New Integrable RG Flows with Parafermions

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Based on

Z. Bajnok & CA [arXiv:2407.06582]

Integrability, Q-systems and Cluster Algebras, Varna, 2024

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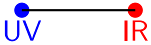

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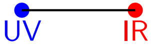

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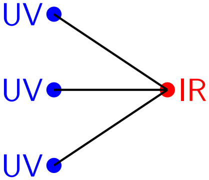

Can we classify all possible 2D QFTs which flow into a given IR CFT?

Plan of Talk

1. New Approach [LeClair & CA (2022)]
   - Classify UV CFTs connected to an IR CFT
   - But only UV CFTs can be identified
2. Identifying QFTs [Bajnok & CA (2024)]
   - Focus on the unitary minimal CFTs $\mathcal{M}_p$
   - Find and identify all QFTs which flow into this
3. Concluding remarks

PART 1. RG flows from IR to UV

Conventional Approach (top down)

- UV CFT+a relevant field $\longmapsto$ IR CFT+ irrelevant fields
    - Only a special relevant field maintains the integrability
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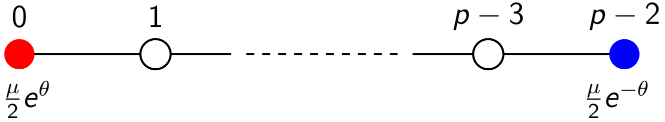

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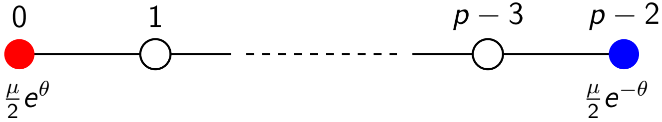

- Many more RG flows have been conjectured
  - have been guessed based on conjectured TBA or NLIE
  - Lagrangians / exact S-matrices are missing

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- Use $T\overline{T}$ as a ladder

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- Exact results possible for even non-integrable theories

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- Exact S-matrices are given by additional CDD factors

$$\mathsf{S}(\theta) = \prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s} \sinh(s\theta)} \cdot \mathsf{S}_0(\theta)$$

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- Singularity occurs also for each $[T\overline{T}]_s$ with $s \geq 1$
- We show that the singularities can be avoided if we fine-tune all α_s

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  - S-matrices between RL, LR are trivial since $\theta_{12} \to \pm\infty$

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- Summary

$$\boxed{\mathsf{S}^{RR}(\theta) = \mathsf{S}^{LL}(\theta) = \mathsf{S}_0, \; \mathsf{S}^{RL}(\theta) = \mathsf{S}^{LR}(-\theta) = \mathsf{S}_{\mathsf{CDD}}(\theta)}$$

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- We can work with UV limit of TBA, namely, "plateaux equations"

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- But central charges are **not enough** to identify the QFTs

PART 2. Identifying UV complete QFTs

with Z. Bajnok

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- S-matrix of kinks $S^{[\mathrm{p}]}_{\mathrm{RSOS}}(\theta)$: non-diagonal [Bernard, LeClair (1990)]

$$K_{ab}(\theta_1) K_{bc}(\theta_2) \quad \to \quad K_{ad}(\theta_2) K_{dc}(\theta_1)$$

$$\mathsf{S}_p(\theta)^{ab}_{dc} = U(\theta)\, (X^{ac}_{db})^{\frac{i\theta}{2\pi}} \left[(X^{ac}_{db})^{\frac{1}{2}} \sinh\left(\frac{\theta}{p}\right) \delta_{db} + \sinh\left(\frac{i\pi - \theta}{p}\right) \delta_{ac} \right]$$

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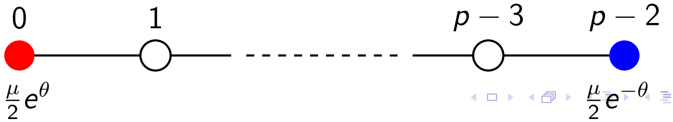
- Derived the conjectured TBA (partially) shown before

$$\epsilon_a(\theta) = \frac{\mu R}{2}(\delta_{a0} e^{\theta} + \delta_{a,p-2} e^{-\theta}) - \sum_{b=0}^{p-2} \mathbb{I}_{ab} \varphi \star \log[1 + e^{-\epsilon_b}](\theta)$$



TBA for $[T\overline{T}]_s$ deformed minimal CFT $\mathcal{M}_p$

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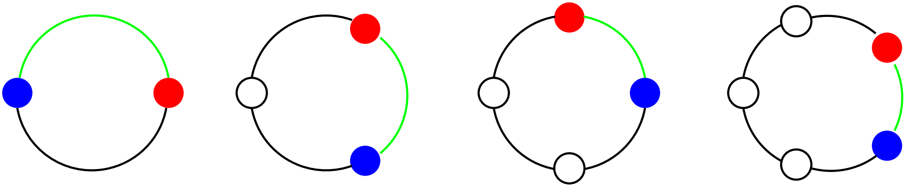
- Being diagonal, $\mathsf{S}_{\mathrm{CDD}}$ introduces additional kernels between R & L



UV complete theories

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- Plateaux equations

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\begin{aligned}
x_n &= (1+x_{n-1})^{1/2}(1+x_{n+1})^{1/2}, \quad n = 1, \cdots, p-3, \\
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\mathrm{S}^{(1)}_{\mathrm{CDD}} = -\tanh\left(\frac{\theta-\beta}{2} - \frac{i\pi}{4}\right) \quad \rightarrow \quad \varphi_{RL} = \frac{1}{\cosh(\theta-\beta)}
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- We also notice that only real β ($p>3$) becomes UV complete and

$$c_{\text{eff}} = 3\,\frac{p-1}{p+1}$$

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$$\mathcal{L} = \mathrm{Kin.} - \frac{i}{2}\psi\overline{\psi}W''(\phi) + 2\pi[W'(\phi)]^2, \quad W(\phi) = -\mu \sinh(b\phi)$$

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$$\begin{aligned} \epsilon^R(\theta) &= \frac{\mu R}{2} e^{\theta} - \varphi^{RL} \star \log[1 + e^{-\epsilon^L}], \\ \varphi^{RL}(\theta) &= \frac{1}{\cosh(\theta - ia)} + \frac{1}{\cosh(\theta + ia)}, \quad a = \frac{1 - b^2}{1 + b^2} \end{aligned}$$

Two theories match when $\beta = 0$ and $a = 0$

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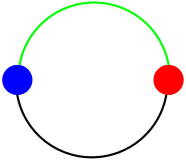

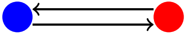

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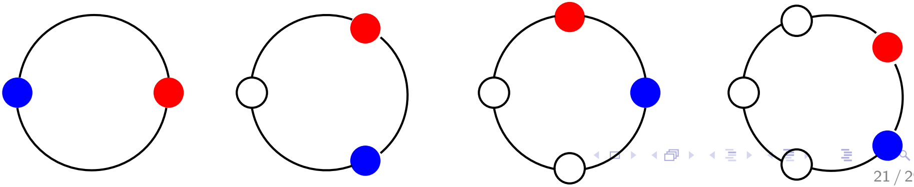

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$$\mathcal{L}_{\mathrm{PShG}} = \mathcal{L}_{\mathrm{PF}} + \frac{1}{4\pi}(\partial_\mu \phi)^2 - \kappa \left(\psi_1 \overline{\psi}_1 e^{2b\phi} + \eta\, \psi_1^\dagger \overline{\psi}_1^\dagger e^{-2b\phi} \right) + \cdots$$

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- These types of QFTs compute c_{eff} from momentum quantization using Reflection amplitudes

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$$e^{2(\frac{Q}{2}+ip)\phi} = \mathcal{R}(p)e^{2(\frac{Q}{2}-ip)\phi}, \qquad \alpha = \frac{Q}{2}+ip,\ Q = b+\frac{1}{kb}$$

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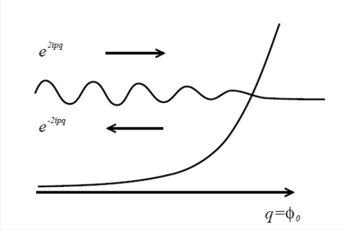

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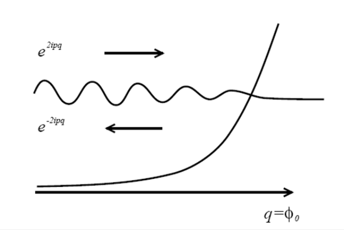

- This amplitude has been computed in the PLFT [Baseilhac, Fateev (1998)]

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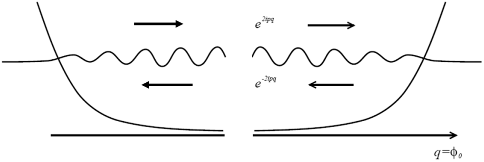

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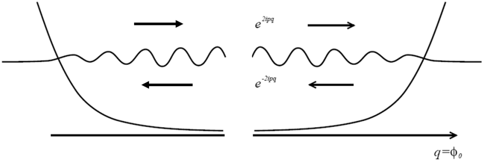

- Scaling function can be obtained from

$$\begin{aligned} c_{\mathrm{eff}} &= \frac{3k}{k+2} - 24p^2 + \mathcal{O}(R) \\ &= \frac{3k}{k+2} - \frac{3\pi^2}{2Q^2 \ln^2 x} - \frac{3\pi^2 \delta_1}{4Q^3 \ln^3 x} - \frac{9\pi^2 \delta_1^2}{32Q^4 \ln^4 x} - \frac{3(2\pi^2\delta_1^3 + \pi^4 \delta_3)}{64Q^5 \ln^5 x} + \ldots \quad (x = \tfrac{R}{2\pi}) \end{aligned}$$

Reflection amplitude vs. massless TBA

- Need to solve numerically with high precision

$$c_{\text{eff}}(r) = \frac{3k}{k+2} + \sum_{n=2} \frac{c_n(k)}{(\log r)^n} + O(r)$$

- Two match very well at self-dual coupling $b = \frac{1}{\sqrt{k}}$

	c_2	c_3	c_4	c_5
$k=2$	7.402199	42.7620	185.218	714.563
	7.402203	42.7628	185.282	717.247
$k=3$	11.10332	77.8573	409.598	1924.84
	11.10330	77.8543	409.425	1919.36
$k=4$	14.8045	119.428	722.797	3898.75
	14.8044	119.4197	722.475	3892.55

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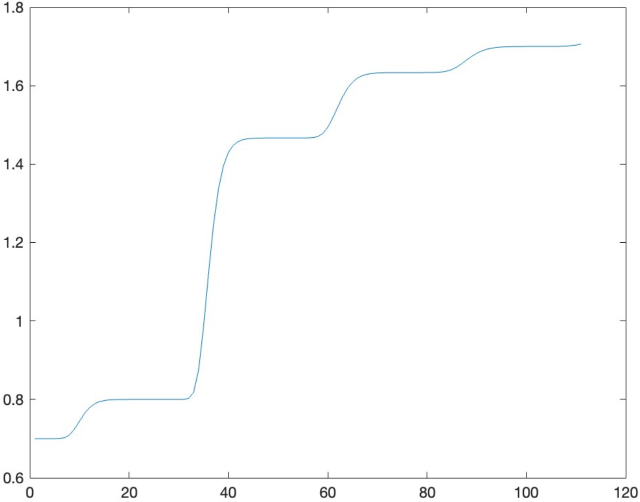

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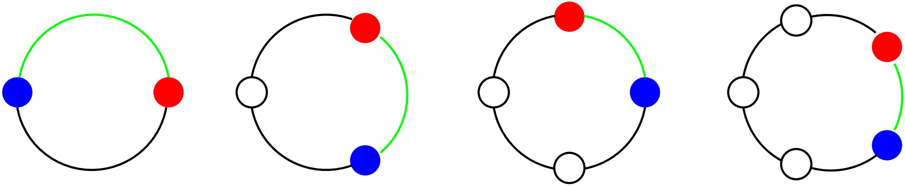

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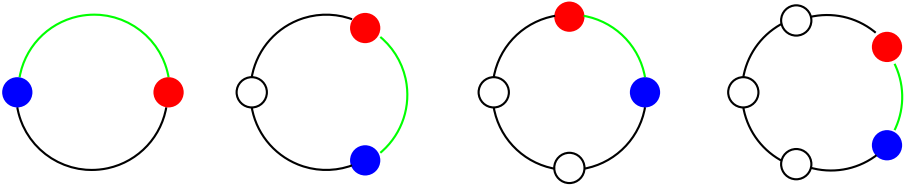

- Each plateaux corresponds to

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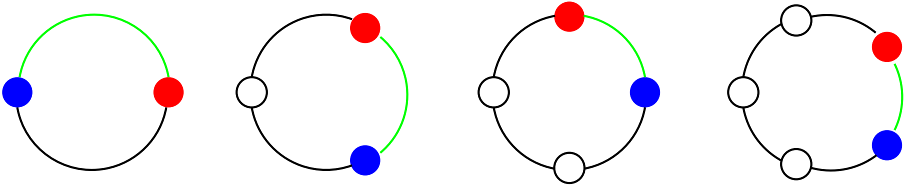

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- Reproduce PF staircase TBA conjectured by [Dorey, Ravanini (1993)]

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- Connection with non-invertible symmetries

Thank you for attention!