^{\ell-1} (-1)^{s(\ell+|J|)} \phi_J^{(-(\ell-s-1)/2)} \mathcal{D}^{-1} \psi_I^{(-s/2)} \tag{343}$$

$$\mathcal{M}^{(-\ell/2)}_{\bar{\mathcal{F}}} = \sum_{I,J=1}^{2(N+r+1)} \bar{\mathcal{F}}^{(-\ell/2)}_{IJ} \sum_{s=0}^{\ell-1} (-1)^{s(\ell+|J|)} \phi_J^{(-(\ell-s-1)/2)} \mathcal{D}^{-1} \psi_I^{(-s/2)} \tag{344}$$

where short-hand notations (326) are used. Here $\bar{\mathcal{A}}^{(-\ell/2)}_{IJ}$ and $\bar{\mathcal{F}}^{(-\ell/2)}_{IJ}$ are constant graded matrices belonging to $\widehat{GL}(N+r+1, N+r+1)$ (cf. (337)–(339)). Furthermore, $\bar{\mathcal{A}}^{(-n)} \neq \mathbb{1}$ in Eqs.(343),(342) since $\mathcal{M}^{(-n)}_{\bar{\mathcal{A}}=\mathbb{1}} = \mathcal{L}^{-2n}$, so that the flow $\delta^{(-n)}_{\bar{\mathcal{A}}=\mathbb{1}}$ identically vanishes according to the first Eq.(342).

Similarly to (340) we find the following infinite-dimensional algebra of the "negative-grade" flows (342):

$$\left[\,\delta^{(-\ell/2)}_{\bar{\mathcal{A}}_1}\,,\,\delta^{(-m/2)}_{\bar{\mathcal{A}}_2}\,\right] = \delta^{(-(\ell+m)/2)}_{[\bar{\mathcal{A}}_1,\,\bar{\mathcal{A}}_2]}$$

$$\left[\,\delta^{(-\ell/2)}_{\bar{\mathcal{A}}}\,,\,\delta^{(-m/2)}_{\bar{\mathcal{F}}}\,\right] = \delta^{(-(\ell+m)/2)}_{[\bar{\mathcal{A}},\bar{\mathcal{F}}]} \;\; (\ell = \text{even}) \;\;,\;\; \left[\,\delta^{(-\ell/2)}_{A}\,,\,\delta^{(-m/2)}_{F}\,\right] = -\delta^{(-(\ell+m)/2)}_{\{A,\,F\}} \;\; (\ell = \text{odd})$$

$$\left\{\delta^{(-\ell/2)}_{\bar{\mathcal{F}}_1},\,\delta^{(-m/2)}_{\bar{\mathcal{F}}_2}\right\} = \pm\delta^{(-(\ell+m)/2)}_{\{\bar{\mathcal{F}}_1,\bar{\mathcal{F}}_2\}} \quad \text{for} \;\; (\ell,m) = (odd, odd)/(even, even)$$

$$\left\{\delta^{(-\ell/2)}_{\bar{\mathcal{F}}_1},\,\delta^{(-m/2)}_{\bar{\mathcal{F}}_2}\right\} = \pm\delta^{(-(\ell+m)/2)}_{[\bar{\mathcal{F}}_1,\bar{\mathcal{F}}_2]} \quad \text{for} \;\; (\ell,m) = (odd, even)/(even, odd) \tag{345}$$

which is isomorphic to $\left(\widehat{GL}'(N+r+1, N+r+1)\right)_-$. The latter is the negative-grade part of $\widehat{GL}(N+r+1, N+r+1)$ (cf.(337)–(339)), where the prime indicates factoring out $\bar{\mathcal{A}}^{(-n)} = \mathbb{1}$ in each integer-grade subspace (recall that the flows $\delta^{(-n)}_{\bar{\mathcal{A}}=\mathbb{1}}$ (342) vanish identically).

We also obtain that in fermionic constrained $SKP_{(R;M_B,M_F)}$ supersymmetric hierarchies (308) "positive-grade" $\left(\widehat{GL}(M_B, M_F)\right)_+$ symmetry flows (anti-)commute with "negative-grade" $\left(\widehat{GL}(N+r+1, N+r+1)\right)_-$ symmetry flows (recall $M \equiv M_B + M_F = 2N+1$) :

$$\left[\,\delta^{(\ell/2)}_{\mathcal{A}}\,,\,\delta^{(-m/2)}_{\bar{\mathcal{A}}}\,\right] = 0 \quad , \quad \left[\,\delta^{(\ell/2)}_{\mathcal{A},\mathcal{F}}\,,\,\delta^{(-m/2)}_{\bar{\mathcal{F}},\bar{\mathcal{A}}}\,\right] = 0 \quad , \quad \left\{\delta^{(\ell/2)}_{\mathcal{F}},\,\delta^{(-m/2)}_{\bar{\mathcal{F}}}\right\} = 0 \tag{346}$$

The construction of superloop superalgebra additional symmetries for $SKP^{N=1}_{(M_B,M_F)}$ hierarchies with bosonic super-Lax operators proceeds along the same lines (for details, see Ref.[C17]). Thus, we conclude:

- Fermionic constrained $SKP_{(R;M_B,M_F)}$ supersymmetric hierarchies (308) (where $R = 2r+1$, $M \equiv M_B + M_F = 2N+1$) possess the following superloop superalgebra symmetries:

$$\left(\widehat{GL}(M_B, M_F)\right)_+ \oplus \left(\widehat{GL}'(N+r+1, N+r+1)\right)_- \tag{347}$$

- Bosonic constrained $SKP_{(R;M_B,M_F)}$ supersymmetric hierarchies (316) (where $R = 2r$, $M \equiv M_B + M_F = 2N$) possess the following superloop superalgebra symmetries:

$$\left(\widehat{GL}(M_B, M_F)\right)_+ \oplus \left(\widehat{GL}'(N+r, N+r)\right)_- \tag{348}$$

4.10.2 Virasoro Symmetry of Constrained SKP Hierarchies

Here we will present the Virasoro additional symmetries for bosonic constrained $SKP_{(R;M_B,M_F)}$ hierarchies (316). This same construction based on the super-pseudo-differential formalism does not, however, carry over to the case of the fermionic part of the full super-Virasoro and the rest of super-$W_{1+\infty}$ symmetries, as well to the case of (super-)Virasoro and super-$W_{1+\infty}$ symmetries for fermionic constrained $SKP^{N=1}_{(M_B,M_F)}$ hierarchies (308).

In Ref.[C17] we find the following explicit form of the action of Virasoro flows on bosonic super-Lax $L \equiv \mathcal{L}_{(2r;M_B,M_F)}$ and super-dressing $\mathcal{W}$ operators:

$$\delta^V_n L = \Big[-\big(\mathcal{W} \overset{(0)}{\mathcal{M}}_n \mathcal{W}^{-1}\big)_- + \mathcal{X}_n \, , \, L \Big] \quad , \quad \delta^V_n \mathcal{W} = \Big(-\big(\mathcal{W} \overset{(0)}{\mathcal{M}}_n \mathcal{W}^{-1}\big)_- + \mathcal{X}_n \Big) \mathcal{W} \, , \tag{349}$$

where $\delta^V_n \simeq -L_{n-1}$ (in terms of standard Virasoro notations). Here $\overset{(0)}{\mathcal{M}}_n$ are the "bare" (undressed) Virasoro operators:

$$\overset{(0)}{\mathcal{M}}_n \equiv \Gamma_0 \partial^n + \frac{n}{2} \Gamma_1 (Q + \Gamma_1 \partial) \partial^{n-1} \tag{350}$$

with $Q = \frac{\partial}{\partial\theta} - \theta\partial$ being the standard super-charge operator and:

$$\Gamma_0 \equiv \sum_{l=1}^{\infty} l t_l \partial^{l-1} + \sum_{n=1}^{\infty} (n - \frac{1}{2}) \theta_n \partial^{n-2} \mathcal{D} - \frac{1}{2} \sum_{n=1}^{\infty} \theta_n \partial^{n-2} Q + \frac{1}{2} \sum_{n,l=1}^{\infty} (n-l) \theta_n \theta_l \partial^{n+l-2} \, , \tag{351}$$

$$\Gamma_1 \equiv \theta + \sum_{n=1}^{\infty} \theta_n \partial^{n-1} \quad , \quad Q + \Gamma_1 \partial = \frac{\partial}{\partial\theta} + \sum_{n=1}^{\infty} \theta_n \partial^n \, . \tag{352}$$

The additional operators $\mathcal{X}_n$ are to be chosen in such a way that the flows (349) define a symmetry, *i.e.*, they must preserve the constrained form of L (316).

For non-negative Virasoro flows ($n \geq 0$ in (349)) we find the following expression for $\mathcal{X}_n$:

$$\mathcal{X}_n \equiv \sum_{s=1}^{n-1} (s - \frac{n}{2}) \Big[\sum_{a=1}^{M_B} \Phi_a^{(n-s)} D^{-1} \widetilde{\Psi}_a^{(s)} + \sum_{b=1}^{M_F} \widetilde{\Phi}_b^{(n-s)} D^{-1} \Psi_b^{(s)} \Big] \, , \tag{353}$$

where the short-hand notations (321) are used. For negative Virasoro flows ($n < 0$ in (349)) we obtain (employing again notations (322)–(323)) :

$$\mathcal{X}_{(-|n|)} = \sum_{b=1}^{N+r} \sum_{j=0}^{|n|} (\frac{|n|}{2} - j) \Big[\Phi_b^{-(|n|-j+1)} \mathcal{D}^{-1} \widetilde{\Psi}_b^{-(j+1)} + \widetilde{\Phi}_b^{-(|n|-j+1)} \mathcal{D}^{-1} \Psi_b^{-(j+1)} \Big] \, . \tag{354}$$

As shown in [C17], the following identity holds for the operators $\mathcal{X}_n$:

$$\begin{aligned} \delta^V_n \mathcal{X}_m - \Big[\big(\mathcal{W} \overset{(0)}{\mathcal{M}}_n \mathcal{W}^{-1}\big)_+ , \mathcal{X}_m \Big]_- - \delta^V_m \mathcal{X}_n + \Big[\big(\mathcal{W} \overset{(0)}{\mathcal{M}}_m \mathcal{W}^{-1}\big)_+ , \mathcal{X}_n \Big]_- - \Big[\mathcal{X}_n \, , \, \mathcal{X}_m \Big] \\ = -(n-m) \mathcal{X}_{n+m-1} \end{aligned} \tag{355}$$

with the help of which it is straightforward to prove the closure of the full Virasoro algebra of additional symmetry flows (349) (without central extension):

$$\Big[\delta^V_n \, , \, \delta^V_m \Big] = -(n-m) \delta^V_{n+m-1} \tag{356}$$

4.10.3 Multi-Component (Matrix) Supersymmetric KP Hierarchies

Let us now consider the following subalgebra of the superloop superalgebra symmetry flows for bosonic $SKP_{(2r;N,N)}$ hierarchies (*i.e.*, $R = 2r,\ M_B = M_F = N$) which are defined as:

$$\delta^{(n)}_{\mathcal{A}=\mathcal{E}_k} \equiv -\partial/\partial \overset{(k)}{t}_n \quad , \quad \delta^{(n-\frac{1}{2})}_{\mathcal{F}=\mathcal{E}_k} \equiv -\overset{(k)}{D}_n \tag{357}$$

$$\mathcal{E}^{(n)}_k = \begin{pmatrix} E_k & 0 \\ 0 & E_k \end{pmatrix} \quad , \quad \mathcal{E}^{(n-\frac{1}{2})}_k = \begin{pmatrix} 0 & E_k \\ E_k & 0 \end{pmatrix} \quad \text{with } E_k \equiv diag(0,\ldots,0,\overset{(k)}{1},0,\ldots,0) \tag{358}$$

where $k = 1,\ldots,N$. The flows (357) span a direct sum of N copies of the original Manin-Radul isospectral flow algebra (292) :

$$\left\{\overset{(k)}{D}_n,\, \overset{(l)}{D}_m\right\} = -\delta_{kl}\partial/\partial \overset{(k)}{t}_{n+m-1} \quad , \quad \text{rest} = 0 \quad ; \quad k,l = 1,\ldots,N \quad , \quad n,m = 1,2,\ldots \tag{359}$$

which justifies their representation in a form similar to (292):

$$\overset{(k)}{D}_n = \partial/\partial \overset{(k)}{\theta}_n - \sum_{s=1}^{\infty} \overset{(k)}{\theta}_s\, \partial/\partial \overset{(k)}{t}_{n+s-1} \tag{360}$$

Now, we can construct the following supersymmetric extended integrable hierarchy built on the original bosonic $SKP_{(2r;N,N)}$ supersymmetric hierarchy (316) by supplementing the latter with the set of additional superloop superalgebra supersymmetric Manin-Radul-like flows (357)–(360) (recall here $L \equiv \mathcal{L}_{(2r;N,N)}$ (316)) :

$$\partial/\partial \overset{(k)}{t}_n L = -\Big[\mathcal{M}^{(n)}_k\,,\, L\Big] \quad , \quad \overset{(k)}{D}_n\, L = -\Big[\mathcal{M}^{(n-\frac{1}{2})}_k\,,\, L\Big] \tag{361}$$

with:

$$\mathcal{M}^{(n)}_k \equiv \sum_{s=0}^{n-1}\Big[L^{n-s-1}(\widetilde{\Phi}_k)\mathcal{D}^{-1}\left(L^s\right)^*(\Psi_k) + L^{n-s-1}(\Phi_k)\mathcal{D}^{-1}\left(L^s\right)^*(\widetilde{\Psi}_k)\Big] \tag{362}$$

$$\mathcal{M}^{(n-\frac{1}{2})}_k \equiv \sum_{s=0}^{n-1} L^{n-s-1}(\Phi_k)\mathcal{D}^{-1}\left(L^s\right)^*(\Psi_k) - \sum_{s=0}^{n-2} L^{n-s-2}(\widetilde{\Phi}_k)\mathcal{D}^{-1}\left(L^s\right)^*(\widetilde{\Psi}_k) \tag{363}$$

where the flow action on the constituent (adjoint) super-eigenfunctions is given by:

$$\partial/\partial \overset{(k)}{t}_n \overset{(\sim)}{\Phi}_a = -\mathcal{M}^{(n)}_k(\overset{(\sim)}{\Phi}_a) \quad , \quad \partial/\partial \overset{(k)}{t}_n \overset{(\sim)}{\Psi}_a = \left(\mathcal{M}^{(n)}_k\right)^*(\overset{(\sim)}{\Psi}_a) \quad , \quad a \neq k \tag{364}$$

$$\partial/\partial \overset{(k)}{t}_n \overset{(\sim)}{\Phi}_k = -\mathcal{M}^{(n)}_k(\overset{(\sim)}{\Phi}_k) + L^n(\overset{(\sim)}{\Phi}_k) \quad , \quad \partial/\partial \overset{(k)}{t}_n \overset{(\sim)}{\Psi}_k = \left(\mathcal{M}^{(n)}_k\right)^*(\overset{(\sim)}{\Psi}_k) - \left(L^n\right)^*(\overset{(\sim)}{\Phi}_k) \tag{365}$$

$$\overset{(k)}{D}_n \overset{(\sim)}{\Phi}_a = -\mathcal{M}^{(n-\frac{1}{2})}_k(\overset{(\sim)}{\Phi}_a) \quad , \quad \overset{(k)}{D}_n \overset{(\sim)}{\Psi}_a = \left(\mathcal{M}^{(n-\frac{1}{2})}_k\right)^*(\overset{(\sim)}{\Psi}_a) \quad , \quad a \neq k \tag{366}$$

$$\overset{(k)}{D}_n\, \Phi_k = -\mathcal{M}^{(n-\frac{1}{2})}_k(\Phi_k) + L^{n-1}(\widetilde{\Phi}_k) \quad , \quad \overset{(k)}{D}_n\, \Psi_k = \left(\mathcal{M}^{(n-\frac{1}{2})}_k\right)^*(\Psi_k) + \left(L^{n-1}\right)^*(\widetilde{\Psi}_k) \tag{367}$$

$$\overset{(k)}{D}_n \widetilde{\Phi}_k = -\mathcal{M}_k^{(n-\frac{1}{2})}(\widetilde{\Phi}_k) + L^n(\Phi_k) \quad , \quad \overset{(k)}{D}_n \widetilde{\Psi}_k = \left(\mathcal{M}_k^{(n-\frac{1}{2})}\right)^* (\widetilde{\Psi}_k) + (L^n)^* (\Psi_k) \tag{368}$$

The above construction is the superspace analog of the construction in subsection 4.5 (cf. Refs.[C12,C15]), where the corresponding ordinary bosonic scalar (one-component) KP hierarchy, supplemented with the flows belonging to the Cartan subalgebra of additional loop-algebra symmetries, was identified as a matrix (multi-component) KP hierarchy. Therefore it is natural to call the supersymmetric extended KP hierarchy defined by Eqs.(361)–(368) *N-component constrained super-KP hierarchy.*

Similarly to the construction described in subsection 4.6, we have shown in Ref.[C17] that the multi-component extension of the constrained supersymmetric $SKP_{(2;2,2)}$ hierarchy contains a supersymmetric version of ordinary Davey-Stewartson system.

4.11 Supersymmetric Darboux-Bäcklund Transformations and "Super-Soliton" Solutions

In Ref.[C17] (see also [C13,C14]) we have developed a systematic approach to superspace Darboux-Bäcklund transformations of $SKP_{(R;M_B,M_F)}$ hierarchies generating Wronskian-type super-determinant solutions. Although the logic of the construction is similar to the ordinary bosonic case (subsection 4.7), there are several non-trivial issues to be solved on the way related to the presence of fermionic isospectral and fermionic additional symmetry flows.

4.11.1 Darboux-Bäcklund Transformations for Constrained $\boldsymbol{SKP}_{(R;M_B,M_F)}$ Hierarchies

In analogy with the ordinary "bosonic" case, DB transformations within the Sato super-pseudo-differential operator approach are defined as "gauge" transformations of special kind on the pertinent super-Lax operator of the supersymmetric integrable hierarchy:

$$\mathcal{L} \quad \to \quad \widetilde{\mathcal{L}} = \mathcal{T}_\phi \mathcal{L} \mathcal{T}_\phi^{-1} \quad , \quad \mathcal{T}_\phi \equiv \phi \mathcal{D} \phi^{-1} \tag{369}$$

with ϕ being a bosonic superfunction, which obey the following requirements:

(A) Super-DB transformations (369) have to preserve the specific constrained form (308) of $\mathcal{L}$ (or (316) for bosonic $SKP_{(R;M_B,M_F)}$ hierarchies), *i.e.*, the transformed super-Lax operator $\widetilde{\mathcal{L}}$ (369) must be again of the form:

$$\widetilde{\mathcal{L}} \equiv \widetilde{\mathcal{L}}_{(R;\widetilde{M}_B,\widetilde{M}_F)} = \mathcal{D}^R + \sum_{j=0}^{R-1} \widetilde{v}_{\frac{j}{2}} \mathcal{D}^j + \sum_{i=1}^{M} \widetilde{\Phi}_i \mathcal{D}^{-1} \widetilde{\Psi}_i \quad , \quad M = \widetilde{M}_B + \widetilde{M}_F \; , \tag{370}$$

where $\widetilde{M}_{B,F}$ are the numbers of DB-transformed bosonic/fermionic (adjoint) super-eigenfunctions $\widetilde{\Phi}_i$, $\widetilde{\Psi}_i$. Let us stress that we require the total number M of negative super-pseudo-differential terms in $\widetilde{\mathcal{L}}$ to be the same as in the initial super-Lax operator $\mathcal{L}$ (308).

(B) Super-DB transformations (369) have to preserve the bosonic (315) and fermionic (310) isospectral evolution equations (in the case of fermionic $SKP_{(R;M_B,M_F)}$ hierarchies). As shown in [C17], the fermionic isospectral flows (310) can be strictly preserved under super-DB transformations only for the subclass $SKP_{(R;1,0)}$ of constrained super-KP hierarchies $SKP_{(R;M_B,M_F)}$ with fermionic super-Lax operators. On the other hand, for $SKP_{(R;M_B,M_F)}$ with bosonic super-Lax operator the fermionic isospectral flows are preserved under super-DB transformations upto an overall sign flip.

Similarly, we can define adjoint-DB transformations:

$$\mathcal{L} \quad \to \quad \widehat{\mathcal{L}} = \left(-\mathcal{T}_\psi^{-1}\right)^* \mathcal{L}\mathcal{T}_\psi^* \quad , \quad \mathcal{T}_\psi \equiv \psi\mathcal{D}\psi^{-1} \tag{371}$$

obeying the same requirements (A) and (B).

Condition (A) above can be satisfied for two different choices of the (adjoint-)DB generating superfunctions ϕ and ψ :

(i) First choice: $\phi = \Phi_{i_0}$ where Φ_{i_0} is some fixed *bosonic* super-eigenfunction entering the negative pseudo-differential part of the original super-Lax operator (308). In this case we obtain:

$$\widetilde{\Phi}_{i_0} = \mathcal{T}_\phi\mathcal{L}(\phi) \quad , \quad \widetilde{\Psi}_{i_0} = \phi^{-1} \quad , \quad \phi \equiv \Phi_{i_0} \tag{372}$$

$$\widetilde{\Phi}_i = \mathcal{T}_\phi(\Phi_i) \quad , \quad \widetilde{\Psi}_i = (-1)^{|i|}\left(\mathcal{T}_\phi^{-1}\right)^*(\Psi_i) \quad , \quad i \neq i_0 \tag{373}$$

Similarly, the first choice for adjoint-DB transformations is $\psi = \Psi_{i_0}$ where Ψ_{i_0} is some fixed *bosonic* adjoint super-eigenfunction entering the negative pseudo-differential part of the original super-Lax operator (308). Accordingly, for the adjoint-DB transformed (adjoint) super-eigenfunctions we have:

$$\widehat{\Phi}_{i_0} = -\psi^{-1} \quad , \quad \widehat{\Psi}_{i_0} = -\mathcal{T}_\psi\mathcal{L}^*(\psi) \quad , \quad \psi \equiv \Psi_{i_0} \tag{374}$$

$$\widehat{\Phi}_i = (-1)^{|i|}\left(\mathcal{T}_\psi^{-1}\right)^*(\Phi_i) \quad , \quad \widehat{\Psi}_i = -\mathcal{T}_\psi(\Psi_i) \quad , \quad i \neq i_0 \tag{375}$$

Let us note that the Grassmann parity of the DB transformed (adjoint) super-eigenfunctions $\widetilde{\Phi}_i$ and $\widetilde{\Psi}_i$ (373) for $i \neq i_0$ changes from $|i|$ to $|i|+1$, and similarly for the adjoint-DB transformed ones (375).

(ii) Second choice: $\phi = L_{N+\frac{1}{2}}(\widetilde{\varphi}_{\alpha_0})$ (for super-DB transformations) and $\psi = \psi_{\alpha_0}$ (for adjoint super-DB transformations) where $L_{N+\frac{1}{2}}(\widetilde{\varphi}_{\alpha_0})$ and ψ_{α_0} are some fixed bosonic (adjoint) super-eigenfunctions (325)–(326) entering the expression (324) for the inverse power of the super-Lax operator $\mathcal{L}$. We get for the (adjoint) DB-transformed (adjoint) super-eigenfunctions:

$$\widetilde{\Phi}_i = \mathcal{T}_\phi(\Phi_i) \quad , \quad \widetilde{\Psi}_i = (-1)^{|i|}\left(\mathcal{T}_\phi^{-1}\right)^*(\Psi_i) \quad , \quad \phi \equiv L_{N+\frac{1}{2}}(\widetilde{\varphi}_{\alpha_0}) \; , \tag{376}$$

$$\widehat{\Phi}_i = (-1)^{|i|}\left(\mathcal{T}_\psi^{-1}\right)^*(\Phi_i) \quad , \quad \widehat{\Psi}_i = -\mathcal{T}_\psi(\Psi_i) \quad , \quad \psi = \psi_{\alpha_0} \; , \tag{377}$$

for all $i = 1, \ldots, M$.

In Ref.[C17] we have also analyzed the interplay between superloop superalgebra additional nonisospectral symmetries and super-DB transformations. We obtain the following results for fermionic $SKP_{(R;M_B,M_F)}$ hierarchies (308) (recall $M_B + M_F = 2N+1$, $R = 2r+1$):

- Super-DB transformations of the first type (i) (372)–(375) preserve (up to an overall sign change of the fermionic flows) the following subalgebra of additional non-isospectral symmetries:

$$\left(\widehat{GL}(M_B-1,M_F)\right)_+ \oplus \left(\widehat{GL}'(N+r+1,N+r+1)\right)_- \tag{378}$$

- Super-DB transformations of the second type (ii) (376)–(377) preserve (up to an overall sign change of the fermionic flows) the following subalgebra of additional symmetries:

$$\left(\widehat{GL}(M_B,M_F)\right)_+ \oplus \left(\widehat{GL}(N+r,N+r+1)\right)_- \tag{379}$$

  (here the positive-grade part includes the Manin-Radul isospectral flows).

For bosonic $SKP_{(R;M_B,M_F)}$ hierarchies (316) (recall $M_B+M_F=2N$, $R=2r$) we obtain similar results with (378) and (379) replaced by:

$$\left(\widehat{GL}(M_B-1,M_F)\right)_+ \oplus \left(\widehat{GL}'(N+r,N+r)\right)_- \tag{380}$$

and

$$\left(\widehat{GL}(M_B,M_F)\right)_+ \oplus \left(\widehat{GL}(N+r-1,N+r)\right)_- \tag{381}$$

respectively.

4.11.2 Iterations of Supersymmetric Darboux-Bäcklund Transformations and Wronskian-Type Super-Determinant Solutions

During iteration of (adjoint) super-DB transformations we encounter Berezinians (super-determinants) whose matrix blocks possess the following special generalized Wronskian-like $k \times (m+n)$ matrix form:

$$\widetilde{W}^{(k;n)}_{k,m+n}\left[\{\varphi\}\mid\{\psi\}\right] \equiv \widetilde{W}^{(k;n)}_{k,m+n}\left[\varphi_0,\ldots,\varphi_{k-1}\mid\psi_{\frac{1}{2}},\ldots,\psi_{n-\frac{1}{2}}\right] =$$
$$= \begin{pmatrix} \varphi_0 & \cdots & \cdots & \varphi_{k-1} \\ \vdots & \ddots & \ddots & \vdots \\ \partial^{m-1}\varphi_0 & \cdots & \cdots & \partial^{m-1}\varphi_{k-1} \\ \mathcal{D}_\theta^{-1}(\varphi_0\psi_{\frac{1}{2}}) & \cdots & \cdots & \mathcal{D}_\theta^{-1}(\varphi_{k-1}\psi_{\frac{1}{2}}) \\ \vdots & \ddots & \ddots & \vdots \\ \mathcal{D}_\theta^{-1}(\varphi_0\psi_{n-\frac{1}{2}}) & \cdots & \cdots & \mathcal{D}_\theta^{-1}(\varphi_{k-1}\psi_{n-\frac{1}{2}}) \end{pmatrix} \tag{382}$$

where $\{\varphi\} \equiv \{\varphi_0,\ldots,\varphi_{k-1}\}$ is a set of k bosonic or fermionic superfunctions whereas $\{\psi\} \equiv \left\{\psi_{\frac{1}{2}},\ldots,\psi_{n-\frac{1}{2}}\right\}$ is a set of n fermionic superfunctions. The generalized Wronskian-like matrix (382) is the supersymmetric generalization of the Wronskian-like block matrices (232) entering the general Darboux-Bäcklund determinant solutions for the tau-functions of ordinary bosonic constrained KP hierarchies $\mathsf{cKP}_{R,M}$ (see subsubsections 4.7.2 and 4.7.4). In the special case of $n=0$ (382) reduces to the rectangular $k \times m$ Wronskian matrix:

$$W_{k,m}\left[\varphi_0,\ldots,\varphi_{k-1}\right] = \begin{pmatrix} \varphi_0 & \cdots & \cdots & \varphi_{k-1} \\ \partial\varphi_0 & \cdots & \cdots & \partial\varphi_{k-1} \\ \vdots & \ddots & \ddots & \vdots \\ \partial^{m-1}\varphi_0 & \cdots & \cdots & \partial^{m-1}\varphi_{k-1} \end{pmatrix} \tag{383}$$

Let us introduce yet another short-hand notations:

$$\{\varphi\} \equiv \{\varphi_0,\ldots,\varphi_{n+m-1}\} \quad , \quad \{\varphi_{(\frac{1}{2})}\} \equiv \{\varphi_{\frac{1}{2}},\ldots,\varphi_{m-\frac{1}{2}}\} \quad , \quad \{\psi\} \equiv \{\psi_{\frac{1}{2}},\ldots,\psi_{n-\frac{1}{2}}\}\ , \tag{384}$$

where $\{\varphi\}$ and $\{\varphi_{(\frac{1}{2})}\}$ are sets of bosonic/fermionic super-eigenfunctions whereas $\{\psi\}$ is a set of fermionic adjoint super-eigenfunctions of the constrained super-KP hierarchy $SKP_{(R;M_B,M_F)}$ (308). Let us recall that all pertinent (adjoint) super-eigenfunctions are of the form (327)–(326). Similarly to the ordinary bosonic case we denote by $\tau^{(k;l)}$ the super-DB-transformed super-tau-function after k super-DB interations (369) and l adjoint-super-DB iterations (371).

With the help of (384) the explicit expressions for the super-tau function of constrained super-KP hierarchies on the general super-DB orbit read:

$$\frac{\tau^{(0;0)}}{\tau^{(n+2m;n)}} = (-1)^{mn+n(n-1)/2} \times$$

$$\mathrm{Ber}\begin{pmatrix} \widetilde{W}^{(n+m;n)}_{n+m,n+m}\left[\{\varphi\}\mid\{\psi\}\right] & | & \widetilde{W}^{(m;n)}_{m,n+m}\left[\{\varphi_{(\frac{1}{2})}\}\mid\{\psi\}\right] \\ -------------- & | & ------------ \\ W_{n+m,m}[\mathcal{D}_\theta\varphi_0,\ldots,\mathcal{D}_\theta\varphi_{n+m-1}] & | & W_{m,m}[\mathcal{D}_\theta\varphi_{\frac{1}{2}},\ldots,\mathcal{D}_\theta\varphi_{m-\frac{1}{2}}] \end{pmatrix} \tag{385}$$

$$\tau^{(n+2m+1;n)}\,\tau^{(0;0)} = (-1)^{mn+n(n-1)/2} \times$$

$$\mathrm{Ber}\begin{pmatrix} \widetilde{W}^{(n+m+1;n)}_{n+m+1,n+m+1}\left[\{\varphi\},\varphi_{n+m}\mid\{\psi\}\right] & | & \widetilde{W}^{(m;n)}_{m,n+m+1}\left[\{\varphi_{(\frac{1}{2})}\}\mid\{\psi\}\right] \\ -------------------- & | & ------------ \\ W_{n+m+1,m}\left[\mathcal{D}_\theta\varphi_0,\ldots,\mathcal{D}_\theta\varphi_{n+m-1},\mathcal{D}_\theta\varphi_{n+m}\right] & | & W_{m,m}\left[\mathcal{D}_\theta\varphi_{\frac{1}{2}},\ldots,\mathcal{D}_\theta\varphi_{m-\frac{1}{2}}\right] \end{pmatrix} \tag{386}$$

where the symbol *Ber* indicates Berezinian (super-determinant). For further details, see Refs.[C14,C17].

4.11.3 Example: Darboux-Bäcklund Orbit of $\boldsymbol{SKP}_{1;1,0}$ Hierarchy and Supersymmetric Toda Lattice

Let us now consider the DB-orbit, *i.e.* the chain of DB-transformations on the simplest non-trivial constrained super-KP hierarchy $SKP_{1;1,0}$ with super-Lax operator $\mathcal{L} \equiv \mathcal{L}_{1;1,0}$

(309) starting from the "free" initial $\mathcal{L}^{(0)} = \mathcal{D}$ (the superscript (k) indicating the step of DB-iteration) [C13]:

$$\mathcal{L}^{(k+1)} = \mathcal{T}^{(k)}\mathcal{L}^{(k)}\left(\mathcal{T}^{(k)}\right)^{-1} = \mathcal{D} + f^{(k+1)} + \Phi^{(k+1)}\mathcal{D}^{-1}\Psi^{(k+1)} \quad , \quad \mathcal{T}^{(k)} = \Phi^{(k)}\mathcal{D}\left(\Phi^{(k)}\right)^{-1} \tag{387}$$

where:

$$f^{(k+1)} = -2\mathcal{D}_\theta \ln \Phi^{(k)} - f^{(k)} \quad , \quad \Psi^{(k+1)} = \left(\Phi^{(k)}\right)^{-1} , \tag{388}$$

$$\Phi^{(k+1)} = \Phi^{(k)}\partial_x \ln \Phi^{(k)} + \Phi^{(k)}\mathcal{D}_\theta f^{(k)} + \left(\Phi^{(k)}\right)^2 \Psi^{(k)} , \tag{389}$$

and where $\Phi^{(0)}$ is a super-eigenfunction of the initial "free" $\mathcal{L}^{(0)} = \mathcal{D}$ given by:

$$\Phi^{(0)} \equiv \Phi_0(t,\theta) = \int d\lambda\, d\eta\; \varphi^{(0)}(\lambda,\eta) \exp\Big\{\sum_{l=1}^{\infty} \lambda^l t_l + \eta\theta - (\eta - \lambda\theta)\sum_{n=1}^{\infty} \lambda^{n-1}\theta_n\Big\} \tag{390}$$

with "spectral" density $\varphi^{(0)}(\lambda,\eta) = \varphi_B(\lambda) + \eta\varphi_F(\lambda)$. Relations (388)–(389) imply:

$$\partial_x \ln \Phi^{(k)} = \frac{\Phi^{(k+1)}}{\Phi^{(k)}} - \frac{\Phi^{(k)}}{\Phi^{(k-1)}} \tag{391}$$

which, upon the substitution $\Phi^{(k)} = e^{\varphi_k}$, acqure the form of equations of motion of a new *supersymmetric Toda lattice model*:

$$\partial_x \varphi_k = e^{\varphi_{k+1}-\varphi_k} + e^{\varphi_k - \varphi_{k-1}} \tag{392}$$

Note, that by acting on (392) with ∂_x we get:

$$\partial_x^2 \varphi_k = e^{\varphi_{k+2}-\varphi_k} - e^{\varphi_k - \varphi_{k-2}} \tag{393}$$

which has the form of the ordinary one-dimensional Toda lattice equations (238) with the following important differences:

(a) Toda lattice variables $\varphi_k = \varphi_k(x, t_2, \ldots; \theta, \theta_1, \ldots)$ are now *superfields*;

(b) The basic equations (392) are *first order* differential equations;

(c) Eqs.(393) indicate *doubled* lattice spacing.

Therefore, as a result of (392)–(393) supersymmetric Toda lattice may be viewed as a "square" root of the standard (bosonic) Toda lattice.

Finally, let us present the Wronskian solutions for the DB-iterated super-eigenfunctions:

$$\Phi^{(2n)} = \frac{W_{n+1}[\Phi_0, \partial_x\Phi_0, \ldots, \partial_x^n\Phi_0]}{W_n[\Phi_0, \partial_x\Phi_0, \ldots, \partial_x^{n-1}\Phi_0]} \quad , \quad \Phi^{(2n+1)} = \frac{W_{n+1}[\partial_x\Phi_0, \partial_x^2\Phi_0, \ldots, \partial_x^{n+1}\Phi_0]}{W_n[\partial_x\Phi_0, \partial_x^2\Phi_0, \ldots, \partial_x^n\Phi_0]} , \tag{394}$$

and the super-tau function:

$$\tau^{(2n)} = \frac{W_n\left[\partial_x\Phi_0, \ldots, \partial_x^n\Phi_0\right]}{W_n\left[\Phi_0, \partial_x\Phi_0, \ldots, \partial_x^{n-1}\Phi_0\right]} \quad , \quad \tau^{(2n+1)} = \frac{W_{n+1}\left[\Phi_0, \partial_x\Phi_0, \ldots, \partial_x^n\Phi_0\right]}{W_n\left[\partial_x\Phi_0, \ldots, \partial_x^n\Phi_0\right]} . \tag{395}$$

Eqs.(394)–(395) yield as a consequence an alternative super-tau-function form of the supersymmetric Toda lattice (392):

$$\partial_x \ln \tau^{(k)}(t,\theta) = \frac{\tau^{(k+1)}(t,\theta)}{\tau^{(k-1)}(t,\theta)} \; . \tag{396}$$

(cf. Eqs.(238) for the ordinary bosonic case).

Let us stress that the super-tau function is now given by *ratios* of Wronskians unlike the ordinary bosonic case and, of course, all entries in the Wronskian determinants are now superfields given by (390).

In particular, choosing for the "spectral" density of $\Phi^{(0)}$ (390) $\varphi_B(\lambda) = \sum_{i=1}^{N} c_i \delta(\lambda - \lambda_i)$, $\varphi_F(\lambda) = \sum_{i=1}^{N} \epsilon_i \delta(\lambda - \lambda_i)$, where c_i, λ_i and ϵ_i are Grassmann-even and Grassmann-odd constants, respectively, and substituting into (395) we obtain the following "super-soliton" solutions for the super-tau function of the simplest constrained supersymmetric KP hierarchy $SKP_{(1;1,0)}$ (309) :

$$\tau^{(2m+1)} = \frac{\sum_{1\leq i_1 < \ldots < i_{m+1} \leq N} \binom{N}{m+1} \widetilde{c}_{i_1} \ldots \widetilde{c}_{i_{m+1}} E_{i_1} \ldots E_{i_{m+1}} \Delta^2_{m+1}(\lambda_{i_1}, \ldots, \lambda_{i_{m+1}})}{\sum_{1\leq j_1 < \ldots < j_m \leq N} \binom{N}{m} \widetilde{c}_{j_1} \ldots \widetilde{c}_{j_m} E_{j_1} \ldots E_{j_m} \lambda_{j_1} \ldots \lambda_{j_m} \Delta^2_m(\lambda_{j_1}, \ldots, \lambda_{j_m})} \; , \tag{397}$$

$$\widetilde{c}_i \equiv c_i + \Bigl(\theta - \sum_{n\geq 1} \lambda_i^{n-1} \theta_n\Bigr)\epsilon_i \quad , \quad E_i \equiv e^{\sum_{l\geq 1} \lambda_i^l (t_l + \theta\theta_l)} \; ,$$

$$\Delta_m(\lambda_{i_1}, \ldots, \lambda_{i_m}) \equiv \det \left\| \lambda_{i_a}^{b-1} \right\|_{a,b=1,\ldots,m} \; . \tag{398}$$

5 Cited Literature

References

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6 LIST of selected scientific papers whose full text is included as part of the present thesis

- Total number of included selected papers: **33** (out of 99 papers in the complete list of author's papers)
- Total number of selected papers published in international scientific journals: **25**
  Phys.Lett. B – 11 ; *Phys.Lett. A* – 3 ; *Nucl.Phys. B* - 1 ; *Comm.Math.Phys.* - 1 ; *Mod.Phys.Lett. A* - 1 ; *Int.Journ.Mod.Phys. A* - 2 ; *Journ.Math.Phys.* – 2 ; *Theor.Math.Phys.* – 2 ; *J.E.T.P. Letters* – 1 ; *Appl. Analysis* – 1
  Total number of selected papers published in Bulgarian scientific journals: **3**
  Bulg.J.Phys. – 2 ; *Comp.Rend.Acad.Sci.Bulg.* – 1
  Total number of selected papers published as full-text articles in proceedings of international conferences: **4**
  Total number of selected papers published as full-text articles in Scientific Reports of Foreign Institutes: **1**
- Total number of independent citations of the included selected papers: **331** (out of 958 independent citations of all author's papers)
- Total impact-factor of the included selected papers: **68.621** (out of the total impact-factor 175.350 of all author's papers)

A1. ANOMALIES OF QUANTUM CURRENTS IN EXACTLY SOLVABLE MODELS. *Theor. Math. Phys.* **29** (1976) 161-170.
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7 Principal Contributions

(A1) Proof of quantum complete integrability for classically integrable two-dimensional field theory models. To this end we have worked out a non-trivial modification and systematic ultraviolet renormalization of the composite operators – quantum analogues of the infinite sets of classical higher local conserved currents, which enable us to prove their conservation on quantum level.

(A2) We have explicitly constructed an infinite system of higher local conserved currents in the classical massive Thirring model (both in its fermionic and its bosonic versions) which subsequently are proved to survive quantization with the help of the general procedure described in (A1).

(A3) We have explicitly constructed higher local quantum conserved currents in the two-dimensional N-component sigma-models within the framework of the non-perturbative $1/N$ expansion. The latter currents significantly differ from their classical counterparts as a result of the pertinent conformal anomaly and dimensional transmutation.

(A4) We work out a systematic generalization of the procedure described in (A1) to the case of *supersymmetric* two-dimensional integrable field theory models. In particular, we show that the supersymmetric Sine-Gordon model is the only member in the class of completely integrable scalar supersymmetric models. We also discover a critical point of phase transition in the supersymmetric Sine-Gordon model – a supersymmetric counterpart of the well-known critical point in the ordinary bosonic Sine-Gordon model.

(B1) We propose for the first time in the literature a general scheme for explicit construction of geometric dynamical systems on coadjoint orbits of arbitrary infinite-dimensional groups with central extensions. This general scheme is based on just only two fundamental geometric objects – the Maurer-Cartan one-form and a one-cocycle on the group (whose physical meaning is that of "integrated quantum anomaly").

(B2) The general scheme from (B1) is worked out in detail within the context of various specific physically relevant geometric models on group coadjoint orbits, presented here for the first time in the literature, which are related to the extended super-Kac-Moody and super-Virasoro algebras, the group of area-preserving diffeomorphisms, $\mathbf{W}_{1+\infty}$ algebra. The latter new geometric models describe indiced two-dimensional supergravity, anomalous quantum effects in membrane theory *etc.*

(B3) We derive explicitly the most general form of the group composition law for arbitrary geometric models on (infinite-dimensional) group coadjoint orbits which generalizes the famous Polyakov-Wiegmann group composition law for Wess-Zumino-Novikov-Witten models. The general group composition law is a succinct expression of all pertinent Noether symmetries. As a basic consequence of the latter we obtain Ward identities allowing for the exact quantum solvability of any geometric model.

(B4) With the help of the general group composition law from (B3) we find the explicit form of the "hidden" Kac-Moody symmetry of any geometric model on group coadjoint orbits. Thanks to the same group composition law we are able to extend the construction from (B1) to the case of arbitrary *gauged* or *coset* geometric models.

(B5) With the help of the general formalism from (B1) we have derived the explicit form of the Poisson structures on arbitrary infinite-dimensional groups with central extensions.

These structures are given in terms of operator kernels connected to non-trivial two-cocycles of the corresponding infinite-dimensional Lie algebras and represent infinite-dimensional analogues of the classical R-matrices satisfying *differential* (instead of algebraic) classical Yang-Baxter equations.

(C1) With the help of Sato pseudo-differential formalism we introduce and systematically study the properties of an important class of reduced integrable hierarchies of soliton equations of Kadomtsev-Petviashvili (KP) type – the so called *constrained KP hierarchies* $\mathsf{cKP}_{R,M}$, from the point of view of their relevance in the context of modern string theory. In particular, we show that the pseudo-differential Lax operators of $\mathsf{cKP}_{R,M}$ hierarchies are given by a ratio of two appropriate ordinary differential operators of orders $R+M$ and M, respectively.

(C2) We prove the equivalence between the Sato pseudo-differential and the algebraic Drinfeld-Sokolov description of the $\mathsf{cKP}_{R,M}$ hierarchies.

(C3) We obtain full description of all representations of the reduced KP hierarchies in terms of finite even number of free bosonic fields and we show how the latter can be derived as well-defined Poisson reduction (w.r.t. the first, or w.r.t. the second Hamiltonian structure, respectively) of the original full unconstrained KP hierarchy in the framework of the R-matrix scheme of Adler-Kostant-Symes-Reyman-(Semenov-Tyan-Shansky). As a byproduct we find series of representations in terms of arbitrary finite even number of free bosonic fields for both the (classical) infinite-dimensional Lie algebra $\mathbf{W}_{1+\infty}$, as well as for the (classical) infinite-dimensional nonlinear algebra $\widehat{\mathbf{W}}_{\infty}$.

(C4) We study systematically the relation between the continuum integrable hierarchies of KP type and the discrete lattice integrable hierarchies of Toda type. It is shown thatDarboux-Bäcklund transformations generating the soliton solutions of the KP-type hierarchies are triggered by the lattice equations of motion for the corresponding Toda-Type discrete hierarchies.

(C5) We propose a new general method for description of integrable KP-type hierarchies - the *squared eigenfunction potential method*. We show its equivalence to the standard approaches (Lax operator formalism, bilinear Hirota identities, *etc*). The concept of *squared eigenfunction potential* turns out to be crucial in providing description of the reduced $\mathsf{cKP}_{R,M}$ hierarchies in the language of universal Sato Grassmannians, finding the explicit form of the additional non-isospectral symmetries and constructing of new *generalized binary* Darboux-Bäcklund transformations whose orbits are shown to correspond to a new class of Toda models on *square* lattices.

(C6) We propose for the first time in the literature and systematically study the properties of a new class of *supersymmetric* reduced KP hierarchies $SKP_{\frac{r}{2},\frac{m}{2}}$ – the supersymmetric generalization of the ordinary bosonic $\mathsf{cKP}_{R,M}$ hierarchies. We formulate a *supersymmetric Darboux-Bäcklund* scheme for a systematic generation of supersymmetric soliton-type solutions, where the super-Darboux-Bäcklund orbit lies on a *supersymmetric Toda* lattice.

(C7) We present for the first time in the literature the full systematic description of all additional non-Hamiltonian (non-isospectral) symmetries of KP-type integrable hierarchies, including their multi-component (matrix) generalizations as well as their supersymmetric extensions. The key result is the non-trivial modification of the standard additional symmetry flows in the ordinary full (unconstrained) KP hierarchy so that they become compatible with the constraints in the reduced KP hierarchies.

(C8) We discover a new type of additional non-isospectral symmetries in the reduced KP hierarchies which span non-trivial infinite-dimensional algebras of Kac-Moody type. This result is generalized for the case of supersymmetric KP hierarchies where the corresponding new additional non-isospectral symmetries span super-Kac-Moody algebras.

(C9) We show that the multicomponent (matrix) KP hierarchies may be viewed as ordinary scalar KP hierarchies supplemented with the Cartan subalgebra of the additional symmetry algebra. Thanks to this newly discovered equivalence we use the standard Darboux-Bäcklund techniques for scalar KP hierarchies to obtain explicitly soliton-type solutions for the matrix KP hierarchies in terms of a new kind of multi-Wronskian tau-functions.

(C10) We show that a number of nonlinear systems of equations with applications in nonlinear hydrodynamics, nonlinear optics and plasma physics are contained as particular special subsystems within the constrained $\mathsf{cKP}_{R,M}$ hierarchies, namely, they arise as particular additional symmetry flows.

(C11) We find explicitly supersymmetric soliton-type solutions (super-tau-functions) of the supersymmetric KP hierarchies in terms of special Berezinians (super-determinants) of Wronskian type which possess the crucial additional property of preserving the non-isospectral symmetries.

(C12) As a byproduct we find an interesting physical interpretation of a new class of tau-functions (soliton-type solutions) of reduced KP hierarchies in the condensed matter language as joint distribution functions resulting from new types of random matrix models where a two-particle *attractive* potential appears between the corresponding quasi-particles.